HOPF CONJECTURE HOLDS FOR ANALYTIC, K-BASIC FINSLER 2-TORI WITHOUT CONJUGATE POINTS

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Abstract. We show that analytic, k-basic Finsler metrics in the two torus without conjugate points are analytically integrable, in the sense that the unit tangent bundle of the metric admits an analytic foliation by invariant Lagrangian graphs. This result, combined with the fact that $C^{1,L}$ integrable k-basic Finsler metrics in the two torus have zero flag curvature (Barbosa-Ruggiero [21]) implies that analytic k-basic Finsler metrics in two tori without conjugate points are flat, a positive answer to the so-called Hopf conjecture for tori without conjugate points. Since there are well known examples of non flat tori without conjugate points (Busemann was the first to show such examples) the Hopf conjecture is not true if we drop the k-basic assumption.

Introduction

The theory of metric structures in the torus all of whose geodesics are global minimizers was totally understood in the Riemannian case after the solution of the so-called Hopf conjecture: every Riemannian metric in the torus without conjugate points is flat. This statement was proved by Hopf [23] in the 1940's and by Burago-Ivanov [10] in the early 1990's. However, if we widen our scope to the family of Finsler metrics the theory still poses many interesting, unsolved problems.

Since Busemann examples [11] of non-flat Finsler metrics in the two torus without conjugate points it is known that the Hopf conjecture is false in the Finsler realm. Nevertheless, Finsler metrics in the torus without conjugate points enjoy many properties in common with flat metrics. One of them is their connection with weakly integrable systems in the sense of [21]: there exists a continuous, invariant foliation by compact leaves of the unit tangent bundle, all of them tori, which are graphs of the canonical projection (see [22], [15], and Section 2). The existence of a C^k Lagrangian invariant foliation in the unit tangent bundle of the Finsler metric is called in [21] C^k integrability of the geodesic flow (a natural extension of this notion to C^0 integrability is also given in [21]). Moreover, in all known examples of smooth Finsler metrics without conjugate points ([11], [28] for instance) such foliation is smooth. In the Riemannian case the smoothness of the foliation follows from the rigidity of the metric: since the metric is flat the Riemannian metric is Euclidean. The smoothness of the foliation is not part of the proof of the Hopf conjecture but one of its consequences.

So two questions arise naturally from the above discussion. Do C^0 integrable Finsler geodesic flows on tori are C^k for some $k \geq 1$? Does the C^k integrability

2010 Mathematics Subject Classification: 53B40, 53C60, 53D25.

Key words and phrases. Finsler metric, two torus, isotropic flag curvature, geodesic flow, invariant foliation, integrability.

The second author was partially supported by CNPq.

of such geodesic flows for $k \geq 1$ play any role in the proof of rigidity results? The first question has been already considered in [15], where it is proved that Lipschitz integrability of the geodesic flow of a Finsler metric on the torus without conjugate points implies C^1 integrability. However, a full answer to the question is still open. The second question has been studied in [21], where it is shown that $C^{1,L}$ integrable geodesic flows of k-basic Finsler metrics on two tori are flat ($C^{1,L}$ means C^1 with Lipschitz first derivatives). The present paper deals with the first question, our main result is the following.

Theorem 1: Let (T^2, F) be an analytic, k-basic Finsler metric without conjugate points in the two torus T^2. Then the geodesic flow is analytically integrable, namely, there exists an analytic foliation by invariant tori of the unit tangent bundle of the metric which are graphs of the canonical projection.

A Finsler metric is called k-basic if the flag curvature $K(p, v)$, which is a function defined on the unit tangent bundle, does not depend on the vertical variable v (see Section 1).

Combining Theorem 1 with the main result of [21] we get a two dimensional version of the Hopf conjecture for k-basic Finsler metrics.

Theorem 2: An analytic k-basic Finsler metric without conjugate points in the two torus is flat.

Theorem 1 is the first result, as far as we know, to show that a Finsler, non-Riemannian metric in the two torus without conjugate points is smoothly integrable without using geometric rigidity. Since Finsler geodesic flows represent high energy levels of Tonelli Hamiltonians, Theorem 1 is actually a Hamiltonian statement. It is remarkable that in the literature about the link between smoothness of invariant foliations of Hamiltonian flows and geometric rigidity, the most common assumption is hyperbolicity (see for instance [25], [14], [17], [18] with results for surfaces of higher genus, and many other higher dimensional results by Kanai, Katok-Feres, Benoit-Foulon-Labourie, etc). Hyperbolic dynamics imposes topological restrictions on the manifold: Anosov geodesic flows act on the unit tangent bundle of manifolds whose fundamental groups have to be Gromov hyperbolic. So most of the ideas applied to such manifolds do not hold on tori.

The methods of the proof of Theorem 1 combine Riemann-Finsler geometry with the theory of Lagrangian graphs, extending the ideas of some recent developments in Finsler rigidity theory [19], [21], [20]. We would like to point out some remarkable consequences of Theorem 2. Flat Finsler surfaces are classified in [1], they are Minkowsky spaces: Finsler spaces obtained as discrete quotients of the plane endowed with a flat Finsler structure. So Theorem 2 not only extends Hopf's conjecture for Riemannian metrics on two dimensional tori but it is meaningful in Finsler geometry: there are genuinely Finsler, non-Riemannian examples of flat Finsler structures. Moreover, in [7] we can find many many examples of Randers metrics, Finsler metrics obtained adding a norm and a one form, defined in T^2, which are not flat. Since Randers metrics are k-basic, Theorem 2 yields that the above mentioned ones have conjugate points. Randers metrics are similar to magnetic fields in many senses, so it is interesting to notice that Theorem 2 applied to Randers metrics provides a Finsler version of a rigidity result for magnetic flows on the two torus due to Bialy [9]: magnetic flows without conjugate points on the two torus are Riemannian and flat.

1. Preliminaries

1.1. **Finsler spaces.** We recall briefly some fundamental notions of Finsler geometry, we follow [6] as main reference.

Let M be a n-dimensional, C^∞ manifold, let T_pM be the tangent space at $p \in M$, and let TM be its tangent bundle. In canonical coordinates, an element of T_xM can be expressed as a pair (x, y), where y is a vector tangent to x. Let $TM_0 = \{(x, y) \in TM;\, y \neq 0\}$ be the complement of the zero section. A C^k $(k \geq 2)$ *Finsler structure* on M is a function $F : TM \to [0, +\infty)$ with the following properties:
(i) F is C^k on TM_0;
(ii) F is positively homogeneous of degree one in y, where $(x, y) \in TM$, that is,

$$F(x, \lambda y) = \lambda F(x, y)\ \forall\ \lambda > 0$$

(iii) The Hessian matrix of $F^2 = F \cdot F$

$$g_{ij} = \frac{1}{2}\frac{\partial^2}{\partial y^i \partial y^j}F^2$$

is positive definite on TM_0.

A C^k *Finsler manifold* (or just a Finsler manifold) is a pair (M, F) consisting of a C^∞ manifold M and a C^k Finsler structure F on M.

Item (iii) implies that the function $L(p, v) = \frac{1}{2}F(x, y)^2$ defines a Tonelli Lagrangian in the tangent space, so Finsler metrics have geodesics and their local theory is just the local theory of existence and uniqueness of solutions of the Euler-Lagrange equation. We shall assume throughout the paper that geodesics have unit speed.

The Lagrangian action gives rise to a notion of Finsler distance: given a Lipschitz continuous curve $c : [a, b] \to M$ its Finsler length is

$$L_F(c) := \int_a^b F(c(t), \frac{dc}{dt}(t))\, dt$$

and from L_F we get a function $d = d_F : M \times M \longrightarrow [0, \infty)$ by

$$d_F(p, q) := \inf_c L_F(c) = \inf_c \int_a^b F(c(t), \frac{dc}{dt}(t))\, dt$$

where the infimum is taken over all Lipschitz continuous curves $c : [0, 1] \to M$ with $c(0) = p$ and $c(1) = q$. This function of course might fail to be symmetric like in Riemannian geometry.

A geodesic $\sigma : [a, b] \longrightarrow \tilde{M}$ is called *forward minimizing*, or simply minimizing if $L_{\tilde{F}}(\sigma) \leq L_{\tilde{F}}(c)$ for all rectifiable curves $c : [a, b] \longrightarrow \tilde{M}$ such that $c(a) = \sigma(a)$, $c(b) = \sigma(b)$ (this implies that $\sigma : [s, t] \longrightarrow \tilde{M}$ is also minimizing for every $a \leq s \leq t \leq b$). The term "forward" is used to stress the facta that a geodesic σ might fail to be minimizing if one reverses its orientation.

For a non-vanishing vector $y \in T_xM$, we shall denote by $\gamma_{(x,y)}(t)$ the geodesic with initial conditions $\gamma_{(x,y)}(0) = x$ and $\gamma'_{(x,y)}(0) = y$. The *exponential map* at x, $exp_x : T_xM \to M$ is defined as usual: $exp_x(y) := \gamma_{(x,y)}(1)$.

The Finsler manifold (M, F) induces naturally a Finsler structure in the universal covering $\tilde{M}$ of M, just by pulling back the Finsler structure F to the tangent space of $\tilde{M}$ by the covering map. Let us denote by $(\tilde{M}, \tilde{F})$ this Finsler manifold.

1.2. **Chern-Rund connection (or Chern connection) and Jacobi fields.** One of the main tools of Finsler geometry used to study geodesics is the so-called Chern-Rund connection that we describe briefly in this subsection. We follow see [6], [27].

There exists a Riemannian metric in the tangent bundle, called in the literature the **fundamental tensor** of the Finsler metric, that is given by the Hessian of the square of the Finsler metric in canonical coordinates divided by $\frac{1}{2}$. This tensor is a Riemannian metric in the cotangent bundle by the convexity assumptions on the Finsler metric. The fundamental tensor has many remarkable properties, the most important is that the orbits of the geodesic flow of the Finsler metric are geodesics of the fundamental tensor. This gives a sort of covariant differentiation for the geodesics of the Finsler metric, the Chern-Rund connection, that we describe next.

A piecewise C^1 variation of a smooth curve $\sigma(t)$ in M is a continuous map

$$\sigma(t, u) : \triangle = \{(t, u); 0 \leq t \leq r, -\varepsilon < u < \varepsilon\} \longrightarrow M$$

which is C^1 on each $[t_{i-1}, t_i] \times (-\varepsilon, \varepsilon)$, and such that $\sigma(t, 0) = \sigma(t)$ for every $t \in [0, r]$. When the variation is piecewise C^2 and the curves $\sigma_u(t) = \sigma(t, u)$, $t \in [0, r]$, are geodesics, then the derivative of the variation is a *Jacobi field* of the Finsler metric.

Lemma 1.1. *Let (M, F) be a C^4 Finsler manifold. For $(x, v) \in TM$, let $g_{ij}(x, v)$ be as defined in item (3) of the definition of Finsler metric. Let σ be a smooth curve and $\sigma(t, u)$ be a variation as before. Then, in the tangent space $T_{\sigma(t,u)}M$ the inner product*

$$g_T := g_{ij(\sigma(t,u),T(t,u))} dx^i \otimes dx^j$$

where $T = T(t, u) := \sigma_* \frac{\partial}{\partial t} = \frac{\partial \sigma}{\partial t}$, satisfies the following properties:

1. *$g_T(T, T) = F^2(T)$.*
2. *σ is a Finslerian geodesic if and only if*

$$\frac{d}{dt} g_T(V, W) = g_T(D_T V, W) + g_T(V, D_T W)$$

where V and W are two arbitrary vector fields along σ. The operator $D_T = \frac{d}{dt}$ is called covariant differentiation with reference vector T.

3. *In particular, Finslerian geodesics satisfy*

$$D_T \left[\frac{T}{F(T)} \right] = 0.$$

The constant speed Finslerian geodesics $F(v) = c$ are the solutions of

$$D_T T = 0,$$

just like Riemannian geodesics.

4. *Assume that σ has unit speed. Then a Jacobi field along σ satisfies*

$$D_T D_T J + R(J, T)T = 0,$$

where R is the Jacobi tensor of the Finsler metric (We shall denote as usual $J'' = D_T D_T J$, $J' = D_T J$). When dim$(M) = 2$,

$$R(y, u)u = K(y)[g_y(y, y)u - g_y(y, u)y], \ y, u \in T_x M \setminus \{0\}$$

where $K(y)$ is the Gaussian curvature, which coincides as well with the flag curvature.

(5) *Let σ have unit speed. Then, if $J(t)$ is a Jacobi field along σ, the component $J_{\perp}(t)$ of $J(t)$ that is perpendicular to $\sigma'(t)$ with respect to g_T satisfies the scalar Jacobi equation*

$$J''_{\perp} + KJ_{\perp} = 0.$$

as in the Riemannian case. Moreover, if $g_T(T, J(t_0)) = g_T(T, J'(t_0)) = 0$ at some point t_0, then $g_T(T, J) = 0$ at every point.

Throughout the paper, all covariant differentiations will be carried out with reference vector T. Lemma 1.1 reduces many Finsler problems concerning Jacobi fields to Riemannian ones. We shall often call the inner product g_T the *adapted Riemannian metric*. The next result is proved in [21] and extends a well known result about bounded Jacobi fields of Riemannian metrics.

Proposition 1.2. *Let $M, F)$ be a compact Finsler surface and let γ be a geodesic without conjugate points. Let $J_0(t)$ be a Jacobi field defined along γ that is perpendicular to $\gamma'(t)$ for every $t \in \mathbb{R}$ with respect to g_t such that $\| J_0(t) \| \leq L$ for every $t \in \mathbb{R}$. Then $J_0(t)$ is the only Jacobi field, up to scalar multiple, satisfying:*

(1) *$\| J(t) \| \leq A$ for some constant A for every $t \in \mathbb{R}$,*
(2) *$J(t)$ is never zero.*

These everywhere bounded Jacobi fields are usually called *central* Jacobi fields. Central Jacobi fields will be important for our purposes.

1.3. **Conjugate points and Riccati equation.** We say that q is *conjugate* to p along a geodesic σ if there exists a nonzero Jacobi field J along σ which vanishes at p and q. We say that (M, F) *has no conjugate points* if no geodesic has conjugate points. The following result taken from [6] (Proposition 7.1.1) has a similar, well known counterpart in Riemannian geometry.

Proposition 1.3. *Let $\sigma(t) = exp_p(tv)$, $0 \leq t \leq r$, be a unit speed geodesic. Then the following statements are mutually equivalent:*

(1) *The point $q = \sigma(r)$ is not conjugate to $p = \sigma(0)$ along σ;*
(2) *Any Jacobi field defined along σ that vanishes at p and q must be identically zero;*
(3) *Given any $V \in T_pM$ and $W \in T_qM$, there exists a unique Jacobi field $J : [0, r] \longrightarrow TM$ defined along σ such that $J(0) = V$, $J(r) = W$;*
(4) *The derivative $(exp_p)_*$ of the exponential map exp_p is nonsingular at rv.*

An important piece of the proof of the main Theorem is the relationship between the existence of codimension one, invariant foliations in T_1M and manifolds without conjugate points. We recall briefly some basic notions concerning the so-called Riccati equation associated to Jacobi fields and Lagrangian subbundles of the geodesic flow (see for instance [26] for a Hamiltonian general setting). Let $\langle\langle , \rangle\rangle$ be the Sasaki-like metric in the unit tangent bundle of (M, F) (for the definition see [6]). Given $\theta \in T_1M$, let us denote by $\mathcal{V}_\theta \subset T_\theta T_1M$ the vertical subspace of $T_\theta T_1M$ (namely, the kernel of $d\pi$, the differential of the canonical projection). Let us denote by $\mathbf{H}_\theta \subset T_\theta T_1M$ the horizontal subspace of $T_\theta T_1M$, and let $X_\theta \in T_\theta T_1M$ be the unit vector tangent to the direction of the geodesic flow, i.e., $\frac{d}{dt}\varphi_t(\theta) = X_{\varphi_t(\theta)}$ for every $t \in \mathbb{R}$. Recall that $X_\theta \in \mathbf{H}_\theta$ for every $\theta \in T_1M$, and that the vertical and the horizontal subspaces at θ are perpendicular with respect to the Sasaki-like metric. Let us consider the subspaces

$$N_\theta = \{v \in T_\theta(T_1M); \langle\langle v,\ X_\theta \rangle\rangle = 0\},$$
$$\mathcal{H}_\theta = \mathbf{H}_\theta \cap N_\theta.$$

The differential of the geodesic flow preserves the bundle of subspaces N_θ, and the the differential Ω of the canonical one-form of the geodesic flow defines a symplectic two-form when restricted to each N_θ, that is also invariant under the geodesic flow.

Definition 1.1. *Let (M, F) be a complete, smooth Finsler manifold of dimension n. A continuous subbundle of subspaces $\theta \to L_\theta$, where $L_\theta \subset N_\theta$, is called Lagrangian if $\Omega(v, w) = 0$ for every $v, w \in L_\theta$, and $dim(L_\theta) = n - 1$, for every $\theta \in T_1M$.*

If the dimension of M is two, every continuous bundle $L_\theta \subset N_\theta$ of dimension one subspaces is Lagrangian. A Lagrangian bundle that is invariant under the action of the differential of the geodesic flow will be called invariant Lagrangian bundle. The next statement is a consequence of the standard theory of Lagrangian subspaces (see for instance [26]).

Lemma 1.4. *Let (M, F) be a compact Finsler surface without conjugate points. Given an orbit $\phi_t(\theta)$ of the geodesic flow and and an ϕ_t-invariant Lagrangian subbundle $\mathfrak{E}(\phi_t(\theta))$ of TT_1M defined along the orbit which never meets the vertical bundle, there exists a continuous function $U_\theta : \mathbb{R} \longrightarrow \mathbb{R}$ such that*

1. *The subspace $\mathfrak{E}(\phi_t(\theta))$ is the graph of the linear map $\bar{U}_{\phi_t(\theta)} : \mathcal{H}_{\phi_t(\theta)} \longrightarrow \mathcal{V}_{\phi_t(\theta)}$ given by $\bar{U}_{\phi_t(\theta)}(Z) = U_\theta(t)Z$ for every $\theta \in T_1M$,*
2. *The function $U_\theta(t)$ is a solution of the Riccati equation*
$$u'(t) + u^2(t) + K(\phi_t(t)) = 0,$$
where the derivatives mean covariant derivatives with respect to the adapted metric.
3. *$| U_{\phi_t(\theta)} | \leq k_0$ for every $t \in \mathbb{R}$, where k_0 is the maximum of the flag curvature of the surface.*

1.4. **Cartan's structural equations.** Here we recall briefly Cartan's structural equations for Finsler metrics, for details we refer to [6]. Like in the Riemannian case, the tangent bundle of T_1M has a natural oriented frame of vectors $e_1 = H$, $e_2 = X$, $e_3 = V$, where $e_2 = X$ is the unit vector tangent to the geodesic flow and $e_3 = V$ is tangent to the vertical bundle.

The vectors e_1, e_2 are chosen in a way that they are orthonormal in each T_pM with respect to the Sasaki-like metric associated to the adapted Riemannian metric $g_T := g_{ij}dx^i \otimes dx^j$. The partial derivatives of a function $f : T_1M \longrightarrow \mathbb{R}$ with respect to the vectors fields e_i will be denoted by f_i.

Proposition 1.5. *The structural equations of the Finsler metric (M, F) are written in terms of the 1-forms ω_i, $i = 1, 2, 3$, in the following way:*

$$\begin{aligned} d\omega_1 &= -I\omega_1 \wedge \omega_3 + \omega_2 \wedge \omega_3 \\ d\omega_2 &= -\omega_1 \wedge \omega_3 \\ d\omega_3 &= K\omega_1 \wedge \omega_2 - J\omega_1 \wedge \omega_3 \end{aligned}$$

The structural equations in terms of the dual basis $\{H, X, V\}$ *are the following:*

$$[V, X] = H$$
$$[H, V] = X + IH + JV$$
$$[X, H] = kV.$$

The scalar K is the *Gauss-Finsler curvature* (or Gaussian curvature) of the Finsler surface, J is called the *Landsberg scalar* and I the Cartan scalar. The three scalars I, J, K are all functions on T_1M.

It is possible to characterize Finsler metrics which are Riemannian in terms of the above functions. For our purposes, the following characterization will be the relevant one: I vanishes everywhere if and only if the Finsler structure is Riemannian.

The next result contains some properties of the Cartan tensor I that will be relevant for our purposes.

Proposition 1.6. *Let* (M, F) *be a* C^∞ *Finsler surface. Then*

(1) $J = XI$,
(2) *(Bianchi identity)* $Vk + kI + XJ = 0$. *In particular, if* $Vk = 0$ *then* I *satisfies*
$$X(XI) + kI = 0.$$

For the proof, it suffices to calculate $d(d\omega_2)$ (for item (1)), and $d(d\omega_3)$ (for item (2)).

2. Finsler tori without conjugate points

The next result is taken from [15].

Theorem 2.1. *Every* C^∞ *Finsler metric* F *in the torus* T^n *without conjugate points is* C^0 *integrable. Namely, there exists a continuous, invariant foliation* $\mathcal{F}$ *of* T_1T^n *by compact, continuous leaves having the following properties:*

(1) *Each periodic orbit is contained in a totally periodic leaf: a leaf where all the orbits are periodic.*
(2) *Each totally periodic leaf is* C^∞*, and if the metric* F *is analytic each one of them is analytic as well.*
(3) *The set of totally periodic leafs is dense.*
(4) *If* $n = 2$*, there is a bijective correspondence between the leaves and the homology classes of the 2-torus, and each point has a local cross section for the geodesic flow where the Poincaré map is a* C^0*-integrable twist map.*
(5) *Each leaf with irrational homology class has a unique limit set where the geodesic flow is transitive.*

Item (4) in Theorem 2.1 has the following inmediate consequence,

Lemma 2.1. *Let* (T^2, F) *be a smooth Finsler metric without conjugate points. Then any continuous compact invariant curve is contained in a leaf of* $\mathcal{F}$.

The following result is based on Proposition 5.1 in [21] and tells us that the Cartan tensor is linked to the dynamics of the geodesic flow and to the foliation $\mathcal{F}$.

Proposition 2.2. *Let (T^2, F) be an analytic k-basic Finsler metric without conjugate points. Then*

(1) *If $I(\theta) \neq 0$, the tangent space of the leaf through θ meets the subbundle N_θ at the graph of $\bar{U}_\theta : H_\theta \longrightarrow V_\theta$, where $\bar{U}_\theta(Z) = \frac{X(I(\theta))}{I(\theta)} Z$.*

(2) *The foliation $\mathcal{F}$ is analytic in the set of points where $I \neq 0$.*

Proof. A similar idea already appears in [19], [21]. We divide the proof in two cases.

First suppose that the leaf through θ is totally periodic. Since a totally periodic leaf is smooth, its tangent space along an orbit $\phi_t(\theta)$ intersects N_θ in a straight line generated by the vector $(J_\theta(t), J'_\theta(t))$, where $J_\theta(t)$ is a Jacobi field perpendicular to the geodesic $\pi(\phi_t(\theta))$ with respect to the adapted metric, and the derivatives represent the Chern-Rund derivatives with respect to the geodesic flow. Since the leaf is a graph of the canonical projection, the Jacobi field never vanishes. Therefore, by Proposition 1.2 and the Bianchi identity the Jacobi field $J_\theta(t)$ coincides with $I(\phi_t(\theta))$ up to a scalar multiple for every t. Hence, the subspace generated by $(J_\theta(t), J'_\theta(t))$, that equals the subspace generated by $(Z, \frac{J'_\theta(t)}{J_\theta(t)} Z)$, $Z \in H_{\phi_t(\theta)}$, coincides with the subspace generated by $(Z, \frac{I'(\phi_t(\theta)}{I(\phi_t(\theta)} Z)$ as claimed.

If the leaf $\mathcal{F}(\theta)$ is not totally periodic, the idea to show that $(Z, (X(I)/I)Z)$ is always tangent to the leaf is suggested by the proof of Lemma 5.5 in [21]. Consider a unit parallel (with respect to Chern-Rund connection) vector field $e_\theta : (-\epsilon, \epsilon) \longrightarrow T_{\pi(\phi_t(\theta))}M$ defined along the geodesic $\pi(\phi_t(\theta))$, that is perpendicular to this geodesic with respect to the adapted metric. Take a neighborhood $V(\theta) \subset T_1M$ where $I \neq 0$, and consider a smooth extension $e_\eta(t)$, for $\eta \in V(\theta)$, of this parallel vector field to parallel fields defined in the geodesics $\pi(\phi_t(\eta))$, $t \in (-\epsilon, \epsilon)$, with $e_\eta(t)$ perpendicular to the geodesic. This smooth extension e_η of e_θ has a lift E_η in the horizontal subspaces H_η.

Now, let us consider the vector field $Y(\eta) = (I(\eta)E_\eta, I'(\eta)E_\eta) \subset N_\eta$. The line field generated by this vector field is invariant by the geodesic flow since $I(\phi_t(\eta))$ is a Jacobi field. The vector field $Y(\eta)$ admits a lift $\tilde{Y}$ to an open neighborhood of a lift $\tilde{\theta}$ of θ in $T_1(\tilde{M})$. If $c(t)$ is the integral curve of $\tilde{Y}(\tilde{\eta})$ with $c(0) = \tilde{\theta}$ the orbits $\phi_t(c(s))$ are bi-asymptotic to the orbit $\phi_t(\tilde{\theta})$:

$$d(\phi_t(c(s)), \phi_t(c(r))) = \int_s^r \parallel \tilde{Y}(\phi_t(c(\rho))) \parallel d\rho \leq K \int_s^r \mid I(\phi_t(c(\rho)) \mid d\rho,$$

where the distance and the norm $\parallel .. \parallel$ are taken with respect to the Sasaki-like metric, and $K = \sqrt{1 + k_0^2}$ is a constant where $I'(\phi_t(\eta)) \leq k_0 I(\phi_t(\eta))$ (this constant k_0 is given by the theory of the Riccati equation). Hence, since I is uniformly bounded in T_1M we conclude that

$$d(\phi_t(c(s)), \phi_t(c(r))) \leq K \max_{\psi \in T_1M} \mid I(\psi) \mid$$

for every $t \in \mathbb{R}$ which implies that the orbits through the points of $c(s)$ are always bi-asymptotic.

Since every orbit of the geodesic flow is a global minimizer, all orbits of $\phi_t(c(s))$ are bi-asymptotic to the orbit $\phi_t(\tilde{\theta})$, their asymptotic homology classes are the same. Therefore, the orbits $\phi_t(c(s))$ must be lifts of orbits in the leaf $\mathcal{F}(\theta)$ because the only orbits of the geodesic flow whose homology classes coincide with the homology class of the orbit of θ are the orbits in $\mathcal{F}(\theta)$. This yields that the tangent space of $\mathcal{F}(\theta)$ is generated by $Y(\theta)$ in a neighborhood of θ, or equivalently, generated by $(Z, (I'/I)Z)$ $Z \in T_\eta \mathcal{F}(\theta)$, $\eta \in V(\theta) \cap \mathcal{F}(\theta)$. This finishes the proof of item (1).

Item (2) is straightforward from item (1) since whenever $I(\theta) \neq 0$ the tangent space of the foliation can be expressed in terms of an analytic family of graphs. □

3. The set of zeroes of the Cartan tensor

The goal of the Section is to show the main Theorem, namely, that the foliation $\mathcal{F}$ is analytic. Let us start with the following remark about the zeroes of the Cartan tensor.

Lemma 3.1. *Let (T^2, F) be a smooth k-basic Finsler metric without conjugate points in the torus. Then the set of zeroes of the Cartan tensor is invariant by the geodesic flow.*

Proof. The Cartan tensor restricted to each geodesic satisfies the Jacobi equation. Since geodesics have no conjugate points, each Jacobi field which vanishes somewhere is either zero or diverges with time. This well known property of Jacobi fields of Riemannian metrics without conjugate points holds as well for Finsler metrics (see [13], [20]). So if $I(\theta) = 0$ it must be zero along the orbit of θ, otherwise it would be unbounded in the orbit which is not possible. □

In this section the analytic character of the Finsler metric will be crucial to study the set of zeroes of the Cartan tensor. Just by continuity we know that the set $I = 0$ is a compact set. Assuming that I is analytic the Weierstrass preparation Theorem and Lojasiewicz's structure Theorem (see for instance [24]) describe quite precisely the geometry of the set of zeroes. We shall state 3-dimensional versions of both Theorems for our convenience.

Theorem 3.1. *(Weierstrass) Let $f : U \longrightarrow \mathbb{R}$ be a real analytic function defined in an open neighborhood U of $0 \in \mathbb{R}^3$. Suppose that $I(0,0,0) = 0$, and that some derivative of f with respect to z at $(0,0,0)$ is not equal to zero. Then there exists an analytic function $Q(x,y,z)$ such that $Q(0,0,0) \neq 0$, a distinguished polynomial*

$$P(x,y,z) = \sum_{i=0}^{m} a_i(x,y) z^i$$

where $a_i(x,y)$ are analytic functions, and an open neighborhood $U' \subset U$ such that

$$f(x,y,z) = P(x,y,z)Q(x,y,z)$$

for every $(x,y,z) \in U'$.

Theorem 3.2. *(Lojasiewicz) In the assumptions of Theorem 3.1, the set of zeroes of f is the union of a finite collection of points, one dimensional and two dimensional analytic subvarieties.*

Therefore, Lojasiewicz's Theorem tells us that the set of zeroes of I is locally a finite collection of analytic sets, of dimension 0, 1 or 2. Actually, by Lemma 3.1 the dimension must be at least one, so the set of zeroes of I is locally a finite collection of analytic curves and surfaces. Lojasiewicz's Theorem gives much more information about the set of zeroes of real analytic functions, the version stated here is in fact a consequence of the complete version.

Lemma 3.2. *Let (T^2, F) be an analytic k-basic without conjugate points. Then the set of zeroes of I is a finite collection of leaves of $\mathcal{F}$ and closed orbits.*

Proof. By Lemma 2.1 we now that if the set of zeroes of I contains a curve that is topologically transversal to the geodesic flow then this curve must be contained in a leaf. Let $I(\theta) = 0$, so $I = 0$ along the orbit of θ.

Claim: If the leaf $\mathcal{F}(\theta)$ has irrational homology class, I vanishes in $\mathcal{F}(\theta)$.

Indeed, if $I(\theta) = 0$ then I vanishes in the limit set of the leaf $L(\theta)$. Let $\theta_0 \in L(\theta)$, and take an analytic cross section Σ of the flow at θ_0. By Lojasiewicz's Theorem, the set of zeroes of I in Σ is a finite collection of points and curves, and if there are no curves in the set of zeroes then the collection of points in this set must be finite. Since $L(\theta)$ is a perfect set θ_0 is an accumulation point of zeroes of I, so $I = 0$ must contain a curve through θ_0. By Lemma 2.1, this curve is analytic and contained in $\mathcal{F}(\theta)$ and hence, $\mathcal{F}(\theta)$ is analytic in a neighborhood of $L(\theta)$. Since every orbit in $\mathcal{F}(\theta)$ approaches $L(\theta)$ then every point has a neighborhood in the leaf where $I = 0$ and the leaf is analytic. Namely, the whole leaf is analytic and $I = 0$ in the entire leaf $\mathcal{F}(\theta)$.

If the homology class of the orbit of θ is rational, then there exists an open neighborhood $V(\theta)$ in a cross section Σ through θ where $I(\eta) = 0$ if and only if $\eta = \theta$, or there exists an infinite sequence of points in Σ of zeroes accumulating θ. In the latter case, as in the claim we have that there exists an analytic curve in $\Sigma \cap \mathcal{F}(\theta)$ where $I = 0$, and by analytic continuation we deduce that I must vanish in the whole leaf.

Finally, the collection of leaves and closed orbits where $I = 0$ is isolated by Lojasiewicz's Theorem. □

The following statement is a consequence of the proof of Lemma 3.2, Proposition 2.2 and Theorem 2.1.

Lemma 3.3. *Under the assumptions of Lemma 3.2 we have*

1. *A leaf $\mathcal{F}(\theta)$ with irrational homology where $I = 0$ at some point in the leaf is analytic and contained in the set of zeroes of I.*
2. *Every leaf of the foliation $\mathcal{F}$ is analytic.*
3. *The family of Riccati operators $\bar{U}_\theta : H_\theta \longrightarrow V_\theta$ associated to the foliation $\mathcal{F}$ is measurable, uniformly bounded by some constant k_0 depending on the maximum value of the flag curvature, and analytic when restricted to each leaf.*

Proof. Leaves with rational homology class are analytic according to Theorem 2.1. A leaf with irrational homology class either contains a zero of I and hence, by Lemma 3.2 it is analytic or it does not contain any zero of I. In this case Proposition 2.2 implies that it is analytic. This yields the proof of items (1) and (2).

Item (3) is a consequence of Lemma 1.4 applied to the tangent bundle of the invariant foliation $\mathcal{F}$. □

Corollary 3.3 is close to what we want to show, the analyticity of the foliation $\mathcal{F}$. However, the analyticity of the foliation depends on the analyticity of the function $f(\theta) = \frac{XI(\theta)}{I(\theta)}$. According to Lemma 3.2, the function f defines the Riccati operator whenever $I \neq 0$. This yields that the Riccati operator is analytic in an open, dense set whose complement is a finite collection of leaves and closed orbits. So the problem of the analyticity of the foliation is somehow similar to a problem of removing the singularities of the Riccati operator.

Lemma 3.4. *Let (T^2, F) be an analytic, k-basic Finsler surface without conjugate points. Then the function $f(\theta) = \frac{XI(\theta)}{I(\theta)}$ has an analytic extension to T_1T^2 and therefore, the foliation $\mathcal{F}$ is analytic.*

Proof. We focus on the set of zeroes of the Cartan tensor. Let $I(\theta) = 0$. By lemma 3.2 either the leaf $\mathcal{F}(\theta)$ is contained in the set of zeroes of I or the orbit of θ is an isolated curve of zeroes. In the first case, we can take analytic coordinates (x, y, z) in a open neighborhood of $\theta \in T_1T^2$ such that x is the geodesic flow parameter, the coordinates of θ are $(0, 0, 0)$, and the leaf $\mathcal{F}(\theta)$ is the surface $z = 0$. So if I is not identically zero (in which case the Finsler structure is Riemannian) there exists some derivative of I with respect to z that is not zero. By Weierstrass preparation Theorem we get that

$$I(x, y, z) = P(x, y, z)Q(x, y, z)$$

where $P(x, y, z) = \sum_{i=0}^{m} a_i(x, y)z^i$ and $Q(x, y, z)$ is an analytic function which does not vanish. Since $I(x, y, 0) = 0$ for every x, y we have that $a_0(x, y) = 0$, so we can write

$$P(x, y, z) = z^k(\sum_{i=0}^{m-k} b_i(x, y)z^i) = z^k G(x, y, z)$$

where $b_0(x, y) \neq 0$. So we have

$$\frac{XI}{I} = \frac{(XP)Q + P(XQ)}{PQ} = \frac{XP}{P} + \frac{XQ}{Q}.$$

The term $\frac{XQ}{Q}$ is analytic since Q has no zeroes in the coordinate neighborhood. The first term is

$$\frac{XP}{P} = \frac{XG}{G} = \frac{X(b_0(x, y)) + \sum_{i=1}^{m-k} X(b_i(x, y))z^i}{b_0(x, y) + \sum_{i=1}^{m-k} b_i(x, y)z^i}.$$

Taking $z = 0$ we get

$$\frac{XP}{P}(x, y, 0) = \frac{X(b_0(x, y))}{b_0(x, y)},$$

at every (x, y) where $b_0(x, y)$ is not identically zero. The set of zeroes of $b_0(x, y)$ is a finite collection of analytic curves in the plane $z = 0$ since $b_0(x, y)$ is analytic. So we have two possibilities: either $b_0(0, 0) \neq 0$ or $b_0(0, 0) = 0$. In the first case the function $\frac{XP}{P}$ is analytic at $(0, 0, 0)$ which are the coordinates of the point θ, so the Lemma is proved at θ. In the latter case we apply again Weierstrass' preparation Theorem to the function $b_0(0, 0)$ to get

$$b_0(x, y) = (\sum_{n=0}^{r} q_n(x)y^n)\sigma(x, y) = (q_0(x) + \sum_{n=1}^{r} q_n(x)y^n)\sigma(x, y),$$

where $\sigma(0, 0) \neq 0$. As before,

$$\frac{X(b_0(x, y))}{b_0(x, y)} = \frac{X(\sum_{n=0}^{r} q_n(x)y^n)\sigma + (\sum_{n=0}^{r} q_n(x)y^n)X(\sigma)}{(\sum_{n=0}^{r} q_n(x)y^n)\sigma}$$

which equals

$$\frac{X(\sum_{n=0}^{r} q_n(x)y^n)}{(\sum_{n=0}^{r} q_n(x)y^n)} + \frac{X(\sigma)}{\sigma}.$$

Again, the second term is nonsingular at $(0,0)$, so we have to look at the first term. Taking $y = 0$ we get

$$\frac{XP}{P}(x,0,0) = \frac{X(q_0(x))}{q_0(x)}$$

at every point x where $q_0(x) \neq 0$.

Claim: $q_0(0)$ is not zero.

Because if $q_0(0) = 0$, we have $q_0(x) = x^\rho h(x)$ for some analytic function $h(x)$ with $h(0) \neq 0$, and therefore $X(q_0(x)) = \rho x^{\rho-1}h(x) + x^\rho X(h(x))$ which implies

$$\frac{X(q_0(x))}{q_0(x)} = \frac{\rho}{x} + \frac{X(h(x))}{h(x)}.$$

Thus, the function $\frac{XP}{P}(x,0,0) = \frac{X(q_0(x))}{q_0(x)}$ has a pole at $(x,0,0)$ which is not possible by Lemma 3.3 item (3): the function $f = \frac{XI}{I}$ is the restriction of the Riccati operator to the set $I \neq 0$ and the Riccati operator is always bounded above by some constant k_0.

We conclude that the function $\frac{XP}{P}$ is always analytic at θ when $I = 0$, a set which includes the leaf $\mathcal{F}(\theta)$. Therefore, the above equations imply that the function $f = \frac{XI}{I}$ is analytic at θ under this assumption.

It remains to look at the case when $I = 0$ just contains the orbit of θ. By Lemma 3.3 the orbit of θ must be periodic. By the Weierstrass preparation Theorem,

$$\frac{XI}{I} = \frac{(XP)Q + P(XQ)}{PQ} = \frac{XP}{P} + \frac{XQ}{Q}$$

we are reduced to look at $\frac{XP}{P}$. As above, we can apply Weierstrass' Theorem to P and obtain

$$\frac{XP}{P} = \frac{X(a_0(x,y)) + \sum_{i=1}^{m-k} X(a_i(x,y))z^i}{a_0(x,y) + \sum_{i=1}^{m-k} a_i(x,y)z^i},$$

and taking $z = 0$,

$$\frac{XP}{P}(x,y,0) = \frac{X(a_0(x,y))}{a_0(x,y)}.$$

Applying Weierstrass' Theorem to $a_0(x,y)$ and taking $y = 0$ we get an analytic function $h(x)$ such that

$$\frac{XP}{P}(x,0,0) = \frac{X(h(x))}{h(x)}$$

so we conclude as before that $h(0) \neq 0$, hence $f(\theta) = \frac{XI}{I}(\theta)$ is analytic as well. □

Lemma 3.4 finishes the proof of the main Theorem.

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