ENTROPY-EXPANSIVENESS FOR PARTIALLY HYPERBOLIC DIFFEOMORPHISMS.

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Abstract. We show that diffeomorphisms with a dominated splitting of the form $E^s \oplus E^c \oplus E^u$, where E^c is a nonhyperbolic central bundle that splits in a dominated way into 1-dimensional subbundles, are entropy-expansive. In particular, they have a principal symbolic extension and equilibrium states.

1. Introduction

In dynamical systems theory one often considers the following three main levels of structure: measure theoretic, topological, and infinitesimal (properties of the derivative). Connections between such different levels have always been of high interest. A paradigmatic example is provided by uniform hyperbolicity that implies a rich structure on the other two levels. This paper is part of a programm that studies how more general infinitesimal "hyperbolic-like" properties (partial hyperbolicity and existence of dominated splittings) force a certain topological and measure-theoretic behavior for the underlying dynamics. Here we will focus on a special type of partial hyperbolicity that will force entropy-expansive behavior as described below.

A diffeomorphism f is α-expansive, $\alpha > 0$, if $\mathrm{dist}(f^n(x), f^n(y)) \leq \alpha$ for all $n \in \mathbb{Z}$ implies $x = y$. Uniform hyperbolicity implies α-expansiveness for some $\alpha > 0$. One can relax this condition requiring entropy-expansiveness. This notion, introduced by Bowen [B], is characterized by the fact that, for every small $\alpha > 0$ and every point $x \in M$, the intersection of the sets $f^{-n}(B(f^n(x), \alpha))$, $n \in \mathbb{Z}$, has zero topological entropy. Here $B(x, \alpha)$ is the ball centered at x of radius α.

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Entropy-expansive maps are not necessarily expansive, but have similar properties to expansive maps in regards to topological and measure theoretic entropy. For instance, entropy-expansive maps always have equilibrium states, [K], and symbolic extensions preserving the entropy structure (called principal extensions), [BFF]. Moreover, considering the weaker version of asymptotically h-expansive introduced by Misiurewicz in [M], there is the following string of implications relating these notions: entropy-expansive $\Rightarrow$ asymptotically h-expansive $\Rightarrow$ existence of symbolic extensions, see [BFF]. For a broad discussion between these notions see [DN, Section 1].

Several results illustrate the interplay between smoothness and entropy-expansive-like properties. First, it follows from [Bz] and [BFF] that C^∞ diffeomorphisms are asymptotically h-expansive. In [DM] it was shown that every C^2 interval map has a symbolic extension. A similar result for C^2 surface diffeomorphisms can be found in [Br]. These results support the conjecture of Downarowicz and Newhouse [DN] that every C^2 diffeomorphism has a symbolic extension. However, this conjecture does not hold for C^1 diffeomorphisms in any manifold of dimension three or higher, [As, DF].

In this paper we adopt a different approach and study the relation between "hyperbolic-like properties" and entropy-expansiveness. Indeed uniformly hyperbolic diffeomorphisms are entropy-expansive. There are also some results available for "weakly hyperbolic" systems. For instance, in [PV$_1$] for a surface diffeomorphisms f and a compact f-invariant set Λ with a dominated splitting it is shown that the map f restricted to Λ is entropy-expansive. See also further related results in [PV$_2$]. Finally, in [CY] it is shown that every partially hyperbolic set with a one-dimensional center direction is entropy-expansive.

Here we continue with the above investigations and consider partially hyperbolic sets whose center bundle is higher dimensional, but splits in a dominated way into one-dimensional subbundles. We prove that such diffeomorphisms are entropy-expansive, see Theorem 1.1. Moreover, as the discussion below shows, the conditions in the theorem are also necessary at least for C^1 generic diffeomorphisms.

Let us observe that, in contrast with our result, C^1 generically diffeomorphisms having a central (non-hyperbolic) indecomposable bundle of dimension at least two are not even asymptotically h-expansive, see [DF, As]. In fact, the proof of these results follows the methods introduced in [DN] for surface diffeomorphisms, where it is shown how homoclinic tangencies prevent asymptotically h-expansiviness. Indeed, the generation of homoclinic tangencies is closely related to the existence of indecomposable central bundles with dimension at least two.

In particular, the hypotheses of Theorem 1.1 prevents the creation of homoclinic tangencies by perturbations, see for instance [W].

1.1. **Definitions and background.** To state precisely our results let us recall the main concepts in the paper; namely, the notions of entropy-expansiveness, equilibrium states, symbolic extensions, and dominated splittings.

In what follows (X, d) is a compact metric space and f is a continuous self-map of X. The d_n metric on X is defined as

$$d_n(x, y) := \max_{0\leq i\leq n-1} d(f^i(x), f^i(y))$$

and is equivalent to d and defined for all $n \geq 0$.

For a set $Y \subset X$, a set $A \subset Y$ is (n, ϵ)-*spanning* if for any $y \in Y$ there exists a point $x \in A$ where $d_n(x, y) < \epsilon$. The minimum cardinality of the (n, ϵ)-spanning sets of Y is denoted $r_n(Y, \epsilon)$. We let

$$(1) \qquad \bar{r}(Y, \epsilon) := \limsup_{n\to\infty} \frac{1}{n} \log r_n(Y, \epsilon) \quad \text{and} \quad \tilde{h}(f, Y) := \lim_{\epsilon\to 0} \bar{r}(Y, \epsilon).$$

To see that the last limit exists see for instance [M]. The *topological entropy* $h_{\text{top}}(f)$ of f is $\tilde{h}(f, X)$.

Given $\epsilon > 0$ let

$$\Gamma^+_\epsilon(x) := \bigcap_{n=0}^{\infty} f^{-n}(B_\epsilon(f^n(x)))$$

and set

$$h^*_f(\epsilon) := \sup_{x\in X} \tilde{h}(f, \Gamma^+_\epsilon(x)).$$

The map f is *entropy-expansive,* or h-*expansive* for short, if there exists some $c > 0$ such that $h^*_f(\epsilon) = 0$ for all $\epsilon \in (0, c)$.

If f is a homeomorphism, then we define

$$\Gamma_\epsilon(x) \colon = \bigcap_{n\in\mathbb{Z}} f^{-n}(B_\epsilon(f^n(x))) \text{ and } h^*_{f,\text{homeo}}(\epsilon) \colon = \sup_{x\in X} \tilde{h}(f, \Gamma_\epsilon(x)).$$

If X is a compact space and f is a homeomorphism, then $h^*_f(\epsilon) = h^*_{f,\text{homeo}}(\epsilon)$, [B].

For an f-invariant measure μ the *measure theoretic entropy of* f measures the exponential growth of orbits under f that are "relevant" to μ and is denoted $h_\mu(f)$, see for instance [KH] for a precise definition. The *variational principle* states that if X is a compact metric space and f is continuous, then $h_{\text{top}}(f) = \sup_{\mu\in\mathcal{M}(f)} h_\mu(f)$, where $\mathcal{M}(f)$ is the space of all invariant Borel probability measures for f.

If f is a homeomorphism and $\varphi \in C^0(X)$, then the *pressure of f with respect to φ and $\mu \in \mathcal{M}(f)$* is

$$P_\mu(\varphi, f) := h_\mu(f) + \int \varphi d\mu.$$

The topological pressure of (X, f), denoted $P(\varphi, f)$, corresponds to a "weighted" topological entropy, see [KH, p. 623]. The *variational principle for pressure* states that if f is a homeomorphism of X and $\varphi \in C^0(X)$, then

$$P(\varphi, f) := \sup_{\mu \in \mathcal{M}(f)} P_\mu(\varphi, f).$$

A measure μ such that $P(\varphi, f) = P_\mu(\varphi, f)$ is called an *equilibrium state*.

A dynamical system (X, f) has a *symbolic extension* if there exists a subshift (Y, σ) and a continuous surjective map $\pi : Y \to X$ such that $\pi \circ \sigma = f \circ \pi$. The system (Y, σ) is called an *extension* of (X, f) and (X, f) is called a *factor* of (Y, σ). Note that the subshift need not be of finite type and the factor map may be infinite-to-one. A nice form of a symbolic extension is a *principal extension*, that is, an extension given by a factor map which preserves entropy for every invariant measure, see [BD].

Through the rest of the paper we assume that M is a finite dimensional, smooth, compact, and boundaryless Riemannian manifold and $f : M \to M$ is a C^1 diffeomorphism. An f-invariant set Λ (not necessarily closed) has a *dominated splitting* if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E \oplus F$ such that

(i) the bundles E and F are both non-trivial,
(ii) the fibers $E(x)$ and $F(x)$ have dimensions independent of $x \in \Lambda$, and
(iii) there exist $C > 0$ and $0 < \lambda < 1$ such that

$$\|Df^n|E(x)\| \cdot \|Df^{-n}|F(f^n(x))\| \leq C\lambda^n,$$

for all $x \in \Lambda$ and $n \geq 0$.

More generally, a Df-invariant splitting

$$T_\Lambda M = E_1 \oplus \cdots \oplus E_k$$

is dominated if for all $i = 1, \ldots, (k-1)$ the splitting $T_\Lambda M = E_1^i \oplus E_{i+1}^k$ is dominated, where $E_j^\ell = E_j \oplus \cdots \oplus E_\ell$, for $1 \leq j \leq \ell \leq k$.

Note that the above definitions imply the continuity of the splittings.

1.2. **Statement of results.** We are now ready to state our results.

Theorem 1.1. *Let f be a diffeomorphism and Λ be a compact f-invariant set admitting a dominated splitting $E^s \oplus E_1 \oplus \cdots \oplus E_k \oplus E^u$, where E^s is uniformly contracting, E^u is uniformly expanding, and all E_i are one-dimensional. Then $f|_\Lambda$ is entropy-expansive.*

In the previous theorem we allow for the bundles E^s and E^u to possibly be empty.

In [BFF] it is shown that every entropy-expansive diffeomorphism has a principal symbolic extension. We then have the next corollary.

Corollary 1.2. *If Λ and f are as in Theorem 1.1, then $f|_\Lambda$ has a principal symbolic extension*

Since every entropy-expansive diffeomorphism has an equilibrium state, see [K], we have the next result.

Corollary 1.3. *For Λ and f as in Theorem 1.1, if $\varphi \in C^0(\Lambda)$, then $f|_\Lambda$ has an equilibrium state associated with φ.*

As domination is a key ingredient in our constructions we have the next natural question.

Question 1.4. *Let f be a diffeomorphism and Λ be a compact f-invariant set with a Df-invariant splitting (not necessarily dominated) $T_\Lambda M = E^s \oplus E_1 \oplus \cdots \oplus E_k \oplus E^u$, with E^s uniformly contracting, E^u uniformly expanding, and $E_1, \ldots, E_k$ one-dimensional. Is f entropy-expansive?*

As we were preparing this paper Liao, Viana, and Yang [LVY] announced that diffeomorphisms far from homoclinic tangencies satisfy Shub's Entropy Conjecture, see [S], and have a principal symbolic extension. This conjecture relates the topological entropy to the spectral radius of the action induced by the system on the homology (see previous partial results in [SX]).

The rest of the paper proceeds as follows. In Section 2 we provide background and relevant facts for local center-stable and center-unstable manifolds. In Section 3 we prove Theorem 1.1.

2. Center-stable and center-unstable local disks

2.1. **Local center manifolds.** As much of the work will depend on using local center stable and unstable disks we review some relevant facts.

We consider a diffeomorphism f and a compact f-invariant set Λ with a dominated splitting $E^s \oplus E_1 \oplus \cdots \oplus E_k \oplus E^u$ as in Theorem 1.1. For $x \in \Lambda$ and $i \in \{0, ..., k\}$ let us denote

$$F_i(x) := E^s(x) \oplus E_1(x) \oplus \cdots \oplus E_i(x) \text{ and}$$
$$G_i(x) := E_i(x) \oplus \cdots \oplus E_k(x) \oplus E^u(x).$$

We also let $G_{k+1} = E^u$. By definition $F_i \oplus G_{i+1}$ is a dominated splitting for Λ and

$$\|Df^n|_{F_i(x)}\| \cdot \|Df^{-n}|_{G_{i+1}(x)}\| \leq C\lambda^n$$

for some $C \geq 1$ and $\lambda \in (0, 1)$, and all $i \in \{0, ..., k\}$, $x \in \Lambda$, and $n \geq 0$.

The next proposition is an immediate consequence of [G, Theorem 1] and will simplify many of the arguments.

Proposition 2.1. *There exists an adapted Riemannian metric $\| \cdot \|_0$, equivalent to the original one and $\lambda \in (0, 1)$ such that*

$$\prod_{j=0}^{n} \|Df|F_i(f^j(x))\|_0 \cdot \|Df^{-1}|G_{i+1}(f^j(x))\|_0 < \lambda^n \tag{2}$$

for all $n \geq 0$, all $x \in \Lambda$, and all $i \in \{0, ..., k\}$.

Proof. Fix $i \in \{0, ..., k\}$. By [G] there exist an adapted Riemannian metric equivalent to the original one and $\lambda_i \in (0, 1)$ such that $\|Df|F_i(x)\|_0 \cdot \|Df^{-1}|G_{i+1}(f(x))\|_0 \leq \lambda_i$. Since the splitting is invariant we conclude that

$$\prod_{j=0}^{n} \|Df|F_i(f^j(x))\|_0 \cdot \|Df^{-1}|G_{i+1}(f^j(x))\|_0 < \lambda_i^n.$$

Letting $\lambda = \max\{\lambda_0, ..., \lambda_k\}$ we prove the lemma. □

Throughout the rest of the paper we assume that the Riemannian metric is an adapted metric.

Theorem 5.5 of [HPS] regards the existence of local foliations of center-stable manifolds. Before stating the appropriate result we need to review some terms in [HPS]; namely, the notions of a pre-lamination and immediate relative ρ pseudo-hyperbolicity.

Let $\text{Emb}^r(D^k, M)$ be the set of C^r embeddings of the k-dimensional unit disk D^k into M. Consider a set Σ and a continuous map

$$\sigma \colon \Sigma \to \text{Emb}^r(D^k, M), \quad D_x \colon = \sigma(x)(D^k).$$

A C^r *pre-lamination indexed by* Σ is a continuous choice of a C^r embedded disc D_x through each point $x \in \Sigma$. Continuity of the choice of the disks means that Σ can be covered by open sets $\{U_i\}_{i \in I}$ where the map $\sigma_i : U_i \cap \Sigma \to \text{Emb}^r(D^k, M)$ is continuous.

To define immediate relative ρ pseudo-hyperbolic recall that if $L : \mathbb{R}^d \to \mathbb{R}^d$ is an isomorphism and $G \subset \mathbb{R}^d$ is a subspace, then the *conorm of L restricted to G* is

$$m(L|G) \colon = \inf_{v \neq 0, v \in G} \frac{\|Lv\|}{\|v\|}.$$

The conorm gives the minimal expansion of L on G.

Let $g \colon M \to M$ be a diffeomorphism and Ω be an g-invariant compact set. A splitting $T_\Omega(M) = E \oplus F$ is *immediate relatively ρ pseudo-hyperbolic* relative to g if there exists a continuous function $\rho \colon \Omega \to \mathbb{R}^+$ such that

$$\|Tg|_{E(x)}\| < \rho(x) < m(Tg|_{F(x)}).$$

Lemma 2.2. *Let f and Λ be as in Theorem 1.1. Then, for every $i = 0, \dots, k$, $F_i \oplus G_i$ is an immediate relatively ρ pseudo-hyperbolic splitting over Λ relative to f for some continuous function $\rho \colon \Lambda \to \mathbb{R}^+$.*

Proof. Proposition 2.1 implies that

$$\|Df|_{F_i(x)}\| \leq \lambda \cdot m(Df|_{G_{i+1}(x)}),$$

for all $x \in M$ and $i \in \{0, ..., k\}$, where $\lambda < 1$. In particular, there is $\rho(x)$ such that

$$\|Df|_{F_i(x)}\| < \rho(x) < m(Df|_{G_{i+1}(x)}).$$

The continuity of the splitting $F_i \oplus G_{i+1}$ implies that the map ρ can be chosen depending continuously on x. This implies the result. □

We then have the following restatement of the results in [HPS].

Proposition 2.3. [HPS, Theorem 5.5] *Let Λ be a compact f-invariant set having a dominated splitting as in Theorem A. Then for every $i \in \{0, ..., k\}$ there exist local f-invariant C^1 pre-laminations $W^{cs,i}(x)$ and $W^{cu,i+1}(x)$ tangent to F_i and G_{i+1}, respectively.*

We refer to the manifolds in the proposition as *center-stable* and *center-unstable local manifolds.* We observe that the proposition implies that $W^{cs,0}(x)$ is a local strong stable manifold and $W^{cu,k+1}(x)$ is a local strong unstable manifold. we let $W^{cs,-1}(x) = \emptyset$.

Note that the dominated splittings $F_i \oplus G_{i+1}$ defined over Λ can be continuously extended to a sufficiently small neighborhood $V(\Lambda)$ of Λ and the extended splitting is "nearly" invariant under f. That is, there are sufficiently small cone fields about the extended splitting and these are mapped into uniformly smaller cone fields. Furthermore, for $V(\Lambda)$ sufficiently small the f-invariant set

$$\Lambda_V = \bigcap_{n\in\mathbb{Z}} f^n(\overline{V(\Lambda)}) \tag{3}$$

has a dominated splitting that extends the splitting on Λ, see for instance [BDV, App. B].

By choosing $V(\Lambda)$ sufficiently small we have that Λ_V also satisfies the hypotheses in Proposition 2.3 and hence the points in Λ_V have local center-stable and center-unstable manifolds as in the proposition.

Throughout the rest of the paper we fix $\rho > 0$ such that for all $i \in \{0, ..., k\}$ and $x \in V(\Lambda)$ there exists a curve $\gamma_i(x)$ of radius ρ tangent to the bundle E_i.

Given $\delta > 0$ and $x \in \Lambda_V$, let $W^{cs,i}_\delta(x) = W^{cs,i}_\delta(x) \cap B_\delta(x)$, where B_δ is the ball of radius δ centered at x in M. Note that for each $x \in \Lambda_V$ there is δ^i_x such that $W^{cs,i}_{\delta^i_x}(x)$ is contained in the interior of $W^{cs,i}(x)$. By compacity of Λ_V and continuity of the center manifolds, there is $\rho^s > 0$ such that $\delta^i_x > \rho^s$ for all $x \in \Lambda_V$ and all $i = 0, \dots, k$. We have analogous properties for the center-unstable manifolds obtaining a constant ρ^u. We let $\rho = \min\{\rho^s, \rho^u\}$. From now on we also assume that ρ is small enough.

Remark 2.4. There is $\bar\tau \in (0, \rho)$ such that for every $\tau \in (0, \bar\tau)$, every $i \in \{0, \dots, k\}$, and every $x \in \Lambda_V$ it holds

$$f(W^{cs,i}_\tau(x)) \subset W^{cs,i}_{\rho/2}(f(x)) \quad \text{and} \quad f^{-1}(W^{cu,i}_\tau(x)) \subset W^{cu,i}_{\rho/2}(f^{-1}(x)).$$

Remark 2.5 (Notation)**.** Given a point $x \in \Lambda_V$ we denote by $\gamma_i(x)$ a curve centered at x contained in $W^{cs,i}_\rho(x)$ that is tangent to the bundle E_i and has radius $\rho/8$. This means that both connected components of $\gamma_i(x) \setminus \{x\}$ have length $\rho/8$ (if ρ is small enough this curve is simple and the radius is well defined). Given to points $y, z \in \gamma_i(x)$ we let $[y, z]_{\gamma_i(x)}$ the segment of the curve $\gamma_i(x)$ bounded by y and z.

Consider a local center manifold $W^{cs,i}_\rho(x)$ and points y and z such that $W^{cs,i-1}_{\rho/2}(y)$ and $\gamma_i(z)$ are contained in $W^{cs,i}_\rho(x)$. The next remark is a standard consequence of the fact that the manifold $W^{cs,i-1}_{\rho/2}(y)$ and the curve $\gamma_i(z)$, for small ρ and all $i = 0, \dots, k$, are tangent to a pair of invariant cone fields that are transverse and the sum of their dimensions is equal to the dimension of $W^{cs,i}_\rho(x)$. A similar comment holds for the center-stable and center-unstable manifolds.

Remark 2.6. There is a small $\epsilon = \epsilon(\rho) > 0$ such that for all $i \in \{0, \dots, k\}$ and for every pair of points $x, y \in W^{cs,i}_{\rho}(x)$ such that

$$x, y, f(x), f(y) \in \Lambda_V, \quad d(x,y) < \epsilon, \quad \text{and} \quad d(f(x), f(y)) < \epsilon$$

the intersection $W^{cs,i-1}_{\rho/2}(y) \cap \gamma_i(x)$ consists of a unique point $\pi_{\gamma_i(x)}(y)$.

Moreover, we have that the intersections $W^{cs,i}_{\rho/2}(x) \cap W^{cu,i+1}_{\rho/2}(y)$ and $W^{cs,i}_{\rho/2}(y) \cap W^{cu,i+1}_{\rho/2}(x)$ are single points.

The fact that the local center manifolds $W^{cs,i-i}_{\rho}(y)$ and the curves $\gamma_i(x)$ are tangent to transverse cone fields implies the following:

Remark 2.7. There are constants $\kappa_0 > 0$ and $\epsilon_0 > 0$ such that if $d(x,y) < \epsilon_0$ then

$$d(\pi_{\gamma_i(x)}(y), x) \le \kappa_0 \, d(x,y) \quad \text{for all } i = 1, \dots, k.$$

The next lemma is a higher dimensional version of [PV$_1$, Lemma 2.2]. Since the proof is analogous to the one there we omit it.

Lemma 2.8. *Let Λ be and f-invariant set as in Theorem 1.1. For any sufficiently small $\rho > 0$ and any $\delta \in (0, \rho/16)$, if $x \in \Lambda_V$, $y \in \Gamma_\delta(x)$, and γ is a curve with endpoints x and y that is contained in $B_\rho(x)$ and tangent to E_i for some $i \in \{1, ..., k\}$ then*

$$\ell(\gamma) < 2\,\delta \text{ and } \gamma \subset \Gamma_{2\,\delta}(x).$$

2.2. **Local product structure and the sets $\Gamma^+_\epsilon(x)$.** Using the notation introduced in Remark 2.5, we have the following lemma implying that local center manifolds $W^{cs,i-1}_{\rho}(y)$ and central-curves $\gamma_i(x)$ intersect in a coherent way.

Lemma 2.9. *There is $\epsilon_1 > 0$ such that for every $x_0 \in \Lambda_V$, every $x \in \Gamma_{\epsilon_1}(x_0)$, every $i \in \{0, \dots, k\}$, and every $n \ge 0$, the following holds*

$$f^n([x_0, \pi_{\gamma_i(x)}(x)]_{\gamma_i(x)}) \subset W^{cs,i}_{\rho/2}(f^n(x_0))$$

and

$$W^{cs,i-1}_{\rho/2}(f^n(x)) \cap f^n([x_0, \pi_{\gamma_i(x)}(x)]_{\gamma_i(x)}) = \{f^n(\pi_{\gamma_i(x)}(x))\}.$$

Proof. Consider $\bar{\tau}$ as in Remark 2.4. Observe that if ϵ_1 is small enough (in particular, it satisfies $\bar{\tau} - \epsilon_1 > \epsilon_1$), then $\pi_{\gamma_i(x)}(x) \in W^{cs,i-1}_{\bar{\tau}}(x)$. Otherwise, we have $d(x, x_0) > \tau - \epsilon_1 > \epsilon_1$. Remark 2.4 now implies that

$$f(\pi_{\gamma_i(x)}(x)) \in f(W^{cs,i-1}_{\bar{\tau}}(x)) \subset W^{cs,i-1}_{\rho/2}(f(x)).$$

A similar argument implies that

$$f(\pi_{\gamma_i(x)}(x)) \in f(W^{cs,i}_{\bar{\tau}}(x_0)) \subset W^{cs,i}_{\rho/2}(f(x_0)).$$

Arguing as above, we have that for ϵ_1 the curve $f([x_0, \pi_{\gamma_i(x)}(x)]_{\gamma_i(x)})$ has length less than $\rho/8$. Otherwise we get a contradiction with the fact that $f(x_0)$ and $f(x)$ are ϵ_1-close. This implies that $f([x_0, \pi_{\gamma_i(x)}(x)]_{\gamma_i(x)})$ is contained in some curve $\gamma_i(f(x_0))$. In particular, Remark 2.6 implies that

$$W^{cs,i-1}_{\rho/2}(f(x)) \cap \gamma_i(f(x_0)) = \pi_{\gamma_i(f(x_0))}(f(x)).$$

Since

$$f(\pi_{\gamma_i(x)}(x)) \in W^{cs,i-1}_{\rho/2}(f(x)) \text{ and}$$
$$f(\pi_{\gamma_i(x)}(x)) \in f([x_0, \pi_{\gamma_i(x)}(x)]_{\gamma_i(x)}) \subset \gamma_i(f(x_0))$$

we get that $\pi_{\gamma_i(f(x_0))}(f(x)) = f(\pi_{\gamma_i(x_0)}(x))$. This proves the lemma for $n = 1$.

The lemma now follows by recurrence noting that $f^n(x_0)$ and $f^n(x)$ satisfy the hypotheses of the first step of the lemma for all $n \geq 0$. □

Corollary 2.10. *There is $\epsilon_1 > 0$ such that for every $\epsilon \in (0, \epsilon_1]$, every $x_0 \in \Lambda_V$, every $i \in \{0, \dots, k\}$, and every $x \in \Gamma^+_\epsilon(x_0) \cap W^{cs,i}_\rho(x_0)$ one has that*

$$[x_0, \pi_{\gamma_i(x_0)}(x)]_{\gamma_i(x_0)} \subset \Gamma^+_{2\,\kappa_0\,\epsilon}(x_0) \cap \Gamma^+_{2\,(1+\kappa_0)\,\epsilon}(x),$$

where

$$\pi_{\gamma_i(x_0)}(x) = W^{cs,i-1}_{\rho/2}(x) \cap \gamma_i(x_0).$$

Proof. It is enough to prove that $\pi_{\gamma_i(x_0)}(x) \in \Gamma_{\kappa_0\,\epsilon}(x_0)$.Then Lemma 2.8 implies that

$$[x_0, \pi_{\gamma_i(x_0)}(x)]_{\gamma_i(x_0)} \subset \Gamma^+_{2\,\kappa_0\,\epsilon}(x_0).$$

Finally, as $x \in \Gamma_\epsilon(x_0)$ this implies that

$$[x_0, \pi_{\gamma_i(x_0)}(x)]_{\gamma_i(x_0)} \subset \Gamma_{2\,(1+\kappa_0)\,\epsilon}(x).$$

To prove that $\pi_{\gamma_i(x_0)}(x) \in \Gamma_{\kappa_0\,\epsilon}(x_0)$ note that the second part of Lemma 2.8 implies that, for $n \geq 0$, we have

$$f^n(\pi_{\gamma_i(x)}(x)) = \pi_{\gamma_i(f^n(x_0))}(f^n(x)).$$

Thus, from Remark 2.7, we get

$$d(f^n(\pi_{\gamma_i(x)}(x)), f^n(x_0)) \leq \kappa_0\, d(f^n(x), f^n(x_0)) < \kappa_0\, \epsilon,$$

ending the proof of the corollary. □

Arguing inductively we get the following straightforward consequence of Corollary 2.10.

Corollary 2.11. *There are* $\epsilon_1 > 0$ *and* $\kappa_1 > 1$ *such that for every* $\epsilon \in (0, \epsilon_1]$, *every* $x_0 \in \Lambda_V$, *every* $i \in \{0, \dots, k\}$, *and every* $x \in \Gamma^+_\epsilon(x_0) \cap W^{cs,i}_\rho(x_0)$ *the following holds. Let*

$$x_1 = \pi_{\gamma_i(x_0)}(x) = W^{cs,i-1}_{\rho/2}(x) \cap \gamma_i(x_0)$$

and, for $j = 0, \dots, i-1$, *define inductively*

$$x_{j+1} = \pi_{\gamma_{i-j}(x_j)}(x) = W^{cs,i-j-1}_{\rho/2}(x) \cap \gamma_{i-j}(x_j).$$

The points x_j *are well defined and satisfy*

$$[x_j, \pi_{\gamma_{i-j}(x_j)}(x)]_{\gamma_{i-j}(x_j)} \subset \Gamma^+_{\kappa_1 \epsilon}(x_m)$$

for all $m = 0, \dots, k$.

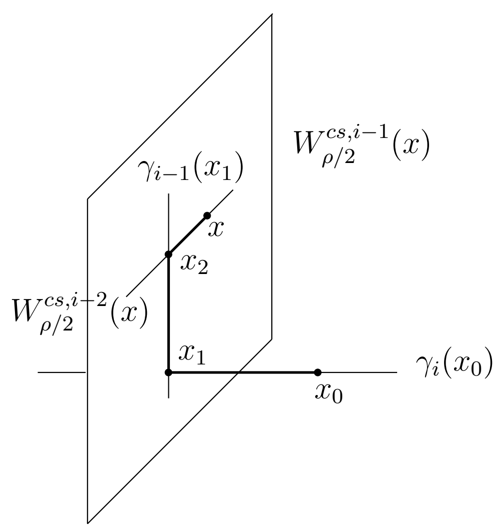

FIGURE 1. Curves contained in $\Gamma^+_\epsilon(x)$

Using Corollary 2.11, we get the following.

Proposition 2.12. *There are* $\epsilon_1 > 0$ *and* $\kappa_1 > 1$ *such that for every* $\epsilon \in (0, \epsilon_1]$, *every* $x_0 \in \Lambda_V$, *every* $i \in \{0, \dots, k\}$, *and every* $x \in \Gamma^+_\epsilon(x_0) \cap W^{cs,i}_\rho(x_0)$ *there is a (piecewise differentiable) curve* $\gamma(x, x_1)$ *joining* x *and* $x_1 = \pi_{\gamma_i(x_0)}(x)$ *defined as the concatenation of the curves*

$$\gamma_{i-j} = [x_j, \pi_{\gamma_{i-j}(x_j)}(x)]_{\gamma_{i-j}(x_j)}.$$

The curve γ_{i-j} *is tangent to* E_{i-j} *and contained in* $\Gamma^+_{\kappa_1 \epsilon}(x_0)$

3. Proof of Theorem

From now on, Λ is an f-invariant sets as in Theorem 1.1. Associated to Λ we have defined the set Λ_V that also satisfies the hypothesis of the theorem.

Remark 3.1 (Choice of constants)**.** We fix some constants:

(a) Fix $\tau > 0$ such that $(1+\tau)\sqrt{\lambda} < 1$, where $\lambda < 1$ is the domination constant in (2).
(b) Fix $\epsilon > 0$ sufficiently small such that $\bigcup_{x\in\Lambda} B_{5\,\epsilon}(x) \subset V(\Lambda)$ and such that if $y, y' \in B_{5\epsilon}(x)$ for some $x \in \Lambda$, then for all $i \in \{0, ..., k\}$ it holds

$$1-\tau < \frac{\|Df^{-1}|_{G_i(y)}\|}{\|Df^{-1}|_{G_i(y')}\|} < 1+\tau \quad \text{and} \quad 1-\tau < \frac{\|Df|_{F_i(y)}\|}{\|Df|_{F_i(y')}\|} < 1+\tau.$$

The next standard fact shows that for all $x \in \Lambda$ and $\epsilon > 0$ sufficiently small the set $\Gamma^+_\epsilon(x)$ is contained in a local center-stable manifold for x. Recall that in Section 2 we fixed small $\rho > 0$ for which the local center-stable and unstable manifolds of size ρ are well defined for every point in Λ_V.

Lemma 3.2. *For $x \in \Lambda$ there exist $i_0(x) = i_0 \in \{0, ..., k\}$ and ϵ_1 such that, for all $\epsilon \in (0, \epsilon_1]$,*

$$\Gamma_\epsilon(x) \subset \Gamma^+_\epsilon(x) \subset W^{cs,i_0}_{\rho/2}(x) \quad \textit{and} \quad \Gamma^+_\epsilon(x) \not\subset W^{cs,i_0-1}_{\rho/2}(x).$$

Proof. Let $y \in \Gamma^+_\epsilon(x)$. Notice that if ϵ is sufficiently small, then there is a local strong-unstable manifold $W^{uu}_\rho(y) = W^{cu,k+1}_\rho(y)$ that intersects $W^{cs,k}_{\rho/2}(x)$ and contains y, recall Remark 2.6. By the uniform expansion of the unstable manifold, it follows that if ϵ is small enough, then y must be contained in $W^{cs,k}_{\rho/2}(x)$ for $f^n(y)$ to stay ϵ close to $f^n(x)$ for all $n \geq 0$.

Now we simply choose i_0 to be the minimum of $\{0, ..., k\}$ such that $\Gamma^+_\epsilon(x) \subset W^{cs,i_0}_{\rho/2}(x)$. This ends the proof of the lemma. □

The next proposition says that for small ϵ the sets $\Gamma_\epsilon(x)$ are essentially one-dimensional and is the main ingredient in the proof of Theorem 1.1. This allows one to adapt the proof of [CY, Proposition 6] where entropy-expansiveness is established for sets with a one-dimensional center bundle.

Proposition 3.3. *Let $x \in \Lambda_V$, $i_0 = i_0(x)$ as in Lemma 3.2, and $\lambda \in (0,1)$ as in Proposition 2.1. Consider the constant ρ in Section 2 and a curve $\gamma_{i_0}(x)$ centered at x of radius $\rho/8$ and tangent to E_{i_0} as in Remark 2.5.*

Then for every $\epsilon > 0$ *small enough there are constants* $C(x) > 0$ *and* $\bar{\lambda} \in (\lambda, 1)$ *such that for all* $z \in \Gamma_\epsilon(x)$ *there is* $z' \in \gamma_{i_0}(x) \cap W^{cs,i_0-1}_{\rho/2}(z)$ *with*

$$d(f^n(z), f^n(z')) \le C(x)\,(\bar{\lambda})^n\, d(z, z').$$

Proof. We use the following reformulation of Pliss Lemma [P] in [Al] that we state for sets Λ_V satisfying the hypotheses of Theorem 1.1.

Lemma 3.4. [Al, P] *Let* $\lambda > 0$ *be as in Proposition 2.1 and* $0 < \lambda < \lambda_1 < \lambda_2 < 1$. *Assume that* $x \in \Lambda_V$, $i \in \{0, ..., k\}$, *and there exists* $n \ge 0$ *such that*

$$\prod_{m=0}^{n} \|Df|_{E_i(f^m(x))}\| \le \lambda_1^n\,. \tag{4}$$

Then there is $N = N(\lambda_1, \lambda_2, f) \in \mathbb{N}$ *and a constant* $c = c(\lambda_1, \lambda_2, f) > 0$ *such that for every* $n \ge N$ *there exist* $\ell \ge c\,n$ *and numbers*

$$0 \le n_1 \le n_2 \le \cdots \le n_\ell \le n$$

such that

$$\prod_{m=n_r}^{h} \|Df|_{E_i(f^m(x))}\| \le \lambda_2^{h-n_r},$$

for all $r = 1, 2, \ldots, \ell$ *and all* h *with* $n_r \le h \le n$.

The next application of Lemma 3.4 provides a lower bound for the expansion of Df^{-1} along the bundle G_i. Recall the definition of the curve $\gamma_i(x)$ and of the segment $[y, z]_{\gamma_i(x)}$ in $\gamma_i(x)$ for points $y, z \in \gamma_i(x)$ in Remark 2.5.

Lemma 3.5. *Consider a small enough* $\epsilon_1 > 0$. *If* $x \in \Lambda_V$, $y \in \Gamma_{\epsilon_1}(x)$, $y \ne x$, $i \in \{1, ..., i_0\}$. *Suppose that* $y \in \gamma_i(x)$. *Then* $[x, y]_{\gamma_i(x)} \subset \Lambda_V$ *and if* $\lambda_1 \in (\lambda, \sqrt{\lambda})$ *then there is* $n_0 > 0$ *such that*

$$\prod_{j=0}^{n} \|Df^{-1}|_{G_i(f^j(y'))}\| > \lambda_1^n$$

for all $y' \in [x, y]_{\gamma_i(x)}$ *and* $n > n_0$.

Proof. By Lemma 2.8, if ϵ is small then the curve $[x, y]_{\gamma_i(x)} \subset \Lambda_V$ and thus the bundles $F_{i-1}(y')$ and $G_i(y')$ are defined along the orbit of any $y' \in [x, y]_{\gamma_i(x)}$. The rest of the proof of the lemma proceeds by contradiction.

Let $\lambda_2 \in (\lambda_1, 1)$ such that $(1+\tau)\,\lambda_2 < 1$, and τ be as in Remark 3.1 (b). Suppose there exist infinitely many $n \in \mathbb{N}$ such that equation (4) holds for n and for some point $w_n \in [x,y]_{\gamma_i(x)}$. Equation (2) and Lemma 3.4 imply that for each such n there exists $0 \le n_1 < n_2 < \ldots < n_\ell \le n$ with $\ell > c\, n$ and

$$\prod_{m=h}^{n_i} \|Df^{-1}|_{G_i(f^m(w_n))}\| \le \lambda_2^{n_i-h},$$

for $n_j \ge h \ge 1$ and $j = 1, \ldots, \ell$.

By Lemma 2.8, for all $j \ge 0$, the curve $f^j([x,y]_{\gamma_i(x)})$ stays $2\,\epsilon$-close to $f^j(x)$, and thus $2\,\epsilon$-close to $f^j(w_n)$. It follows from Remark 3.1 (b) that

$$\prod_{m=h}^{n_\ell} \|Df^{-1}|_{G_i(f^m(y'))}\| \le ((1+\tau)\,\lambda_2)^{n_\ell-h+1}$$

for all $y' \in [x,y]_{\gamma_i(x)}$.

Note that $[x,y]_{\gamma_i(x)} = f^{-n_\ell}([f^{n_\ell}(x), f^{n_\ell}(y)]_{\gamma_i(f^{n_\ell}(x))})$. Then, choosing $h = 1$, we have that

$$\ell([x,y]_{\gamma_i(x)}) \le ((1+\tau)\,\lambda_2)^{n_\ell}\ \ell([f^{n_\ell}(x), f^{n_\ell}(y)]_{\gamma_i(f^{n_\ell}(x))}).$$

Letting $n_\ell \to +\infty$ and observing that $\ell([f^{n_\ell}(x), f^{n_\ell}(y)]_{\gamma_i(f^{n_\ell}(x))})$ is bounded by $2\,\epsilon$, we get that $x = y$, a contradiction. □

We now return to the proof of Proposition 3.3. Fix small ϵ with $\kappa_1\,\epsilon < \epsilon_1$, κ_1 as in Proposition 2.12, and $\lambda_1 \in (\lambda, \sqrt{\lambda})$ and let $\bar\lambda = (1+\tau)\,\lambda_1$. Recall that inequality (2) states that

$$\prod_{m=0}^{n} \|Df|_{F_{i_0}(f^{m-1}(w))}\| \cdot \|Df^{-1}|_{G_{i_0+1}(f^m(w))}\| < \lambda^n$$

for all $w \in \Lambda_V$ and all $n \ge 0$. Since $1 > \lambda_1^2 > \lambda$, Lemma 3.5 (applied to G_{i_0}) implies that there is $n_0 > 0$ such that

$$\prod_{m=0}^{n} \|Df|_{F_{i_0-1}(f^m(y))}\| \le (\bar\lambda)^n, \quad \text{for all } n \ge n_0 \text{ and all } y \in \Gamma_{\epsilon_1}(x). \tag{5}$$

Now we are ready to finish the proof of the proposition. For $z \in \Gamma_\epsilon(x)$ consider the point

$$z_1 = \pi_{\gamma_{i_0}(x)}(z) \in \gamma_{i_0}(x) \cap W^{cs,i_0-1}_{\rho/2}(z).$$

By Proposition 2.12 there is a curve $\gamma(z, z_1)$ joining z and z_1 that is contained in $\Gamma^+_{\kappa_1\,\epsilon}(z) \subset \Gamma^+_{\epsilon_1}(z)$ and tangent to F_{i_0-1}. By equation (5) there is $C(x)$ such that

$$d(f^n(z'), f^n(z)) < C(x)\,\bar\lambda^n\, d(z', z) \text{ for all } n \ge 0 \text{ and } z' \in \gamma(z, z_1).$$

Note that n_0 depends on x. We choose $C(x)$ sufficiently large so that any expansion that occurs in the F_{i_0-1} direction under the first n_0-iterates is '"compensated" by $C(x)$. This constant $C(x)$ also compensates the factor 2 relating distances and lengths of the curves tangent to the bundles E_i (recall Lemma 2.8). This completes the proof of the proposition. □

Proof of Theorem 1.1 Fix $x_0 \in \Lambda$ and let $\gamma_{i_0}(x_0)$ be the curve centered at x_0 tangent to E_{i_0} ($i_0 = i_0(x)$ as in Lemma 3.2) of radius $\rho/8$ defined in Remark 2.5. We fix small ϵ with $\kappa_1\,\epsilon < \epsilon_1 << \rho/100$.

We will prove that $\tilde{h}(f, \Gamma^+_\epsilon(x_0)) = 0$ for all $x_0 \in \Lambda$, thus the restriction of f to Λ is entropy-expansive. The next proposition immediately implies that $\tilde{h}(f, \Gamma^+_\epsilon(x_0)) = 0$ proving Theorem 1.1. To state this proposition recall the definition of $\bar{r}(\Gamma^+_\xi(x), \alpha)$ in equation (1).

Proposition 3.6. $\bar{r}(\Gamma^+_\epsilon(x_0), \alpha) = 0$ *for all* $\alpha > 0$ *sufficiently small.*

Proof. Fix $\alpha > 0$ sufficiently small. To control the rate of growth of (n, α)-spanning sets for $\Gamma^+_\epsilon(x_0)$ we first analyze the growth of (n, α)-spanning sets for $\Gamma^+_\epsilon(x_0) \cap \gamma_{i_0}(x_0)$. This last calculation is only one-dimensional. A key step is the following folklore fact, see for instance [BFSV, Lemma 4.2].

Fact 3.7. *Let* γ *be a compact curve such that the lengths of all its forward iterates* $f^n(\gamma)$, $n \geq 0$, *are bounded by a constant. Then* $h(f, \gamma) = 0$. *In particular, the growth rate of any spanning set of* γ *is at most subexponentially.*

We now provide the details of the proof of the proposition. For $z \in \Gamma^+_\epsilon(x_0)$ consider the point

$$z' = \pi_{\gamma_{i_0}(x_0)}(z) \in \gamma_{i_0}(x) \cap W^{cs, i_0-1}_{\rho/2}(z)$$

given by Remark 2.6. From Corollary 2.11, $z' \in \Gamma_{\kappa_1\,\epsilon}(x_0)$. Therefore, by Proposition 3.3, we have that

$$d(f^n(z'), f^n(z)) < C(x_0)\,\bar{\lambda}^n\,d(z', z), \quad \text{for all } n \geq 0.$$

Thus there is $N \in \mathbb{N}$ such that

$$d(f^n(z), f^n(z')) < \alpha/8, \quad \text{for all } n \geq N. \tag{6}$$

First, for simplicity, let us assume that $C(x_0) = 1$ and thus $N = 1$. Let $\kappa_2 = 3\,\kappa_1^2 + 1$. Equation (6) implies the following:

Claim 3.8. *Fixed* $\epsilon > 0$ *small enough, there is* $L \in N$ *such that if* $z \in \gamma_{i_0}(x_0)$ *then the set*

$$\left\{ z \in \Gamma^+_{\kappa_2\,\epsilon}(x_0) \colon W^{cs, i_0-1}_{\rho/2}(z) \cap \gamma_{i_0}(x_0) = \{z'\} \right\}$$

has a $(1,\alpha/8)$*-spanning set of cardinality at most* L*.*

We have the next lemma.

Lemma 3.9. *There exists* $\zeta\in(0,\alpha)$ *such that for every*

$$x,y\in\Lambda_V\cap\gamma_{i_0}(x_0)\quad\text{and}\quad d_n(x,y)<\zeta$$

the following property holds: If there is $z_x\in W^{cs,i_0-1}_{\rho/2}(x)\cap\Gamma^+_\epsilon(x_0)$ *then there exists* $z_y\in W^{cs,i_0-1}_{\rho/2}(y)\cap\Gamma^+_{\kappa_2\,\epsilon}(x_0)$ *with* $d_n(z_x,z_y)<\alpha/8$*, where* $\kappa_2=3\,\kappa_1^2+1$*.*

Proof. Observe that Corollary 2.11 implies that $x\in\Gamma^+_{\kappa_1\,\epsilon}(x_0)$ and Lemma 2.8 implies that $y\in\Gamma^+_{2\,\kappa_1\,\epsilon}(x_0)$ and thus $y\in\Gamma^+_{3\,\kappa_1\,\epsilon}(z_x)$

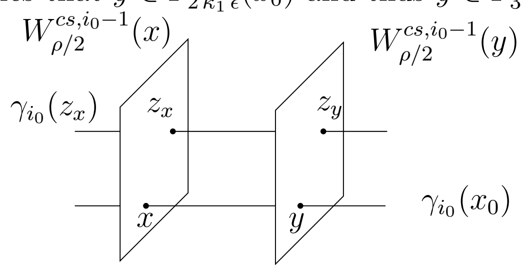

FIGURE 2. The point z_y

Consider the curve $\gamma_{i_0}(z_x)$ of radius $\rho/8$ centered at z_x contained in $W^{cs,i_0}_{\rho/2}(x_0)$ and tangent to E_{i_0}. By transversality, if ζ is small enough and $d(x,y)<\zeta$ then

$$z_y=\pi_{\gamma_{i_0}(z_x)}(y)=\gamma_{i_0}(z_x)\cap W^{cs,i_0-1}_{\rho/2}(y),$$

with $d(z_x,z_y)<\alpha/8$. By Corollary 2.11 we have that $z_y\in\Gamma^+_{3\,\kappa_1^2\,\epsilon}(z_x)$ and thus $z_y\in\Gamma^+_{(3\,\kappa_1^2+1)\,\epsilon}(x_0)=\Gamma^+_{\kappa_2\,\epsilon}(x_0)$.

It remains to check that $d_n(z_x,z_y)<\alpha/8$. By Lemma 2.9, for all $j\geq 0$ we have that

$$f^j(z_x)\in W^{cs,i_0-1}_{\rho/2}(f^j(x)),\quad f^j(z_y)\in W^{cs,i_0-1}_{\rho/2}(f^j(y))$$

and $f^j([z_x,z_y]_{\gamma_{i_0}(z_x)})$ is tangent to E_{i_0}. Note $f^j(x)$ and $f^j(y)$ are ζ-close for all $j=0,\dots,n$. Thus if ζ is small enough this implies that the length of the segment $f^j([z_x,z_y]_{\gamma_{i_0}(z_x)})$ is less that $\alpha/8$. This completes the proof of the lemma. □

Returning to the proof of the proposition, for $n\in\mathbb{N}$ let $M(n)$ be the minimum cardinality of an $(n,\zeta/4)$-spanning set of $\gamma_{i_0}(x_0)\cap\Gamma^+_{2\,\kappa_1\,\epsilon}(x_0)$. Since this set is contained in a closed curve with uniformly bounded length under forward iterates, Fact 3.7 implies that $M(n)$ grows subexponentially. Recall now the definition of L in Claim 3.8.

Lemma 3.10. *There is an (n,α)-spanning set of $\Gamma_\epsilon^+(x_0)$ with cardinality at most $L\,M(n)$.*

As $M(n)$ growths subexponentially this lemma implies that, for all sufficiently small α, $\bar{r}(\Gamma_\epsilon^+(x_0),\alpha)=0$, proving the proposition. We now prove the lemma.

Proof. Consider an $(n,\zeta/4)$-spanning set Y of $\gamma_{i_0}(x_0)\cap\Gamma^+_{2\,\kappa_1\,\epsilon}(x_0)$ with cardinality at most $M(n)$ and list y_i its elements. By Claim 3.8, for each y_i there is an $(n,\alpha/8)$-spanning subset Z_i of $\Gamma^+_{\kappa_2\,\epsilon}(x_0)\cap W^{cs,i_0-1}_\rho(y_i)$ with cardinality at most L. List $z_{i,j}$ the elements of Z_i and let Z be collection of all points $z_{i,j}$. The cardinality of Z is at most $L\,M(n)$.

Claim 3.11. *For every point $x\in\Gamma_\epsilon^+(x_0)$ there is $z\in Z$ such that $d_n(z,x)<\alpha/3$.*

This claim provides an (n,α)-spanning subset of $\Gamma_\epsilon^+(x_0)$ with cardinality less than $L\,M(n)$. Consider the subset $\tilde{Z}$ of Z of points $z_{i,j}$ such that there is some point $x_{i,j}\in\Gamma_\epsilon(x_0)$ with $d_n(x_{i,j},z_{i,j})<\alpha/3$. By construction, the set X formed by the points $x_{i,j}$ is an (n,α)-spanning subset of $\Gamma_\epsilon^+(x_0)$ with cardinality less than $L\,M(n)$.

Proof of the claim. Take any point $w\in\Gamma_\epsilon^+(x_0)$ and let

$$\bar{w}=\pi_{\gamma_{i_0}(x_0)}(w)=W^{cs,i_0-1}_{\rho/2}(w)\cap\gamma_{i_0}(x_0).$$

By Corollary 2.11, $\bar{w}\in\Gamma_{\kappa_1\,\epsilon}(x_0)\cap\gamma_{i_0}(x_0)$. Thus there is some $y_i\in Y$ such that $d_n(\bar{w},y_i)<\zeta/4$. Applying Lemma 3.9 to the pair $\bar{w},y_i$ and the point w we get a point

$$\bar{w}_i\in W^{cs,i_0-1}_{\rho/2}(y_i)\cap\Gamma_{\kappa_2\,\epsilon}(x_0)$$

with $d_n(w,\bar{w}_i)<\alpha/8$. Since there is $z_{i,j}\in Z$ such that $d_n(z_{i,j},\bar{w}_i)<\zeta/4$ it follows that $d_n(w,z_{i,j})<\zeta/2$. This completes the proof of the claim. □

This completes the proof of the lemma. □

When $C(x_0)\neq 1$ then we have uniform contraction in the “fibers” $W^{cs,i_0-1}(x_0)$ for $n\geq N$. In this case we let L be the maximal cardinality of an $(N,\alpha/8)$-spanning set in the “fiber”, instead of $(1,\alpha/8)$-spanning set. The proof now follows identically to above. Therefore, $\bar{r}(\Gamma^+_{\xi/4}(x_0),\alpha)=0$. □

The proof of Theorem 1.1 is now complete. □

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