Some Properties of the Volume Distance to Hypersurfaces

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Abstract. In this paper we consider the volume distance from a point to a convex hypersurface $M \subset \mathbb{R}^{N+1}$, which is defined as the minimum $(N+1)$-volume of a region bounded by M and a hyperplane through the point. We describe some of its properties, among them the centroid property, which says that any point is the centroid of its minimal section. We discuss the differentiability of the volume distance and show that it is smooth in a certain neighborhood of M. Besides, we obtain a formula for the hessian of the volume distance which makes possible to prove that the normalized hessian converges to the affine Blaschke metric when we approximate the hypersurface along a curve whose points are centroids of parallel sections. We also show that the rate of this convergence is given by a bilinear form associated with the shape operator of M. These convergence results provide a geometric interpretation of the Blaschke metric and the shape operator in terms of the volume distance.

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1. Introduction

For a convex hypersurface $M \subset \mathbb{R}^{N+1}$, p in the convex side of M and a unitary vector $n \in S^N$, let $U(n,p)$ be the region bounded by M and a hyperplane $\pi(n,p)$ orthogonal to n through p and $V(n,p)$ the volume of $U(n,p)$. The *volume distance* $v(p)$ of p to M is defined as the minimum of $V(n,p)$, $n \in S^N$.

The volume distance is an important object in computer vision which has been extensively studied in the planar case $n = 1$ ([1]) and was also considered in the case $n = 2$ ([5]). In this paper we discuss some of its properties in the N-dimensional case. Among the properties considered, the centroid property plays a central role. It says that any point is the centroid of its

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minimizing section. Another fact concerning area distance (volume distance in the planar case) is that the determinant of its hessian matrix equals -4 ([3],[4]). Although this property is not extended to higher dimensions, we shall prove in this paper some other properties of the hessian matrix of the volume distance in arbitrary dimensions.

A pair (n, p) is called *minimizing* when n is the minimum of $V(n, p)$ with p fixed. A minimizing pair is necessarily *critical*, i.e., $\frac{\partial V}{\partial n}(n, p) = 0$. We prove in this paper that

$$\frac{\partial V}{\partial n}(n, p) = -b(n, p)\left(\overline{p}(n, p) - p\right),$$

where $b(n, p)$ denotes the N-dimensional volume of the region $R(n, p) \subset \pi(n, p)$ bounded by M and $\overline{p}(n, p)$ denotes the centroid of the same region. As a consequence we have the centroid property: if (n, p) is a critical pair, then p is the centroid of $R(n, p)$. This fact was already pointed out in [2].

In order to obtain $n = n(p)$ implicitly defined by $\frac{\partial V}{\partial n}(n, p) = 0$, the second derivative $\frac{\partial^2 V}{\partial n^2}(n, p)$ must be non-degenerate. In section 2, we give an explicit formula for this second derivative which shows that it is positive definite in a *half-neighborhood* of M, i.e., the part of a neighborhood of M contained in its convex side. In fact, we prove that there exists a half-neighborhood D_0 of M such that, for $p \in D_0$, there exists a unique $n(p)$ that minimizes the map $n \to V(n, p)$. Moreover, the map $p \to n(p)$ is smooth and consequently $v(p) = V(n(p), p)$ is also smooth. We also calculate explicit formulas for Dv and D^2v in this half-neighborhood.

In section 3, we shall only consider points in D_0. For $p \in D_0$, we can define the *normalized hessian* $Hv(p)$ of v as the restriction of the bilinear form $-b(n(p), p)^{-1}D^2v(p)$ to the hyperplane $\pi(n(p), p)$. The results of section 2 say that $Hv(p)$ is positive definite in D_0 and we shall study the behavior of $Hv(p)$ when p converges to the hypersurface M.

For $q \in M$, denote by $T_qM = \pi(n(q), q)$ the tangent plane and, for $t > 0$, define $\gamma_q(t)$ as the centroid of the region $R(n(q), q + t\xi(q))$, where $\xi(q)$ is the affine normal to M at q. We shall consider two symmetric bilinear forms defined on T_qM: the Blaschke metric h which is positive definite and h_S defined as $h_S(X, Y) = h(X, SY)$, where S is the shape operator. For any $t > 0$, the normalized hessian $Hv(\gamma_q(t))$ defines also a symmetric bilinear form in T_qM. The main result of section 3 says that

$$Hv(\gamma_q(t)) = h(q) + th_S(q) + O(t^2),$$

where $O(t^k)$ indicates a quantity such that $\lim_{t\to 0} \frac{O(t^k)}{t^{k-\epsilon}} = 0$, for any $\epsilon > 0$. This result can be regarded as a geometric interpretation of the Blaschke metric and the shape operator in terms of the volume distance.

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2. Volume Distances

Consider a strictly convex hypersurface $M \subset \mathbb{R}^{N+1}$, possibly with a non-empty boundary ∂M. Denote by $\pi(n,p)$ the oriented hyperplane of $\mathbb{R}^{N+1}$ passing through $p \in \mathbb{R}^{N+1}$ with normal $n \in S^N$. For $p \in \mathbb{R}^{N+1}$, denote by $E(p) \subset S^N$ the set of unitary vectors n whose corresponding hyperplane $\pi(n,p)$ intersects $M - \partial M$ transversally at a closed hypersurface $\Gamma(n,p) \subset \pi(n,p)$ bounding a region $R(n,p) \subset \pi(n,p)$ containing p in its interior and such that the region $U(n,p)$ bounded by $R(n,p)$ and M, with n pointing outwards, has finite volume $V(n,p)$ (see figure 1). Denote by $D \subset \mathbb{R}^{N+1}$ the set of $p \in \mathbb{R}^{N+1}$ such that $E(p) \neq \emptyset$ and the infimum $\inf\{V(n,p)|\ n \in E(p)\}$ is attained at $E(p)$. When $n \in E(p)$ attains this minimum, we call the pair (n,p) minimizing and $v(p) = V(n,p)$ the volume distance to M. We remark that if M is a closed hypersurface enclosing a convex region, then the domain D of the volume distance is all the enclosed region.

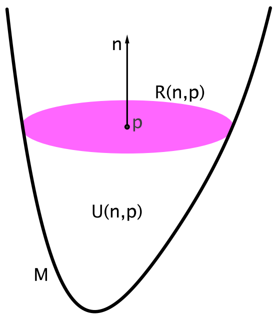

Figure 1. The section $R(n,p)$ and the enclosed region $U(n,p)$.

For $q \in M$, denote by $\xi(q)$ the affine normal vector pointing to the convex side of M. Along this paper, we shall call a half-neighborhood of M any set of the form

$$\{q + t\xi(q)|\ q \in M, 0 \leq t < T(q)\},$$

where $T(q) > 0$ is some smooth function of q. The main objective of section 2 is to prove the following result:

Theorem 2.1. *There exists a half-neigborhood $D_0 \subset D$ of M such that for any $p \in D_0$ there exists a unique $n = n(p)$ such that the pair $(n(p), p)$ is minimizing, the map $p \to n(p)$ is smooth and thus $v(p) = V(n(p), p)$ is also smooth. Moreover, for any $p \in D_0$,*

$$Dv(p) = b(n(p), p)n(p)$$

and $D^2v(p)$ restricted to $\pi(n(p), p)$ is negative definite.

2.1. Centroid property

Close to a pair (n_0, p_0), consider cartesian coordinates $(x, z) \in \mathbb{R}^N \times I$, $I = (-\epsilon, \epsilon)$ such that $p_0 = (0, 0)$ and $n_0 = (0, 1)$. To describe the hypersurface M in a neighborhood of $\pi(n_0, p_0)$, consider the cylindrical coordinates (r, η, z), where $x = r\eta$, $\eta \in S^{N-1}$, $r > 0$. Then M is described by $r = r(\eta, z)$, for some smooth function r (see figure 2). We write

$$r(\eta, z) = g(\eta) + m(\eta)z + O(z^2), \tag{2.1}$$

for z close to 0, where $g(\eta) = r(\eta, z)$ and $m(\eta) = r_z(\eta, 0)$.

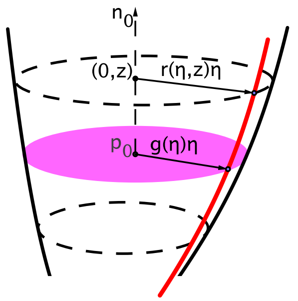

FIGURE 2. The curve $r = r(\eta, z)$ with fixed $\eta \in S^N$.

Close to $n_0 = (0, 1)$, we parameterize S^N by

$$n = \frac{1}{\sqrt{||a||^2 + 1}}(-a, 1),$$

with a in a neighborhood of $0 \in \mathbb{R}^N$. A simple calculation shows that the derivative $\frac{dn}{da}(a) : \mathbb{R}^N \to \mathbb{R}^{N+1}$ satisfies

$$\frac{dn}{da}(a)w = (-w, -a \cdot w) + O(||a||^2), \tag{2.2}$$

for any $w \in \mathbb{R}^N$, where $a \cdot w$ corresponds to the canonical inner product of $\mathbb{R}^N$. Denote by $V(a) = V(n(a), p_0)$ the volume in a-coordinates, with p_0 fixed. The derivative $\frac{\partial V}{\partial n}(n, p_0)$ can be regarded as a linear functional on $T_n S^N \approx \pi(n, p_0)$ and we write

$$\frac{dV}{da}(a) = \frac{\partial V}{\partial n}(n(a)) \cdot \frac{dn}{da}(a). \tag{2.3}$$

Identifying $T_{n_0} S^N$ with $\mathbb{R}^N$ we have that $\frac{dn}{da}(0) = -Id$ and thus

$$\frac{dV}{da}(0) = -\frac{\partial V}{\partial n}(n_0, p_0). \tag{2.4}$$

The equation of $\pi(a) = \pi(n(a), p_0)$ is

$$z = a \cdot x = r a \cdot \eta. \tag{2.5}$$

By eliminating z in equations (2.1) and (2.5), we obtain the equation of the projection on $\mathbb{R}^N$ of the intersection of M and the hyperplane $\pi(a)$. We have

$$r = g(\eta) + m(\eta)ra \cdot \eta + O(z^2)$$

and since $O(z) \le O(||a||)$ we obtain

$$r(a, \eta) = g(\eta)(1 - m(\eta)a \cdot \eta)^{-1} + O(||a||^2). \tag{2.6}$$

Let $l = l(a) = \pi(a) \cap \pi(0)$ and denote by $R_i(a)$, $i = 1, 2$, the two subsets of $R(a)$ determined by l (see figure 3). Also, denote by d_l the orthogonal distance to l and by

$$J_i(a) = \int_{R_i(a)} d_l \; d\lambda$$

the momentum of the region $R_i(a)$ with respect to l, where λ is the N-dimensional Lebesgue measure of $\pi(a)$. Let $\eta_l = \frac{a}{||a||}$ be orthogonal to l pointing outwards $R_1(0)$. Thus, for $a = 0$, $d_l(x) = (-1)^i x \cdot \eta_l$ and so

$$J_i(0) = (-1)^i \int_{R_i(0)} x \cdot \eta_l \; d\lambda(x).$$

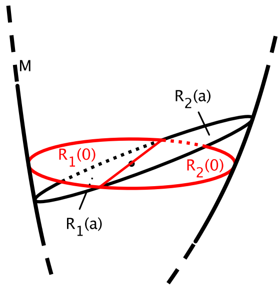

FIGURE 3. The regions $R_1(a)$ and $R_2(a)$.

Lemma 2.2. *We have that*

$$J_i(a) - J_i(0) = O(||a||).$$

Proof. The volume element of $\pi(0)$ in (r, η)-coordinates is $r^{N-1} dr d\eta$, where $d\eta$ denotes the volume element of S^{N-1}. Thus the volume element of $\pi(a)$ is $r^{N-1} dr d\eta \sqrt{1 + ||a||^2}$. Also, in the plane $\pi(a)$, $d_l = (-1)^i x \cdot \eta_l \sqrt{1 + ||a||^2}$. Thus

$$J_i(a) = (-1)^i (1 + ||a||^2) \int_{S^{N-1}} \int_{s=0}^{r(a,\eta)} s\eta \cdot \eta_l s^{N-1} ds d\eta,$$

where $r(a, \eta)$ is given by (2.6). Thus

$$J_i(a) = \frac{(-1)^i (1 + ||a||^2)}{N+1} \int_{S^{N-1}} g^{N+1}(\eta)(1 - m(\eta)a \cdot \eta)^{-(N+1)} \eta \cdot \eta_l \; d\eta + O(||a||^2).$$

Since $(1-s)^{-(N+1)} = 1 + (N+1)s + O(s^2)$, we conclude that

$$J_i(a) - J_i(0) = (-1)^i \int_{S^{N-1}} g^{N+1}(\eta) m(\eta)(a \cdot \eta)\eta \cdot \eta_l \, d\eta + O(||a||^2),$$

thus proving the lemma. □

Proposition 2.3. *Denote by $\overline{p}(n,p)$ the center of gravity of $R(n,p)$ and by $b(n,p)$ the N-dimensional volume of the region $R(n,p)$. Then*

$$\frac{\partial V}{\partial n}(n,p) = -b(n,p)\left(\overline{p}(n,p) - p\right). \tag{2.7}$$

Thus, a pair (n,p) is critical if and only if $\overline{p}(n,p) = p$.

Proof. By equation (2.4), $\frac{\partial V}{\partial n}(n_0,p_0) = -\frac{dV}{da}(0)$. Consider the family of hyperplanes $\pi(ta)$, $0 \le t \le 1$, all containing l. The hyperplane $\pi(a)$ determine two regions of R^{N+1} bounded by $\pi(0)$, $\pi(a)$ and M, the first one contained in $U(n_0,p_0)$, whose volumes will be denoted by $V_1(a)$ and $V_2(a)$, respectively. The new enclosed volume $V(a)$ is then $V(0) + V_2(a) - V_1(a)$.

Now the volume element in $\mathbb{R}^{N+1}$ is $d_l \cdot d\theta d\lambda$, where λ denotes the Lebesgue measure of $\pi(ta)$ and $\tan\theta = t||a||$. Thus

$$V_i(a) = \int_{\theta=0}^{\alpha} \int_{R_i(ta)} d_l \, d\lambda d\theta = \int_0^{\alpha} J_i(ta) d\theta,$$

where $\tan\alpha = ||a||$. By lemma 2.2, $J_i(ta) = J_i(0) + O(\theta)$ and thus we conclude that

$$V_i(a) = \alpha J_i(0) + O(\alpha^2).$$

Since

$$J_2(0) - J_1(0) = b(0)\overline{p}(0) \cdot \eta_l,$$

and $\alpha = ||a|| + O(||a||^3)$, we conclude that

$$V(a) - V(0) = b(0)\overline{p}(0) \cdot ||a||\eta_l + O(||a||^2) = b(0)\overline{p}(0) \cdot a + O(||a||^2).$$

This shows that

$$\frac{dV}{da}(0) = b(0)\overline{p}(0),$$

which proves the proposition. □

The following lemma will be useful below:

Lemma 2.4. *We have that*

$$\frac{\partial V}{\partial p}(n,p) = b(n,p)n. \tag{2.8}$$

Proof. Since $p \to V(n,p)$ is constant along the hyperplane $\pi(n,p)$, we conclude that $\frac{\partial V}{\partial p}(n,p)$ is parallel to n. Also, for t small,

$$V(n, p+tn) - V(n,p) = tb(n,p) + O(t^2),$$

and thus the lemma is proved. □

2.2. Some more derivatives and the critical set

The hessian $\frac{\partial^2 V}{\partial n^2}(n,p)$ can be seen as a linear operator of $T_nS^N \approx \pi(n,p)$. The same hessian in a-coordinates, denoted $\frac{d^2V}{da^2}(a)$, can be seen as a linear operator of $\mathbb{R}^N$. Differentiating (2.3) and taking into account that $\frac{\partial^2 n}{\partial a_i \partial a_j}(0)$ is orthogonal to $T_{n_0}S^N$, for any $1 \leq i,j \leq N$, we obtain

$$\frac{d^2V}{da^2}(0) = \left(\frac{dn}{da}(0)\right)^* \cdot \frac{\partial^2 V}{\partial n^2}(n_0,p_0) \cdot \frac{dn}{da}(0),$$

where $\left(\frac{dn}{da}(0)\right)^*$ is the adjoint of $\frac{dn}{da}(0)$. Considering the identification $T_{n_0}S^N \approx \mathbb{R}^N$, we have that $\frac{dn}{da}(0) = -Id_N$. Thus we conclude that

$$\frac{\partial^2 V}{\partial n^2}(n_0,p_0) = \frac{d^2V}{da^2}(0).$$

Proposition 2.5. *Denote* $\mathbf{M}_N$ *the symmetric positive definite* $N \times N$ *matrix* $\eta \cdot \eta^t$, *where* η *is a column vector and* η^t *its transpose. We have that*

$$\frac{\partial^2 V}{\partial n^2}(n_0,p_0) = \int_{S^{N-1}} g^{n+1}(\eta)m(\eta)\mathbf{M}_N d\eta. \tag{2.9}$$

Proof. From the above remark, we must calculate $\frac{d^2V}{da^2}(0)$. Writing $\overline{p}(a) = (\overline{x}(a), a \cdot \overline{x}(a))$ and using equations (2.2), (2.3) and (2.7) we obtain

$$\frac{dV}{da}(a) = -b(a)\left(-\overline{x}(a) - (a \cdot \overline{x}(a))a\right) + O(||a||^2).$$

Thus

$$\frac{dV}{da}(a) = b(a)\overline{x}(a) + O(||a||^2),$$

which, by equation (2.6), can be written as

$$\frac{dV}{da}(a) = \frac{1}{(n+1)} \int_{S^{N-1}} g^{n+1}(\eta)(1 - m(\eta)a \cdot \eta)^{-(n+1)}\eta d\eta + O(||a||^2).$$

Since $(1-s)^{-(n+1)} = 1 + (n+1)s + O(s^2)$, we obtain

$$\frac{dV}{da}(a) - \frac{dV}{da}(0) = \int_{S^{N-1}} g^{n+1}(\eta)m(\eta)(a \cdot \eta)\eta d\eta + O(||a||^2),$$

thus proving the proposition. □

Below we shall prove that close to M, the right hand side of (2.9) is positive definite. For now, we just assume that $\frac{\partial^2 V}{\partial n^2}$ is non-degenerate at a critical pair (n_0,p_0). Then $\frac{\partial V}{\partial n} = 0$ defines n implicitly as a function of p in a neighborhood of this pair.

Close to a critical pair (n_0,p_0) define $v_1(p) = V(n(p),p)$. If $(n(p),p)$ are minimizing pairs, then v_1 equals the volume distance v. The derivative of v_1 is given by

$$Dv_1(p) = \frac{\partial V}{\partial p}(n(p),p) = b(n(p),p)n(p), \tag{2.10}$$

where we have used formula (2.8).

From

$$\frac{\partial^2 V}{\partial n^2}(n,p)\frac{dn}{dp}+\frac{\partial^2 V}{\partial n\partial p}(n,p)=0,$$

we obtain

$$\frac{dn}{dp}=-b(n,p)\left[\frac{\partial^2 V}{\partial n^2}(n,p)\right]^{-1}. \tag{2.11}$$

The hessian of v_1 restricted to $\pi(n,p)$ is then

$$D^2v_1=-b^2(n,p)\left[\frac{\partial^2 V}{\partial n^2}(n,p)\right]^{-1}. \tag{2.12}$$

2.3. Smoothness of the volume distance v in a half-neighborhood of M

If $m(\eta)>0$, for any $\eta\in S^{n-1}$, then formula (2.9) implies $\frac{\partial^2 V}{\partial n^2}(n_0,p_0)$ is positive definite. We shall now prove theorem 2.1, which says that, for p in a half-neighborhood of M, there is a unique minimizer $n(p)$ and $m(p,n(p))>0$.

Proof. (of theorem 2.1). Given $q\in M$ consider a neighborhood W of q in M with the following property: for any pair (n,p) such that $\Gamma(n,p)\subset W$, $m(n,p)$ is strictly positive. For p fixed, denote by $E_1(p)=\{n\in S^{N-1}|\ \Gamma(n,p)\subset W\}$.

There is a half-neighborhood $U(q)$ of q such that for any $p\in U(q)$, there exists a minimizing $n(p)\in E_1(p)$ and any minimizing pair $n(p)$ must be in $E_1(p)$. Since the map $n\in E_1(p)\to V(n,p)$ is convex, the minimizer $n(p)$ is unique. Considering $D_0=\cup_{q\in M-\partial M}U(q)$, one completes the proof of the theorem. □

3. Hessian of the volume distance close to the hypersurface

In this section, we shall study the asymptotic behavior of the hessian of the volume distance, for points close to the hypersurface M. Thus we may assume that all points considered are in D_0, i.e., they satisfy the conditions of theorem 2.1. We define the normalized hessian of the volume distance by

$$Hv(p)=-\frac{D^2v(p)}{b(p)} \tag{3.1}$$

restricted to $\pi(p)$, where $b(p)=b(n(p),p)$. Formulas (2.9) and (2.12) imply that, in the coordinates introduced in section 2,

$$Hv(p)=\left[\frac{1}{b(p)}\int_{S^{N-1}}g^{n+1}(\eta)m(\eta)\mathbf{M}_n(\eta)d\eta\right]^{-1}. \tag{3.2}$$

We shall denote by C^{-1} the right hand side of the above expression.

3.1. Asymptotic behavior of $Hv(p)$

We shall now study the behavior of $Hv(p)$ for p close to $q \in M$. Applying a suitable affine transformation, we may assume that $q = (0,0)$, the tangent plane T_qM is $z = 0$ and the affine normal at q is $(0,1)$. Then, close to q, the surface M is defined by an equation of the form

$$z = \frac{r^2}{2} + \frac{r^3}{6} P_3(\eta) + \frac{r^4}{24} P_4(\eta) + O(r^5), \tag{3.3}$$

where $P_3(\eta)$ and $P_4(\eta)$ are homogeneous polynomials of degrees 3 and 4, respectively. We shall use the following notation:

$$B_3 = \int_{S^{N-1}} P_3^2(\eta) d\eta, \quad B_3(i,j) = \int_{S^{N-1}} P_3^2(\eta) \eta_i \eta_j d\eta,$$

$$B_4 = \int_{S^{N-1}} P_4(\eta) d\eta, \quad B_4(i,j) = \int_{S^{N-1}} P_4(\eta) \eta_i \eta_j d\eta.$$

Proposition 3.1. *Consider the $N \times N$ symmetric matrix A given by*

$$A = \frac{N}{36\lambda} \left((N+4) B_3 - 3B_4 \right) I_N + A_1, \tag{3.4}$$

where

$$A_1(i,j) = \frac{N(N+4)}{36\lambda} \left(3B_4(i,j) - (N+6) B_3(i,j) \right) \tag{3.5}$$

and I_N is the $N \times N$ identity matrix. Then

$$C^{-1}(z) = I + zA + O(z^2).$$

We begin the proof of the above proposition with the following lemma:

Lemma 3.2. *We can write*

$$r(\eta, z) = \sqrt{2} z^{1/2} - \frac{P_3(\eta)}{3} z + \beta z^{3/2} + O(z^2), \tag{3.6}$$

where

$$\sqrt{2}\beta = \frac{5P_3^2(\eta) - 3P_4(\eta)}{18}.$$

Proof. Since

$$\lim_{r \to 0} \frac{r}{\sqrt{2} z^{1/2}} = 1,$$

we can write $r = \sqrt{2} z^{1/2} + O(z)$. Then write $r = \sqrt{2} z^{1/2} + \alpha z + O(z^{3/2})$ and substitute in equation 3.3, obtaining

$$z = z + \sqrt{2} \alpha z^{3/2} + \frac{\sqrt{2}}{3} P_3(\eta) z^{3/2} + O(z^2).$$

Thus $\alpha = -\frac{P_3(\eta)}{3}$. Finally write $r = \sqrt{2} z^{1/2} - \frac{P_3(\eta)}{3} z + \beta z^{3/2} + O(z^2)$ and substitute in equation 3.3, obtaining

$$0 = \left(\frac{P_3^2(\eta)}{18} + \sqrt{2}\beta \right) z^2 - \frac{P_3^2(\eta)}{3} z^2 + \frac{P_4(\eta)}{6} z^2 + O(z^{5/2}),$$

and so we have proved the lemma. □

Now we prove proposition 3.1.

Proof. Straightforward calculations from (3.6) show that

$$\frac{r^N}{N2^{\frac{N}{2}}} = \frac{1}{N}z^{\frac{N}{2}} - \frac{P_3}{3\cdot 2^{\frac{1}{2}}}z^{\frac{N+1}{2}} + \left[\frac{(N+4)P_3^2}{36} - \frac{P_4}{12}\right]z^{\frac{N+2}{2}} + O(z^{\frac{N+3}{2}}).$$

Differentiating $\frac{r^{N+2}}{N+2}$ with respect to z leads to

$$\frac{r^{N+1}r_z}{2^{\frac{N}{2}}} = z^{\frac{N}{2}} - \frac{(N+3)P_3}{3\cdot 2^{1/2}}z^{\frac{N+1}{2}} + \left[\frac{(N+6)P_3^2}{36} - \frac{P_4}{12}\right](N+4)z^{\frac{N+2}{2}} + O(z^{\frac{N+3}{2}}).$$

By symmetry, the integral of $P_3\eta_i\eta_j$ over S^{N-1} is always zero. Also, the integral of $\eta_i\eta_j$ over S^{N-1} is equal to $\frac{\lambda}{N}\delta(i,j)$, where $\lambda = \lambda(N)$ is the Lebesgue measure of S^{N-1} and $\delta_{ij} = 1$, if $i = j$, and 0, if $i \neq j$. Thus the integral $L(i,j)$ of $r^{N+1}r_z\eta_i\eta_j$ satisfies

$$\frac{L(i,j)}{2^{\frac{N}{2}}} = \frac{\lambda\delta_{ij}}{N}z^{\frac{N}{2}} + \frac{(N+4)}{12}\left[\frac{(N+6)}{3}B_3(i,j) - B_4(i,j)\right]z^{\frac{N+2}{2}} + O(z^{\frac{N+3}{2}}).$$

Also, calculating $b(z)$ as the integral of r^N/N over S^{N-1} we obtain

$$\frac{b(z)}{2^{\frac{N}{2}}} = \frac{\lambda}{N}z^{\frac{N}{2}} + \frac{B_3(N+4) - 3B_4}{36}\, z^{\frac{N+2}{2}} + O(z^{\frac{N+3}{2}}).$$

Thus

$$2^{N/2}b(z)^{-1} = \frac{N}{\lambda}z^{-\frac{N}{2}} + \frac{N^2}{36\lambda^2}\left(3B_4 - B_3(N+4)\right)z^{\frac{-N+2}{2}} + O(z^{\frac{-N+3}{2}}).$$

and so $C(z)(i,j) = b(z)^{-1}L(i,j) = \delta_{ij} - A(i,j)z + O(z^{\frac{3}{2}})$, where $A(i,j)$ is given by 3.4 and 3.5. □

For $q \in M$, consider the centroid line $\gamma_q(t),\ t > 0$ defined by the property $\pi(\gamma_q(t))$ parallel to T_qM. The normalized hessian $H(\gamma_q(t))$ and the Blaschke metric $h(q)$ are symmetric bilinear forms defined in T_qM. Next corollary says that in fact $H(\gamma_q(t))$ is converging to $h(q)$ when t goes to 0.

Corollary 3.3. *We have that*

$$Hv(\gamma_q(t)) = h(q) + O(t). \tag{3.7}$$

Proof. We may assume that M is described by equation (3.3) and $q = (0,0)$. Then $\xi(q) = (0,1)$ and we choose $t = z$. Since $\xi(q)$ is tangent to the centroid line ([7], p.52), we have that $\overline{x}(z) = O(z^2)$, where $\gamma_q(z) = (\overline{x}(z), z)$. Thus $H(\gamma_q(z))$ is $O(z^2)$-close to $C^{-1}(0,z)$. But proposition 3.1 says that

$$C^{-1}(0,z) = I + O(z),$$

thus proving the corollary. □

Example. Consider the surface $M \subset \mathbb{R}^3$ described by the equation

$$z = \frac{1}{2}\left(x^2 + y^2\right) + \frac{c}{6}\left(x^3 - 3xy^2\right) + \frac{a}{24}x^4.$$

Then direct calculations shows that $I_3 = c^2\pi$, $I_4 = \frac{3\pi a}{4}$, $I_3(1,1) = \frac{c^2\pi}{2}$, $I_3(2,2) = \frac{c^2\pi}{2}$, $I_4(1,1) = \frac{a5\pi}{8}$, $I_4(2,2) = \frac{a\pi}{8}$ and, for $i \neq j$, $I_3(i,j) = I_4(i,j) = 0$. Thus A is a diagonal matrix with $A(1,1) = \frac{a}{4} - \frac{c^2}{2}$ and $A(2,2) = -\frac{c^2}{2}$.

On the other hand, we can calculate the shape operator of M at the origin following [6], p.47. We have that

$$D^2 z = \begin{bmatrix} 1 + cx + \frac{a}{2}x^2 & -cy \\ -cy & 1 - cx \end{bmatrix}.$$

Thus $\phi^4 = \det(D^2 z) = 1 - c^2x^2 + \frac{a}{2}x^2 - c^2y^2$ and so $4\phi^3\phi_x = -2c^2x + ax$, $4\phi^3\phi_y = -2c^2y$. Since $(D^2z)^{-1} = I + O(r)$, we conclude that

$$\xi(x,y) = (-\frac{c^2}{2}x + \frac{a}{4}x, -\frac{c^2}{2}y, 1) + O(r^2).$$

Thus the shape operator at $(0,0,0)$ coincides with A.

It is possible to prove as in the above example that, for any hypersurface M, A is the shape operator of M at $(0,0)$, but the general calculations are lengthy and tedious. So we prefer to prove this fact by a more conceptual way.

3.2. Convergence to the shape operator

Along this section, we shall use the following notation: let $f : M \subset \mathbb{R}^N \to \mathbb{R}^{N+1}$ be the inclusion map and denote by ξ its normal vector field pointing to the convex part of M. For $X, Y \in \mathcal{X}(U)$, we write

$$\begin{aligned} D_X f_*(Y) &= f_*(\nabla_X Y) + h(X,Y)\xi \\ D_X \xi &= -f_*(SX), \end{aligned}$$

where ∇ denotes the Blaschke connection, h is the positive definite Blaschke metric and S is the shape operator. Denote by $\nu : M \to \mathbb{R}_{N+1}$ the corresponding co-normal immersion.

Close to the hypersurface M, we write $p = \gamma_q(t)$, $q \in M$, $t \in [0,T)$, where $\gamma_q(t)$ is the centroid of the section through $q + t\xi(q)$ parallel to T_qM. Then p is not necessarily on the normal line $q + t\xi(q)$, but we can write

$$p = q + t\xi(q) + Z, \tag{3.8}$$

for some $Z = Z(q,t) \in T_qM$, with $Z = O(t^2)$ (see [7], p.52). Differentiating (3.8) with respect to t gives

$$\frac{\partial p}{\partial t} = \xi(q) + Z_t, \tag{3.9}$$

for some $Z_t \in T_qM$, with $Z_t = O(t)$. We conclude that

$$v_t(p) = Dv(p) \cdot (\xi(q) + Z_t) = Dv(p) \cdot \xi(q),$$

where the last equality comes from the fact that $Dv(p)$ is orthogonal to $\pi(p)$ (lemma 2.4). We have thus proved the following lemma:

Lemma 3.4. *The derivative of v is given by*

$$Dv(p) = v_t(p)\ \nu(q), \tag{3.10}$$

where $\nu(q)$ is the co-normal vector at $q \in M$.

Lemma 3.5. *We have that, for any $X \in T_qM$,*

$$\lim_{t\to 0} \frac{1}{v_t} \cdot D^2v(X,\xi) = 0.$$

Proof. Differentiate equation (3.10) with respect to t and use (3.9) to obtain

$$D^2v(\xi(q) + Z_t) = v_{tt}\nu(q).$$

Thus, for any $X \in T_qM$,

$$D^2v(\xi(q) + Z_t, X) = 0.$$

So $D^2v(X,\xi) = -D^2v(X,Z_t)$ and hence

$$\frac{1}{v_t} \cdot D^2v(X,\xi) = Hv(\gamma_q(t))(X,Z_t).$$

By corollary 3.3, $Hv(\gamma_q(t))$ is converging to h and since $Z_t = O(t)$, we conclude that this last expression converges to 0, thus proving the lemma. □

Proposition 3.6. *The rate of convergence of the normalized hessian $Hv(\gamma_q(t))$ to $h(q)$ is $h_S(q)$, i.e.,*

$$\lim_{t\to 0} \frac{Hv(\gamma_q(t))(X,Y) - h(q)(X,Y)}{t} = h_S(q)(X,Y).$$

for any $q \in M$, $X,Y \in T_qM$.

Proof. Observe first that if we differentiate (3.8) in the direction $X \in T_qM$, we obtain

$$D_X(p) = (I - tS)X + \nabla_X Z + h(X,Z)\xi(q), \tag{3.11}$$

with $\nabla_X Z = O(t^2)$ and $h(X,Z) = O(t^2)$. Then differentiate equation (3.10) in the direction of $X \in T_xM$ to obtain

$$D^2v(D_X(p)) = v_t\nu_X(q) + X(v_t)\nu(q).$$

Thus, for $Y \in T_qM$,

$$D^2v(D_X(p),Y) = v_t\nu_X(q)(Y) = -v_t h(X,Y)$$

(see [6], p.57, for the last equality). Expanding this equation using (3.11) and dividing by v_t we obtain

$$Hv(\gamma_q(t))(I-tSX,Y)-h(X,Y) = -Hv(\gamma_q(t))(\nabla_X Z,Y)+h(X,Z)\frac{D^2v(\xi,Y)}{v_t}.$$

Now, from lemma 3.5 and corollary 3.3, we conclude that

$$\lim_{t\to 0} \frac{Hv(\gamma_q(t))(X,Y) - h(X,Y)}{t} = h(SX,Y),$$

thus proving the proposition. □

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