

# Manifolds that are not leaves of codimension one foliations

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## Abstract

We present new open manifolds that are not homeomorphic to leaves of any  $C^0$  codimension one foliation of a compact manifold. Among them are simply connected manifolds of dimension  $d \geq 5$  that are non-periodic in homotopy, namely in their 2-dimensional homotopy groups.

**keywords:** foliations, non-leaves, non-periodic manifolds, codimension one foliations

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## 1 Introduction

In this paper we construct a large class of open (non-compact) manifolds that cannot be leaves of codimension one foliations on compact manifolds

(see Theorems A, B, and C). They are constructed as “sum-manifolds” (see Definition 2.4) by forming connected sums of closed connected manifolds (called “blocks”), replacing each vertex of an infinite tree by a block and performing a connected sum for each edge.

These non-leaves include simply connected open manifolds that we call **non-periodic in homotopy** (or in homology, see Definition 4.1 for the precise definitions), since the second (or higher) homotopy group (or homology group) is not periodic at infinity. These non-periodic sum-manifolds give the first examples of simply-connected 5-manifolds that cannot be leaves in codimension one. There are uncountably many homotopy types of such non-leaves.

**Proposition 1.1.** *For every  $d \geq 5$  there exist simply connected  $d$ -dimensional sum-manifolds that are non-periodic in homotopy and others that are non-periodic in homology.*

**Theorem B.** *Sum-manifolds that are non-periodic in homotopy (or in homology) are not homeomorphic to any leaf of a foliation of codimension one on a compact manifold.*

This Theorem follows from Theorem A, which includes and generalizes the results of Ghys [4] and Inaba *et al.* [6], who used the fundamental group of certain open manifolds of dimension  $d \geq 3$  to show that they cannot be homeomorphic to leaves of any codimension one foliation of a compact manifold.

**History.** The problem of deciding when a manifold is homeomorphic or diffeomorphic to a leaf of a foliation (the “realizability problem”) is well-known. It was initially proposed by J. Sondow in 1975 [12]. Since every manifold is a leaf in a product foliation, one considers non-compact manifolds and asks whether they can be leaves in foliations of compact manifolds. Cantwell and Conlon [2] showed that every open surface is diffeomorphic to a leaf of a foliation on a compact 3-manifold, and as already mentioned, Ghys [4] and Inaba, Nishimori, Takamura, and Tsuchiya [6] showed that this does not hold in higher dimensions. Attie and Hurder [1], using the first Pontrjagin class, constructed simply connected examples of dimension 6 or greater that are not leaves in codimension one. There are also various constructions of open Riemannian manifolds that are not quasi-isometric to leaves of foliations on a compact manifold [1, 14, 9, 10].

In Section 2 we give several definitions and outline the construction of sum-manifolds that are non-periodic in homotopy (and in homology) using Proposition 3.1, which is proven by constructing certain blocks in Section

5. The proof of our main result, Theorem A, is given in Section 6 using two lemmas proven in Section 2. The proof follows the proof of Ghys [4] in the initial steps, but the final part of the proof is considerably simpler. The precise definition of manifolds that are non-periodic in homotopy or homology is given in Section 4. In this Section and the following one it is shown that Theorem A implies Theorem B. Theorem C (in Section 2) is a version of Theorem A that applies directly to the construction of sum-manifolds.

Some interesting open questions remain. Is it possible for a leaf in a codimension one foliation of a compact manifold to have an isolated non-periodic end? Are there any simply connected manifolds of dimension 3 or 4 that are not homeomorphic to leaves in codimension one foliations of a compact manifold?

This work is based on part of the doctoral thesis [13] of the first author at PUC-Rio de Janeiro under the supervision of the second author.

## 2 Blocks, sum-manifolds, and results

**Definition 2.1.** *A **block** (of dimension  $d$ ) is a compact connected  $d$ -dimensional manifold without boundary. If  $B$  is a block, then a **deleted  $B$ -block** is a manifold homeomorphic to  $B$  less the interiors of finitely many disjoint collared closed  $d$ -balls. A block is **trivial** if it is homeomorphic to a sphere, and it is **prime** if it cannot be decomposed as a connected sum of two non-trivial blocks, or, equivalently, if every collared embedded  $(d-1)$ -sphere bounds a  $d$ -ball.*

**Proposition 2.2.** *Every closed connected manifold is homeomorphic to a connected sum of a finite number of prime blocks.*

The proof of this Proposition is given in Section 3. Note that the decomposition is not unique for any dimension  $d \geq 2$ , since the connected sum of a projective plane and a torus is homeomorphic to the connected sum of a projective plane with a Klein bottle and this example generalizes to higher dimensions. On the other hand, when we consider oriented manifolds (so that the connected sum operation is well defined) Milnor [8] has shown that the decomposition is unique for oriented 3-manifolds, but not for oriented 4-manifolds.

**Example 2.3.** ([8], p. 6) If  $P$  is the complex projective plane with the usual orientation and  $P'$  is the same manifold with the opposite orientation, then  $P \# P' \# P'$  is diffeomorphic to  $(S^2 \times S^2) \# P'$ , while  $S^2 \times S^2$  is not homeomorphic to  $P$  or  $P'$ , and they are prime 4-dimensional blocks.

Now let  $B_n$  be disjoint  $d$ -dimensional blocks,  $n = 1, 2, \dots$ , and let  $G$  be an infinite connected graph (always supposed to be locally finite) whose set of vertices is  $\{v_n\}_{n \in \mathbb{N}}$  and whose set of edges is  $\{e_m\}_{m \in \mathbb{N}}$ . For each edge  $e_m$  of  $G$  with endpoints  $v_{n_1}$  and  $v_{n_2}$ , perform a connected sum operation between  $B_{n_1}$  and  $B_{n_2}$  to obtain a non-compact connected manifold  $W$  as in [6]. An analogous construction can be done using a finite graph.

**Definition 2.4.** *The manifold  $W$  constructed in this way is a **sum-manifold patterned on the graph  $G$**  (See Figure 1).*

We recall that a tree is a connected and simply connected locally finite graph and observe that the examples of codimension one non-leaves given by Ghys [4] and Inaba *et al.* [6] are sum-manifolds patterned on infinite trees.

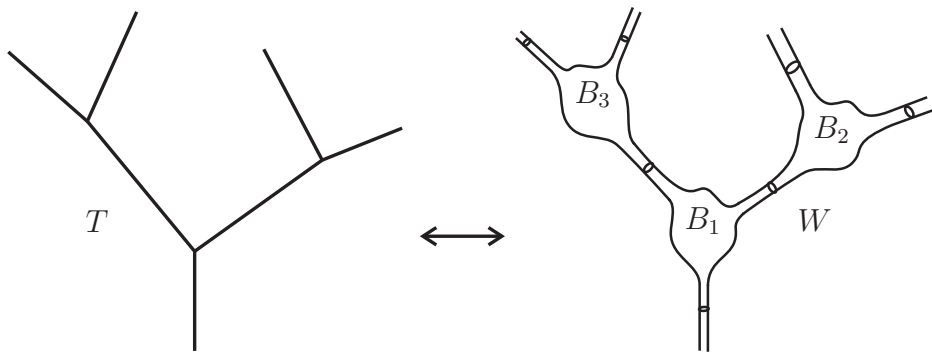


Figure 1: A connected sum patterned on the tree  $T$ .

**Definition 2.5.** *A block  $B$  repeats finitely in a manifold  $W$  (both of dimension  $d$ ) if  $W$  contains a deleted  $B$ -block, but  $W$  does not contain an infinite family of disjoint deleted  $B$ -blocks. The block  $B$  repeats infinitely if  $W$  does contain an infinite family of disjoint deleted  $B$ -blocks.*

The following theorem, proven in §6, is our main result.

**Theorem A.** *Let  $W$  be a sum-manifold patterned on an infinite tree such that the fundamental group of each block is generated by torsion elements of odd order (or is trivial) and infinitely many non-homeomorphic blocks repeat finitely. Then  $W$  is not homeomorphic to any leaf of a codimension one foliation of a compact manifold.*

In order to get blocks that repeat finitely, we shall use sets of blocks with certain properties, which we now define.

**Definition 2.6.** *A set  $\mathcal{S}$  of blocks of dimension  $d \geq 3$  is **non-repeating** if the two following conditions are satisfied:*

1. If  $B_1, B_2 \in \mathcal{S}$  are distinct blocks and  $\pi_1(B_1)$  and  $\pi_1(B_2)$  are expressed as free products of prime groups, then  $\pi_1(B_1)$  and  $\pi_1(B_2)$  have no isomorphic prime factors;
2. If a block  $B_1 \in \mathcal{S}$  is simply connected, then there exist a number  $r, 1 < r < d$  and a cyclic group  $\mathbb{Z}_{p^j}$  of prime power order which is isomorphic to a direct summand of  $H_r(B_1)$ , but not to a direct summand of  $H_r(B)$  for any other block  $B \in \mathcal{S}$ .

**Proposition 2.7.** *If  $W$  is a sum-manifold patterned on a tree using only blocks from a non-repeating set, and a block  $B$  is used for only a finite number of vertices, then the block  $B$  repeats finitely in  $W$ .*

This proposition will be proven in §3. It should be noted that it is not obvious, since even if  $B$  is an prime block, a deleted  $B$ -block may occur in a sum-manifold in which no vertex was replaced by a block homeomorphic to  $B$ , as in Example 2.3, where  $S^2 \times S^2 \setminus \text{Int}(D^4)$  is a deleted prime block which embeds in  $P \# P' \# P'$ . Theorem A and this proposition together imply the following result.

**Theorem C.** *Let  $W$  be a sum-manifold patterned on an infinite tree using only blocks from a non-repeating set  $\mathcal{S}$  such that each block has a fundamental group generated by torsion elements of odd order. If the set  $\mathcal{T} \subseteq \mathcal{S}$  of blocks that are each used a finite non-zero number of times to replace vertices of the tree in the construction of  $W$  as a sum-manifold is infinite, then the blocks in  $\mathcal{T}$  repeat finitely, and consequently  $W$  is not homeomorphic to any leaf of a codimension one foliation of a compact manifold.*

### 3 Prime blocks

In this section we prove Propositions 2.2 and 2.7, but first we observe that many prime blocks exist.

**Proposition 3.1.** *1. A 3-dimensional block  $B$  whose fundamental group  $\pi_1(B)$  is not isomorphic to a free product of non-trivial groups (for example, a lens space) is prime.*

*2. For every dimension  $d \geq 5$  and every positive integer  $n$ , there is a prime 1-connected  $d$ -dimensional block  $B$  with  $\pi_2(B)$  isomorphic to  $Z_n$  if  $d \geq 6$ , or to  $Z_n$  or  $Z_n \oplus Z_n$  if  $d = 5$ .*

Milnor [8] remarks that the first assertion follows easily from the Poincaré conjecture that a simply connected closed connected 3-manifold is homeomorphic to the 3-sphere. It has been proven by G. Perelman. If a 3-dimensional block is a connected sum of two non-trivial blocks  $B_1$  and  $B_2$ , the Seifert-van Kampen Theorem shows that its fundamental group must be isomorphic to the non-trivial free product  $\pi_1(B_1)*\pi_1(B_2)$ . The proof of the second assertion is given in §5.

In the proof of Proposition 2.2 we shall use the Grushko-Neumann Theorem ([7], vol. 2, p. 58):

**Theorem 3.2.** (Grushko-Neumann) *Every finitely generated group can be represented as a free product of a finite number of groups that are indecomposable as free products, and the decomposition is unique up to isomorphism and the order of the factors.*

*Proof of Proposition 2.2.* Let  $B$  be a block of dimension  $d$  that is homeomorphic to  $B_1\#\dots\#B_k$  for some  $k$  with all blocks  $B_i$  non-trivial. Since the Proposition is well-known for dimensions 1 and 2, we assume that  $d \geq 3$ . Reorder the factors so that the blocks  $B_i$  for  $1 \leq i \leq \ell$  are the ones which are not simply connected. Now  $\pi_1(B)$  is isomorphic to the free product  $\pi_1(B_1)*\dots*\pi_1(B_\ell)$  by the Seifert-van Kampen Theorem. Since these groups are all finitely generated, the Grushko-Neumann Theorem shows that each has a unique decomposition as a free product of a finite number of groups that are indecomposable as free products. If  $\pi_1(B)$  has  $m$  free factors, then  $\ell \leq m$ , since each  $\pi_1(B_i)$  contributes at least one free factor.

Now the direct sum of the homology groups  $\bigoplus_{r=2}^{d-1} H_r(B)$  is a finitely generated abelian group, so it is a direct sum of a finite number, say  $n$ , of cyclic groups of infinite or prime power order. If  $B$  is a connected sum of two blocks, so that  $B$  is the union of deleted blocks  $B'$  and  $B''$  with  $B' \cap B'' = S^{d-1}$ , then it follows from the Mayer-Vietoris exact sequence of homology groups

$$H_{r-1}(S^{d-1}) \rightarrow H_r(B') \oplus H_r(B'') \rightarrow H_r(B) \rightarrow H_r(S^{d-1})$$

that  $H_r(B) \approx H_r(B') \oplus H_r(B'')$  for  $1 < r < d$ , since the first and last terms vanish. By induction it follows that  $H_r(B) = \bigoplus_{i=1}^k H_r(B_i)$  for every  $r, 1 < r < d$ , so at most  $n$  blocks  $B_i$  can have  $H_r(B_i) \neq 0$  for some  $r, 1 < r < d$ . By the generalized Poincaré conjecture, every  $B_i$  with  $i > \ell$  must have such a non-trivial homology group, for otherwise it would be a  $d$ -sphere, which is excluded. Thus  $k - m \leq n$  and  $k$  cannot be greater than  $m + n$ . If we choose the connected sum decomposition so that  $k$  is maximal, then each summand  $B_i$  must be prime.  $\square$

We shall use the following result.

**Proposition 3.3.** *Any compact set  $K$  contained in a sum-manifold  $W$  must be contained in the union of a finite number of the deleted blocks whose union is  $W$ .*

*Proof.* Let the deleted  $B_n$ -blocks whose union is  $W$  be  $B'_n, n \in \mathbb{N}$ , so that  $W = \bigcup_{n=1}^{\infty} B'_n$  and the interiors of the  $B'_n$ 's are disjoint. Let  $A_n$  be the interior of  $B'_1 \cup \dots \cup B'_n$  in  $W$ . The sets  $A_n$  form a nested open cover of  $K$  in  $W$ . Since  $K$  is compact there exists an  $n_0 \geq 1$  such that  $K$  is contained in  $A_{n_0}$ , but no  $B'_n$  with  $n > n_0$  can meet  $A_{n_0}$ .  $\square$

**Lemma 3.4.** *If a deleted block  $B$  of dimension  $d \geq 3$  is contained in a connected  $d$ -manifold  $M$ , then  $\pi_1(B)$  is isomorphic to a free factor of  $\pi_1(M)$ , i.e., there exists a finitely generated group  $H$  such that  $\pi_1(M) \approx \pi_1(B) * H$ .*

*Proof.* Note that  $\partial B = S_1 \sqcup \dots \sqcup S_\ell$  is a disjoint union of  $(d-1)$ -spheres. Let  $B, L_1, \dots, L_r$  be the connected components obtained from  $M$  by cutting it along the spheres  $S_1, \dots, S_\ell$ . Let us glue the pieces  $L_i$  to  $B'$  successively along  $r$  of the spheres to obtain a connected manifold  $L$ , with the remaining spheres (if any) as boundary components of  $L$ . Observe that repeated application of the Seifert-van Kampen Theorem shows that  $\pi_1(L) \approx \pi_1(B) * G$  where  $G = \pi_1(L_1) * \dots * \pi_1(L_r)$ .

Now glue the remaining boundary spheres (if any) together to obtain  $M$ . If there remain  $s$  pairs of spheres to be identified to obtain  $M$  from  $L$ , then  $\pi_1(M) \approx \pi_1(L) * \mathbb{Z}^s$ , since each pair that is glued changes the fundamental group by forming a free product with  $\mathbb{Z}$ . Then  $\pi_1(M) \approx \pi_1(B) * G * \mathbb{Z}^s$ , as desired.  $\square$

**Lemma 3.5.** *If  $B_1, \dots, B_k$  are disjoint  $d$ -dimensional deleted blocks contained in a manifold  $W$  of dimension  $d \geq 3$  (possibly with boundary) then  $\pi_1(W)$  is isomorphic to the free product of two subgroups,  $\pi_1(W) = G * H$ , with  $G \approx \pi_1(B_1) * \dots * \pi_1(B_k)$ .*

The proof of this lemma is similar to that of the previous lemma and is omitted. The following lemma is proven in a similar way, using the Mayer-Vietoris exact sequence as in the proof of Proposition 2.2, instead of the Seifert-van Kampen Theorem.

**Lemma 3.6.** *If  $B_1, \dots, B_k$  are disjoint  $d$ -dimensional deleted blocks contained in a manifold  $W$  of dimension  $d \geq 3$  (possibly with boundary) and  $2 \leq r < d$ , then  $H_r(W)$  is isomorphic to the direct sum of two subgroups,  $H_r(W) = G \oplus H$ , with  $G \approx H_r(B_1) \oplus \dots \oplus H_r(B_k)$ .*



*Proof of Proposition 2.7.* Let  $B'_1, \dots, B'_\ell$  be disjoint deleted non-trivial  $B$ -blocks contained in a sum-manifold  $W$  patterned on a tree  $T$  and suppose that only  $k$  deleted  $B$ -blocks are used in constructing  $W$  and all the blocks used are taken from a non-repeating set. By Proposition 3.3, the compact disjoint union  $B'_1 \sqcup \dots \sqcup B'_\ell$  is contained in a compact sum-manifold  $W_1$  patterned on a finite subtree  $T_1$  using  $k_1 \leq k$  deleted  $B$ -blocks. Let  $v_1, \dots, v_n$  be all the vertices of  $T_1$ , and let  $B_i$  replace the vertex  $v_i$  in the construction of  $W_1$ . We distinguish two cases.

Case 1.  $\pi_1(B) \neq 1$ . By the Seifert-van Kampen Theorem  $\pi_1(W_1)$  is the free product  $\pi_1(B_1) * \dots * \pi_1(B_n)$ . By property (1) of Definition 2.6, some indecomposable factor of  $\pi_1(B)$  appears only  $k_1$  times in the free product decomposition of  $\pi_1(W_1)$  into indecomposable factors. On the other hand, by Lemma 3.5, this group must appear at least  $\ell$  times, so  $\ell \leq k_1 \leq k$ .

Case 2.  $\pi_1(B)$  is trivial. Since the simply connected block  $B$  is non trivial, by property (2) of Definition 2.6, for some dimension  $r$  with  $2 \leq r < d$ ,  $H_r(B)$  is non-trivial and contains a non-trivial direct summand  $\mathbb{Z}_{p^s}$ , so that no other block in the non-repeating set has a summand isomorphic  $\mathbb{Z}_{p^s}$  in its  $r$ th homology. By Lemma 3.6,  $H_r(W_1)$  contains at least  $\ell$  direct summands isomorphic to  $\mathbb{Z}_{p^s}$ . Now by the Mayer-Vietoris exact sequence  $H_r(W)$  is the direct sum of the  $r$ th homology groups of the blocks used in constructing  $W_1$ . Hence the group  $\mathbb{Z}_{p^s}$  occurs exactly  $k_1$  times in the direct sum decomposition of  $H_r(W)$  into cyclic groups, so again  $\ell \leq k_1 \leq k$ .  $\square$

## 4 Manifolds that are non-periodic in homotopy and homology

**Definition 4.1.** A  $(k - 1)$ -connected manifold  $M$  is **non-periodic in homotopy** in dimension  $k \geq 2$  if its  $k$ th homotopy group  $\pi_k(M)$  is isomorphic to the direct sum of cyclic groups of odd prime power order, where for an infinite number of odd prime powers  $p^k$ , the number of summands of order  $p^k$  is finite but non zero. A  $k$ -manifold is **non-periodic in homology** in dimension  $k \geq 2$  if its  $k$ th homology group  $H_k(M)$  is isomorphic to a direct sum of cyclic groups satisfying the same property.

**Definition 4.2.** A  $(k - 1)$ -connected manifold  $M$  is **non-periodic in homotopy** in dimension  $k \geq 2$  if its  $k$ th homotopy group  $\pi_k(M)$  is isomorphic to the direct sum of finite abelian  $p_n$ -groups  $G_n, n = 1, \dots$ , where  $(p_n)_{n \in \mathbb{N}}$  is a sequence of odd prime numbers such that the set  $R_i = \{n \in \mathbb{N}; p_n = p_i\}$  is finite for infinitely many indices  $i \geq 1$ . A  $k$ -manifold is **non-periodic in homology** in dimension  $k \geq 2$  if its  $k$ th homology group  $H_k(M)$  is isomor-



phic to the direct sum of finite abelian  $p_n$ -groups  $G_n, n = 1, \dots$ , satisfying the above properties.

Using blocks obtained in Proposition 3.1 it is easy to construct many  $(k - 1)$ -connected sum-manifolds that are non-periodic in homotopy (and in homology) of dimension  $k \geq 2$ .

*Proof of Proposition 1.1.* Let  $(p_n)_{n \in \mathbb{N}}$  be a sequence of prime numbers such that infinitely many primes occur finitely many times in the sequence and let  $T$  be a countable tree with vertices  $\{v_n\}_{n \in \mathbb{N}}$ . Given  $d \geq 5$ , let  $B_n$  ( $n \in \mathbb{N}$ ) be blocks of dimension  $d$  such that  $\pi_2(B_n)$  is  $\mathbb{Z}_{p_n} \oplus \mathbb{Z}_{p_n}$  or  $\mathbb{Z}_{p_n}$ , as obtained in Proposition 3.1. Now  $H_2(B_n) \approx \pi_2(B_n)$  by the Hurewicz Theorem. Since the deleted blocks are identified along  $(d - 1)$ -spheres with  $d \geq 5$ , the Mayer-Vietoris sequence shows that  $H_2(W)$  is the direct sum of the groups  $H_2(B_n)$  and this sum is isomorphic to  $\pi_2(W)$ . (This argument is given in more detail in the proof of Proposition 4.3 below.)

The  $d$ -manifold  $W$  constructed as a connected sum patterned on  $T$  using the blocks  $B_n$  is clearly non-periodic in homotopy and in homology.  $\square$

**Proposition 4.3.** *Sum-manifolds that are non-periodic in homotopy or in homology satisfy the hypotheses of Theorem A.*

The proof of this proposition will be given in Section 5. Together with Theorem A it establishes Theorem B.

## 5 Construction of blocks and proof of Proposition 3.1

In this section we construct simply connected blocks with given homology groups in dimension 2 to prove the second part of Proposition 3.1.

*Proof of Proposition 3.1.* In case the dimension  $d$  is 5, the main theorem of Smale [11] states that for every finitely generated free abelian group  $F$  and finite abelian group  $G$  there is a 1-connected smooth closed 5-manifold  $B$  with  $\pi_2(B) \approx F \oplus G \oplus G$ . If we take  $F$  to be trivial and set  $G = \mathbb{Z}_p$  then  $\pi_2(B) \approx \mathbb{Z}_p \oplus \mathbb{Z}_p$ , as desired.

For  $d \geq 6$  let  $f : S^1 = \partial D^2 \rightarrow S^1$  be a smooth non-singular map of degree  $p$ , where  $D^m$  denotes the closed  $m$ -dimensional disk, so that  $H_1(S^1 \cup_f D^2) \approx \mathbb{Z}_p$ . The 2-dimensional complex  $S^1 \cup_f D^2$  can be embedded in  $\mathbb{R}^5$  and so its suspension  $K$  can be considered a subset of  $\mathbb{R}^d$ ,

$$K = S^2 \cup_{Sf} D^3 = S(S^1 \cup_f D^2) \subset \mathbb{R}^6 \subseteq \mathbb{R}^d.$$

Take a closed tubular neighborhood  $V$  of  $K$  in  $\mathbb{R}^d$  and identify the boundaries of two copies of  $V$ , say  $V_1$  and  $V_2$ , to obtain the double of  $V$ , a compact 1-connected  $d$ -manifold which we denote by  $B$ . Note that  $\partial V$  is 1-connected, since a loop in  $\partial V$  has a contraction in  $V$ , and the contracting singular surface can be pushed off  $K$  and into  $\partial V$ . The inclusion  $\partial V \hookrightarrow V_i$  induces a surjective homomorphism on the second homology  $H_2(\partial V) \rightarrow H_2(V_i)$ ,  $i = 1, 2$ , for the same dimensional reason. Then the Mayer-Vietoris sequence

$$\cdots \rightarrow H_2(\partial V) \rightarrow H_2(V_1) \oplus H_2(V_2) \rightarrow H_2(B) \rightarrow H_1(\partial V) \rightarrow \cdots$$

shows that  $H_2(B) \approx \mathbb{Z}_p$ , so by the Hurewicz theorem,  $\pi_2(B) \approx \mathbb{Z}_p$ .

Now it is not clear that the blocks constructed with either  $\mathbb{Z}_{p^k}$  or  $\mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^k}$  as second homology are prime, but by Proposition 2.2 they can be expressed as connected sums of a finite number of prime blocks, one of which must have  $H_2(B)$  isomorphic to  $\mathbb{Z}_{p^k}$  or  $\mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^k}$ .  $\square$

The proof of the Proposition can easily be adapted to show the existence of prime  $(k - 1)$ -connected blocks of dimension  $k + 3$  or greater whose  $k$ th homotopy group is  $\mathbb{Z}_{p^k}$ .

*Proof of Proposition 4.3.* Let  $W$  be a connected sum of blocks  $B_n$ ,  $n \in \mathbb{N}$ , patterned on a tree  $T$ , such that  $\pi_2(B_n)$  is isomorphic to  $G_n$ , where these groups satisfy the conditions of Definition 4.1. Let  $i$  be one of the infinitely many indices for which  $R_i = \{n \in \mathbb{N}; p_n = p_i\}$  is finite. When  $d \geq 6$ , take  $k$  to be greater than the number of elements in  $R_i$ . We claim that  $W$  cannot contain  $k$  deleted blocks corresponding to  $B_i$ , so that  $B_i$  repeats finitely in  $W$ .

Suppose, to obtain a contradiction, that  $W$  contains  $k$  disjoint submanifolds with boundary, say  $W_1, \dots, W_k$ , each homeomorphic to  $B_i$  minus a finite set of open  $d$ -balls, where  $k$  is greater than the number of elements of  $R_i$ . Let  $Y$  be the closure of  $W \setminus (W_1 \cup \dots \cup W_k)$ . Thus,  $Y \cap (W_1 \cup \dots \cup W_k)$  is a disjoint union of  $(d - 1)$ -spheres. Applying the exact sequence of Mayer-Vietoris to  $W = Y \cup (W_1 \cup \dots \cup W_k)$  we have:

$$0 \rightarrow H_2(Y) \oplus H_2\left(\bigcup_{l=1}^k W_l\right) \xrightarrow{\varphi} H_2(W)$$

so the homomorphism  $\varphi$  is injective. In particular,

$$H_2\left(\bigcup_{l=1}^k W_l\right) \simeq \underbrace{\mathbb{Z}_{p_i} \oplus \dots \oplus \mathbb{Z}_{p_i}}_{k\text{-times}}$$

injects into  $H_2(W)$ , but that is a contradiction because  $H_2(W)$  has less than  $k$  copies of  $\mathbb{Z}_{p_i}$ .

For the case  $d = 5$ , each group  $H_2(B_n)$  with  $n \in R_i$  is isomorphic to  $\mathbb{Z}_{p_i} \oplus \mathbb{Z}_{p_i}$ , so a similar argument works if we take  $k$  greater than twice the number of elements of  $R_i$ .  $\square$

We note that for  $k \geq 3$  there is an analogous construction for  $(k - 1)$ -connected blocks  $B$  of dimension  $d \geq k + 4$  with  $\pi_k(B) \approx \mathbb{Z}_p$  by using the  $(k - 1)$ st suspension of the embedding  $\mathbb{S}^1 \cup_f D^2 \hookrightarrow \mathbb{R}^5$ . We obtain open  $(k + 4)$ -manifolds that are non-periodic in homotopy (and in homology) of dimension less than or equal to  $k$ , possibly mixing homotopy and homology groups in distinct dimensions. It is easy to adapt the proof of Theorem B to show that manifolds that are non-periodic in homotopy and homology of higher dimensions cannot be codimension one leaves in compact manifolds.

## 6 Non-leaves of foliations of codimension one

In this section we give the proof of Theorem A which implies Theorems B and C. The proof is a small adaptation of the proof of Ghys' main theorem in [4] (which it generalizes), although the final arguments, involving blocks that repeat finitely, are simpler.

Throughout this section we let  $L$  be a manifold that satisfies the hypotheses of Theorem A, so it is a connected sum of blocks patterned on a countable tree. Furthermore infinitely many blocks repeat finitely and every block has a fundamental group generated by torsion elements with no 2-torsion (possibly the block may be simply connected). We suppose that  $L$  is a leaf of a  $C^0$  foliation  $\mathcal{F}$  of codimension one of a compact manifold  $M$ , in order to arrive at a contradiction.

Since every block has a fundamental group generated by torsion elements with no 2-torsion (or possibly trivial), each block must have trivial holonomy in  $\mathcal{F}$ , since every torsion element of odd order has trivial holonomy and then the holonomy of the whole leaf  $L$  must be trivial. We may assume that  $\mathcal{F}$  is transversely oriented, by passing to a double cover if necessary. Under these conditions the leaf  $L$  is proper. In fact, as in Lemma 4.3 of Ghys [4], we can take a block of  $L$  which repeats finitely and as the holonomy of  $L$  is trivial the Reeb Stability Theorem provides us with an embedding of a product neighborhood of this block on  $M$ . Thus, each transversal section induced by the embedding intercepts the leaf  $L$  in a finite number of points.

Fix a one-dimensional foliation  $\mathcal{N}$  transverse to  $\mathcal{F}$  (see [5], Theorem 1.1.1, pp. 2-3). Given an open saturated set  $U \subset M$ , there is a manifold  $\widehat{U}$  with boundary and corners, called the **completion** of  $U$ , and an immersion  $i : \widehat{U} \rightarrow M$  such that  $U$  is the interior of  $\widehat{U}$  and  $i$  restricted to  $U$  is the

inclusion of  $U$  in  $M$  (see [5] pp. 87–88 for the explicit construction of  $\widehat{U}$ ). The foliations  $i^*\mathcal{F}$  and  $i^*\mathcal{N}$  agree with  $\mathcal{F}$  and  $\mathcal{N}$  on  $U$ .

**Theorem 6.1** (Dippolito [3], Hector [5]). *Under the above hypotheses, there is a compact manifold with boundary and corners  $K$  in  $\widehat{U}$  such that  $\partial K = \partial^{tg} \cup \partial^{tr}$  where*

1.  $\partial^{tg} \subset \partial\widehat{U}$ ;
2.  $\partial^{tr}$  is saturated by the foliation  $i^*\mathcal{N}$ ;
3. the complement of the interior of  $K$  in  $\widehat{U}$  is a finite union of connected non-compact submanifolds  $B_i$  with boundary and corners, and there are non-compact manifolds with boundary  $S_i$  so that each  $B_i$  is homeomorphic to  $S_i \times [0, 1]$  by a homeomorphism  $\phi_i : S_i \times [0, 1] \rightarrow B_i$  that takes  $\{*\} \times [0, 1]$  to a leaf of  $i^*\mathcal{N}$ .

The compact set  $K$  is called the **kernel** of  $\widehat{U}$  and the submanifolds  $B_i$  are the **branches** of  $\widehat{U}$ . The foliation  $i^*\mathcal{F}$  restricted to a branch  $B_i$  is defined by the suspension of a representation of the fundamental group of  $S_i$  into the group of orientation preserving homeomorphisms of the interval  $[0, 1]$ .

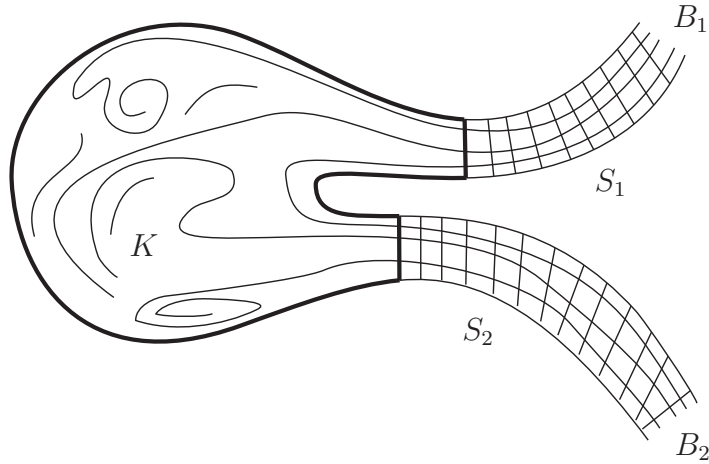


Figure 2: The completion of a saturated open set.

**Proposition 6.2.** *(An extension of Reeb stability) Let  $L$  be a leaf in the boundary  $\partial\widehat{U}$ , where  $U$  is a connected saturated open set of a transversely oriented codimension one foliation of a compact manifold. If  $L$  is a connected*

sum of blocks (a sum-manifold as in Definition 2.1), and each block has a fundamental group that is trivial or generated by torsion, then  $L$  has a one-sided product foliated neighborhood in  $\widehat{\partial U}$ .

*Proof.* Note that only a finite number of blocks can meet a neighborhood of the kernel  $K$ , according to Proposition 3.3. Hence we can enlarge the nucleus  $K$  a little so that  $L \cap K$  is a union of a finite number of deleted blocks. Then for each branch  $B_i$ , the intersection  $B_i \cap L$  is also a connected sum of blocks, so its fundamental group is trivial or generated by torsion elements. Since the holonomy of  $\mathcal{F}|_{B_i}$  is globally defined, the foliation  $\mathcal{F}|_{B_i}$  is a product foliation. Similarly, by the usual Reeb stability theorem, the compact set  $L \cap K$  has trivial holonomy and therefore has a one-sided product foliated neighborhood in  $K$ . Combining the product foliated neighborhoods in the kernel  $K$  and the blocks  $B_i$  gives a one-sided product foliated neighborhood of  $L$  in  $\widehat{U}$ .  $\square$

**Lemma 6.3.** *The leaf  $L$  possesses an open neighborhood saturated by  $\mathcal{F}$  which is homeomorphic to  $L \times (-1, 1)$  by a homeomorphism that takes  $L \times \{*\}$  to leaves of  $\mathcal{F}$  and  $\{*\} \times (-1, 1)$  into leaves of  $\mathcal{N}$ .*

*Proof.* The proof uses Dippolito's Theorem and follows the proof of Lemma 4.4 of Ghys [4]. Let  $\tau : [0, 1) \rightarrow M$  be a positive transversal that meets  $L$  only in  $\tau(0)$ , and let  $U$  be the saturation of  $\tau(0, 1)$ . Then one of the leaves in the boundary of the completion  $\widehat{U}$  is homeomorphic to  $L$  and has trivial holonomy, so by Proposition 6.2,  $L$  has a one-sided saturated product neighborhood. The same argument applies to the other side of  $L$  to give the desired product neighborhood.  $\square$

*Proof of Theorem A.* Let  $\Omega$  be the union of all the leaves of  $\mathcal{F}$  that are homeomorphic to  $L$  and let  $\Omega_1$  be the connected component of  $\Omega$  that contains  $L$ . The previous lemma shows that every leaf in  $\Omega$  has an  $\mathcal{F}$ -saturated open neighborhood foliated as a product by the two foliations  $\mathcal{F}$  and  $\mathcal{N}$ . These product neighborhoods fit together to show that  $\Omega_1$  is an open saturated set that fibers over a Hausdorff manifold of dimension one, either an open interval or a circle, with the leaves of  $\mathcal{F}$  as fibers. Consider the completion  $\widehat{\Omega}_1$ .

The proof of Lemma 4.5 of Ghys [4] shows the following result.

**Lemma 6.4.** *The set  $\widehat{\Omega}_1$  is noncompact and  $\partial\widehat{\Omega}_1$  is nonempty. Every leaf of  $\mathcal{F}$  contained in  $\partial\widehat{\Omega}_1$  has an infinite cyclic holonomy group generated by a contraction.*

*Proof.* The set  $\Omega_1$  is not compact. In fact, if  $\Omega_1$  is compact, then  $\mathcal{F}|_{\Omega_1}$  possesses a minimal set  $\mu$ . Let  $F_0$  be a leaf of  $\mathcal{F}|_{\Omega_1}$  in  $\mu$ . Then  $F_0$  is dense in  $\mu$ , but given a point  $x \in F_0$  there is a neighborhood  $V$  of  $x$  in  $M$  that intercepts only one plaque of the leaf  $F_0$ , because  $F_0$  is proper. Then  $F_0$  is open in  $\mu$ , so  $\mu \setminus F_0$  is closed and therefore  $\mu \setminus F_0 = \emptyset$ . Then  $F_0$  is closed and therefore compact, but this is a contradiction. Therefore  $\Omega_1$  is not compact and  $\partial\widehat{\Omega}_1$  must be nonempty.

Let  $F$  be a leaf contained in  $\partial\widehat{\Omega}_1$  and suppose that the holonomy group of  $F$  is trivial, to get a contradiction. Since the leaves in  $\Omega$  are homeomorphic under translation along the transverse foliation  $\mathcal{N}$ , any loop in  $F \cap B_i$  must have globally trivial holonomy in  $B_i$ . In  $F \cap K$  the compact set  $F \cap N$  will have a one-sided product foliated neighborhood, and hence  $F$  must have a product neighborhood in  $\widehat{\Omega}_1$  and will be homeomorphic to  $L$ , but since  $F$  belongs to the boundary of  $\widehat{\Omega}_1$  it is not homeomorphic to  $L$ . Hence the holonomy of  $F$  must be nontrivial. The leaves in the interior of  $\widehat{\Omega}_1$  are proper, so the holonomy group of  $F$  acts discretely; it must be infinite cyclic and one of its two generators will be a contraction. Hence there is a small open neighborhood  $V$  of  $\partial\widehat{\Omega}_1$  in  $\widehat{\Omega}_1$  such that every block of  $L$  contained in  $V$  repeats infinitely.

Suppose that  $\widehat{\Omega}_1$  is compact and let  $C = \widehat{\Omega}_1 \setminus V$ . Then  $C \cap L$  is closed in  $\widehat{\Omega}_1$ , so it is a compact set contained in  $L$ . The proper leaf  $L$  has the induced topology and so  $C \cap L$  is compact in the topology of  $L$ . By Proposition 3.3 only finitely many blocks of  $L$  meet  $C \cap L$ . Therefore some block of  $L$  that repeats finitely in  $L$  is contained in  $V$ , so it must repeat infinitely, which gives a contradiction.  $\square$

If  $\Omega_1$  were fibered over an open interval  $(0, 1)$ , then it would be homeomorphic to a foliated product  $L \times (0, 1)$  and the completion would be  $\widehat{\Omega}_1 \approx L \times [0, 1]$ , which is impossible since the boundary leaves of  $\widehat{\Omega}_1$  are not homeomorphic to  $L$ .

Hence  $\Omega_1$  must fiber over the circle. The remainder of our proof simplifies considerably the argument of Ghys in Section 5 of [4]. Let  $h : L \rightarrow L$  be the monodromy map that takes a point  $x$  in  $L$  to its first return to  $L$  in the positive direction along the leaf of  $\mathcal{N}$  that contains  $x$ . The local product structure given by Lemma 6.3 shows that  $h$  is defined on all of  $\Omega_1$ . If a point  $x$  of  $L$  is in a branch  $B_i$ , then  $h(x) > x$  on the interval  $\{*\} \times [0, 1]$  in the leaf of  $\mathcal{N}$  that contains  $x$ , so  $x$  is neither fixed nor periodic. If  $x$  is in a sufficiently small open neighborhood  $V$  of  $\partial\widehat{\Omega}_1$ , then again  $h(x) > x$  and  $x$  is neither fixed nor periodic. Set  $K' = K \setminus V$  and observe that  $K' \cap L$  is compact, so by Proposition 3.3,  $K'$  meets finitely many blocks of  $L$ . Hence some block  $C$  that repeats finitely on  $L$  must be contained in  $L \cap (V \cup \bigcup B_i)$ ,

which contains no periodic points of  $h$ . Therefore there is an integer  $r > 0$  such that the compact sets  $h^{nr}(C)$ ,  $n \in \mathbb{N}$ , are pairwise disjoint, so  $C$  must repeat infinitely. This contradiction completes the proof of Theorem A.  $\square$

The manifold constructed by Ghys [4] and similar constructions patterned on an arbitrary (locally finite) tree cannot be homeomorphic to leaves in codimension one; this follows from Theorem C. Inaba *et al.* [6] show that the manifolds that they construct are not  $C^2$  diffeomorphic to leaves in codimension one, but Theorem C shows that they are not even homeomorphic to leaves in codimension one. Theorem C also gives examples of non-leaves that mix blocks of various types—with finite fundamental groups or varying higher homotopy groups—provided that infinitely many blocks repeat finitely.

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