SEMISIMPLE COADJOINT ORBITS AND COTANGENT BUNDLES

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ABSTRACT. Semisimple (co)adjoint orbits through real hyperbolic elements are well-known to be symplectomorphic to cotangent bundles. We provide a new proof of this fact based on elementary results on both Lie theory and symplectic geometry, thus shedding some new light on the symplectic geometry of semisimple (co)adjoint orbits.

1. INTRODUCTION

The coadjoint representation has Poisson nature: the Lie bracket of a Lie algebra $\mathfrak{g}$ canonically induces a linear Poisson bracket on its dual $\mathfrak{g}^*$. The symplectic leaves of the linear Poisson structure are the coadjoint orbits. The induced symplectic structure on a coadjoint orbit is the so called Konstant-Kirillov-Souriau (KKS) symplectic structure.

For any Poisson structure the understanding of the symplectic structure of any of it leaves is fundamental; for duals of Lie algebras this understanding is even more important, for has deep implications on Hamiltonian group actions and representation theory [7, 9].

When $\mathfrak{g}$ is (semisimple) of compact type, coadjoint orbits –which are the classical flag manifolds from complex geometry– are compact, and therefore more tools are available for the study of their symplectic geometry [7].

Global aspects of the symplectic geometry of non-compact coadjoint orbits are much harder to grasp. The first result in that direction is due to Arnold. In [1] he proved that the regular (complex) coadjoint orbit of $\mathrm{SL}(n+1,\mathbb{C})$ endowed with the imaginary part of the KKS holomorphic symplectic structure is symplectomorphic to the cotangent bundle of the variety of full flags in $\mathbb{C}^{n+1}$, if and only if all the eigenvalues of some (and hence any) matrix in the orbit are pure imaginary. Later on, Azad, van den Ban and Biswas [2] discovered that Arnold's result had a far reaching generalization for semisimple real hyperbolic orbits, which we briefly discuss:

Let G be a connected, non-compact semisimple Lie group with finite center, and let $\mathfrak{g}$ denote its Lie algebra. The Killing form $\langle \cdot, \cdot \rangle$ intertwines the coadjoint and adjoint actions, and it is used to transfer the symplectic structure from a coadjoint orbit to the corresponding adjoint one, so one can speak of the KKS symplectic structure of an adjoint orbit. An element $H \in \mathfrak{g}$ is real hyperbolic if the operator $\mathrm{ad}(H)$ diagonalizes with real eigenvalues; if an Iwasawa decomposition $G = KAN$ has been fixed, then real hyperbolic elements are those conjugated to elements in the closure of the fixed positive Weyl chamber $\mathrm{Cl}(\mathfrak{a}^+) \subset \mathfrak{a}$ of the Lie algebra of A.

Theorem 1 ([2]). *Let G be a connected, non-compact semisimple Lie group with finite center and let $G = KAN$ be any fixed Iwasawa decomposition. Then for any*

real hyperbolic element $H \in \mathfrak{g}$, there exist a canonical symplectomorphism between the adjoint orbit $\mathrm{Ad}(G)_H \subset \mathfrak{g}$ with its KKS symplectic structure, and the cotangent bundle of the real flag manifold $\mathrm{Ad}(K)_H$ with its standard Liouville symplectic structure.

The existing proofs of Theorem 1 are far from elementary. They rely either on deep results on integrable systems [2], or on non-trivial integrability results for Lie algebra actions [6].

The purpose of this note is to revisit Theorem 1 and provide a proof based on elementary facts on both Lie theory and symplectic geometry, thus shedding some new light on the symplectic geometry of semisimple (co)adjoint orbits.

The key ingredients in our strategy are the full use of the *canonical ruling* of the adjoint orbit and the description of new aspects the *symplectic geometry of the 'Iwasawa projections'*. In what follows we briefly discuss the main ideas behind our approach, and compare it with [2, 6]:

We assume without loss of generality that $H \in \mathrm{Cl}(\mathfrak{a}^+)$. The Iwasawa decomposition defines a well-known canonical ruling on the adjoint orbit $\mathrm{Ad}(G)_H$. The ruling, together with the Killing form, determine a diffeomorphism

$$i \colon T^* \mathrm{Ad}(K)_H \to \mathrm{Ad}(G)_H \tag{1}$$

extending the inclusion of real flag manifold $\mathrm{Ad}(K)_H \hookrightarrow \mathrm{Ad}(G)_H$.

Of course, the diffeomorphism (1) appears both in [2, 6], but it is not fully exploited.

- In [2] non-trivial theory of complete Lagrangian fibrations is used to construct a symplectomorphism $\varphi \colon (T^* \mathrm{Ad}(K)_H, \omega_{\mathrm{std}}) \to (\mathrm{Ad}(G)_H, \omega_{\mathrm{KKS}})$, with the property of being the unique symplectomorphism which (i) extends the identity on $\mathrm{Ad}(K)_H$ and (ii) is a morphism fiber bundles, where $\mathrm{Ad}(G)_H$ has the bundle structure induced by i in (1).

  In fact, it can be checked that φ coincides with i (1) (the uniqueness statement is also a consequence of the absence of non-trivial symplectic automorphisms of the cotangent bundle preserving the zero section and the fiber bundle structure).

- In [6] a complete Hamiltonian action of $\mathfrak{g}$ on $(T^* \mathrm{Ad}(K)_H, \omega_{\mathrm{std}})$ is built. The momentum map $\mu \colon (T^* \mathrm{Ad}(K)_H, \omega_{\mathrm{std}}) \to (\mathrm{Ad}(G)_H, \omega_{\mathrm{KKS}})$ is the desired symplectomorphism; the authors also show that the momentum map μ matches i in (1).

Both in [2, 6] a global construction on a non-compact symplectic manifold is performed, something which always presents technical difficulties. [1]

Our strategy is much simpler: we shall take full advantage of the ruling structure to prove the equality

$$i_* \omega_{\mathrm{std}} = \omega_{\mathrm{KKS}}. \tag{2}$$

In fact, this is the approach sketched by Arnold [1].

Basic symplectic linear algebra [11] implies that to prove (2) at $x \in \mathrm{Ad}(G)_H$ it is enough to find $L_v, L_h \subset T_x \mathrm{Ad}(G)_H$ such that: (i) L_v, L_h are Lagrangian subspaces for both symplectic structures; (ii) $L_v \cap L_h = \{0\}$; (iii) $i_* \omega_{\mathrm{std}}(x)(Y, Z) = \omega_{\mathrm{KKS}}(x)(Y, Z)$, $\forall Y \in L_v$, $Z \in L_h$.

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[1] In [9], Corollary 1, a proof of the isomorphism of a regular coadjoint orbit with the cotangent bundle is presented, but it is not correct as the completeness issues are entirely ignored.

As the notation suggests, L_v will be the vertical tangent space coming from the fiber bundle structure, which is trivially Lagrangian for $i^*\omega_{\mathrm{std}}$ and easily seen to be Lagrangian for ω_{KKS} [2]. Transitivity of the adjoint action implies the existence of $g \in G$ so that

$$x \in \mathrm{Ad}(g)(\mathrm{Ad}(K)_H) := \mathrm{Ad}(K)_H^g.$$

The 'horizontal' subspace L_h will be the tangent space to $\mathrm{Ad}(K)_H^g$ at x; because the zero section $\mathrm{Ad}(K)_H$ is also Lagrangian w.r.t. ω_{KKS} [2], G-invariance of ω_{KKS} implies that L_h is a Lagrangian subspace w.r.t ω_{KKS}. If $\mathrm{Ad}(K)_H^g$ it is to be Lagrangian w.r.t to ω_{std}, it should correspond to a closed 1-form in $\mathrm{Ad}(K)_H$. In fact, it will be the graph of an exact 1-form, and the 'projections' associated to the Iwasawa decomposition will play a crucial role to determine a potential.

The 'Iwasawa projection' $H \colon G \colon \to \mathfrak{a}$ is defined by $x \in K \exp H(x) N$. A pair $H \in \mathfrak{a}$, $g \in G$ determines a function

$$F_{g,H} \colon K \to \mathbb{R}, \ k \mapsto \langle H, H(gk) \rangle.$$

Under the assumption $H \in \mathrm{Cl}(\mathfrak{a}^+)$ the function descends to the real flag manifold $\mathrm{Ad}(K)_H \cong K/Z_K(H)$, where $Z_K(H)$ is the centralizer of H in K [5]. The functions $F_{g,H}$ are well-studied, and they play a prominent role in Harmonic analysis and convexity theory [5, 8, 3, 4].

Our main technical result is:

Proposition 1. *Let G be a connected, non-compact semisimple Lie group with finite center, let $G = KAN$ be any fixed Iwasawa decomposition and let $H \in \mathrm{Cl}(\mathfrak{a}^+)$. Then for any $g \in G$ the submanifold*

$$\mathrm{Ad}(K)_H^g \subset \mathrm{Ad}(G)_H \overset{i^{-1}}{\cong} T^*\mathrm{Ad}(K)_H$$

is the graph of the exterior differential of $-F_{g,H} \in C^\infty(\mathrm{Ad}(K)_H)$.

Proposition 1 completes the description of the 'horizontal Lagrangians'. The equality

$$i_*\omega_{\mathrm{std}}(x)(Y,Z) = \omega_{\mathrm{KKS}}(x)(Y,Z), \ \forall Y \in L_v, \ Z \in L_t$$

will follow from computations analogous to those used to establish proposition 1, this providing a proof of Theorem 1 which only appeals to basic symplectic geometry and Lie theory.

2. PROOF OF THEOREM 1

In this section we fill in the details of the proof of Theorem 1 sketched in the introduction.

Let us fix a Cartan decomposition $G = KP$ associated to an involution θ, and let $\mathfrak{k}, \mathfrak{p}$ denote the respective Lie algebras. A choice maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$ and positive Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$ (or root ordering) gives rise to an Iwasawa decomposition $G = KAN$, with $\mathfrak{n}$ the Lie algebra of the nilpotent factor.

We shall denote the adjoint action of g on $X \in \mathfrak{g}$ by X^g.

We may pick without any loss of generality $H \in \mathrm{Cl}(\mathfrak{a}^+)$ and consider the corresponding adjoint orbit $\mathrm{Ad}(G)_H$. The orbit is identified with the homogeneous space $G/Z(H)$, where $Z(H)$ denotes the centralizer of H. Under this identification $\mathrm{Ad}(K)_H$ is mapped to a submanifold canonically isomorphic to $K/Z_K(H)$, where $Z_K(H) = K \cap Z(H)$ is the centralizer of H in K. At the infinitesimal level the

tangent space at $H \in \mathrm{Ad}(K)_H$ is identified with the quotient space $\mathfrak{k}/\mathfrak{z}_K(H)$, where $\mathfrak{z}_K(H)$ is the Lie algebra of $Z_K(H)$.

2.1. The ruling and the identification $T^*\mathrm{Ad}(K)H \stackrel{i}{\cong} \mathrm{Ad}(G)_H$.

The contents we sketch in this subsection are rather standard. We refer the reader to [2] for a throughout exposition.

Let $\mathfrak{n}(H)$ be the sum of root subspaces associated to positive roots not vanishing on H. We have the θ-orthogonal decomposition

$$\mathfrak{g} = \theta\mathfrak{n}(H) \oplus \mathfrak{z}(H) \oplus \mathfrak{n}(H),$$

where $\theta\mathfrak{n}(H) = \mathfrak{n}^-(H)$ are the root spaces corresponding to the negative roots which are non-trivial on H.

The affine subspace $H + \mathfrak{n}(H)$ is tangent to $\mathrm{Ad}(G)_H$ and complementary to $\mathrm{Ad}(K)_H$ at H. Even more, the adjoint action of the subgroup $N(H)$ integrating the nilpotent Lie algebra $\mathfrak{n}(H)$ maps $N(H)$ diffeomorphically into $H+\mathfrak{n}(H) \subset \mathrm{Ad}(G)_H$. This induces the well-known ruling of $\mathrm{Ad}(G)_H$.

As any ruled manifold, $\mathrm{Ad}(G)_H$ becomes an affine bundle. Since $\mathrm{Ad}(K)_H$ is transverse to the affine fibers, the structure can be reduced to that of a vector bundle with zero section $\mathrm{Ad}(K)_H$. As to which vector bundle this is, a vector tangent to the fiber over H belongs to $\mathfrak{n}(H)$; the map $X \mapsto X + \theta X$ is a monomorphism from $\mathfrak{n}$ to $\mathfrak{k}$. Since the image of $\mathfrak{n}(H)$ has trivial intersection with $\mathfrak{z}_K(H)$, it is isomorphic to $T_H\mathrm{Ad}(K)_H$. Therefore the pairing $\langle\cdot,\cdot\rangle \colon \mathfrak{n}(H) \times \mathfrak{k}/\mathfrak{z}_K(H) \to \mathbb{R}$ –which is well defined– is also non-degenerate, and this provides the canonical identification of the fiber at H with $T_H^*\mathrm{Ad}(K)_H$. Since the Killing form and Lie bracket are Ad-invariant, for any $k \in K$ we have the analogous statement for

$$\langle\cdot,\cdot\rangle \colon \mathfrak{n}(H^k) \times \mathfrak{k}/\mathfrak{z}_K(H^k) = \mathfrak{n}(H)^k \times \mathfrak{k}/\mathfrak{z}_K(H)^k \to \mathbb{R},$$

this giving the identification

$$i \colon T^*\mathrm{Ad}(K)_H \longrightarrow \mathrm{Ad}(G)_H.$$

2.2. The symplectic forms ω_{std} and ω_{KKS}.

From now on we shall omit the map i in the notation, so we have $\omega_{\mathrm{std}}, \omega_{\mathrm{KKS}}$ two symplectic forms on $\mathrm{Ad}(G)_H$ whose equality we want to check.

For the purpose of fixing the sign convention, we take the standard symplectic form of the cotangent bundle ω_{std} to be $-d\lambda$, where $\lambda = \xi dx$ and ξ, x are the momentum and position coordinates, respectively.

The tangent space at $H^g \in \mathrm{Ad}(G)_H$ is spanned by vectors of the form $[X^g, H^g]$, $X \in \mathfrak{g}$. The formula

$$\omega_{KKS}(H^g)([X^g, H^g], [Y^g, H^g]) = \langle H, [X, Y]\rangle$$

is well defined on $\mathfrak{g}/\mathfrak{z}(H)$, and gives rise to an $\mathrm{Ad}(G)$-invariant symplectic form on the orbit [9].

As discussed in the introduction, to prove the equality $\omega_{\mathrm{std}}(H^g) = \omega_{\mathrm{KKS}}(H^g)$, we shall start by finding complementary Lagrangian subspaces for both symplectic forms.

2.3. **The vertical Lagrangian subspaces.** At H^g we define $L_v(H^g) = H^g + \mathfrak{n}(H)^g$, i.e. the tangent space to the ruling. Of course, this space is Lagrangian for ω_{std}. It is also Lagrangian for ω_{KKS} [2]. We include the proof of this fact to illustrate the kind of arguments we will use in our computations:

Two vectors in $L_v(H^g)$ are of the form $[X^g, H^g], [Y^g, H^g]$, where $X, Y \in \mathfrak{n}(H)$. Therefore

$$\omega_{KKS}(H^g)([X^g, H^g], [Y^g, H^g]) = \langle H^g, [X^g, Y^g] \rangle = \langle H, [X, Y] \rangle = 0,$$

where the vanishing follows because $[X, Y] \in \mathfrak{n}(H)$ and the subspaces $\mathfrak{a}$ and are $\mathfrak{n}(H)$ are orthogonal w.r.t. the Killing form (following from the orthogonality w.r.t. the inner product $\langle \cdot, \theta \cdot \rangle$ used in the Iwasawa decomposition).

2.4. **The horizontal Lagrangian subspaces.** We consider $\mathrm{Ad}(K)_H^g$ the conjugation by g of $\mathrm{Ad}(K)_H$ and we define $L_h(H^g) = T_{H^g} \mathrm{Ad}(K)_H^g$. We shall prove that $\mathrm{Ad}(K)_H^g$ is a Lagrangian submanifold for both symplectic forms, so in particular $L_h(H^g)$ is a Lagrangian subspace.

The KKS symplectic form is $\mathrm{Ad}(G)$-invariant. Therefore it suffices to prove that $\mathrm{Ad}(K)_H$ is Lagrangian w.r.t ω_{KKS} to conclude that for all $g \in G$ the submanifold $\mathrm{Ad}(K)_H^g$ is Lagrangian w.r.t ω_{KKS}.

At H^k two vectors tangent to $\mathrm{Ad}(K)_H$ are of the form $[X^k, H^k], [Y^k, H^k]$, where $X, Y \in \mathfrak{k}$. Hence

$$\omega_{KKS}(H^k)([X^k, H^k], [Y^k, H^k]) = \langle H^k, [X^k, Y^k] \rangle = \langle H, [X, Y] \rangle = 0,$$

where the vanishing follows from $[X, Y] \in \mathfrak{k}$ and the orthogonality of $\mathfrak{a} \subset \mathfrak{p}$ and $\mathfrak{k}$ w.r.t. the Killing form (see also [2]).

To describe the behavior of $\mathrm{Ad}(K)_H^g$ w.r.t. to ω_{std}, we need a formula for the projection map $\mathrm{pr} \colon \mathrm{Ad}(G)_H \to \mathrm{Ad}(K)_H$ defined by the bundle structure. To that end we introduce all the 'Iwasawa projections'

$$K \colon G \to K, \; A \colon G \to A, \; N \colon G \to N,$$

characterized by $x \in K(x)AN$, $x \in KA(x)N$, $x \in KAN(x)$, respectively (note that the 'Iwasawa projection' cited in the Introduction is $H = \log A$).

Lemma 1. *The 'Iwasawa projection' $K \colon G \to K$ descends to the bundle projection*

$$\mathrm{pr} \colon \mathrm{Ad}(G)_H \cong G/Z(H) \to \mathrm{Ad}(K)_H \cong K/Z_K(H)$$

associated to the ruling.

Proof. Let us write $g = K(g)A(g)N(g)$. Then it follows that

$$H^g = H^{K(g)A(g)N(g)K(g)^{-1}K(g)},$$

where $K(g)A(g)N(g)K(g)^{-1} \in \mathfrak{n}(H)^{K(g)}$, which proves our assertion. $\square$

To understand the bundle projection infinitesimally we also need information on the differential of the 'Iwasawa projections'. This information can be found for H or A in [5] (for higher order derivatives as well; see also [4]). The result for the three projections is presented below; the proof is omitted since it is a straightforward application of the chain rule.

Lemma 2. *For any $X \in \mathfrak{g}$ and $g \in G$ we have*

$$X^{AN(g)} = K(X, g) + A(X, g)^{A(g)} + N(X, g)^{AN(g)} \tag{3}$$

written as sum of vectors in $\mathfrak{k}, \mathfrak{a}, \mathfrak{n}$, where $K(X,g), A(X,g), N(X,g)$ stand for the left translation to the identity on K, A, N of the vector field represented by the curves $K(g\exp(tX)), A(g\exp(tX)), N(g\exp(tX))$, respectively, and $AN(g)$ denotes $A(g)N(g)$.

Proof of Proposition 1. The submanifold $\mathrm{Ad}(K)_H^g$ is the graph of a 1-form $\alpha_{g,H} \in \Omega^1(\mathrm{Ad}(K)_H)$, which we evaluate now: given any $k \in K$, according to Lemma 1 the point $H^{gk} \in \mathrm{Ad}(K)_H^g$ projects over $H^{K(gk)} \in \mathrm{Ad}(K)_H$. The tangent space $T_{H^{K(gk)}}\mathrm{Ad}(K)_H$ is spanned by vectors of the form $L_{K(gk)^{-1}*}K(X,gk)$, where $X \in \mathfrak{k}$. By definition of $\alpha_{g,H}$ we have:

$$\alpha_{g,H}(K(gk))(L_{K(gk)^{-1}*}K(X,gk)) = \langle (H^{gk} - H^{K(gk)})^{K(gk)^{-1}}, K(X,gk)\rangle.$$

Because $\mathfrak{k}$ and $\mathfrak{a} \subset \mathfrak{p}$ are orthogonal, we deduce

$$\langle (H^{gk} - H^{K(gk)})^{K(gk)^{-1}}, K(X,gk)\rangle = \langle H^{AN(gk)}, K(X,gk)\rangle.$$

By (3)

$$\langle H^{AN(gk)}, K(X,gk)\rangle = \langle H^{AN(gk)}, X^{AN(gk)} - A(X,gk) - N(X,g)^{AN(g)}\rangle.$$

Because $H^{AN(gk)} \in \mathfrak{a} + \mathfrak{n}$ and because $\mathfrak{a}$ and $\mathfrak{n}(H)$ are $\langle \cdot, \cdot \rangle$-orthogonal

$$\langle H^{AN(gk)}, X^{AN(gk)} - A(X,gk) - N(X,g)^{AN(g)}\rangle = -\langle H^{AN(gk)}, A(X,gk)\rangle,$$

and therefore

$$\alpha_{g,H}(K(gk))(L_{K(gk)^{-1}*}K(X,gk)) = -\langle H, A(X,gk)\rangle. \tag{4}$$

Now consider the function $F_{g,H}\colon K \to \mathbb{R}$, $k \mapsto \langle H, H(gk)\rangle$. By [5], Proposition 5.6, it descends to a function $F_{g,H} \in C^\infty(\mathrm{Ad}(K)_H)$. According to [5], Corollary 5.2,

$$-DF_{g,H}(L_{K(g)*}K(X,g)) = -\langle X^{AN(g)}, H\rangle,$$

and by equation (3)

$$-DF_{g,H}(L_{K(g)*}K(X,g)) = -\langle H, A(X,g)\rangle. \tag{5}$$

Hence by equations (4) and (5) we conclude

$$\alpha_{g,H} = -DF_{g,H},$$

as we wanted to prove. $\qquad\square$

2.5. **The equality $\omega_{\mathrm{KKS}} = \omega_{\mathrm{std}}$.** We just need to prove the equality at any point H^g on pairs of vectors $[X^g, H^g], [Y^g, H^g]$, where $X \in \mathfrak{k}$ and $Y \in \mathfrak{n}(H)$.

By definition of the KKS form $\omega_{KKS}(H^g)([X^g, H^g], [Y^g, H^g]) = \langle H, [X,Y]\rangle$.

As for the standard form

$$\omega_{\mathrm{std}}(H^g)([X^g, H^g], [Y^g, H^g]) = \langle [Y^g, H^g]^{K(g)^{-1}}, K(X,g)\rangle = \langle [Y^{AN(g)}, H^{AN(g)}], K(X,g)\rangle.$$

By equation (3)

$$\langle [Y^{AN(g)}, H^{AN(g)}], K(X,g)\rangle = \langle [Y,H], X\rangle - \langle [Y^{AN(g)}, H^{AN(g)}], A(X,g)^{A(g)} + N(X,g)^{AN(g)}\rangle,$$

which equals $\langle [Y,H], X\rangle = \langle H, [X,Y]\rangle$ since in the second summand the first entry belongs to $\mathfrak{n}$ and the second to $\mathfrak{a} + \mathfrak{n}$.

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