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Visual Contours and Planar Sections for Affine Immersions

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1 Introduction

In this paper, we consider affine immersions $q: M \to \mathbb{R}^{n+1}$ of a n-dimensional manifold M into the $(n+1)$-dimensional affine space. More precisely, we shall consider convex affine immersions, or equivalently, affine immersions whose Blaschke metric is positive definite.

The paper deals mainly with two geometric objects, the planar sections and the visual contours. A planar section is a set of points of M whose corresponding points on $q(M)$ belong to the same hyperplane. A visual contour is a set of points of M whose corresponding points on $q(M)$ form a contour observed by a point p of $\mathbb{R}^{n+1}$ outside $q(M)$, i.e., the tangent planes at the points of the image of the contour pass through p. The concept of visual contour is widely used in computational vision ([1]).

To be more precise, for $x_0 \in M$ and $s > 0$, define the planar section $S(x_0, s)$ as the set of points $x \in M$ whose image $q(x)$ is contained in the plane parallel to the tangent plane at $q(x_0)$ and passing through $q(x_0) - s\xi(x_0)$, where $\xi(x_0)$ is the Blaschke affine normal vector at x_0 pointing "outwards" $q(M)$. Similarly, for $t > 0$, define the visual contour $T(x_0, t)$ as the set of points in $x \in M$ such that the tangent plane at $q(x)$ passes through $q(x_0) + t\xi(x_0)$.

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For s and t small, we can consider that s and t are real functions defined in a neighborhood of x_0 in M.

In the first part of the paper, we study the relation between the planar section and the visual contour functions. The main result that we show is that the third order Taylor expansion of $s - t$ at x_0 is exactly the cubic form $C(x_0)$. This result gives a nice geometric interpretation of the cubic form.

An affine immersion $q : M \to \mathbb{R}^3$ is called a proper affine sphere when all the Blaschke affine normal vectors ξ meet at a point, or equivalently, when the shape operator S is a multiple of the identity. When the vectors ξ are all parallel, or equivalently $S = 0$, we call the immersion an improper affine sphere. For affine spheres, we may re-scale the planar section function to obtain a duality property between planar sections and visual contours. In particular, when the affine sphere is a quadric, we have $s = t$.

In the second part of the paper, we consider parameterizations of two dimensional affine immersions whose coordinate lines are geodesics, in one direction, and both planar sections and visual contours in the other direction. More explicitly, let M be 2-dimensional and $q : M \to \mathbb{R}^3$ an affine immersion. A parameterization $x : D \subset \mathbb{R}^2 \to M$ is called parallel-meridian if the coordinate lines $u \to x(u, v_0)$ are geodesics for both the first and second connections and the coordinate lines $v \to x(u_0, v)$ are both visual contours and planar sections. It is worth mentioning that revolution surfaces admit parallel-meridian coordinates in a natural way.

A pole $p \in M$ is an isolated singularity of a parallel-meridian parameterization such that the parallel coordinate lines are both visual contours and planar sections based on p, while the meridian coordinate lines are geodesics for both the first and second connections starting at p. At a pole, since the planar sections coincide with the visual contours, necessarily the cubic form is zero. The main result of the second part of the paper says that if an affine sphere admits a parallel-meridian parameterization with a pole, then it is necessarily a quadric. We also consider parallel-meridian parameterizations of a quadric. Since parallel-meridian coordinates can be re-parameterized to be isothermal, and any change of isothermal parameters is conformal, we are in conditions to describe all the parallel-meridian of a quadric.

The paper is organized as follows: In section 2 we review basic concepts of affine immersions and isothermal coordinates. In section 3 we define precisely the planar section and visual contour functions and prove the theorem that relates them to the cubic form. In section 4 we study the properties of parallel-meridian parameterizations, in particular proving the theorem that affine spheres admiting parallel-meridian coordinates with a pole is necessarily a quadric.

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2 Background on affine immersions

2.1 Fundamental equations

We shall follow the notation of [2]. Let $q : M \to \mathbb{R}^{n+1}$ be an affine immersion and ξ its Blaschke normal vector field. For $X, Y \in \mathcal{X}(M)$, we write

$$\begin{aligned} D_X q_*(Y) &= q_*(\nabla_X Y) + h(X,Y)\xi \\ D_X \xi &= -q_*(SX). \end{aligned}$$

where ∇ denotes the first connection, h is the fundamental quadratic form and S is the shape operator. For the co-normal immersion $\nu : M \to \mathbb{R}^3$, we have

$$D_X \nu_*(Y) = \nu_*(\overline{\nabla}_X Y) - h(SX, Y)\nu$$

where $\overline{\nabla}$ denotes the second connection. The cubic form is defined as

$$C(X, Y, Z) = X(h(Y, Z)) - h(\nabla_X Y, Z) - h(\nabla_X Z, Y).$$

It can also be calculated by $C(X, Y, Z) = -2h(K_X Y, Z)$, where $2K_X Y = \nabla_X Y - \overline{\nabla}_X Y$.

2.2 Isothermal coordinates

For affine immersions $q : M \to \mathbb{R}^3$, of a two-dimensional manifold M into the 3-dimensional affine space and a parameterization $x : D \subset \mathbb{R}^2 \to M$, we shall use the following notation:

Notation.Denoting the coordinates in D by (u, v), we shall write $U = \frac{\partial x}{\partial u}$ and $V = \frac{\partial x}{\partial v}$. Also, we shall use the notation q_u for $q_*(U)$ and ν_u for $\nu_*(U)$, with the corresponding notation for the v coordinate. Similarly, we shall write q_{uu} for $D_U q_*(U)$ and ν_{uu} for $D_U \nu_*(U)$ with a corresponding meaning for q_{uv}, q_{vv}, ν_{uv} and ν_{vv}.

We say that the parameterization x is isothermal if $h(U, U) = h(V, V)$ and $h(U, V) = 0$. The Gauss equations of an affine immersion in isothermal coordinates can be written as

$$\begin{cases} q_{uu} = \dfrac{\Omega_u + a}{2\Omega} q_u - \dfrac{\Omega_v + b}{2\Omega} q_v - \Omega\xi \\ q_{vv} = -\dfrac{\Omega_u + a}{2\Omega} q_u + \dfrac{\Omega_v + b}{2\Omega} q_v - \Omega\xi \\ q_{uv} = \dfrac{\Omega_v - b}{2\Omega} q_u + \dfrac{\Omega_u - a}{2\Omega} q_v, \end{cases}$$

where $\Omega = h(U, U) = h(V, V)$, $a = C(U, U, U) = C(U, V, V) = a$ and $b = C(U, U, V) = C(V, V, V)$. One can verify that $[x_u, x_v, x_{uu}] = [x_u, x_v, x_{vv}] = \Omega^2$ and $[x_u, x_v, x_{uv}] = 0$, where $[A, B, C]$ denotes the determinant of the matrix whose columns are A, B and C. We write also

$$\begin{cases} \xi_u = \left(-H + \dfrac{a_u - b_v}{2\Omega^2} \right) q_u - \dfrac{a_v + b_u}{2\Omega^2} q_v \\ \xi_v = -\dfrac{a_v + b_u}{2\Omega^2} q_u - \left(H + \dfrac{a_u - b_v}{2\Omega^2} \right) q_v, \end{cases}$$

where $H = S - J$ is the affine mean curvature, the Pick invariant J is defined by $2\Omega^3 J = a^2 + b^2$ and the scalar curvature S is defined by $2\Omega^3 S = \Omega_u^2 + \Omega_v^2 - \Omega(\Omega_{uu} + \Omega_{vv})$. For affine spheres, the affine mean curvature H is constant and $a_u = b_v = 0$. For the derivation of these formulas, see [3].

3 Planar sections and visual contours

3.1 Planar sections and visual contours for a general immersion

We fix an initial point $x_0 \in M$ and denote by $T(x_0,t)$ the contour based on $q_0+t\xi_0$, where $q_0=q(x_0), \nu_0=\nu(x_0), \xi_0=\xi(x_0)$. Thus, a point $x\in M$ is in the contour $T(x_0,t)$ if and only if

$$\nu(x)\cdot(q(x)-q_0-t\xi_0)=0. \tag{1}$$

Differentiating the above equation in the direction X we obtain

$$\nu(x)\cdot(q_*(X)-X(t)\xi_0)+\nu_*(X)\cdot(q(x)-q_0-t\xi_0)=0.$$

which can be simplified to

$$X(t)\nu(x)\cdot\xi_0=\nu_*(X)\cdot(q(x)-q_0-t\xi_0). \tag{2}$$

Since $t(x_0)=0$ and $\nu(x_0)\cdot\xi_0=1$, we obtain $X(t)(x_0)=0$. Since this holds for any $X\in\mathcal{X}(M)$, we conclude, by the implicit function theorem, that t is defined implicitly as a function of x by equation (1), in a neighborhood of x_0. We shall call $t(x)$ the *visual contour function* around x_0.

A *planar section* is the intersection of a plane with the affine sphere M. Denote by $S(x_0,s)$ the section of M by the plane parallel to the tangent plane at q_0 and passing through $q_0-s\xi_0$. A point $x\in M$ is in the planar section $S(x_0,s)$ if and only if $\nu_0\cdot(q(x)-q_0+s\xi_0)=0$, or equivalently,

$$\nu_0\cdot(q(x)+s\xi_0)=\nu_0\cdot q_0. \tag{3}$$

Differentiating the above equation in the direction X we obtain

$$\nu_0\cdot(q_*(X)+X(s)\xi_0)=0 \tag{4}$$

We conclude that $X(s)(x_0)=0$, for any $X\in\mathcal{X}(M)$. Thus, by the implicit function theorem, s can be implicitly defined by (3) as a function of x in a neighborhood of x_0. We shall call $s(x)$ the *planar section function* around x_0.

Lemma 1 *The second derivates of t and s satisfy $YX(t)(x_0)=YX(s)(x_0)=-h(X,Y)(x_0)$. As a consequence, the second order Taylor expansion of the difference between the planar section and visual contour functions is zero.*

Proof By differentiating (2) we obtain

$$YX(t)\nu(x)\cdot\xi_0+X(t)\nu_*(Y)\cdot\xi_0=\nu_*(X)\cdot(q_*(Y)-Y(t)\xi_0)+D_Y\nu_*(X)\cdot(q(x)-q_0-t\xi_0). \tag{5}$$

Substituting the point x_0, we obtain $YX(t)(x_0)=\nu_*(X)\cdot q_*(Y)(x_0)=-h(X,Y)(x_0)$.

By differentiating (4) we obtain

$$YX(s)=-\nu_0\cdot D_Yq_*(X)=-\nu_0(q_*(\nabla_YX)+h(X,Y)\xi) \tag{6}$$

Substituting the point x_0, we obtain $YX(s)(x_0)=-h(X,Y)(x_0)$.

Next theorem gives a nice geometric interpretation of the cubic form:

Theorem 1 *The third order Taylor expansion of the difference between the visual contour and planar section functions at x_0 is equal to the cubic form at this point.*

Proof Differentiating (5) we obtain

$$[ZXY(t)\nu + YX(t)\nu_*(Z) + ZX(t)\nu_*(Y) + X(t)D_Z\nu_*(Y)] \cdot \xi_0 =$$
$$D_Z\nu_*X \cdot (q_*(Y) - Y(t)\xi_0) + \nu_*(X) \cdot (D_Zq_*(Y) - ZY(t)\xi_0) +$$
$$+D_Y\nu_*(X) \cdot (q_*(Z) - Z(t)\xi_0) + D_ZD_Y\nu_*(X) \cdot (q(x) - q_0 - t\xi_0).$$

At x_0, we obtain $ZXY(t) = -Z(h(X,Y)) + D_Y\nu_*(X) \cdot q_*(Z)$. Since $D_Y\nu_*(X) = \nu_*(\overline{\nabla}_Y X) - h(SX,Y)\nu$, we obtain

$$D_Y\nu_*(X) \cdot q_*(Z) = -h(\overline{\nabla}_Y X, Z).$$

We conclude that

$$ZYX(t) = -Z(h(X,Y)) - h(\overline{\nabla}_Y X, Z). \tag{7}$$

Now differentiate (6). We obtain

$$ZXY(s) = -\nu_0 \cdot [\nabla_Z q_*(\nabla_Y X) + h(\nabla_Y X, Z)\xi + Z(h(X,Y))\xi - h(X,Y)q_*(SZ)]$$

Thus, at x_0,

$$ZXY(s) = -Z(h(X,Y)) - h(\nabla_Y X, Z). \tag{8}$$

From (7) and (8)

$$ZXY(t) - ZXY(s) = h(\nabla_Y X, Z) - h(\overline{\nabla}_Y X, Z) = 2h(K_X Y, Z) = -C(X,Y,Z),$$

thus proving the theorem.

3.2 Re-scaling of planar sections in the case of proper affine spheres

In the case of proper affine spheres, we can define the planar sections with a slightly different scaling. The objective is to have $s = t$ for quadrics and also a duality property between planar sections and visual contours.

Let $q : M \to \mathbb{R}^3$ be a proper affine sphere. By a dilation of the affine 3-dimensional space, we may assume that $S = Id$ and $\xi = q$. In this case, $q_0 + t\xi_0 = q_0(1+t)$ and thus $T(x_0,t)$ is given by $\nu(x) \cdot (q(x) - q_0(1+t)) = 0$, that can be re-written as

$$\nu(x) \cdot q(x_0)(1+t) = 1. \tag{9}$$

On the other hand, we define the planar section function to be the a re-scaling of the planar section function defined above, with $r + 1 = \frac{1}{1-s}$. Thus $x \in S(x_0,s)$ if and only if $\nu(x_0) \cdot (q(x_0)(1+s)^{-1} - q(x)) = 0$, which can be re-written as

$$\nu(x_0) \cdot q(x) = (1+s)^{-1}. \tag{10}$$

In the context of proper affine spheres, we shall maintain the notation $S(x_0,s)$ for this re-scaled planar section, hoping that this will cause no confusion.

It is easy to verify that, if the affine sphere is a quadric, $S(x_0,t) = T(x_0,t)$. Another interesting consequence of the re-scaling is the following duality relation:

Proposition 1 *Consider an affine sphere M and a point $x_0, x_1 \in M$. Then $x_1 \in T(x_0, t)$ if and only if $x_0 \in S(x_1, t)$.*

Proof For proper affine spheres, the statement is an immediate consequence of (9) and (10). For improper affine spheres, (1) is equivalent to

$$t = \nu(x) \cdot (q(x) - q(x_0)),$$

and (3) is equivalent to

$$s = \nu(x_0) \cdot (q(x_0) - q(x)).$$

From these equations it is immediate to see that the proposition holds also for improper affine spheres.

Remark 1 We remark that this re-scaling does not affect the validity of theorem 1. In fact, we have that , for $X, Y, Z \in \mathcal{X}(M)$, $X(r) = \frac{1}{(1-s)^2} X(s)$, and thus $X(r) = 0$ at $s = 0$. Also $YX(r) = \frac{2}{(1-s)^3} X(s)Y(s) + \frac{1}{(1-s)^2} YX(s)$, and thus $YX(r) = YX(s)$ at $s = 0$. Finally, differentiating the latter formula one obtains $ZYX(r) = ZYX(s)$ at $s = r = 0$.

4 Parallel-meridian coordinates

From now on, we shall restrict ourselves to affine immersions $q : M \to \mathbb{R}^3$ of two dimensional manifolds M into the three dimensional affine space.

4.1 Definition and basic properties

Consider a parameterization $x : D \subset \mathbb{R}^2 \to M$, and denote $x_u = U, x_v = V$.

- For a fixed v_0, the line $u \to x(u, v_0)$ is called a (pre)-geodesic for ∇ if $\nabla_U U$ is parallel to U. Similarly, $u \to x(u, v_0)$ is called a $\overline{\nabla}$ (pre)-geodesic if $\overline{\nabla}_U U$ is parallel to U.
- For a fixed u_0, the line $v \to x(u_0, v)$ is called a visual contour based on $p(u_0) \in \mathbb{R}^3$ if, for each v, $q(x(u_0, v)) - p(u_0)$ is parallel to $q_*(U)$. Dually, the line $v \to x(u_0, v)$ is called a planar section with normal $n(u_0) \in (\mathbb{R}^3)*$ if, for each v, $\nu(x(u_0, v)) - n(u_0)$ is parallel to $\nu_*(U)$.

*Definition.*We shall call a parameterization $x : D \subset \mathbb{R}^2 \to M$ of an affine immersion $q : M \to \mathbb{R}^3$ *parallel-meridian* if the lines $u \to x(u, v_0)$ are both ∇ and $\overline{\nabla}$ geodesics and the lines $v \to x(u_0, v)$ are both visual contours and planar sections.

Our first lemma gives conditions to verify whether or not a given parameterization is parallel-meridian.

Lemma 2 *A parameterization $x : D \subset \mathbb{R}^2 \to M$ is parallel-meridian if and only if there exist functions a, b, c, d, e, f and g that does not depend on v such that*

$$\begin{cases} q_{uu} = a(u)q_u + b(u)\xi \\ q_{vv} = c(u)q_u + d(u)\xi \\ q_{uv} = e(u)q_v. \end{cases} \tag{11}$$

and

$$\begin{cases} \xi_u = f(u)q_u \\ \xi_v = g(u)q_v. \end{cases} \tag{12}$$

Proof Assume first that the coordinates (u, v) are parallel-meridian. Since the lines $u \to x(u, v_0)$ are ∇-geodesics, we can write

$$q_{uu} = aq_u + b\xi, \tag{13}$$

for some real functions a, b. Now the condition that the lines $v \to x(u_0, v)$ are visual contours implies that the net is conjugate, i.e.,

$$q_{uv} = \alpha q_u + eq_v. \tag{14}$$

Thus $\nu \cdot q_{uv} = 0$, and then $\nu_v \cdot q_u = \nu_u \cdot q_v = 0$. Now we use the hypothesis of the lines $u \to x(u, v_0)$ being $\overline{\nabla}$-geodesics to write $\nu_{uu} \cdot q_v = 0$, which implies that $\nu_u \cdot q_{uv} = 0$. Thus if we multiply (14) by ν_u we obtain $\alpha = 0$. Since $v \to x(u_0, v)$ is a visual contour, this implies that $e_v = 0$. From the hypothesis of the lines $v \to x(u_0, v)$ being planar sections, we obtain $\nu_{uv} \cdot \xi = 0$. This implies that $\nu_u \cdot \xi_v = \nu_v \cdot \xi_u = 0$, and thus we can write $\xi_v = gq_v$ and $\xi_u = fq_u$.

Computing q_{uvu} from (14) we observe that $q_{uvu} = (e_u + e^2)q_v$. Then compute q_{uuv} from (13) to obtain

$$q_{uuv} = a_v q_u + aeq_v + b_v\xi + bgq_v$$

and conclude that $a_v = b_v = 0$ and $ae + bg = e_u + e^2$. Thus also $g_u = 0$. From $\xi_{uv} = \xi_{vu}$ we conclude that also $f_v = 0$. Finally write

$$q_{vv} = cq_u + \beta q_v + d\xi. \tag{15}$$

Since the net is conjugate, some linear combination of q_{uu} and q_{vv} must be in the direction of ξ, which implies that $\beta = 0$. On the other hand, comparing q_{uvv} calculated from (14) and q_{vvu} calculated from (15), we conclude that

$$\begin{cases} c_u = ce - ca - df \\ d_u = de - cb. \end{cases}$$

Since c and d are solutions of the above system of differential equations, we conclude that $c = c(u)$ and $d = d(u)$, and so equations (11) and (12) hold.

Now assume that equations (11) and (12) hold. They imply immediately that the lines $u \to x(u, v_0)$ are ∇-geodesics and that the lines $v \to x(u_0, v)$ are visual contours. Since $\nu \cdot q_{uv} = 0$, we have $\nu_u \cdot q_v = \nu_v \cdot q_u = 0$. And since $\nu_u \cdot q_{uv} = 0$ we have that $\nu_{uu} \cdot q_v = 0$, and so the lines $u \to x(u, v_0)$ are $\overline{\nabla}$-geodesics.

To show that the lines $v \to x(u_0, v)$ are planar sections, we write $\nu_{uv} = \alpha\nu_u + \beta\nu_v + \gamma\nu$. Since $\nu_{uv} \cdot q_u = -\nu_v \cdot q_{uu} = 0$, we conclude that $\alpha = 0$. Also, $\nu_{uv} \cdot \xi = -\nu_u \cdot \xi_v = 0$ and thus $\gamma = 0$. Finally, $\nu_{uv} \cdot q_v = -\nu_u \cdot q_{vv} = -c(u)\nu_u \cdot q_u = b(u)c(u)$. So $-\beta(u, v)d(u) = b(u)c(u)$ and thus $\beta_v = 0$. Hence $\nu_{uv} = \beta(u)\nu_v$, which implies that $\nu_u - \beta(u)\nu = n(u)$, for a certain vector $n(u)$, thus proving that the lines $v \to x(u_0, v)$ are planar sections.

Example 1 The most typical examples of parallel-meridian parameterizations are that of revolution surfaces. A general revolution surface can be parameterized as

$$q(u, \theta) = (r(u)\cos\theta, r(u)\sin\theta, z(u)) .$$

By a change of variables $w = w(u)$, we may assume that $z_{uu}r_u - r_{uu}z_s = rz_u$. Under this condition the coordinates (u, θ) become isothermal. One can verify that $\Omega = r\sqrt{z_u}$ and $\xi = -\frac{q_{uu}+q_{\theta\theta}}{2\Omega}$. The Gauss equations are then

$$\begin{cases} q_{uu} = \dfrac{z_{uu}}{2z_u}q_u - \Omega\xi \\ q_{\theta\theta} = -\dfrac{z_{uu}}{2z_u}q_u - \Omega\xi \\ q_{u\theta} = \dfrac{r_u}{r}q_\theta . \end{cases}$$

By lemma 2, (u, θ) are parallel-meridian coordinates.

4.2 Parallel-meridian isothermal coordinates

Next lemma shows that any parallel-meridian parameterization can be re-parameterized to become isothermal without changing its coordinate lines.

Lemma 3 *Assume $x : D \subset \mathbb{R}^2 \to M$ is a parallel-meridian parameterization of the affine immersion $q : M \to \mathbb{R}^3$. Then there is a change of variables $w = w(u)$ such that $q(w, v)$ is isothermal.*

Proof From lemma 2, one can see that $([q_u, q_v, \xi])_v = 0$, which implies that $L = [q_u, q_v, q_{uu}]$ and $N = [q_u, q_v, q_{vv}]$ does not depend on v. After a change of variables $u = u(w)$, we have $q_w = u_w q_u$ and $q_{ww} = u_{ww}q_u + u_w^2 q_{uu}$. Thus, for the new parameterization, $L' = u_w^3 L(u)$ and $N' = u_w N(u)$. Thus take $w(u)$ satisfying

$$w_u^2 N(u) = L(u)$$

to obtain an isothermal parameterization.

4.3 Parallel-meridian coordinates with a pole

Suppose that the image of a parallel-meridian parameterization $x : D \subset \mathbb{R}^2 \to M$ is a disk minus a point $x_0 \in M$.

We say that $x_0 \in M$ is a *pole* of x if the following conditions hold:

- For any v_0, the meridian $u \to x(u, v_0)$ is contained in a ∇ geodesic that starts at x_0.
- For any v_0, the meridian $u \to x(u, v_0)$ is contained in a $\overline{\nabla}$ geodesic that starts at x_0.
- For any u_0, t is constant along the parallel $v \to x(u_0, v)$, or, equivalently, any parallel is a visual contour based on $q(x_0) + t\xi(x_0)$, for some t.
- For any u_0, s is constant along the parallel $v \to x(u_0, v)$, or, equivalently, any parallel is a planar section with normal $\nu(x_0)$.

Lemma 4 *Suppose that a parallel-meridian parameterization $x : D \subset \mathbb{R}^2 \to M$ of an affine immersion $q : M \to \mathbb{R}^3$ admits a pole $x_0 \in M$. Then the cubic form is zero at x_0.*

Proof This is a direct consequence of theorem 1 and the definition of pole.

4.4 Parallel-meridian coordinates with a pole on an affine sphere

Next theorem shows that, between affine spheres, only quadrics admit parallel-meridian coordinates.

Theorem 2 *Suppose that the affine sphere* $q : M \to \mathbb{R}^3$ *admits a parallel-meridian parameterization* $x : D \subset \mathbb{R}^2 \to M$ *with a pole. Then* $q(M)$ *is contained in a quadric.*

Proof By lemma 3, we may assume that the parallel-meridian coordinates are also isothermal. We can thus use Gauss equations of an isothermal parameterization to conclude that $\Omega_v = b = 0$ and $a_v = 0$ (see section 2.2). Since for affine spheres $a_u = 0$, we conclude that a is constant. But from lemma 4, $a = 0$ at the pole. We conclude that $a = 0$ which implies that the cubic form is zero, and thus $q(M)$ is contained in a quadric.

Corollary 1 *Suppose that* $q : M \to \mathbb{R}^3$ *is an affine sphere of revolution with a point in the axis of revolution. Then* $q(M)$ *is contained in a quadric.*

Proof Any revolution surface admit a parallel-meridian parameterization whose points in the axis of revolution are poles. The result follows then from the theorem.

4.5 Parallel-meridian coordinates on a quadric

In this section we shall describe all the parallel-meridian coordinates on a quadric. We begin with a general lemma that says that a change of variables between isothermal coordinates must be holomorphic or anti-holomorphic.

Lemma 5 *Suppose that* $x : D_1 \subset \mathbb{R}^2 \to M$ *and* $y : D_2 \subset \mathbb{R}^2 \to M$ *are parameterizations associated with an affine immersion* $q : M \to \mathbb{R}^3$*, with* y *isothermal. Denote by* $W \subset M$ *the intersection of the images of* x *and* y*,* $D_1' = x^{-1}(W)$*,* $D_2' = y^{-1}(W)$*,* $g = y^{-1} \circ x : D_1' \to D_2'$*. Then* x *is isothermal if and only if* g *is conformal.*

Proof Denote by (u, v) the coordinates in D_1' and by (w, z) the coordinates in D_2'. We write $(w, z) = g(u, v)$, $q(u, v) = q(x(u, v))$ and $q(w, z) = q(y(w, z))$. We have

$$\begin{cases} q_u = w_u q_w + z_u q_z \\ q_v = w_v q_w + z_v q_z, \end{cases}$$

and so $q_u \times q_v = (w_u z_v - w_v z_u) q_w \times q_z$. Also

$$\begin{cases} q_{uu} = w_{uu} q_w + z_{uu} q_z + w_u^2 q_{ww} + 2 w_u z_u q_{wz} + z_u^2 q_{zz} \\ q_{vv} = w_{vv} q_w + z_{vv} q_z + w_v^2 q_{ww} + 2 w_v z_v q_{wz} + z_v^2 q_{zz} \\ q_{uv} = w_{uv} q_w + z_{uv} q_z + w_u w_v q_{ww} + (w_u z_v + z_u w_v) q_{wz} + z_u z_v q_{zz}. \end{cases}$$

Since (w, z) are isothermal coordinates, we have that $[q_w, q_z, q_{wz}] = 0$ and $[q_w, q_z, q_{ww}] = [q_w, q_z, q_{zz}] = \Omega$. So

$$\begin{cases} [q_u, q_v, q_{uu}] = (w_u z_v - w_v z_u) \Omega \left(w_u^2 + z_u^2 \right) \\ [q_u, q_v, q_{vv}] = (w_u z_v - w_v z_u) \Omega \left(w_v^2 + z_v^2 \right) \\ [q_u, q_v, q_{uv}] = (w_u z_v - w_v z_u) \Omega \left(w_u w_v + z_u z_v \right). \end{cases}$$

Thus x is isothermal if and only if $w_u^2 + z_u^2 = w_v^2 + z_v^2$ and $w_u w_v + z_u z_v = 0$. But these conditions are equivalent to the conformality of the Jacobian matrix.

We now assume that M is a quadric and that the immersion $q : M \to \mathbb{R}^3$ is the identity. Possibly after an affine transformation of $\mathbb{R}^3$, we may also assume that M is the unit sphere $x^2+y^2+z^2 = 1$, the convex hyperboloid $z = \sqrt{1+x^2+y^2}$ or the paraboloid $z = x^2+y^2$. Consider the following isothermal parameterizations of M: For the sphere, let

$$ms(u, v) = (\cos\phi(u)\cos v, \cos\phi(u)\sin v, \sin\phi(u)),$$

where $\phi = \phi(u)$ satisfies $\phi_u = \cos\phi$, $\phi(0) = 0$. This map from the plane to the sphere is known as the Mercator's map. It is straightforward to verify that me is an isothermal parallel-meridian of the sphere with poles $(0, 0, \pm 1)$. For more details on parallel-meridian coordinates of the sphere and the Mercator's map, see [4]. For the convex hyperboloid, let

$$mh(u, v) = (\sinh\psi(u)\cos v, \sinh\psi(u)\sin v, \cosh\psi(u)),$$

where $\phi = \phi(u)$ satisfies $\psi_u = \sinh\psi$, $\psi(0) = 0$. It is also straightforward to verify that mh is an isothermal parallel-meridian of the convex hyperboloid with pole $(0, 0, 1)$. For the paraboloid,

$$mp(u, v) = (\exp(u)\cos v, \exp(u)\sin v, \exp(2u))$$

defines isothermal parallel-meridian coordinates with pole $(0, 0, 0)$.

Let $x : D \subset \mathbb{R}^2 \to M$ be a parallel-meridian parameterization with pole N. By an equi-affine transformation of $\mathbb{R}^3$, we may assume that $N = (0, 0, 1)$, if M is the sphere, $N = (0, 0, 1)$, if M is the convex hyperboloid, and $N = (0, 0, 0)$ if M is the paraboloid. By lemma 3, we may assume also that x is isothermal. Next proposition describe all possible isothermal parallel-meridian parameterizations of a quadric with a given pole:

Proposition 2 *Let $x : D \subset \mathbb{R}^2 \to M$ be an isothermal parallel-meridian parameterization of a quadric M with pole N as above. Then, possibly after a translation and a dilation of D, x must be equal to ms, mh or mp, according to M being a sphere, a hyperboloid or a paraboloid.*

Proof Let $g : D \subset \mathbb{R}^2 \to \mathbb{R}^2$ denote the map $g = m^{-1} \circ x$, where $m = me$, $m = mh$ or $m = mp$ in each case. It is clear that g takes u and v coordinate lines into u and v coordinate lines, respectively. Thus we can write $g(u, v) = (g_1(u), g_2(v))$, for some one variable functions g_1 and g_2. From lemma 5, we know that g is conformal. Thus $g_1'(u) = g_2'(v)$, which implies that $g_1(u) = au + b_1, g_2(v) = av + b_2$, for some $a, b_1, b_2 \in \mathbb{R}$. We conclude that $g(u, v) = a \cdot (u, v) + (b_1, b_2)$, which proves the proposition.

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