BIRKHOFF FIRST THEOREM FOR LAGRANGIAN, INVARIANT TORI IN DIMENSION 3

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Abstract. We show that C^2 Lagrangian, minimizing torus that is invariant by the geodesic flow of a Riemannian metric g in the three torus T^3 is a graph of the canonical projection. This result known as the first Birkhoff Theorem for Lagrangian, invariant tori was proved in dimension two by Bialy and Polterovich in the early 1980's and since then, no further progress in higher dimensions has been made. As a consequence of our arguments, we get the proof of some partial versions of the first Birkhoff Theorem in any dimension.

Introduction

The theory of Lagrangian submanifolds that are invariant by a Hamiltonian flow has two well known statements, called Birkhoff Theorems for Lagrangian invariant tori, which can be viewed as higher dimensional versions of the celebrated Birkhoff Theorems for invariant curves of measure preserving twist maps of the annulus.

Let $L : TM \longrightarrow \mathbb{R}$ be a convex, superlinear Lagrangian (or optic Lagrangian) defined in the tangent space TM of an n dimensional C^∞ manifold M. Let $\pi : TM \longrightarrow M$, $\pi(p, v) = p$, be the canonical projection. The first Birkhoff Theorem asserts that each continuous, Lagrangian torus $W \subset TT^n$ that is invariant by the Euler-Lagrange flow of L and that is minimizing, i.e., each orbit in W projects into a global minimizer of the Lagrangian action of L (see Section 1 for details), is in fact a graph of the canonical projection. Namely, the canonical projection restricted to W is a homeomorphism. The second Birkhoff Theorem states that a continuous, Lagrangian torus W, invariant by the Euler-Lagrange flow of L and that is homologous to the zero section must be a graph of the canonical projection. Notice that both problems seek to show that the set of points in the Lagrangian torus where the canonical projection is not injective is empty. If we assume in addition that the Lagrangian torus in the statements is smooth, this is equivalent to show that the set of singular points of the canonical projection restricted to the Lagrangian torus is empty.

The above statements are actually conjectures, and the best understood of them is the second one. Let us make a brief account of the results known up to date. For two dimensional tori, Bialy [4] proved the second Birkhoff Theorem assuming that the Lagrangian is a reversible Finsler metric and that the Lagrangian, invariant torus has no periodic orbit. Carneiro-Ruggiero [11] showed the second Birkhoff Theorem for two dimensional reversible Finsler metrics and Lagrangian, invariant

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tori with closed orbits. So the second Birkhoff Theorem is a "true " theorem for Lagrangian invariant tori of reversible Finsler metrics of T^2. In higher dimensions, Bialy-Polterovich [6], [7], Polterovich [26] showed the second Birkhoff Theorem provided that the Lagrangian, invariant torus is C^3 and that the Lagrangian flow in the Lagrangian torus is chain recurrent. The smoothness assumption on the Lagrangian torus is very important, it allows the application of powerful tools of symplectic topology to study the singular set of the restriction of the canonical projection to the Lagrangian torus. The proof in this case relies strongly on a result proved by Viterbo [28] which essentially implies that the Maslov cycle of a chain recurrent, C^3, Lagrangian, invariant torus is trivial. Without the chain recurrence assumption, the second Birkhoff Theorem for C^3 Lagrangian, invariant tori remains an open problem. The first Birkhoff Theorem is much less explored in the literature. It was proved for Lagrangian invariant tori of Riemannian metrics in T^2 by Bialy and Polterovich [5] in the early 1980's; it was proved for Lagrangian, invariant, nonsingular tori in the Mañé critical level of an optical Lagrangian in T^2 by Carneiro and Ruggiero [10] (2004). And as far as we know, no further progress has been made in the subject (even assuming smoothness of the Lagrangian invariant tori) in higher dimensions. Although the assumptions in both theorems are of course related, the assumption in the first Theorem is different in nature to the assumption in the second one. The minimizing assumption in the first Theorem comes from calculus of variations, while the homological assumption on the second Theorem is more topological. Indeed, tools of symplectic topology, notably the Maslov index, proved to be very fruitful to tackle the second Birkhoff Theorem. While a mix of direct calculus of variations and two-dimensional dynamics of the geodesic flow in an invariant two torus gives the approach to study the first Theorem. This is perhaps why the second Theorem is better understood than the first. The dynamics of nonsingular flows in the two torus is quite simple, in particular, the nature of the nonwandering set is well understood. And since the work of Morse and Hedlund it is known that accumulation properties of globally minimizing geodesics determine their intersection properties (notice that an invariant minimizing torus is a graph if the canonical projection of its orbits provide a flow by geodesics of the corresponding metric). On the other hand, in higher dimensions the nonwandering set of a nonsingular flow in the torus might be extremely more complicated than in dimension two, giving no clue at all about the accumulation properties of minimizing orbits in an invariant torus. It is not difficult to show that a C^1, Lagrangian, invariant torus that is a graph must be minimizing. So if the second Theorem is true, every C^1 Lagrangian, invariant torus homologous to the zero section would be a minimizing one (in particular, C^3 Lagrangian, invariant tori homologous to the zero section and chain recurrent dynamics are minimizing).

The Maslov cycle is closely related to the set of singular points of the canonical projection in smooth Lagrangian, invariant tori. Since it is an homological invariant it vanishes when the Lagrangian torus is homologous to the zero section. From this fact and the chain recurrence assumption on the dynamics it is not hard to show the second Birkhoff Theorem (see [25] for instance). This is roughly speaking, the main idea in [26]. We might think that the same idea applies to study minimizing tori. However, the minimizing assumption in the first Birkhoff Theorem does not give any hint about the triviality of the Maslov cycle. The main result of the paper is the following:

Theorem A: Let (T^3, g) be a C^∞ Riemannian metric in the 3-torus, and let $T \subset T_1T^3$ be a C^2 invariant, Lagrangian torus that is minimizing. Then the canonical projection $\pi : T_1T^3 \longrightarrow T^3$ is a diffeomorphism.

The proof of Theorem A can be extended to Finsler metrics in the torus, therefore to Lagrangians in sufficiently high energy levels of Tonelli Lagrangians. The whole point of the paper is to show that the assumptions on Theorem A imply that the set of singular points of the canonical projection in the Lagrangian torus is empty.

Let us make a brief account of the main ideas of the proof. Let us suppose that a C^1, Lagrangian, invariant torus W in T_1M is not a graph of the canonical projection. We study first the topological structure of the singular set, we show that it consists on a finite collection of C^0, codimension 1 submanifolds of T_1M which are topologically transverse to the geodesic flow in W. Since the torus W is minimizing, an orbit can meet the singular set at most at one point and hence recurrent orbits in W are contained in regular components.

As a byproduct of this idea we get that C^1, Lagrangian, invariant minimizing tori which are transitive are graphs of the canonical projection, in any dimension. This statement is interesting from the point of view of Aubry-Mather theory, which is considered nowadays as the main body of results about intersection properties of minimizing orbits. The so-called Mather graph theorem yields that if the support of a minimizing measure is dense in a Lagrangian, invariant torus, then such a torus is a graph of the canonical projection. If the geodesic flow in a Lagrangian, minimizing, invariant torus is transitive, it is not clear that the whole torus is the support of a minimizing measure. So Mather graph theorem might not apply straightforwardly to transitive, Lagrangian, minimizing tori.

In the general case, the nonwandering set of the invariant torus is a proper subset. With a little help of Aubry-Mather theory we deduce that the degree of the canonical projection in each regular component of the projection is one whenever $M = T^n$. So the geodesic flow in W is a gradient-like flow (according to Conley [12]) equipped with a finite number of transversal sections which isolate the nonwandering set of the flow. Moreover, the canonical one form of the geodesic flow restricted to each regular component gives rise to a closed one form in the canonical projection of the component. This fact reveals a rigid transversal structure of the geodesic flow in regular components, similar to the transversal structure of nonsingular flows defined by closed one forms. Now, combining the dynamics in regular components with direct calculus of variations, and assuming W of class C^2, we show that the dimension of the intersection of the tangent space of the torus W with the vertical subspace in $T_\theta T_1T^n$ at a point θ in the singular set is neither maximal nor minimal. This is the more subtle part of the paper, perhaps our main contribution to the study of the first Birkhoff Theorem. The Maslov cycle, that is the most common tool in the study of singularities of invariant, Lagrangian submanifolds, is not the main tool of the proof. Since for $n = 3$ the maximal and the minimal dimensions are just $n = 2$ and $n = 1$, we conclude that the singular set must be empty. Therefore, the whole of W is regular and since the degree of the canonical projection in regular components is one, Theorem A holds.

1. Topological structure of the singular set of the canonical projection

Let us introduce some notations. TM is the tangent bundle of a C^∞ manifold M, T_pM is the tangent space of M at a point p, the unit tangent bundle of a Riemannian manifold (M,g) is T_1M, and the canonical projection $\pi : TM \longrightarrow M$ is the map $\pi(p,v) = p$, where (p,v) is a point in TM in canonical coordinates. We shall always assume that M is complete.

The geodesic flow of (M,g) in T_1M will be denoted by $\phi_t : T_1M \longrightarrow T_1M$. The canonical one form of the geodesic flow will be α, the pull back by the Legendre transform of the Liouville form on T_1^*M and the canonical two-form $\omega = d\alpha$. Both forms are invariant by ϕ_t and ω is symplectic: it is non-degenerate and closed. The tangent space $T_\theta T_1M$ is the orthogonal (with respect to the Sasaki metric) sum of the horizontal subspace H_θ and the vertical subspace $V_\theta = Ker(d_\theta \pi)$. The subspace $N_\theta \in T_\theta M$ of vectors which are orthogonal to the geodesic vector field $X(\theta)$ is preserved by $d\phi_t$ for every $t \in \mathbb{R}$, and we have $N_\theta = \mathcal{H}_\theta \bigoplus V_\theta$, where $\mathcal{H}_\theta = H_\theta \cap N_\theta$.

Many forthcoming definitions and results have natural generalizations to Tonelli Lagrangians: a C^∞ Lagrangian $L : TM \to \mathbb{R}$ is a Tonelli Lagrangian if it is strictly convex and superlinear (i.e., $\lim_{\|v\| \to +\infty} \frac{L(p,v)}{\|v\|} = +\infty$ for every $v \in T_pM$, for every $p \in M$) when restricted to each tangent space T_pM, $p \in M$.

1.1. Lagrangian submanifolds and the Riccati equation.

Suppose that the dimension of M is n, so the dimension of T_1M is $2n-1$. A subspace L_θ of N_θ is called Lagrangian if its dimension is $(n-1)$ and the restriction of ω to L_θ is vanishes. A continuous vector subbundle L of TT_1M is called invariant if it is is invariant by the action of $d\phi_t$ for every $t \in \mathbb{R}$.

A smooth submanifold $W \subset TM$ is called Lagrangian if its dimension is n and the two-form ω in T_pW vanishes for every $p \in W$. There is a natural notion of Lagrangian submanifolds in T_1M although its dimension is odd: $W \subset T_1M$ is called Lagrangian if the intersections $T_\theta W \cap N_\theta$ are all Lagrangian subspaces in N_θ. A smooth Lagrangian submanifold W is called a **graph** if the canonical projection π restricted to W is a diffeomorphism. Let us denote by S, the singular set, the set of points $\theta \in W$ where the canonical projection is singular, namely, where $T_\theta W \cap V_\theta$ is a nontrivial subspace.

The singular set S is a closed set in W. Moreover, given an orbit of the geodesic flow, the set of singularities is discrete (for a proof see [25]). We shall give a geometric proof of this fact assuming that the submanifold W is *minimizing*, always bearing in mind that we want to show that the singular set is actually empty.

We recall the definition of minimizing Lagrangian submanifold.

The action of the Lagrangian L in an absolutely continuous curve $c : I \to M$ is $A_L(c) = \int_I L(c(t), c'(t))dt$. Local minimizers of the action are of course the solutions of the Euler-Lagrange equation of L, in particular when $L(p,v) = \frac{1}{2} g_p(v,v)$ such local minimizers are the geodesics of (M,g).

An absolutely continuous curve $\gamma : \mathbb{R} \longrightarrow M$ is a minimizer of the Lagrangian action if for every $a < b \in \mathbb{R}$, given an absolutely continuous curve $\delta : [a,b] \longrightarrow M$ with $\delta(a) = \gamma(a)$, $\delta(b) = \gamma(b)$, we have that the action $A_L(\delta) = \int_a^b L(\delta(t), \delta'(t))dt$ of the Lagrangian L in δ is at least the action of the Lagrangian in $\gamma[a,b]$: $A_L(\delta) \geq A_L(\gamma)$.

The abelian covering M_a of a manifold M is the quotient of the universal covering by the subgroup of $\pi_1(M)$ generated by commutators $xyx^{-1}y^{-1}$. The abelian covering projection $\pi_a : M_a \longrightarrow M$ is a local homeomorphism and the first integer homology group has a natural representation in the set of automorphisms of M_a. Any Lagrangian $L : TM \longrightarrow \mathbb{R}$ lifts to a Lagrangian $L_a : TM_a \longrightarrow \mathbb{R}$ in a natural way. In particular, if (M, g) is a Riemannian manifold, the metric g lifts to a metric g_a in M_a that is locally isometric to g. As in the case of the universal covering, the first homology group acts a discrete subgroup of isometries of (M_a, g_a). Of course, if the fundamental group is abelian, M_a coincides with $\tilde{M}$. A geodesic in (M, g) is a global minimizer in the abelian covering if any of its lifts in (M_a, g_a) is a minimizer of the g_a-length. A global minimizer in the abelian covering is always a a minimizer of the universal covering, the converse of this statement is not true in general.

Definition 1.1. An invariant submanifold W is called a minimizing if any lift to the abelian cover of the projection of every trajectory in W is a global minimizer.

Let us start with a well known result linking geometry and invariant Lagrangian bundles (see [25] for instance): there is a close relationship between Lagrangian, invariant subbundles and self-adjoint operators. Many relevant geometric properties of Lagrangian submanifolds arise from this relationship.

Lemma 1.2. *Suppose that $K_0 < 0$ is a lower bound for the sectional curvatures of (M, g). Let $W \subset T_1M$ be a C^k, $k \geq 1$, complete, invariant, Lagrangian, minimizing submanifold . Then for each $\theta \in W$ such that $T_\theta W \cap V_\theta = \{0\}$, there exists a family of C^{k-1} symmetric linear maps $U_\theta : \mathcal{H}_\theta \longrightarrow V_\theta$ defined for every θ outside the singular set S such that*

1. *The subspace $T_\theta W \cap N_\theta$ is the graph of U_θ.*
2. *Taking a parallel frame $\{e_i(t)\}$, $i = 1, 2, .., n-1$ of vector fields along the geodesic $\pi(\phi_t(\theta))$, which are orthogonal to the geodesic, the matrix expressions of the operators U_θ satisfy the matrix Riccati equation*
$$U'_\theta(t) + U^2_\theta(t) + K(t) = 0,$$
where the derivatives are covariant derivatives of (M, g) taken along the geodesics $\pi(\phi_t(\theta))$, $U_\theta(t) = U_{\phi_t(\theta)}$, and $K(t)$ is the matrix of sectional curvatures of $\pi(\phi_t(\theta))$ in the basis $\{e_i(t)\}$.
3. *The family of operators $U_\theta(t)$ is defined for every $t \in \mathbb{R}$ if and only if $T_{\phi_t(\theta)}W \cap V_{\phi_t(\theta)} = \{0\}$ for every $t \in \mathbb{R}$. Moreover, in this case we have*
$$\| U_\theta(t) \|_\infty \leq \sqrt{-K_0},$$
for every $t \in \mathbb{R}$.
4. *The solution $U_\theta(t)$ has at most one singularity, namely, it is defined either in $\mathbb{R}$ or in $\mathbb{R} - \{t_0\}$ for some t_0. Singularities correspond to the points $\phi_{t_0}(\theta)$ where the tangent space of W meets the vertical space.*
5. *Given $\epsilon > 0$ there exists $L = L(\epsilon) > 0$ such that for every $\theta \in W$, if t_0 is a singularity of $U_\theta(t)$ and $| t - t_0 | > \epsilon$ then*
$$\| U_\theta(t) \|_\infty \leq L,$$
$$\| U_\theta(t) \|_\infty \leq L.$$

We just recall some highlights of the proof for the sake of completeness, the subject is quite known in geometry and classical mechanics (see the work of Eberlein [17] for manifolds without conjugate points and the work of Contreras-Iturriaga [15] for energy levels of optical Lagrangians without conjugate points). Since every orbit in W is minimizing, there are no conjugate points in the orbit. This implies that there exists a solution $\bar{U}_\theta(t)$ of the Riccati equation that is defined for every $t \in \mathbb{R}$. The trace $v_\theta(t) = trace(\bar{U}_\theta(t))$ is a real valued function that satisfies the one-dimensional Riccati equation

$$\frac{d}{dt} trace(\bar{U}_\theta(t)) + trace(\bar{U}^2_\theta(t)) + R(t) = 0,$$

where $R(t)$ is the scalar curvature at the point $\pi(\phi_t(\theta))$. The solutions of this equation have the same analytic properties of the solutions of the Riccati equation

$$u'(t) + u^2(t) + R(t) = 0.$$

The existence of a solution of $\frac{d}{dt} trace(\bar{U}_\theta(t)) + trace(\bar{U}^2_\theta(t)) + R(t) = 0$ defined for every $t \in \mathbb{R}$ implies the existence of a solution of $u'(t) + u^2(t) + R(t) = 0$ defined for every $t \in \mathbb{R}$. And the solutions of the above equation have the properties stated in Lemma 1.2. Eberlein in [17] shows that such properties extend to $v_\theta(t)$.

The singularities of $U_\theta(t)$ correspond to the points of $\phi_t(\theta)$ where the tangent space of W intersects the vertical bundle. A number t_0 is a singularity of the trace of $U_\theta(t)$ if and only if it is a singularity of $U_\theta(t)$. So Lemma 1.2 yields that along any given orbit of W there is at most one singularity of the restriction to W of the canonical projection.

1.2. C^0 **regularity of the Singular set.** The main goal of the paper is to study the set of points $\theta \in W$ where the canonical projection is singular, namely, where $T_\theta W \cap V_\theta$ is a nontrivial subspace. The study of singularities of the canonical projection restricted to Lagrangian submanifolds is classic and important in mechanics. In the case of invariant Lagrangian submanifolds, many relevant properties of such singularities come from the structure of the Riccati equation defined in the previous subsection (or equivalently, from the structure of Jacobi fields).

Through the subsection, (M, g) will be a compact C^∞ Riemannian manifold, and $W \subset T_1M$ will be a C^1 compact, Lagrangian, minimizing submanifold that is invariant by the geodesic flow. Let $S \subset W$ be the set of singular points of the restriction of the canonical projection $\pi : W \longrightarrow M$. The following statement is the best we can say about the topological structure of the singular set assuming that W is of class C^1.

Lemma 1.3. *The singular set S consists of a finite collection of compact, C^0 codimension 1 submanifolds which are topologically transversal to the geodesic flow.*

Proof. First we show that each connected component of S is a compact, C^0 submanifold. We cannot expect to get more regularity under the assumptions of Lemma 1.3 since the regularity of S is linked to the regularity of the Maslov cycle. By Lemma 1.2, each orbit in W has at most one singular point. We know that the set S is closed, let us show that

Claim 1: The set of orbits with singular points is open in W.

For suppose that an orbit $\phi_t(\theta)$ with a singular point $\phi_{t_0}(\theta)$ is accumulated by orbits without singular points. Assume that θ is singular and let $\theta_n \to \theta$ be points in regular orbits coverging to θ. Then the tangent spaces $T_{\theta_n}W$ converge to the tangent space $T_\theta W$, and since each tangent space $T_{\theta_n}W$ is the graph of $U(\theta_n)$ we have that the limits of the graphs of the operators $U(\theta_n)$ is $T_\theta W$. But now, item (3) of Lemma 1.2 says that the supremum norms of the operators $U(\theta_n)$ are uniformly bounded by some universal constant. So the limit of their graphs is also a graph of some linear operator defined in N_θ. This graph has to be $T_\theta W$ by the smoothness of W, which contradicts the fact that θ is a singular point.

So we get that there exists an open neighborhood B of θ in W where every point belongs to an orbit with a singularity, proving the Claim.

Claim 2: The singularity $\phi_{t_0}(\theta)$ of the orbit of θ depends continuously on θ.

The argument follows from Lemma 1.2 item (4). Indeed, assume that θ is singular, let B be an open neighborhood in W where every point η belongs to an orbit with a singular point $\phi_{t(\eta)}(\eta)$. Define the function $\eta \to t(\eta)$ the smallest (in absolute value) t such that $\phi_t(\eta) \in S$ The function $t(\eta)$ is continuous, otherwise we would get some $\epsilon > 0$ and a sequence $t(\eta_n)$ such that $\mid t(\eta_n) \mid > \epsilon$. This implies that there exists a sequence t_n with $\mid t_n - t(\eta_n) \mid > \frac{\epsilon}{2}$ such that $\lim_{n\to\infty} \phi_{t_n}(\eta_n) = \theta$. The tangent space of W at the points $\phi_{t_n}(\eta_n)$ is the graph of $U(\phi_{t_n}(\eta_n))$, and there is an upper bound L for the supremum norms of the operators $U(\phi_{t_n}(\eta_n))$ by item (4) of Lemma 1.2. The same argument used to show Claim 1 gives a contradiction in this case.

Now, take a local transversal section $\Sigma \subset B$ for the geodesic flow, suppose for convenience that $\theta \in \Sigma$. Let us consider the function $f_\Sigma : \Sigma \longrightarrow S$, $f_\Sigma(\eta) = \phi_{t(\eta)}$. The function f_Σ is continuous and 1-1, so it is a local homeomorphism from Σ to the singular set S, there is a $n-1$ dimensional open set $B_S \subset S$ parametrized by the graph of f_Σ. Since S is closed and W is compact, S is compact and we can cover S with a finite number of open neighborhoods B_i of the form

$$B_i = \cup_{|t|<a}\{\phi_t(\Sigma_i)\},$$

where Σ_i is a local cross section of the geodesic flow. In each of the neighborhoods there is a subset of S parametrized by the graph S_i of a continuous bijection, which implies that S is a continuous submanifold of codimension 1. Since the graphs S_i separate the neighborhoods B_i, such graphs are topologically transversal to the geodesic flow. This finishes the proof of Lemma 1.3. □

1.3. **Regular sets and dynamics.** The structure of the singular set S of the Lagrangian submanifold W gives substantial information about the dynamics of the geodesic flow in W.

Corollary 1.4. *If an orbit of the invariant submanifold W intersects the singular set then it is wandering. In this case, the intersection is just one point in the orbit.*

Proof. We know that each orbit $\phi_t(\theta)$ of W has no conjugate points, which means that the subspaces $Z(t) = D_\theta\phi_t(V_\theta)$ can meet the vertical bundle at most once. By Lemma 1.3, if $\theta \in S$ then θ is an isolated intersection of its orbit with S. And since $V_\theta \cap T_\theta W \neq \emptyset$ the subspaces $Z(t)$ will never meet the vertical subbundle outside $t = 0$. Therefore, the orbit of $\theta \in S$ intersects S at the point θ, the only point of this intersection. Moreover, if the orbit of $\theta \in S$ was non-wandering, the point θ would be accumulated by points $\phi_{t_n}(\theta)$, $t_n \to +\infty$. Since the singular set is a continuous transversal section of the geodesic flow, the orbit of θ will cross the singular set infinitely many times which is impossible. □

An immediate consequence of Corollary 1.4 is the following partial version of the main Theorem.

Corollary 1.5. *Let W be a compact C^1, minimizing, Lagrangian submanifold of T_1M that is invariant by the geodesic flow of (M, g). If there is a dense orbit in W then W is a graph.*

As we already mentioned in the Introduction, the graph property for transitive, C^3 Lagrangian, invariant tori homologous to the zero section was proved by Bialy-Polterovich [6], [7], [26].

Let $\Omega(W)$ be the set of non-wandering orbits of W, it is a compact set. The set of wandering orbits is then an open invariant set.

Corollary 1.6. *Each connected component of the set of wandering orbits has either a global cross section in the singular set or is regular for the canonical projection.*

Proof. This is simply because each connected component of S is a connected compact C^0 submanifold. So each set of the form $\bigcup_{t\in\mathbb{R}} \phi_t(C)$, where C is a connected component of the singular set S, is a connected open set. Thus, if a connected component of wandering points contains singular points, it must be one of the above sets. □

2. Aubry-Mather theory and local injectivity of the canonical projection

The purpose of the section is to show that the main theorem holds indeed in each regular component of the Lagrangian, minimizing submanifold W. This fact will have strong consequences on the geometry and the topology of regular components.

In the context of Aubry-Mather theory, the search for injectivity of the canonical projection leads us naturally to look at recurrent orbits. Let us explain briefly some basic results of Aubry-Mather theory of Tonelli Lagrangians shown by Mañé [20], [21] (see also [16]).

2.1. Minimizing measures and graph properties of their supports.

Let $\mathcal{M}(L)$ be the set of invariant probability measures of the Euler-Lagrange flow of a convex, superlinear Lagrangian L. The **action** of L in $\mathcal{M}(L)$ is defined by

$$A_L(\mu) = \int L d\mu.$$

The **homology class** (Mather) $\rho(\mu)$ of the measure μ is given by

$$< \rho(\mu), \omega >= \int \omega d\mu,$$

where ω is a closed 1-form. (Recall that the homology group $H_1(M, \mathbb{R})$ is the dual of the cohomology group $H^1(M, \mathbb{R})$).

Definition 2.1. A measure $\mu \in \mathcal{M}(L)$ is called minimizing in its homology class if

$$A_L(\mu) = \inf\{A_L(\nu), \rho(\nu) = \rho(\mu)\}.$$

Definition 2.2. A **minimizing** measure μ is defined by

$$A_L(\mu) = \inf\{A_L(\nu), \nu \in \mathcal{M}(L)\}.$$

The union of the supports of all minimizing measures is called the **Mather set** for L. Global minimizers in the abelian covering are closely related to supports of minimizing measures, the following two results describe important links between them.

Theorem 2.3. *((Mather, [22]), Mañé [21]) Let $L : TM \longrightarrow \mathbb{R}$ be a C^∞ convex, superlinear Lagrangian defined in the tangent bundle of a compact C^∞ manifold M. Then the support of a minimizing measure in its homology class is a Lipschitz graph over an invariant set of global minimizers of the action in the abelian covering.*

Proposition 2.4. *(Mather, [22]) Let (M, g) be a compact, C^∞ Riemannian manifold and let γ be a geodesic of (M, g) that is globally minimizing in the abelian covering of (M, g). Suppose that the orbit $(\gamma(t), \gamma'(t))$ of the geodesic flow supports an invariant measure. Then this measure is a minimizing measure in a nontrivial homology class.*

Since the image of projetion of orbits contained in W lift to global minimizers in $(\tilde{M}, \tilde{g}) = (M_a, g_a)$, this proposition implies that *any invariant probability with support contained in W is a minimizing measure.*

We recall that by Krylov-Bogolyubov theorem, this set is not empty. Finally, we are ready to prove the main result of the subsection.

Lemma 2.5. *Let (T^n, g) be a compact C^∞ Riemannian manifold, and let W be a compact, C^1, Lagrangian minimizing submanifold. Then the canonical projection restricted to each connected component of the regular set of W is a diffeomorphism.*

Proof. Let R be a regular component of W, and let Ω be the set of non-wandering orbits of R. There is an invariant measure for the geodesic flow supported in a subset X of Ω, Since the orbits in X are global minimizers in $(\tilde{M}, \tilde{g}) = (M_a, g_a)$, the set X is the support of a minimizing measure by Proposition 2.4. Therefore, Theorem 2.3 implies that the canonical projection restricted to X is injective. This yields that the degree of the canonical projection in the regular component R is one since the degree is constant in each regular component. □

2.2. **Local Weak KAM solutions in regular components.** The above lemma is a partial version of the main theorem, regular components of the canonical projection are graphs. This is not enough of course to show that the Lagrangian submanifold W is a graph. We have to show that the singular set is empty for this purpose, but we have to work a lot more to get there. The next statement has a flavor of the theory of the Hamilton-Jacobi equation.

Lemma 2.6. *Let $R \subset W$ be a regular component of the restriction of $\pi : TM \longrightarrow M$ to W. Then there exists a regular, codimension one foliation $\mathcal{F}$ in R that is transversal to the geodesic flow. This foliation has the following properties:*

1. *$\mathcal{F}$ projects by π into a foliation $\mathfrak{F}$ of $\pi(R)$ by locally equidistant leaves.*
2. *The leaves of $\mathcal{F}$ are preserved by the flow, namely, if N is a submanifold in $\mathcal{F}$ and $A \subset N$ satisfies $\phi_t(A) \subset R$ for some t, then $\phi_t(A)$ is contained in a leaf of $\mathcal{F}$.*

(3) *Each leaf of the transverse foliation $\mathcal{F}$ has trivial homotopy relative to R in the following sense: given any leaf $\mathcal{F}(\theta)$, then any closed curve γ in $\mathcal{F}(\theta)$ that is contractible in R is contractible in $\mathcal{F}(\theta)$ as well.*

Proof. The existence of the transverse foliation in a regular component of the canonical projection is an application of standard elementary arguments of symplectic topology.

By hypothesis, the set R is a Lagrangian graph, so the restriction of the canonical one-form α to R projects into a closed one-form $\pi_*(\alpha)$ in $\pi(R)$ that is locally exact.

Namely, given an open, simply connected region $A \subset \pi(R)$ there exists a smooth function $f : A \longrightarrow \mathbb{R}$ such that $\pi_*(\alpha) = df$. Since the difference between two primitives of $\pi_*(\alpha)$ is a constant, the level sets of the primitives are the same for all the primitives. Therefore, we can cover R by open simply connected regions each of which has a codimension one foliation by smooth level sets of primitives of α, and clearly the local foliations extend to a global one $\mathfrak{F}$ in $\pi(R)$. Observe that if X is the geodesic vector field then $\alpha(X) \neq 0$. Therefore, we can normalize X in order to get $\alpha(X) = 1$. Hence the leaves of $\mathfrak{F}$ area locally equidistant. The canonical projection of the geodesic flow restricted to R gives a (possibly non-complete) flow in $\pi(R)$ that is the Reeb flow of $\pi_*(\alpha)$. The orbits are minimizing geodesics in (T^n, g), and the foliation $\mathfrak{F}$ of level sets is perpendicular to this flow. Hence the leaves of $\mathfrak{F}$ area locally equidistant. Then, the first variation formula implies that the leaves of $\mathfrak{F}$ are locally equidistant (see [27]). The foliation in $\pi(R)$ has a natural lift to R, giving rise to a locally invariant foliation $\mathcal{F}$. This finishes the proof of items (1) and (2) in the lemma.

The proof of item (3) requires well known tools of the theory of codimension one foliations. Let $\mathcal{F}(\theta)$ be a leaf of $\mathcal{F}$ containing a point $\theta \in R$. The foliation $\mathcal{F}$ is locally equidistant too in T_1T^n with respect to the Sasaki metric. Let $\gamma \subset \mathcal{F}(\theta)$ be a smooth close curve whose homotopy class in R is trivial. Suppose that the homotopy class of γ in the leaf $\mathcal{F}(\theta)$ is not trivial. Then, according to the theory of codimension one foliations (see [9] for instance), there exists a two-dimensional disk $D \subset R$ whose boundary is γ that is in general position with respect to the foliation $\mathcal{F}$. Namely, the intersection of each leaf with D is a flow with a finite number of singularities. The set of singularities cannot be empty, and each singularity must be of index one, so there exists a unique singularity corresponding to the intersection of a leaf $\mathcal{F}(\theta_0)$ with D that is tangent to D. Saturating by the geodesic flow the leaf $\mathcal{F}(\theta_0)$ we get a flow in D by closed curves, each of which is the intersection of a leaf in $\mathcal{F}$ with D. Such curves are usually called vanishing cycles since the famous Novikov's theorem about the existence of Reeb components in codimension one foliations in the 3-sphere. But since this foliation is equidistant, and the disk D is compact, there exists $T > 0$ such that the iterates $\phi_t(\mathcal{F}(\theta_0))$ for $\mid t \mid \leq T$ cover the disk D. This yields that the curve γ is contained in $\phi_t(\mathcal{F}(\theta_0))$, for some $\mid t \mid \leq T$, and that is homotopic to a vanishing cycle in this leaf. Therefore, γ is null homotopic in the leaf as we wanted to show. □

Lemma 2.6 can be generalized to Finsler metrics and therefore to Lagrangian, minimizing submanifolds in supercritical energy levels of Tonelli Lagrangians in T^n.

3. A closer look at the singular set

In this section we obtain more properties of the singular set, in any dimension.

3.1. **The Maslov cycle and the stratification of the singular set.** The singular set $S \subset T_1T^n$ is closely related to the Maslov cycle of T_1T^n that is a stratified set.

Definition 3.1. Let (M, g) be a C^∞ complete n-dimensional Riemannian manifold, let N be the fiber bundle of TT_1M formed by the subspaces N_θ, $\theta \in T_1M$. The Maslov cycle Λ_V of T_1M is defined by

$$\Lambda = \bigcup_{k\in[1,n-1]} \Lambda_k(T_1M)$$

where $\Lambda_k(T_1M)$ is the bundle of Lagrangian subspaces of N whose intersection with the vertical subspace has dimension k.

The Maslov cycle Λ_V is a subset of the "Grassmanian $\mathcal{G}$ of Lagrangian subspaces of the fiber bundle N. Each of the sets $\Lambda_k(TM)$ is a smooth submanifold of codimension $\frac{k(k+1)}{2}$, so Λ is a stratified manifold. This definition of Maslov cycle in some sense adapted to T_1M (that is not a symplectic manifold) is well suited to study Lagrangian bundles that are invariant by the geodesic flow. The usual definition of the Maslov cycle considers symplectic manifolds and Lagrangian subspaces of the tangent space of such manifolds.

There is a natural lift of the geodesic flow ϕ_t to the Grassmanian $\mathcal{G}$. Namely, if $(\theta, S) \in \mathcal{G}$, where $\theta \in T_1M$, $S \subset N_\theta$ is Lagrangian, then $\phi_t^G(\theta, S) = (\phi_t(\theta), D_\theta\phi_t(S))$ defines a flow in $\mathcal{G}$. This flow commutes with the projection $P : \mathcal{G} \longrightarrow T_1M$ given by $P(\theta, S) = \theta$: $P \circ \phi_t^G = \phi_t \circ P$. A C^k invariant Lagrangian submanifold $\Sigma \subset T_1M$ lifts to a C^{k-1} ϕ_t^G-invariant submanifold $\tilde{\Sigma}$ of $\mathcal{G}$.

Notice that the intersections of $\tilde{\Sigma}$ with the different strata of the Maslov cycle yield a decomposition of the singular set $S(\Sigma)$ of the invariant manifold Σ in a union of subsets $S_i(\Sigma)$, $i = 1, 2, .., n-1$, where i indicates the dimension of the intersection between $T\Sigma$ and the vertical bundle. If $n = 3$ then we only have $i = 1$ or $i = 2$. Moreover, if Σ is of class C^2 then it is know that the intersection of $\tilde{\Sigma}$ with $\Lambda_1(T_1M)$ is transversal (see [1]). So we have

Lemma 3.2. *Let Σ be a C^2 Lagrangian, invariant submanifold of T_1M. Then the set $S_1(\Sigma)$ is a C^1 submanifold.*

3.2. **A minimizing Lagrangian submanifold avoids strata of minimal dimension.** The goal of this subsection is to show that the dimension of the intersection of $T_\theta W$ with the vertical space $V(\theta)$ is never maximal. The key tool of the proof is a short cut argument for "almost crossing minimizing geodesics, an idea that already appeared in [10], Section 3. Since in [10] the statements refer to surfaces, we shall state and proof for any dimension for the sake of completeness.

Lemma 3.3. *Let (M, g) be a compact, C^∞ Riemannian manifold. Suppose that there exists a C^2 variation by geodesics $f : (-\epsilon, \epsilon) \times \mathbb{R} \longrightarrow \tilde{M}$ satisfying*

1. *$f(s, t) = f_s(t) = \bar{\pi}(\phi_t(\theta(s)))$ is a minimizing geodesic of $\tilde{M}$ for every $s \in (-\epsilon, \epsilon)$, where $\theta(s)$ is a C^2 embedded curve of points in T_1M and $\bar{\pi} : T_1\tilde{M} \longrightarrow \tilde{M}$ is the canonical projection,*
2. *The curve $\theta(s)$ has unit speed and is tangent to the vertical subspace at $\theta(0)$.*

Then there exists $\epsilon_0 > 0$, $a > 0$ depending on the C^2-norm of $\theta(s)$ at $\theta(0)$, such that for every $\mid s \mid\leq \epsilon_0$ we have

$$\inf_{t>0}\{d(\bar{\pi}(\phi_t(\theta(s)), \bar{\pi}(\phi_{[0,+\infty)}(\theta(0))), d(\bar{\pi}(\phi_t(\theta(0)), \bar{\pi}(\phi_{[0,+\infty)}(\theta(s))) \geq a \mid s \mid$$

for every $t > 0$.

Proof. By the assumptions, the curve $\theta(s)$ is tangent to a C^2 vertical curve $c(s)$ contained in the unit sphere of the point p at $s = 0$, so $c(0) = \theta(0)$. By the Taylor expansion of the Sasaki distance $\bar{d}(\theta(s), c(s))$, we get an interval $(-b, b) \subset (-\epsilon, \epsilon)$ and a constant $C > 0$ depending on the norm of the second derivatives of $\theta(s)$ at $s = 0$ such that

$$\bar{d}(\theta(s), c(s)) \leq Cs^2$$

for every $s \in (-b, b)$. Let $\theta(s) = (p(s), v(s))$, and let us denote by $\gamma_{\theta(s)}(t)$ the geodesic whose initial conditions are $\gamma_{\theta(s)}(0) = p(s)$, $\gamma'_{\theta(s)}(0) = v(s)$. The geodesics $\gamma_{c(s)}(t)$ meet all at $t = 0$, $\gamma_{c(s)}(0) = p(0) = p$ for every s. The Gronwall inequality applied to the geodesic flow implies that there exists a constant $\bar{C}$ depending on C, and $0 < \delta \leq b$ such that

$$d(\gamma_{\theta(s)}(t), \gamma_{c(s)}(t)) = d(\gamma_{\theta(s)}(0), p) \leq \bar{C}s^2$$

for every $s \in (-\delta, \delta)$ and $t \in [-1, 1]$.

Let $1 \geq \rho > 0$ be a normal radius for (M, g), say less than half of the injectivity radius, and let $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \longrightarrow \mathbb{R}$ be a positive function outside $\theta = 0$ defined by the following condition: if $\beta_1 : [-r, 0] \longrightarrow \tilde{M}$, $\beta_2 : [0, r] \longrightarrow \tilde{M}$ are two unit speed geodesics satisfying

(1) $\beta_1(0) = \beta_2(0)$,
(2) The angle between $\beta'_1(0)$ and $\beta'_2(0)$ is θ,

then $d(\beta_1(-r), \beta_2(r)) \leq 2r - f(\theta)$. The existence of $f(\theta)$ is a straightforward consequence of the first variation formula. Moreover, there exists a constant $L > 0$ depending on the metric g such that $f(\theta) \geq L\sin(\theta)$. Let $\sigma > 0$ such that $\sin(x) \geq \frac{1}{2}x$ for every $x \in [0, \sigma]$.

Now, let us take $\mid s \mid \leq \sigma$ and consider the geodesics $\gamma_{\theta(s)}(t)$, $\gamma_{\theta(0)}(t)$. Let $\tau(s) = \liminf_{t\to+\infty} d(\gamma_{\theta(s)}(t), \gamma_{\theta(0)}([0, +\infty))$. Consider the broken geodesic Γ_s formed by the union of

(1) The minimizing geodesic β joining $\gamma_{\theta(0)}(-r)$ to some point $\gamma_{\theta(s)}(r(s))$, where $r(s) > 0$ is defined by $d(p, \gamma_{\theta(s)}(r(s)) = r$,
(2) $\gamma_{\theta(s)}[r(s), +\infty)$.

The length of the subset $\Gamma_s(t)$ of Γ_s joining $\gamma_{\theta(0)}(-r)$ to $\gamma_{\theta(s)}(t)$, $t > r(s)$, satisfies

$$l(\Gamma_s(t)) \leq 2r - f(\theta) + (t - r(s)).$$

By the choice of $r(s)$ we have that $\mid r(s) - r \mid \leq \bar{C}s^2$, so we get

$$l(\Gamma_s(t)) \leq r + t - f(\theta) + \bar{C}s^2 \leq r + t - \frac{L}{2} \mid s \mid + \bar{C}s^2,$$

for every $t \geq r(s)$. Let $\epsilon_0 > 0$ be such that

$$\frac{L}{2} \mid s \mid - \bar{C}s^2 \geq \frac{L}{4} \mid s \mid$$

for every $\mid s \mid \leq \epsilon_0$.

Notice that $r + t$ is the length of $\gamma_{\theta(0)}[-r, t]$. Since $\gamma_{\theta(0)}$ is minimizing this yields that the number $\tau(s)$ cannot be less than $\frac{L}{4} \mid s \mid$ for every $t \geq r(s)$. Interchanging the roles of $\gamma_{\theta(0)}$ and $\gamma_{\theta(s)}$ in the above argument we get the proof of the lemma. □

Lemma 3.4. *Let W be a C^2 minimizing, Lagrangian, invariant torus of T_1T^n. Then the set S_{n-1} is empty.*

Proof. For suppose by contradiction that S_{n-1} is not empty. Let $\theta = (p, v) \in S_{n-1}$ and let $B_r(\theta) \subset W$ be an open, $n-1$-ball in W of radius r in the Sasaki metric around θ that is tangent to $T_\theta W \cap V_\theta = V_\theta$. Since W is of class C^2 the ball $B_r(\theta)$ has bounded C^2-norm, so the geodesics of $B_r(\theta)$ through θ with respect to the restriction of the Sasaki metric have uniformly bounded C^2-norm.

Hence, for r suitably small we can apply Lemma 3.3 to the $B_r(\theta)$-geodesics through $\theta = \theta(0)$. So we get that there exist $\bar{a} > 0$ depending on the constant $a > 0$ given in Lemma 3.3, $0 < \epsilon_1 \leq \min\{\epsilon_0, r\}$ such that

$$inf_{t,s>0} d(\phi_t(\theta), \phi_s(\psi)) \geq ad(\theta), \psi),$$

for every $\psi \in B_{\epsilon_1}(\theta)$, where d is the Sasaki distance in T_1T^n.

This yields that there exists $\delta > 0$ such that for every $t > 1$ the ball $B_\delta(\phi_t(\theta))$ of radius δ around $\phi_t(\theta)$ in the Sasaki metric is contained in the set

$$C = \bigcup_{t>0} \{\phi_t(\eta),\ \eta \ \in \ (B_r(\theta))\},$$

that is embedded in W. So C contains a subset diffeomorphic to a cylinder whose volume is not finite, clearly contradicting the compactness of W. □

So the dimension of the intersection of $T_\theta W$ with the vertical space $V(\theta)$ is never maximal.

Next, we analyze the case when this intersection has always minimal dimension, that is the stratum S_1. By Lemma 3.2, we have that the critical set S is a smooth hypersurface in W if we assume that W is of class C^2.

Lemma 3.5. *There is no open subset $U \subset S_1$ where the subbundle $TW \cap V$ is everywhere transversal to S.*

Proof. Suppose by contradiction that there exists such an open neighborhood U in S. The subspaces $\sigma(\theta) = T_\theta W \cap V_\theta$ are all transversal to U for every $\theta \in U$ and form a 1-dimensional subbundle of TW. It is possible to construct a local, smooth, unit vector field Z with integral flow ψ_s, $s \in (-\epsilon, \epsilon)$, in an open neighborhood of U in W, such that

1. $Z(\theta) \in T_\theta W \cap V_\theta$ for every $\theta \in U$,
2. There exists a regular component $R \subset W$ such that $\psi_s(\theta) \in R$ for every $s \in (0, \epsilon)$.

Then the set

$$A = \bigcup_{p \in U,\ s \in (0,\epsilon)} \psi_s(p)$$

is an n-dimensional open subset of W. But now, if we consider the geodesic flow in the regular component R, and suppose without loss of generality that $\phi_t(U) \subset R$ for every $t > 0$, we get that the set

$$\bigcup_{t>0} \phi_t(A) = \bigcup_{p \in U,\ s \in (0,\epsilon)} \phi_t(\psi_s(p))$$

is homeomorphic to an open $n+1$-dimensional set, which clearly contradicts the dimension of W. □

The above result is remarkable from the point of view of the generic theory of Lagrangian singularities: Lemma 3.5 is highly non generic.

4. The singular set in dimension 3 and the proof of the main Theorem

The rest of the article is devoted to show that the canonical projection restricted to a C^2 Lagrangian invariant torus in T_1T^3 that is minimizing is actually a diffeomorphism. Or equivalently,

Theorem 4.1. *Let (T^3, g) be a C^∞ Riemannian metric in the torus and let $W \subset T_1T^3$ be a C^2 minimizing, Lagrangian invariant torus. Then the critical set of W is empty.*

The results of the previous section imply that the critical set S of the Lagrangian minimizing torus W has dimension 2, the intersection $T_\theta W \cap V(\theta)$ has always dimension 1 $\forall \theta \in S$ and therefore, there exists a vertical line field everywhere tangent to the critical set.

Proposition 4.2. *Let W be a C^2, invariant, Lagrangian, minimizing torus of the geodesic flow of (T^3, g). Then singular set S of W does not admit a vertical line field that is everywhere tangent to S.*

We shall subdivide the proof of Proposition 4.2 in several steps.

Lemma 4.3. *Let C be a connected component of S that is always tangent to a unit, vertical vector field $v : C \longrightarrow V$. Let $v^\perp$ be a unit vector field tangent to C that is perpendicular to v with respect to the Sasaki metric. Then,*

1. *The integral orbits of v are all closed.*
2. *The integral orbits of $v^\perp$ are closed.*
3. *C is a two torus whose canonical projection in T^3 is a smooth closed curve Γ and all integral orbits of $v^\perp$ are isometric to Γ.*

Proof. The fact that C is either a two torus or a Klein bottle follows from the existence of a nonsingular smooth line field that is always tangent to C. The orientation of the geodesic flow in W then implies that C is in fact a two torus. The smooth unit field v comes from the vertical line bundle that is tangent to C. Restricting the Sasaki metric to C we get a smooth unit vector field $v^\perp$ tangent to C that is always perpendicular to v. The vector field $v^\perp$ is never vertical and therefore, the differential of the canonical projection restricted to TC applied to $v^\perp$ is always nonzero. For all $\theta in C$, $T_\theta W$ is generated by $v(\theta)$, $v^\perp(\theta)$ and $X(\theta)$, the unit geodesic vector field.

Observe that if $\eta(s, \theta) \in C$ denotes the local flow generated by $v(\theta)$, that is, $\frac{d}{ds}\eta(s, \theta) = v(\eta(s, \theta)$ then $\pi(\eta(s, \theta))$ is constant. Hence, each orbit $\eta(s, \theta)$ is contained in a single vertical fibre of T_1T^3. Moreover, since the intersection of W with each vertical fibre is a C^2 closed curve we get that the orbits $\eta(s, \theta)$ of the integral flow of the vertical field v are all closed, just proving item (1).

Since the rank of the restriction to S of the canonical projection is constantly equal to 1 (and therefore a submersion), we have that the curves $\pi(\mu(t, \theta))$ coincide with a single smooth (by the local form of submersions), closed curve $\Sigma \subset T^3$. This yields item (3).

To show item (2), let us notice that

Claim: The vector field $v^\perp$ is horizontal.

In fact, the vertical subspace V_θ at each point of T_1T^3 has dimension 2, the horizontal subspace H_θ has dimension 3, the direction $X(\theta)$ of the geodesic flow is horizontal and transverse to TC, and $T_\theta C$ contains the vertical line field generated by v at each $\theta \in C$. If H_θ was transverse to $T_\theta C$ at some point $\theta \in C$, then $H_\theta \oplus V_\theta = T_\theta T_1 M$ would be independent of the direction $v^\perp$ that is in $T_\theta C$ and is not vertical. This is clearly a contradiction, so $T_\theta C \cap H_\theta$ must contain at least a one dimensional vector space for every $\theta \in C$. Now, since $T_\theta C$ always contains the vertical line field generated by v we get that the dimension of $T_\theta C \cap H_\theta$ must be one. To finish the proof of the claim, just recall that H_θ and V_θ are perpendicular with respect to the Sasaki metric.

The above claim implies that the differential of the canonical projection restricted to the vector field $v^\perp$ is an isometry, since it is horizontal. Moreover, $v^\perp$ is non-singular and complete as a vector field, and hence its orbits are locally isometric coverings of Σ. Let $\mu(t,\theta)$ be the local flow generated by $v^\perp$. We have just proven that $\pi(\eta(s,\mu(t,\theta))$ is independent of s, and the horizontality of $v^\perp$ implies that this flow preserves the vertical orbits $\eta(s,.)$: t is the arc length of Σ.

This yields that every integral curve of $\mu(t,.)$ is closed: the horizontal lift of Σ at each point $\theta \in C$ is tangent to the vector field v and hence an integral curve of such vector field. This shows item (2). □

The proof of Lemma 4.3 has many interesting consequences.

Corollary 4.4. *Let C be a connected component of S that is always tangent to a vertical line field. Then each integral curve of a unit vertical vector tangent to $TW \cap V$ is closed and contained in a vertical fibre. In particular, the integral curves of this vector field are all contractible in both T_1T^3 and W. Moreover, if $\pi_1(C)$ is the first free homotopy group of C, then $\pi_1(C) = H_1(C,\mathbb{Z}) \cong \mathbb{Z}$.*

Proof. The fact that each orbit of the vertical vector field v is closed and contained in a vertical fibre is already proved in Lemma 4.3. Since the vertical fibres are two-dimensional spheres, each closed curve of the vertical vector field v is simply connected and contractible in T_1T^3. This implies that $\pi_1(C)$ is the fundamental group of the closed ntegral curves of the flow $v^\perp$. Since all those curves project diffeomorphically by the canonical projection into one single closed curve, they all have the same homotopy class. The same holds for the first homology group. □

Lemma 4.5. *Let $W \subset T_1T^3$ be a Lagrangian, invariant minimizing torus for the geodesic flow of (T^3,g). Suppose that the critical set S is always tangent to a line bundle contained in $TW \cap V$. Then a connected component C of S has the following properties:*

1. *The fundamental group of C in W is isomorphic to $\mathbb{Z}$.*
2. *C bounds a solid torus in W.*

Proof. Notice that item (1) is an improvement of Corollary 4.4. Indeed, the fact that $\pi_1(C)$ in T_1T^3 is isomorphic to $\mathbb{Z}$ might not imply that the fundamental group of C in W is isomorphic to $\mathbb{Z}$.

To show item (1), let us consider the image of C by small iterates of the geodesic flow: let $C_t = \phi_t(C)$, $t > 0$. By Lemma 2.5 the canonical projection $\pi(C_t)$ restricted to C_t is diffeomorphic to C_t for every $t \neq 0$. By Corollary 4.4, $\pi(C)$ is a closed curve $\pi(C) : S^1 \longrightarrow T^3$. Hence, the continuity of π and ϕ_t imply that $\pi(C_t)$ is contained in a small tubular neighborhood of $\pi(C)$ if t is suitably small. Moreover, the map $f : \pi(C) \times [0, \epsilon] \longrightarrow W$ given by $f(\theta, t) = \pi(\phi_t(\theta))$ provides a homotopy from C to C_ϵ. Therefore, $\pi(C)$ and $\pi(C_\epsilon)$ have the same free homotopy groups for every ϵ, and since C_t and $\pi(C_t)$ are diffeomorphic, they have as well the same homotopy groups up to isomorphisms. So the fundamental group of C_t in W is isomorphic to $\mathbb{Z}$, and since C and C_t are diffeomorphic in W, the fundamental group of C in W is isomorphic to $\mathbb{Z}$ as we claimed.

Item (2) is straightforward from item (1). Since the fundamental group of C in W is isomorphic to $\mathbb{Z}$, a lift of C in the universal covering of W (that is diffeomorphic to $\mathbb{R}^3$) is a solid cylinder which implies that C bounds a solid torus in W.

□

Proof of Proposition 4.2

So let us assume that the singular set S of the minimizing torus W has an everywhere tangent vertical vector field. Let C be a connected component of S.

Lemma 4.6. *In the assumptions of Proposition 4.2 all the connected components of the singular set S are homotopic in W, and the solid tori bounded by such components are totally ordered by inclusion.*

Proof. Let R be a regular component of the canonical projection, and let C be a connected component of the critical set S in the boundary of R. By Lemma 4.5 the set C is the boundary of a solid torus T_C in W. This yields that the homotopy group of every connected component C' of the singular set contained in T_C is a subgroup of finite index of $\pi_1(T_C)$. Because the homotopy group of T_C is isomorphic to $\mathbb{Z}$ as well as the homotopy group of C' by Lemma 4.4. Since $\pi_1(C')$ is a subgroup of $\pi_1(T_C) \simeq \pi_1(C)$, it must have finite index in $\pi_1(T_C)$.

For the components of the singular set which are outside T_C, let us consider a regular component $\hat{R}$ outside T_C whose boundary contains C. Let $\hat{C}$ be a connected component of the singular set in the boundary of $\hat{R}$ that is different from C. Since $\hat{R} \cup T_C$ is an open neighborhood of T_C, and the boundary of $\hat{R} \cup T_C$ is $\hat{C}$, we have that the solid torus $T_{\hat{C}}$ bounded by $\hat{C}$ in W is just $\hat{C} \cup \hat{R} \cup T_C$. By induction in the (finite) collection of connected components of the singular set, we get a total order for the solid tori bounded by connected components of the singular set.

It remains to show that in fact, the connected components of the critical set and the solid tori have the same (free) homotopy group. To see this we look at the foliation by cross sections of the flow in each regular component.

So let R be a regular component whose boundary contains two connected components C_1, C_2 of the singular set. Suppose that $T_{C_1} \subset T_{C_2}$. For $\epsilon > 0$ we have that either $\phi_\epsilon(\eta) \in R$ for every $\eta \in C_1$ or $\phi_{-\epsilon}(\eta) \in R$ for every $\eta \in C_1$. The same assertion holds for C_2, assume without loss of generality that $\phi_\epsilon(\eta) \in R$ for every $\eta \in C_i$, $i = 1, 2$. Let us denote

$$C_{i,\epsilon} = \cup_{\eta \in C_i} \{\phi_\epsilon(\eta)\},$$

$$\phi_\epsilon(T_{C_i}) = \cup_{\eta \in T_{C_i}} \{\phi_\epsilon(\eta)\},$$

$$R_\epsilon = \phi_\epsilon(T_{C_2}) - (\phi_\epsilon(T_{C_1}))^o,$$

where B^o is the interior of a subset $B \subset W$. Notice that $C_{i,\epsilon}$ is a torus transverse to the foliation $\mathcal{F}$ of equidistant leaves of R for each $i = 1, 2$. By Lemma 2.6, each leaf of $\mathcal{F}$ has trivial homotopy in R. Moreover, given $\theta \in R$, and $T > 0$ a positive return time of the leaf $\mathcal{F}(\theta)$ for the geodesic flow, the closed set R_ϵ is diffeomorphic to the product $\mathcal{F}(\theta) \cap R_\epsilon \times S^1_T$, where S^1_T is a circle of length T. Let R_Q be the quotient space of R_ϵ by the foliation $\mathcal{F}$. The homotopy sequence of quotient spaces implies that the first homotopy groups of both $C_{1,\epsilon}$ and $C_{2,\epsilon}$ relative to R are the same, but each $C_{i,\epsilon}$ has the first homotopy group of C_i. This shows that the solid tori to T_{C_1} and T_{C_2} have the same free homotopy group.

Finally, an induction argument on the ordered solid tori T_{C_i} yields that all of them have the same free homotopy group as claimed.

□

Lemma 4.6 yields that, under the assumptions of Proposition 4.2, we can order the connected components C_i of the singular set S by means of the solid tori T_{C_i} they bound in W. Let us consider an ordered, maximal chain of such tori, $T_{C_i} \subset T_{C_{i+1}}$ for every $i = 1, 2, .., m-1$. For a subset $B \subset W$, let B^c be the complement of B in W.

Claim 1: The closure of the complement $(T_{C_m})^c$ of the solid torus T_{C_m} is a solid torus too.

This is because the set $(T_{C_m})^c$ is a regular component, there are no more singular points inside this set. Therefore, $(T_{C_m})^c$ is a regular component whose boundary coincides with C_m, and hence Lemma 4.5 implies that its closure is a solid torus. This finishes the proof of the claim.

Claim 2: The union of the tori T_{C_i}, $i = 1, 2, .., m$ and $(T_{C_m})^c$ is the whole W.

In fact, it is easy to check that this union is open and closed in W, so by connectedness we get that W must coincide with the above union of tori.

Now, let C be a connected component of S in the boundary of an invariant solid torus T_C, and suppose that $\phi_t(\theta) \in T_C$ for every $t > 0$ and $\theta \in C$. For $\epsilon > 0$ small, let

$$A^+ = \phi_{-\epsilon}(T_C),$$

and

$$A^- = \phi_\epsilon((T_C)^c).$$

By Lemma 4.6 the first homotopy (and hence homology) group of A^+ is isomorphic to $\mathbb{Z}$. Moreover, by reversing the orientation of the flow in W we can apply Lemma 4.6 to $(T_C)^c$ to deduce that it is a solid torus too and that A^+ and A^- are homotopic in W.

Since $A^+ \cap A^-$ is a tubular neighborhood of C, and $A^+ \cup A^- = W$, the Mayer-Vietoris homology sequence tells us that the first homology group of W is a quotient of

$$H_1(A^+, \mathbb{Z}) \oplus H_1(A^-, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$$

that is clearly impossible because W is a three torus and its first homology group is $\mathbb{Z}^3$. This concludes the proof of Proposition 4.2

Proof of Theorem 4.1

Theorem 4.1 follows from Lemma 3.4, Lemma 3.5 and Proposition 4.2.

Since the singular set of W is empty, the canonical projection is regular everywhere, and since its degree in regular components is one (Lemma 2.5) we get Theorem A.

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