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On the Dynamic Markov–Dubins Problem: from Path Planning in Robotics and Biolocomotion to Computational Anatomy

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Abstract—Andrei Andreyevich Markov proposed in 1889 the problem (solved by Dubins in 1957) of finding the twice continuously differentiable (arc length parameterized) curve with bounded curvature, of minimum length, connecting two unit vectors at two arbitrary points in the plane. In this note we consider the following variant, which we call the *dynamic Markov–Dubins problem (dM-D)*: to find the *time-optimal* C^2 trajectory connecting two velocity vectors having possibly *different norms*. The control is given by a force whose norm is bounded. The acceleration may have a tangential component, and corners are allowed, provided the velocity vanishes there. We show that for almost all the two vectors boundary value conditions, the optimization problem has a smooth solution. We suggest some research directions for the dM-D problem on Riemannian manifolds, in particular we would like to know what happens if the underlying geodesic problem is completely integrable. Path planning in robotics and aviation should be the usual applications, and we suggest a pursuit problem in biolocomotion. Finally, we suggest a somewhat unexpected application to "dynamic imaging science". Short time processes (in medicine and biology, in environment sciences, geophysics, even social sciences?) can be thought as tangent vectors. The time needed to connect two processes via a dynamic Markov–Dubins problem provides a notion of distance. Statistical methods could then be employed for classification purposes using a training set.

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1. INTRODUCTION

In 1889 Andrei Andreyevich Markov posed the following question [43]: given two unit vectors in the plane, find the shortest piecewise C^2, continuously differentiable curve $\gamma(s)$ with $|d\gamma/ds| \equiv 1$ (so s is the arc length) connecting them, under the restriction that the curvature (defined everywhere except for exceptional points) is bounded:

$$\kappa = |\gamma''(s)| \leqslant R^{-1} . \tag{1.1}$$

Lester Dubins showed in 1957 that the solution consists "of not more than three pieces, each of which is either a straight line segment or an arc of a circle of radius R" [17]. There is nowadays a vast literature in the theme. We just mention a few references, firstly with respect to theoretical developments: Boissonnat, Cérézo and Leblond, allowed solutions with cusps [10]; for the M–D problem on the sphere and hyperbolic plane, see Monroy–Pérez and Mittenhuber [44, 46, 47].

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Following up on work by Sussmann, Mittenhuber showed that Dubin's problem is intrinsically three dimensional [45, 55].

Concerning to applications in robotics and locomotion inspired by the Markov–Dubins problem, we refer to Thomaschewski [58] where many references until c. 2000 are given. For a few recent ones, see [18] for rotational (besides positional) control in the plane. Recently, motivated by path planning of unmanned aereal vehicles [59], several Markov–Dubins versions of the "traveling salesman problem" (visiting a set of locations, in order or not) have been studied, see e.g. [48, 53].

Contents of the paper. The main results are summarized in a Theorem on section 1.4. The dynamic Markov–Dubins Hamiltonian in Euclidean spaces are described in Propositions 1–4. Proposition 5 is just a motivation, describing the Fermat-Huygens' geometric optics via control theory. The same ideas allowed us to obtain the Dynamic Markov–Dubins Hamiltonian on an embedded surface, Proposition 6. Proposition 7 presents the dynamic M–D equations in coordinates. For the simplest "case study", the Euclidian metrics, we show that the time-optimal problem with specified initial and terminal velocity vectors is easily solved by quadratures. We present some basic considerations about the dynamic Markov–Dubins problem on manifolds, avoiding the differential geometric machinery. Several research directions are suggested, interspersed in the text and in the final comments. Besides the natural applications in path planning for locomotion and robotics[1)], we we suggest a potential application in imaging science that we hope could attract interest.

1.1. Dynamic Markov–Dubins Problem on the Plane

In a sense the Markov–Dubins problem is purely geometric. Hence is seems reasonable to consider a *dynamic* modification. Namely, to find a *minimum time* trajectory $\gamma(t)$ connecting two velocity vectors having in general *different norms*. Here the control is in the acceleration vector. The direction of the control force at the animal or engineer's disposal is arbitrary, but its intensity is bounded, say by 1. We refer to this as the dynamic version of Markov–Dubins problem (for short, dM–D). *It is allowed (and in fact to the planner's advantage) to make use also of the tangential acceleration.*

Following [10], the class of curves we consider are parameterized curves that are piecewise C^2 with respect to a time parameter. *At the exceptional points the velocity vector must vanish* so that these admissible curves are still C^1 at these points, even though *corners with variable angles are allowed.* This choice seems reasonable, because of a trade-off: *there is a higher cost in time due to the reduction of velocity near a corner, in exchange for the ability to change direction abruptly.*

Definition. *We denote this class $C^{3/2}$ (we apologize for a strange notation).*

Sussmann studied a simplified dynamic case where the scalar velocity remains constant, the control being in the curvature rate of change [56]. Although the planar dynamic Markov–Dubins problem seems quite natural, the only authors that we have found are Aneesh and Bhat [1–3, 8]. They found some explicit solutions for the planar dM–D problem, both with time optimal or energy dissipation criteria. They considered however *special cases where the terminal direction is specified, but not the terminal position.*

1.2. Dynamic M–D Problem on Manifolds

Since the pioneering work by Jurdjevic [33, 34], several authors have shown how to extend Pontryagin maximum principle to manifolds [7, 13, 16]. The dM–D problem makes sense on any Riemannian manifold (Q, g). The acceleration is given by the Levi–Civita affine connection D of g:

Definition. *The dynamic Markov–Dubins problem on manifolds (for curves in the class $C^{3/2}$):*

$$T(v_{q_o}, v_{q_1}) = \min \int dt, \quad \text{subject to} \quad D_{\dot\gamma}\,\dot\gamma = \; u \in T_\gamma\, Q, \;\; |u| \leqslant 1, \quad \dot\gamma(0) = v_{q_o},\, \dot\gamma(T) = v_{q_1}. \qquad (1.2)$$

[1)] A commercially oriented mathematician could patent the dynamic Markov–Dubins curves for race tracks designs. Actually, in Markov 1889 paper it was already suggested an application to railroads.

The dynamic M-D problem fits into the class of *mechanical control problems*, consisting of a Lagrangian of mechanical type $T - V$ (nonholonomic constraints may also be present). To control the system the engineer or organism applies a force (abstractly speaking, a semi-basic 1-form from an available control set $\mathcal{C} \subset TQ$),

$$\frac{d}{dt}\,\partial L/\partial \dot{q} - \partial L/\partial q = \lambda \in \mathcal{C}. \tag{1.3}$$

Using the metric to lower the indices, the Euler–Lagrange equations become a spray in $T(TQ)$. This is physically appealing since the control force will then be represented as a vectorfield $u(t)$ along the solution curve $\gamma(t)$ as in (1.2). In this setting, it turns out that it becomes quite a challenge to present an intrinsic formulation of Pontryagin's maximum principle. We refer to Lewis [41], where he considers, more generally, as the drift system a geodesic spray of an arbitrary *affine connection* [35].

The terminology *affine (mechanical) control systems* was introduced by Lewis and Murray [36, 38–40] in the mid 1990's. In these references one can find the main properties of such systems from the control viewpoint. The most natural affine connection is of course the Levi–Civita's. This terminology is a bit misleading, since the world "affine" in control theory usually means any drift term. In our case, the drift is the spray of an affine connection.

New approaches have been recently applied to the study of control systems of mechanical type. The inverse problem of characterizing when a generic control system $\dot{x} = f(x) + \sum_i u_i g_i(x)$ is equivalent to a mechanical control system was posed by Lewis [37] and addressed recently by Ricardo and Respondek [50–52] using differential geometric techniques (reminiscent of Cartan's equivalence method). The general properties of control problems of mechanical type have been studied by several authors, see Barbero and Munoz–Lecanda [5, 6], and Lewis and Bullo [13] for the intrinsic approach that we do not pursue here.

1.3. Imaging Science: Classification of Fast Processes

Computational anatomy has flourished in the last fifteen years, after this research area was inaugurated by the work of Grenander and Miller [20]. Theidea for this program is to define suitable Riemannian metrics on an (infinite dimensional) space of embeddings (modulo reparameterizations and re-scalings). The main task is to find the length of the geodesic that provides the "metamorphosis" path connecting two images[2)]. In practice, finite dimensional approximations are used, for instance, matching some relevant *landmarks* [9]. Combined with tools of machine learning, one can use a training set from a database for classification purposes. For the state-of-the-art see Holm, Trouvé and Younes [26]. Influential geometers working in the area include D. Mumford, D. Holm, T.Ratiu, and P. Michor.

The novel point we want to call the attention here is that *processes* that evolve in short time are common in biology and medicine (for just two recent references in embryology, see [42, 60]). Dynamic images can be thought as tangent vectors on an abstract manifold. How can one compare two such processes? The dynamic Markov–Dubins problem could provide one such criterion. Applications of dynamic imaging and dynamic signal processing to other areas such as geophysics (earthquake warning) and environmental sciences (deforestation, ice melting) are also conceivable.

1.4. Main Results

We anticipate the most relevant results. Let (x, v, y, z) be coordinates in the cotangent bundle of the tangent bundle of a manifold Q. Let $\Gamma^k_{ij}(x)$ be the Christoffel symbols of the Levi–Civita connection associated to a Riemannian metric g in Q.

[2)] The Center for Imaging Science (http://cis.jhu.edu/) at John Hopkins University Medical School, leaded by M. Miller, has developed a software for these large scale diffeomorphic deformations, and the Medical Image Computing and Computer Assisted Intervention Society (The MICCAI Society, http://www.miccai.org/) has organized a series of conferences `http://www-sop.inria.fr/asclepios/events/MFCA11/`.

Theorem. *Coordinate version:*
i) The optimal Hamiltonian for the dynamic Markov–Dubins problem is

$$(H^*(x,v,y,z)\,,\,\Omega = \sum_i dy_i \wedge dx_i + dz_i \wedge dv_i) \tag{1.4}$$

with

$$H^*(x,y,v,z) = -1 + (y,v) + \sqrt{(z,G^{-1}z)} - \sum_{k,i,j} z^k\ \Gamma^k_{ij}(x)v_iv_j. \tag{1.5}$$

The state equations are

$$\dot{x}_k = v_k,\ \ \dot{v}_k = -\sum_{i,j} \Gamma^k_{ij}v_iv_j + u^*_k,\ \ u^*_k = \frac{\sum_i g^{ik}(x)z_k}{\sqrt{(z,G^{-1}z)}} \tag{1.6}$$

and the costate equations $\dot{y}_j = -\frac{\partial H}{\partial x_j}$, $\dot{z}_j = -\frac{\partial H}{\partial v_j}$ *are given more explicitly by*

$$\dot{y}_j = \sum_{k,m,n} z_k \frac{\partial \Gamma^k_{mn}}{\partial x_j}(x)\, v_m v_n \ \ ,\ \ \dot{z}_j = -y_j + \sum_{k,i} z_k \Gamma^k_{ij}(x)v_i. \tag{1.7}$$

ii) Extrinsic version (for hypersurfaces). Let

$$\Sigma^n : g(x) \equiv 0\ ,\ \ g : \Re^{n+1} \to \Re$$

an hypersurface and let $B(x)$ *the Hessian matrix of* g. *Then the dynamic Markov–Dubins Hamiltonian in the variables* $(x,v,p_x,pv) \in \Re^{4(n+1}$ *(there are four automatic Casimirs) is given by*

$$H^* = -1 + (p_x,v) - \frac{(B(x)v,v)}{(\nabla g(x),\nabla g(x))}(p_v,\nabla g(x)) - |p_v - \frac{(p_v,\nabla g(x))}{(\nabla g(x),\nabla g(x))}\nabla g(x)|\ . \tag{1.8}$$

2. SOLUTION OF THE DYNAMIC MARKOV–DUBINS PROBLEM IN EUCLIDEAN SPACES

2.1. Mathematical Set Up

It is convenient to describe the original (geometrical) M–D problem as a dynamic problem on the class $C^{3/2}$ of time parametrized curves with *cusps allowed*, in the spirit of [10]:

$$\min T = \int dt \quad \text{subject to} \quad \dot{x} = v\ \ ,\ \ \dot{v} = u \in \Re^2\ ,\ \ |u| \leqslant R^{-1}\ \ ,\ \ \text{and}\ \ \ u \perp v \tag{2.1}$$

and boundary conditions

$$x(0) = x_o\ \ ,\ \ \dot{v}(0) = v_o\ ,\ (|v_o| = 1)\ \ \ ,\ \ \ x(T) = x_1\ \ ,\ \ v(T) = v_1\ ,\ (|v_1| = 1)\,. \tag{2.2}$$

The condition $u \perp v$ guarantees that $|v(t)| \equiv 1$ at all times, so that time optimal, length optimal, and energy optimal problems all coincide. Introduce adjoint momenta p_x, p_v of Pontryagin's method. In order to obtain the optimal Hamiltonian H^*, we need to maximize the function

$$H = -1 + (p_x,v) + (p_v, iv\alpha)$$

with respect to the real valued control variable α, $|\alpha| \leqslant R^{-1}$. It is readily seen that

$$(p_v, iv) >< 0\ \ \Rightarrow\ \ \alpha = \pm R^{-1} \quad \text{(arcs of circle)}$$

while the case $(p_v, iv) = 0$ requires a detailed scrutiny, since the solution may remain on a straight line for a while. The analysis is subtle. It is necessary to examine carefully the adjoint differential equations, and one obtains extra solutions with cusps in addition to Dubins', see [10].

The point we want to stress is this: dropping out the condition $u \cdot \dot{\gamma} = 0$ provokes a huge difference — the acceleration can now retain a tangential component. This feature changes dramatically the character of the solutions.

Since the control must remain the unit disk, Pontryagin's principle leads us to maximize with respect to $u \in D(0,1)$ the function

$$H = -1 + (p_x, v) + (p_v, u).$$

Proposition 1. *Dynamic Markov–Dubins Hamiltonian in Euclidean spaces (see Proposition 6 below for embedded hypersurfaces and Proposition 7 for the expression in coordinates for a general metric). As long as $p_v \neq 0$, we have the governing Hamiltonian*

$$H^* = -1 + (p_x, v) + |p_v| \tag{2.3}$$

and the equations of motion in $\{(x, v, p_x, p_v) \in \Re^{4n}\ ,\ p_v \neq 0\}$ are

$$\dot{x} = v,\ \ \dot{v} = p_v/|p_v|,\ \ \dot{p_x} = 0,\ \ \dot{p_v} = -p_x. \tag{2.4}$$

Note that H^* is not smooth for $p_v = 0$, and (2.4) is not defined for this value. Singular situations are common in control theory, and often produce drastic dynamical consequences. Here *for a generic initial condition the solutions remain smooth for all time.* We will shortly show that the set of initial conditions for which $p_v(t_*) = 0$ at certain transition times t_* has codimension $n(n-1)/2$.

Let us describe a (very) special situation where $p_v(t_*) = 0$ at a certain time instant. Suppose that $v_o, v_1, x_1 - x_o$ in (2.2) are all parallel. In this case we get a classic textbook problem ("landing in the Moon" or "controlling the position and speed of a train"). It is well known that in this 1-dimensional case the control is bang-bang, $u = \pm 1$ with a single transition time (see, eg. Intriligator [19]). On a complete Riemannian manifold one may have the same special 1-d control subproblem along any geodesic. We discuss the effect of the global topology in section 3.1.

Remark. Other cost functionals. Special solutions. Aneesh and Bhat [1–3, 8] considered cost functionals depending on the velocity,

$$\int |v|^\kappa\, dt\ ,\ \ \kappa \in [0, \infty)\ . \tag{2.5}$$

These functionals may be interpreted as energy dissipation on a viscous medium. They considered prescribed initial position and velocity, a prescribed terminal velocity but *free terminal position.* In this special case, they verify that $p_x(t) \equiv 0$ and they found an additional integral of motion. Analytic solutions are obtained for all κ. They observe that for this terminal condition the time optimal problem ($\kappa = 0$) is trivial: one just takes the maximum possible (constant) acceleration opposite to the undesired velocity component.

There are other natural cost functionals depending on the control (minimizing internal energy expenditure), say

$$\int |u|^\ell\, dt\ ,\ \ \ell \in [0, \infty)\ . \tag{2.6}$$

For $\ell > 0$ one may (or not) assume $u \in B(0,1)$. These functionals can also be combined with (2.5).

Remark. A general comment about reachability. We can use the one dimensional bang-bang solutions to argue that on a (geodesically) complete manifold, any vector v_{q_2} can be reached from v_{q_2} by a $C^{3/2}$ curve with constrained acceleration. The following choice is very sub-optimal. Along the trace of the geodesic determined by v_{q_1} consider the bang-bang solution so that it *soft lands* at q_1 itself (use all the control to decrease the speed of $q(t)$ until it stops and reverses direction, its speed starts to increase again; at a certain transition time we reverse the control so that it will soft land at q_1). Along the geodesic trace from q_1 to q_2 it is also easy to find the minimum time solution connecting these points with zero velocity at the end points. Finally, along the geodesic determined by v_{q_2}, starting with zero velocity at q_2 find the 1-d solution with final condition v_{q_2} .

2.2. Time Optimal Problem: Integrability by Quadratures

Here we will consider the time optimal problem ($\kappa = 0$ in (2.5)) for the *full* two point boundary values. We show that the problem is integrable, requiring only elementary functions. Although the integration is straightforward, the solutions curves have interesting features. It is readily seen that the solution $T(x_o, v_o; x_1, v_1)$ must satisfy the Hamilton–Jacobi–Bellman equation

$$-1 = (\partial T/\partial x_o) \cdot v_o - |\partial T/\partial v_o| \tag{2.7}$$

or, if one thinks backwards,

$$1 = (\partial T/\partial x_1) \cdot v_1 - |\partial T/\partial v_1|.$$

Equations (2.4) keep the same format in any Euclidean space $\Re^n$:

$$\ddot{x} = z/|z| \ , \ \ \ddot{z} = 0, \tag{2.8}$$

where $z(= p_v) \in \Re^n$ is now an auxiliary variable. These equations are readily integrated by quadratures. Let $p_x(t) \equiv y_o$.

$$z = z_o - y_o t \ \ \Rightarrow \ \ u(t) = \tfrac{z_o - y_o t}{|z_o - y_o t|}$$

$$v(t) = v_o + \textstyle\int_0^t \tfrac{z_o - y_o t'}{|z_o - y_o t'|}\, dt' \ \ , \ \ x(t) = x_o + \textstyle\int_0^t v(t') dt' \tag{2.9}$$

so, in summary

Proposition 2. *The solution for the dynamic Markov–Dubins equation in Euclidean space is*

$$x(t) = x_o + v_o t + \int_{t'=0}^{t} \int_{t''=0}^{t'} \frac{z_o - y_o t''}{|z_o - y_o t''|}\, dt'' dt'. \tag{2.10}$$

2.3. General Features

It is clear that $p_v = z_o - y_o t$ never vanishes for most solutions. The explicit formulae for the generic smooth solutions are given in (2.15) below. On a solution curve, p_v may vanish at most once, and this happens only when $y_o || z_o$. Notice that in these special solutions, the component of the initial velocity v_o that is perpendicular to the direction $\ell : [y_o || z_o]$ remains constant.

If this component is nonzero, in the planar case $\Re^2$ it follows that the direction ℓ of the acceleration vector cannot change at t_*. In other words, the component of the acceleration vector in the direction of ℓ reverses. *These solutions are formed by two concatenated parabolas with the same axis direction, but with opposite concavities* (looking as a "syphon"). As $|t| \to \infty$, the scalar velocity increases essentially at a linear rate, $|v| = O(|t|)$. The interesting dynamics occurs in the "middle" region.

What happens when the component of the velocity vector perpendicular to the direction ℓ vanishes (or in the three dimensional dM–D problem)? In this case one must examine the possibility of special solutions consisting of two semi-lines meeting at a corner. For these solutions the velocity vector would decreases linearly, vanishing at the corner, where the trajectory would suffer a sudden arbitrary change of direction. However, we show in the appendix that obtuse angles greater than 126.8 degrees can be short routed by a circular arc; wedges with angles greater than 120 degrees can be short routed by a paraboloidal arc, but circles can be traversed faster. Therefore, we may conjecture that semi-lines with angles less than 120 degrees are extremals. We make this query explicit in the following

Problem 1. *Special solutions.*

1. *How the general smooth solutions* (2.15) *in* R^8 *approach those special ones as the initial conditions approach the singular invariant set* $\mathcal{S}$*?*

2. *Recall our choice of the* $C^{3/2}$ *class of admissible curves. Are the semi-lines for angles smaller than 120 degrees true solutions?*

3. *We suspect that circles of unit radius 1 with uniform angular velocity $\omega = 1$ minimize the time of transfer between the tangent vectors. Are other abnormal solutions present?*

At any rate, all these special solutions live on an invariant set $\mathcal{S}$ of codimension $n(n-1)/2$ given by $p_v = \lambda p_x,\ p_x \neq 0$, together with $p_x = 0,\ p_v \neq 0$. p_v is arbitrary ($\neq 0$), and remains constant. The latter consist of straight line solutions, without a transition time.

We finish this section with a comment about the asymptotic behavior. Provided $y_o \neq 0$, the direction of the acceleration vector

$$u(t) = \frac{z_o - y_o t}{|z_o - y_o t|}$$

for times $t \to \pm\infty$ asymptotically reverses,

$$u(\mp\infty) = \pm y_o/|y_o|\ .$$

This observation tempts one to predict that the smooth solutions of (2.10) tend, asymptotically, to paraboloidal arcs for $t \to \pm\infty$, both axis aligned with $p_x = y_o$. But it is not quite so. These asymptotic parabolas drift logarithmically in time, see (2.14) below.

2.4. Explicit Solution with Elementary Functions

The integrals in (2.9) and (2.10) have the same form for all dimensions n. Every component of the integrand for $v(t)$ in (2.9) is a linear function of t divided by the square root of a quadratic polynomial $(y_o, y_o)t^2 - 2(z_o, y_o)t + (z_o, z_o)$. Its discriminant is $\leqslant 0$, vanishing only in the special cases alluded above. These integrals are elementary, involving that square root, a logarithm and multiplication by t. The second quadrature for the components of $x(t)$ also remains in the field of elementary functions.

Moreover, it suffices to do the cases of $n = 2$ and $n = 3$. The reason is that solutions remain for all times in the 3-dimensional affine subspace of $\Re^n$ with origin placed at x_o and generators $x_1 - x_o, v_o, v_1$, as in [45]. To simplify the description below, we will use convenient parameters. The price to pay in matching the boundary value problem is that the initial conditions (x_o, v_o) will not be immediate.

In the two dimensional case there will be *seven* parameters

$$A_1, A_2, B_1, B_2, \theta, t_o, a.$$

Adding the time t we have *eight* quantities to match the boundary conditions. The parameter θ refers to a rotation angle, t_o to a time shift, and a refers to a velocity parameter. In the three dimensional case, there will be twelve quantities, among which three Euler angles.

Let us now describe the general smooth solution. After a rotation, a dilation (we only need $z/|z|$) and a time shift, we may assume that

$$z(t) = (1, at)\ ,\ \ a \neq 0\,.$$

We observe that can do this for any R^n, with the extra components (for $n \geqslant 3$) set to zero. The elementary integrals we need are

$$\begin{aligned}
f(t;a) &= \int_o^t \frac{1}{\sqrt{1+a^2t^2}}\,dt = \frac{\sinh^{-1}(at)}{a},\\
F(t;a) &= \int_o^t f(t)\,dt = \frac{t\sinh^{-1}(at)}{a} - \frac{\sqrt{1+a^2t^2}}{a^2} + \frac{1}{a^2},\\
g(t;a) &= \int_o^t \frac{at}{\sqrt{1+a^2t^2}}\,dt = \frac{\sqrt{1+a^2t^2}}{a} - \frac{1}{a},\\
G(t;a) &= \int_o^t g(t)\,dt = \frac{at\sqrt{1+a^2t^2} + \sinh^{-1}(at)}{2a^2} - \frac{t}{a}.
\end{aligned} \tag{2.11}$$

We solve first for the special trajectories with

$$\ddot{x}_1 = \frac{1}{\sqrt{1+a^2t^2}}, \ \ \ddot{x}_2 = \frac{at}{\sqrt{1+a^2t^2}}. \tag{2.12}$$

Integrating once,

$$\dot{x}_1 = f(t) + \text{const}, \ \ \dot{x}_2 = g(t) + \text{const}. \tag{2.13}$$

Since f is odd and g is even, and both tend to infinity as $t \to \infty$, it is easy to depict the qualitative features of the solution curves $(x_1(t), x_2(t))$.

Proposition 3. *Qualitative description (depending on the values of the constant of integration. The first component x_1 always turns around, but only once (say at $t = a$). The second component x_2 does not turn around or does it twice (say at $t = b, c$), depending on the values of the constant of integration. Overall, there are four types of solutions, three of them depending on the position of a relative to the interval (b, c), and a forth (U-turn type) if the interval does not exist. Typically a solution therefore will have one or two "meanders".*

Remark. For large $|t|$ the dominant terms are

$$x_1 \sim t \sinh^{-1}(at)/a \sim (|t|/a) \ln |t| \ , \ \ x_2 \sim t|t|/2. \tag{2.14}$$

With a slight poetic licence, we may say that each branch behaves as an elusive "virtual parabola". The curvature at the virtual vertex decreases inversely to $\ln(|t|)$.

Integrating (2.13) again, we get the general solution for $n = 2$ is given by this set of eight equations

Proposition 4. *Explicit solution.*

$$\begin{aligned} v_1(t) &= B_1 + \cos(\theta) f(t; a) - \sin(\theta) g(t; a), \\ v_2(t) &= B_2 + \sin(\theta) f(t; a) + \cos(\theta) g(t; a), \\ x_1(t) &= A_o + B_1(t - t_o) + \cos(\theta) F(t; a) - \sin(\theta) G(t; a), \\ x_2(t) &= A_2 + B_2(t - t_o) + \sin(\theta) F(t; a) + \cos(\theta) G(t; a). \end{aligned} \tag{2.15}$$

Note that the rotation and time shift were undone after the second quadrature.

The following task is in order for practical use: to implement a numerical method to match the eight boundary values (2 position and two velocity vectors).

Problem 2. *What would be a reasonable initial guess so that an iteration converges? (One possibility could be to project the velocities in the direction of the vector joining the base points and use de one dimensional solution.) The minimal time $T(x_o, v_o; x_1, v_1)$ is $t - t_o$, defined implicitly by this set of eight nonlinear equations. Is it possible to check Bellman's equation (2.7)? (This would probably require a tour-de-force with implicit differentiations.) What can be said about the properties of the map from the space of parameters $(A_1, A_2, B_1, B_2, \theta, t_o, a) \in \mathcal{P} \subset \Re^8$ to the space of boundary conditions $(x_1, v_1, x_2, v_2) \in \Re^8$? Can one characterize the singular sets? Are closed loops with $x_o = x_1$, $v_o = v_1$ special?*

2.5. Symmetries and Conserved Quantities

Sussmann [57] observed that group symmetries in control problems may have special features, specially along the nonsmooth solutions. However, we encountered here an unusual situation in Hamiltonian mechanics: we have explicitly integrated (in a straightforward way) a four degrees of freedom system, without caring to first find other integrals of motion, besides the trivial one $p_x = \text{const}$.

The planar dM–D problem in $\Re^8$ has obviously the $SE(2)$ symmetry induced by its original action in $\Re^2$. It is interesting to write down the momentum map for the lifted action, a slightly unfamiliar map $\mathcal{J} : \Re^8 \to \Re^3$, given by

$$\mathcal{J}(px, pv) = (p_x, x \wedge p_x + v \wedge p_v). \tag{2.16}$$

In the second component of the momentum map we interpret the vector product in $\Re^2$ as a number. In $\Re^n$ that component would be the element on $sO(n)^*$ given by

$$A \in sO(n) \mapsto p_x \cdot Ax + p_v \cdot Av. \tag{2.17}$$

More generally, if the group G acts on a configuration space Q, this action lifts in a natural way to a G-action on TQ via $g \cdot v_q = (\Phi_g)_*(v_q)$, where $\Phi_g : Q \to Q$, $\Phi_g(q) = g \cdot q$ is the translation map. To describe the momentum map $\mathcal{J} : T^*(TQ) \to \mathcal{G}^*$, one would need a bit of abstract nonsense. It is perhaps easier to work case by case. Formula (2.17) gives the momentum map relative to the $SO(n)$ action on the sphere S^{n-1}, where $(x, v) \in TS^{n-1}$, ie, $|x| = 1$ and $v \perp x$.

In the planar Euclidean case, it is easy to verify directly that the extended angular momentum

$$\mathcal{L} = x \wedge p_x + v \wedge p_v \tag{2.18}$$

is an integral of motion. $\mathcal{L}$ inherits the same commutation relations with p_x

$$\{\{x \wedge p_x + v \wedge p_v, p_{x_1}\}\} = p_{x_2},\ \{\{x \wedge p_x + v \wedge p_v, p_{x_2}\}\} = -p_{x_1}$$

where p_x, x and and p_v, v are conjugate. This means that we have the three Poisson commuting integrals of motion H^*, $\mathcal{L}$, and $p_{x_1}^2 + p_{x_2}^2$.

Problem 3. *Is there a fourth Poisson commuting integral of motion in $\Re^8$? Must the singular codimension one variety $p_{x_1} p_{v_2} - p_{x_2} p_{v_1} = 0$ be removed from the domain?*

Remark. What may be somewhat "demoralizing" in the case of the planar Dubins problem is that the rotational symmetry has in fact no influence for the integrability. After reducing the translational symmetry (x ignorable, $p_x =$ const.) we end up with a 2 degrees of freedom in the conjugate variables v, p_v. The Hamiltonian is a linear function of v plus $|p_v|$. Hence the system will remain integrable if we replace the latter by any function involving only the components of p_v. In this sense, perhaps the planar system can be called super-integrable.

We take the liberty to ask a trivial question perhaps:

Problem 4. *Let (p, q) canonical conjugate variables in $\Re^n \times \Re^n$, and a Hamiltonian of the very special form*

$$H(p, q) = -a \cdot q + f(p).$$

It is manifestly integrable by quadratures,

$$p(t) = p_o + at\ ,\ \ q(t) = q_o + \int_0^t \nabla f(p_o + at')dt'.$$

Does this simple situation exemplify an integrable system without integrals? Or, in the light of Arnold–Liouville theorem, do we expect the n Poisson commuting integrals? What could be the integrals of motion?

3. DYNAMIC MARKOV–DUBINS PROBLEM ON MANIFOLDS

In this section we content ourselves with some generalities, in an attempt to draw attention to the dynamic Markov–Dubins problem on a Riemannian manifold.

3.1. Desiderata

It was quite easy to integrate the dM–D problem in $\Re^n$ with the Euclidean metric. Nonetheless the solution curves display an interesting behavior. We observed a "shadow" of the underlying geodesic system, in the sense that as $t \to \infty$ the solutions have a preferred direction (the axis of the "drifting parabola").

On a general Riemannian manifold it is not clear how much can we extrapolate these observations. After all, geodesics either spread or converge according to the local curvature.

We also recall that Jacobi's "remarkable transformation" allowing the integration of geodesics in a problem without group symmetries, the triaxial ellipsoid [30, 31], that opened up the fascinating subject of integrable geodesic flows on Riemannian manifolds [11, 12].

Problem 5. *Dynamic Markov–Dubins for integrable geodesic problems:*

1. *For which Riemannian manifolds is the dynamic M–D problem completely integrable? What would be the nature of the obstructions to integrability?*

2. *Following Jurdjevic's viewpoint* [33]*, the most interesting dynamic M-D control problems would be those on Lie groups (say with a left invariant metric), and on homogeneous spaces (sphere, hyperbolic plane). Which ones would be completely integrable?*

3. *A simple dimension count shows that for the 2-dimensional sphere ($SO(3)$ symmetry) and the hyperbolic plane ($Sl(2,R)$ symmetry) a fourth integral of motion would be desired. For a surface of revolution (S^1 symmetry) two additional commuting integrals are needed.*

4. *Is the dynamic Markov–Dubins problem in the triaxial ellipsoid integrable? Three additional commuting conserved quantities besides the Hamiltonian are needed*[3)]*.*

Consider the general setting (1.2). We observe first that although $T(v_{q_o}, v_{q_1})$ vanishes on the diagonal of $TQ \times TQ$, it *must* be discontinuous there. Suppose for instance that q_o is ahead of q_1 in the geodesic determined by v_{q_1}, and v_{q_o} is a tangent to this geodesic. Since we cannot "back up", the (positive) time to joint $v_{q_o} \to v_{q_1}$ is bounded away from zero. This should not be a surprise since the standard Markov–Dubins "metric" used in path-planning is also discontinuous [54].

Problem 6. *What would be the generic properties of the $T(v_{q_o}, v_{q_1})$ level values?*

One dimensional bang-bang problems are imbedded in the higher dimensional dM–D problems. By 1-d we mean controls of the form

$$D_{\dot\gamma}\,\dot\gamma = \alpha\dot\gamma/|\dot\gamma|,\ \alpha = \pm 1$$

which allow connecting vectors (in general of different norms) v_{q_o} and v_{q_1} that are tangent to the *same* geodesic curve. We saw, however, that in the $\Re^2$ case the dynamics drifts logarithmically for slightly perturbed initial conditions.

Furthermore we observe that *global topology must come into play.* Consider for instance a genus 1 torus with the flat metric. Consider a geodesic on it, an irrational one, densely filling the torus. Two base points may be "very far" if one is obliged to traverse along the geodesic, by actually they can be close by in the torus. Identify their tangent planes and solve the Markov–Dubins' problem in $\Re^2$. Most likely this R^2 solution is faster that the bang-bang solution along the geodesic.

Problem 7. *Is there an unicity result for time-minimal $C^{3/2}$ dM-D problem in every (pinned extremes) homology class?*

3.2. Extrinsic Formulation for the Dynamic Markov–Dubins Problem

For the Pontryagin's principle in embedded submanifolds, that we implicitly use in the sequel, see [16]. We now show how to derive the equations of motion for the dynamic Dubins problem in any imbedded manifold in an euclidian space. For concreteness we consider an hypersurface Σ^n defined by $g(x) = 0\,,\ x \in \Re^{n+1}$. It would be a rather simple extension to consider the more general case of codimensions $\geqslant 2$.

As a warm-up we recall how to get geodesics via Fermat's geometric optics. For background on the geometric optics approach to geodesics, see eg. [4] or [28]. We denote the normal vector by $n(x) = \nabla g(x)/|\nabla g(x)|$.

Definition. *Fermat-Huygens' principle in Σ^n as a control problem:*

$$\min T = \int dt \quad \textit{such that} \quad \dot x = u \quad \textit{with} \quad |u| = 1, \quad \textit{and} \quad u \perp n(x).$$

[3)] If true, that result for the triaxial ellipsoid would be "straight from The Book".

The conditions for u define the "indicatrix" (see Arnold, section 49) at $x \in \Re^3$, as the limit (as the minor axis tendo to zero) of an oblate ellipsoid of revolution, whose vanishing anisotropic axis in the direction of ∇g. Clearly the velocity of light in that medium is tangent to Σ . We can find Fermat's rays via Pontryagin's principle. We may assume $u \in \Re^3$, and we take the Hamiltonian

$$H = -1 + (p_x, u).$$

Then

$$u^* = \text{argmax } H \ \text{ s.t. } |u| = 1, \ u \perp \nabla g.$$

It is readily seen that

$$u^* = u^*(x, p_x) = \frac{p_x - \frac{(p_x, \nabla g)}{(\nabla g, \nabla g)} \nabla g}{\left| p_x - \frac{(p_x, \nabla g)}{(\nabla g, \nabla g)} \nabla g \right|}.$$

Hence

Proposition 5. *Fermat–Huygens ray optics in Σ^n via control theory:*

$$H^* = -1 + \left| p_x - \frac{(p_x, \nabla g(x))}{(\nabla g(x), \nabla g(x))} \nabla g(x) \right| \tag{3.1}$$

and the equations of motion are

$$\dot{x} = u^*(x, p_x) \ , \quad \dot{p_x} = -\frac{\partial H}{\partial x}. \tag{3.2}$$

We do not bother to expand the right hand side for $\dot{p_x}$, just noting that it involves the second derivatives of $g(x)$. An equivalent approach uses the ideas in [16] directly. We change the state equation to

$$\dot{x} = u - \frac{(u, \nabla g)}{(\nabla g, \nabla g)} \nabla g \tag{3.3}$$

so that the right hand side is automatically tangent to the surfaces $g = \text{const.}$ for an *arbitrary* choice of $u \in \Re^3$. We can them apply the standard Pontryagin principle with $x \in \Re^3$, $u \in B(0,1) \subset \Re^3$, obtaining

$$\begin{aligned} H &= -1 + p_x \cdot \left(u - \frac{(u, \nabla g)}{(\nabla g, \nabla g)} \nabla g \right) \\ &= -1 + p_x \cdot u - \frac{(u, \nabla g)}{(\nabla g, \nabla g)} (p_x, \nabla g) \\ &= -1 + u \cdot \left(p_x - \frac{(\nabla g, p_x)}{(\nabla g, \nabla g)} \nabla g \right). \end{aligned}$$

Then $u^* = \arg\max H$ for $|u| < 1$ $u \in \Re^3$, and a short calculation gives

$$u^* = p_x - \frac{(\nabla g, p_x)}{(\nabla g, \nabla g)} \nabla g / \left| p_x - \frac{(\nabla g, p_x)}{(\nabla g, \nabla g)} \nabla g \right|$$

so that

$$H^* = -1 + \left| p_x - \frac{(\nabla g, p_x)}{(\nabla g, \nabla g)} \nabla g \right|$$

which is the same result we got via Fermat's indicatrix.

We can now tackle the extrinsic formulations for the dynamic Markov–Dubins problem. Let D the Levi–Civita connection of Σ. Our requirement that

$$D_{\dot{\gamma}} \dot{\gamma} = u \in T_\gamma \Sigma$$

means that

$$\ddot{x} = u + \alpha n\,, \quad u \perp \nabla g. \tag{3.4}$$

It is well known from basic physics that the normal acceleration component is given by

$$\alpha = -(B(x)v, v)/|\nabla g(x)| \quad (B = \text{hessian of } g). \tag{3.5}$$

[In fact: $(\dot{x}, \nabla g) \equiv 0$ implies $(\ddot{x}, \nabla g) + (\dot{x}, B(x)\dot{x}) \equiv 0$ so that $(u + \alpha n, \nabla g) + (\dot{x}, B(x)\dot{x}) \equiv 0$.]

We can write the state equations as

$$\dot{x} = v, \quad \dot{v} = u - \frac{(B(x)v, v)}{(\nabla g(x), \nabla g(x))}\, \nabla g(x) \tag{3.6}$$

where the initial conditions must be restricted to

$$x_o \in \Sigma\,, \; v_o \perp \nabla g.$$

With this state equations is automatic that $g(x(t)) \equiv 0$ for any choice of controls $u(t) \perp \nabla g(x)$. In the same way as we did for the geometric optics, we apply the "extrinsic" version of Pontryagin's principle:

$$H = -1 + (p_x, v) + (p_v, u) - \frac{(B(x)v, v)}{(\nabla g(x), \nabla g(x))}(p_v, \nabla g(x)) \tag{3.7}$$

with

$$u^* = \operatorname{argmax}\,(p_v, u) \;\text{ subject to } |u| \leqslant 1\,,\; u \perp \nabla g(x). \tag{3.8}$$

It follows that

$$u^* = u^*(x, p_v) = p_v - \frac{(p_v, \nabla g)}{(\nabla g, \nabla g)}\nabla g \Big/ \left| p_v - \frac{(p_v, \nabla g)}{(\nabla g, \nabla g)}\nabla g \right|. \tag{3.9}$$

The optimal Hamiltonian in $\Re^{4(n+1)}$ contains the conjugate pairs p_x with x, and p_v with v, vectors in $\Re^{n+1}$. It is given by

Proposition 6. *Dynamic Markov–Dubins Hamiltonian on an hypersurface* Σ^n : $g(x(t)) \equiv 0$:

$$H^* = -1 + (p_x, v) - \frac{(B(x)v, v)}{(\nabla g(x), \nabla g(x))}(p_v, \nabla g(x)) - (p_v, u^*) \tag{3.10}$$

where

$$(p_v, u^*) = \left| p_v - \frac{(p_v, \nabla g(x))}{(\nabla g(x), \nabla g(x))}\nabla g(x) \right|. \tag{3.11}$$

Remark. This proposition reduces to the Euclidian case (Proposition 1) with $g(x) = x_{n+1}$. In comparison with the euclidian metric there is an explicit dependence on x via $\nabla g(x)$ and $B(x)$.

Problem 8.

1. *By dimension count, we must we have four Casimirs. It would be interesting to derive a modified Poisson bracket and a simpler Hamiltonian to represent the equations of motion.*

2. *Numerical experiments are in order. The first example is the sphere* $g(x) = |x|^2 - r^2 \equiv 0$. *Here* r *is a true parameter since we fixed the force constraint* $|u| \leqslant 1$. *For* $n = 2$ *the momentum mao of the* $SO(3)$ *action yields two commuting integrals with the Hamiltonian, so a fourth integral of motion is in order.*

3. *As we commented above, the holy grail would be the integrability of the dM–D for the ellipsoids* $g(x) = (Ax, x) - 1$. *Numerical experiments could provide insights on this possibility.*

3.3. Dynamic Markov Dubins: Pontryagin's Principle in Coordinates

We will not attempt an "abstract nonsense" effort here. A. Lewis and F. Bullo [13] and M. Barbero and M. Munoz-Lecanda [6] have started a research program for this and related problems, using the *intrinsic viewpoint.* Namly, they consider $T^*(TQ)$ with the canonical symplectic structure. Their goal is to write intrinsecally the optimal Hamiltonian. One can also try to take advantage of the double bundle structure to describe the dynamics in terms of a Whitney sum. At a point v_q in the base TQ, the fiber consists of two copies of of T_qQ. The question is to compute (using the metric) the pullback of the canonical symplectic form in $T^*(TQ)$ to this bundle. The optimal Hamiltonian would be simply $H^*(v_q, \lambda_q, \mu_q) = -1 + g(v_q, \lambda_q) + \sqrt{g(\mu_q, \mu_q)}$.

Rather, we will adopt the most naive approach possible: we will write the differential equations in coordinates. This is quite reasonable for a manifold covered by a single chart, such as the hyperbolic space, or the peakon metric (4.1) below.

Recall that given a Riemannian manifold (Q, g), the problem is to connect in minimum time, by a $C^{3/2}$ path, two vectors of possibly different lengths, $(q_o, v_o) \in TQ$ to $(q_1, v_1) \in TQ$, subject to

$$\nabla_{\dot q}\, \dot q = u \in T_qQ\,, \quad g(u, u) \leqslant 1. \tag{3.12}$$

Let x_i coordinates in Q so that the metric writes as

$$ds^2 = g_{ij}\, dx_i \otimes dx_j. \tag{3.13}$$

We consider the matrices $G = (g_{ij}) \quad G^{-1} = (g^{ij})$. We will be careless about upper and lower indices, but whenever needed will recall the correct covariant vs. contravariant transformation rules. We further denote:

1. The corresponding coordinates in $M = TQ$: $\quad x_i\ ,\ v_i$.
2. The corresponding coordinates in T^*M: $\quad x_i,\ v_i\ ,\quad y_i\ ,\ z_i$.
3. The canonical 1-form in T^*M: $\sum y_i dx_i + z_i dv_i$.

   We recall the Levi–Civita connection in coordinates. Given $q(t)$ a curve in Q,

4. $\dot q = \sum v_i\, X_i\ ,\ X_i = \frac{\partial}{\partial x_i}$.
5. $\nabla X_k X_j = \sum_i \Gamma^i_{jk} X_i\ , \quad \Gamma^i_{jk} = \frac{1}{2}\sum_m g^{im}\left(g_{mj,k} + g_{mk,j} - g_{jk,m}\right).$
6. $\nabla_{\dot q}\, \dot q = \sum_k \left(\frac{d^2 x_k}{dt^2} + \sum_{i,j} \Gamma^k_{ij} \frac{dx_i}{dt}\frac{dx_j}{dt}\right) X_k$. More generally, let $A(t) = \sum a_i(t) X_i$ a vectorfield along $q(t)$, then:
7. $\nabla_{\dot q}\, A(t) = \sum_k \left(\frac{da_k}{dt} + \sum_{i,j} \Gamma^k_{ij} a_i(t) \frac{dx_j}{dt}\right) X_k.$
8. Parallel transport is defined by the linear time dependent ODE

$$\nabla_{\dot q}\, A(t) \equiv 0\ . \tag{3.14}$$

We are now able to write Pontryagin's principle for the dynamic dubins problem in coordinates as a sequence of simple steps. The state space is TM. The state equations are given by a spray in $T(TM)$:

$$\dot x_k = v_k,\ \dot{v_k} = -\sum_{i,j} \Gamma^k_{ij} v_i v_j + u_k \tag{3.15}$$

where the control variables satisfy $\sum_{ij} g_{ij}\, u_i u_j = 1$. Therefore the u-Hamiltonians are

$$H(x, y, v, z; u) = -1 + \sum_k \left[y_k v_k + z_k \left(u_k - \sum_{i,j} \Gamma^k_{ij} v_i v_j \right) \right] \tag{3.16}$$

with canonical 2 form given by

$$\Omega = \sum_i dy_i dx_i + dz_i dv_i \ . \tag{3.17}$$

Lemma 1. *Maximizing the Hamiltonian with respect to the control:*

$$u^*(x, y, v, z) = \arg\max_u H(x, y, v, z; u) = \frac{G^{-1}z}{\sqrt{(z, G^{-1}z)}}. \tag{3.18}$$

Proof. It is a just a static optimization problem with one constraint:

$$\text{Max} \sum_k z_k u_k \ \text{ s.t. } \ \sum_{ij} g_i j u_i u_j = 1.$$

In terms of column vectors: $u \ z$ we can write as

$$\text{Max}\,(u, z) \ \text{ s.t. } \ (Gu, u) = 1.$$

Introducing the Lagrange multiplier and Lagrangian

$$\mathcal{L} = (u, z) + \lambda(1 - (Gu, u))$$

we get

$$z - 2\lambda G u = 0, \ (Gu, u) = 1$$

from which we get the answer:

$$u^* = \frac{G^{-1}z}{\sqrt{(z, G^{-1}z)}}, \ \ 2\lambda^* = \sqrt{(z, G^{-1}z)} \ .$$

Proposition 7. *In coordinates, the optimal Hamiltonian for the dynamic Markov–Dubins problem is*

$$(H^*(x, y, v, z)\,,\ \Omega = \sum_i dy_i \wedge dx_i + dz_i \wedge dv_i)$$

where

$$H^*(x, y, v, z) = -1 + (y, v) + \sqrt{(z, G^{-1}z)} - (z, \sum_{ij} \Gamma^{\circ}_{ij}(x) v_i v_j) \tag{3.19}$$

where Γ°_{ij} *is the vector with components* $\circ = 1, 2, \ldots$ *The state equations are*

$$\dot{x}_k = v_k, \ \ \dot{v}_k = -\sum_{i,j} \Gamma^k_{ij} v_i v_j + u^*_k \ , \ \ u^*_k = \frac{\sum_i g^{ik}(x) z_k}{\sqrt{(z, G^{-1}z)}} \tag{3.20}$$

and the costate equations $\dot{y}_j = -\frac{\partial H}{\partial x_j}, \ \dot{z}_j = -\frac{\partial H}{\partial v_j}$ *are given explicitly by*

$$\dot{y}_j = \sum_{k,m,n} z_k \frac{\partial \Gamma^k_{mn}}{\partial x_j}(x)\, v_m v_n \ , \ \ \dot{z}_j = -y_j + \sum_{k,i} z_k \Gamma^k_{ij}(x) v_i. \tag{3.21}$$

Problem 9. *This proposition also reduces to the Euclidean example (Proposition 2.3). The following questions are in order:*

1. (*Easy, see Appendix* 2) *Show the invariance of the Hamiltonian under a point transformation* $x \to \overline{x}$*:*

$$H = H(x, y, v, z) = H(\overline{x}, \overline{y}, \overline{v}, \overline{z}).$$

2. (*Hard*) *Add and subtract and collect terms in the Hamiltonian and symplectic form* (H, Ω) *so that there is an intrinsic interpretation for the explicit expressions.*

3. (*Harder*) *Express everything in terms of instrinsic geometric constructions on vectorfields along the solutions* $x(t)$ *(following Lewis/Bullo or Barbero/Lecanda).*

4. FINAL REMARKS

The classical Markov–Dubins problem is a starting point for several path planning strategies in robotic locomotion. Current research includes domains with boundaries/holes, stochastic interference, collective motions, and graph theoretic version (Markov–Dubins traveling salesman problem).

In this paper we have discussed a dynamic variant where curves have variable velocity. Sharp changes in direction are in principle allowed provided the scalar velocity vanishes at these points. The objective is to connect two tangent vectors of different sizes in minimum time. We did not address more general mechanical type control problems on manifolds with more general cost functionals, nor and bounded controls on convex regions. In this generality, "chattering" (Fuller's phenomenon) and other complications will certainly occur.

4.1. Sasaki and Other Natural Metrics on TQ

There are several natural metrics in TQ associated to a metric on Q, such as the Sasaki and the Cheeger–Gromoll metrics [21] that could also be used to measure the distance between tangent vectors. The Sasaki distance associated to the Euclidian metric combines a quadratic fuel cost and a quadratic dissipation in the medium. That research direction could be a possibility for future studies, since the functional can be interpreted in a nice way. Perhaps this approach could be also useful for a research project on dynamic imaging. Recall that the Levi–Civita connection corresponds to a splitting of $T_{v_q}(TQ)$ in a "horizontal" direction together with the canonical vertical subspace isomorphic to T_qQ. One declares the horizontal and vertical subspaces orthogonal, giving each subspace its natural inner product.

Problem 10. *For which completely integrable metrics on a riemannian manifold* [11, 12] *are the natural metrics on the tangent bundle also completely integrable? As we mentioned above we proposed the same for the associated classic and the dynamic Dubins versions.*

We finish with comments on the two topics alluded in the title.

4.2. Differential Games of Pursuit

A problem that makes good sense in the biological realm is an extension of the "homocidal pursuit game" proposed by Rufus Isaacs in 1951 (see [29] and [49] for the history of this problem). In the original setting, the pursuer moves with constant speed higher than the maximal speed attainable by the evader. The evader can do sharp turns, while the pursuer's rate of turn is bounded, as in the classical Markov–Dubins problem. There are many possibilities of changing the problem, for instance allowing tangential accelerations for both evader and the pursuer. The following problem is probably not yet perfectly stated:

Problem 11. *Assume for instance that both agents* (P) *and* (E) *obeys a constraint of the form* $A_P u_T^2 + B_P v^4\kappa^2 \leqslant 1,\ A_E u_T^2 + B_E v^4\kappa^2 \leqslant 1$, *where the* $A's, B's$ *are "biological" parameters,* u_T *the tangential acceleration and* $v^2\kappa$ *the normal acceleration. Each agent can control its instantaneous pair* (u_T, κ)*. Smaller ratio* B/A*, indicates more maneuverability within the constraint. The evader has more maneuverability, while the pursuer has more power.*

4.3. Fast Processes: Landmark Matching

In their seminal work, Camassa and Holm [14] introduced the finite dimensional "peakon" Hamiltonian, which governs singular solutions for the 1d CH-equation. It is a completely integrable geodesic system in $\Re^n$ with Hamiltonian $H = \frac{1}{2}\sum g^{ij}p_ip_j$ where

$$g^{ij} = \exp(-|q_i - q_j|/\alpha) \tag{4.1}$$

is the peakon metric. The soliton structure was described numerically in Holm and Staley [22]. The generalization of the 1d-CH to n dimensions is the so called EPDiff partial differential equation which is basic for Computational Anatomy [27]. Singular solutions along landmark points, curves and surfaces have been studied numerically by Staley[4]. A few references in the subject are [9, 15, 20, 23, 24, 26, 27, 32].

Problem 12. *Study numerically via Proposition 7 the dynamic Markov–Dubins problem associated to the peakon Hamiltonian* (4.1)*. An quite ambitious project in Computational Anatom would be to consider the corresponding dynamic Markov–Dubins in* 2 *or* 3*d, concentrating the singularities on landmarks (points, curves, surfaces).*

APPENDIX A. SOME HEURISTIC CONSIDERATIONS

Consider a particle moving on a straight line, breaking with acceleration $-a$ until it stops. Then the particle starts to accelerate at rate a on another direction at an angle $\alpha \in [0, \pi)$.

Can this corner be short routed by a circular arc so that the normal acceleration b is $\leqslant a$?

At a point distant $d = v^2/2a$ from the corner, the velocity is v and the time remaining to reach it is v/a. Thus the total time to transfer is $2v/a$. On the circular arc, the normal acceleration is $b = v^2/R$ where the radius of curvature is

$$R = d \tan(\alpha/2) \ .$$

Therefore

$$b = \frac{v^2}{(v^2/2a)\tan(\alpha/2)} = 2a\cot(\alpha/2),$$

where there is no dependence on where the turn starts. So $b \leqslant a$ implies

$$\cot(\alpha/2) \leqslant 1/2 \quad \Rightarrow \quad \alpha \geqslant 2\arcsin(2/\sqrt{5}) \ .$$

The time τ of transfer along the circular arc is obviously smaller, since it is a shorter route with higher velocity. To quantify the difference, we compute

$$v\tau = R(\pi - \alpha) = (\pi - \alpha)\tan(\alpha/2)(v^2/2a)$$

so that the time advantage is

$$(v/a)\left[2 - \frac{\pi - \alpha}{2}\tan(\alpha/2)\right] \ .$$

Curiously, the limit of this expression in brackets as $\alpha \to \pi$ is 1, so there is a residual v/a time advantage bypassing a shallow corner.

Let now experiment to short route the wedge with a paraboloidal arc. Here the acceleration vector points in the direction of the mediatrix of the corner. The perpendicular component of the velocity does not change, $v\sin(\alpha/2)$. The distance between the base points is $2d\sin(\alpha/2)$, hence the transfer time along the parabola should be

$$\frac{2(v^2/2a)\sin(\alpha/2)}{v\sin(\alpha/2)} = v/a$$

which is *half* the time going on the semi-lines.

Let us compute the acceleration b needed along the mediatrix so that the “cannon ball” reaches the target point correctly. We must have

$$(v\cos(\alpha/2))(v/a) - \frac{1}{2}b(v/a)^2 = 0 \quad \Rightarrow \quad b = 2a\cos(\alpha/2)\,,$$

so that we must have

$$\cos(\alpha/2) \leqslant 1/2 \quad \text{i.e} \quad \alpha \geqslant 120 \text{ degrees} \ .$$

Let us compare the time needed for the paraboloidal transfer with the circular transfer. The function

$$1 - \frac{\pi - \alpha}{2}\tan(\alpha/2) \quad \alpha \in [0, \pi]$$

decreases monotonically from 1 to zero, and this means that the circular transfer beats the paraboloidal. This suggests the presence of abnormal circular solutions.

[4] See his web page http://math.lanl.gov/~staley/ and also the demos in http://demonstrations.wolfram.com/1DAnd2DSingularWavefronts/.

APPENDIX B. INVARIANCE UNDER COORDINATE TRANSFORMATIONS

The intrinsic formulation of the dynamic Markov–Dubins problem in terms of natural geometric objects seems somewhat an unfinished project. At this point the expressions still look quite complicated, see e.g. [13]. Nonetheless, it was not hard to get the equations in coordinates. It would be desirable to show that the formulae behavecorrectly under the contravariant/covariant transformation rules in the tangent and cotangent bundles, respectively. We outline this calculation but leave it as an exercise. Let $\overline{x} = \overline{x}(x)$.

Write as column vectors: v $\overline{v}$ coordinates in T_qQ end as row vectors: p $\overline{p}$ coordinates in T_q^*Q. Then

$$\overline{v} = J(x)v \quad \text{(contravariant)}, \quad \overline{p} = p(J(x))^{-1} \quad \text{(covariant)}$$

where the Jacobian matrix is given by

$$J(x) = \left(\frac{\partial \overline{x}_i}{\partial x_j}\right).$$

One applies these recipies to the tangent bundle itself. The change of charts is

$$(x, v) \in \Re^n \times \Re^n \to (\overline{x}, \overline{v}) \in \Re^n \times \Re^n,$$

$$\overline{x} = \overline{x}(x)\,, \;\; \overline{v} = J(x)v.$$

The Jacobian is

$$\tilde{J}(x, v) = \begin{pmatrix} J(x) & 0 \\ A & J(x) \end{pmatrix}$$

where

$$A_{ij} = \sum_k \frac{\partial \overline{x}_i}{\partial x_j \partial x_k} v_k.$$

Lemma 2. *Cotangent bundle of tangent bundle transformation rules. Let* (y, z) *and* $(\overline{y}, \overline{z})$ *(row vectors) (coordinates in* $T^*_{(q,\dot{q})}(TQ)$*)*

$$(\overline{y}, \overline{z}) = (y, z)(\tilde{J}(x, v))^{-1},$$

$$(\tilde{J}(x, v))^{-1} = \begin{pmatrix} J^{-1}(x) & 0 \\ -J^{-1}AJ^{-1} & J^{-1}(x) \end{pmatrix}.$$

In more detail.

$$J(x) = (\partial \overline{x}_i / \partial x_j),$$

$$A_{ij} = \sum_k \frac{\partial \overline{x}_i}{\partial x_j \partial x_k} v_k \;.$$

$$\begin{aligned}
\overline{x} &= \overline{x}(x) \;\; \Rightarrow \;\; d\overline{x} = J(x)dx, \\
\overline{v} &= J(x)v \;\; \Rightarrow \;\; d\overline{v} = J(x)dv + A(x, v)dx, \\
\overline{z} &= zJ^{-1}(x), \\
\overline{y} &= yJ^{-1}(x) - zJ^{-1}(x)A(x, v)J^{-1}(x).
\end{aligned}$$

Exercise. (Easy) Verify that (abstract nonsense guarantees it is true!)

$$\overline{y}d\overline{x} + \overline{z}d\overline{v} = ydx + zdv$$

hence

$$\Omega = \overline{dy}d\overline{x} + \overline{dz}d\overline{v} = dydx + dzdv.$$

Exercise. (Not so easy, but essentially routine) Verify that the Hamiltonian

$$H = -1 + (y, v) + \sqrt{(z, G^{-1}z)} - \sum_{ijk} \Gamma^k_{ij}(x) z_k v_i v_j$$

has the same result with the "overlined" objects. This is subtle since $(y, v) \neq (\overline{y}, \overline{v})$. Only the whole expression is invariant. In the verification, use also the well known transformation rules for

$$G(x) \to \overline{G}(\overline{x}),$$

$$\Gamma^k_{ij}(x) \to \overline{\Gamma}ij^k(\overline{x}) \ .$$

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