INTERNAL PERTURBATIONS OF HOMOCLINIC CLASSES: NON-DOMINATION, CYCLES, AND SELF-REPLICATION

CH. BONATTI, S. CROVISIER, L. J. DÍAZ, AND N. GOURMELON

ABSTRACT. Conditions are provided under which lack of domination of a homoclinic class yields robust heterodimensional cycles. Moreover, so-called viral homoclinic classes are studied. Viral classes have the property of generating copies of themselves producing wild dynamics (systems with infinitely many homoclinic classes with some persistence). Such wild dynamics also exhibits uncountably many aperiodic chain recurrence classes. A scenario (related with non-dominated dynamics) is presented where viral homoclinic classes occur.

A key ingredient are adapted perturbations of a diffeomorphism along a periodic orbit. Such perturbations preserve certain homoclinic relations and prescribed dynamical properties of a homoclinic class.

1. INTRODUCTION

There are two sort of cycles associated to periodic saddles that are the main mechanism for breaking hyperbolicity of systems:

• **Homoclinic tangencies:** A diffeomorphism f has a *homoclinic tangency* associated to a transitive hyperbolic set K if there are points X and Y in K whose stable and unstable manifolds have some non-transverse intersection. The homoclinic tangency is C^r*-robust* if there is a C^r-neighborhood $\mathcal{N}$ of f such that the hyperbolic continuation K_g of K has a homoclinic tangency for every $g \in \mathcal{N}$.

• **Heterodimensional cycles:** A diffeomorphism f has a *heterodimensional cycle* associated to a pair of transitive hyperbolic sets K and L of f if their stable bundles have different dimensions and their invariant manifolds meet cyclically, that is, $W^s(K) \cap W^u(L) \neq \emptyset$ and $W^u(K) \cap W^s(L) \neq \emptyset$. The heterodimensional cycle is C^r*-robust* if there is a C^r-neighborhood $\mathcal{V}$ of f

Date: October 6, 2010.

2000 *Mathematics Subject Classification.* Primary:37C29, 37D20, 37D30.

Key words and phrases. chain recurrence class, dominated splitting, heterodimensional cycle, homoclinic class, homoclinic tangency, linear cocycle, C^1-robustness, wild dynamics.

This paper was partially supported by CNPq, Faperj, Pronex (Brazil), the ANR project DynNonHyp BLAN08-2_313375, and the Agreement Brazil-France in Mathematics. LJD thanks the kind hospitality of LAGA at Univ. Paris 13 and the Institut de Mathématiques of Université de Bourgogne. SC and LJD thanks the financial support and the warm hospitality of Institute Mittag-Leffer. We are grateful for comments by K. Shinohara that allowed us to improve the presentation of Section 9.

such that the continuations K_g and L_g of K and L have a heterodimensional cycle for every $g \in \mathcal{V}$.

Given a closed manifold M consider the space $\mathrm{Diff}^r(M)$ of C^r-diffeomorphisms defined on M endowed with the usual C^r-topology. There is the following conjecture about hyperbolicity and cycles:

Conjecture 1 (Palis' density conjecture, [32])**.** *Any diffeomorphism $f \in \mathrm{Diff}^r(M)$, $r \geq 1$, can be C^r-approximated either by a hyperbolic diffeomorphism (i.e. satisfying the Axiom A and the no-cycles condition) or by a diffeomorphism that exhibits a homoclinic tangency or a heterodimensional cycle.*

This conjecture was proved for C^1-surface diffeomorphisms in [35]. For some partial progress in higher dimensions see [12, 20].

Besides this conjecture one also aims to understand the dynamical phenomena associated to homoclinic tangencies and heterodimensional cycles and the interplay between them. We discuss these topics in the next paragraphs.

Homoclinic tangencies of C^2-diffeomorphisms are the main source of non-hyperbolic dynamics in dimension two, see [33, 30]. Namely, as a key mechanism a homoclinic tangency of a surface C^2-diffeomorphism yields C^2-robust homoclinic tangencies and generates open sets of diffeomorphisms where the generic systems display infinitely many sinks or sources, [28, 29]. This leads to the first examples of the so-called *wild dynamics* (i.e. systems having infinitely many elementary pieces of dynamics with some persistence, see [16, Chapter 10] for a discussion and precise definitions). Moreover, these homoclinic tangencies also yield infinitely many regions containing robust homoclinic tangencies associated to other hyperbolic sets (this follows from [29] and [18], see also the comments in [16, page 33]). Using the terminology in [6], this means that, for surface diffeomorphisms, the existence of C^2-robust tangencies is a *self-replicating* or *viral* property, for more details see Section 1.6.

Comparing with the C^2-case, C^1-diffeomorphisms of surfaces do not have hyperbolic sets with robust homoclinic tangencies, see [27] and also [6, Corollary 3.5] for a formal statement. However, in higher dimensions C^1-diffeomorphisms can display robust tangencies, see for instance [38, 5].

In higher dimensions, the first examples of robustly non-hyperbolic dynamics were obtained by Abraham and Smale in [4] by constructing diffeomorphisms with robust heterodimensional cycles (although this terminology is not used there). Moreover, the diffeomorphisms with heterodimensional cycles in [4] also exhibit robust homoclinic tangencies (this follows from [13]).

In the C^1-setting, the generation of homoclinic tangencies is a quite well understood phenomenon that is strongly related to the existence of non-dominated splittings, [39, 17, 24]. Contrary to the case of tangencies, the generation of heterodimensional cycles is not well understood and remains the main difficulty for solving Palis conjecture in the C^1-case. In contrast with the case of C^1-homoclinic tangencies, heterodimensional cycles yield C^1-robust cycles after small C^1-perturbations, [12]. However, in dimension $d \geq 3$, we do not know "when and how" homoclinic tangencies may occur C^1-robustly. In fact, all known examples of C^1-robust tangencies also exhibit C^1-robust heterodimensional cycles[1]. For further discussion see [6, Conjecture 6].

These comments lead to the following strong version of Palis' conjecture (in fact, this reformulates [12, Question 1]):

Conjecture 2 ([6, Conjecture 7])**.** *The union of the set of hyperbolic diffeomorphisms (i.e. satisfying the Axiom A and the no-cycle condition) and of the set of diffeomorphisms having a robust heterodimensional cycle is dense in* $\mathrm{Diff}^1(M)$*.*

This conjecture holds in two relevant C^1-settings: the conservative diffeomorphisms in dimension $d \geq 3$ and the so called *tame* systems (diffeomorphisms whose chain recurrence classes are robustly isolated), see [19] and [12, Theorem 2]. See also previous results in [1, 23].

1.1. **Some informal statements and questions.** In what follows we focus on C^1-diffeomorphisms defined on closed manifolds of dimension $d \geq 3$. We now briefly and roughly describe some of our results and the sort of questions we will consider (the precise definitions and statements will be given throughout the introduction).

A) *When do homoclinic tangencies yield heterodimensional cycles?* In terms of *dominated splittings,* Theorem 1 and Corollary 2 give a natural setting where homoclinic tangencies generate heterodimensional cycles after arbitrarily small C^1-perturbations.

B) *What are obstructions to the occurrence of heterodimensional cycles? Sectional dissipativity* prevents the "coexistence" of periodic saddles with different indices and hence the occurrence of heterodimensional cycles. For homoclinic classes that do not have dominated splittings, we wonder if this is the only possible obstruction for the generation of heterodimensional cycles. Corollary 4 shows that sectional dissipativity is indeed the only obstruction for the occurrence of heterodimensional cycles in homoclinic classes without any dominated splitting.

C) *Is it possible to turn the lack of domination into a robust property?* For homoclinic classes, Theorem 5 shows that the non-existence of a *dominated splitting of index* i can always be made a robust property when the class contains some saddle of stable index different from i.

D) *Which are the dynamical features associated to robust non-dominated dynamics?* In contrast to the case of surfaces, homoclinic tangencies and

[1]The converse is false: there are diffeomorphisms (of partially hyperbolic type with one dimensional central direction) that display robust heterodimensional cycles but cannot have homoclinic tangencies, see for instance [26, 9].

"some" lack of domination do not always lead to wild dynamics. A homoclinic tangency corresponds to the lack of domination of some index. For homoclinic classes containing saddles of several stable indices, Theorem 7 and Corollary 8 claim that the robust lack of any domination leads to wild dynamics. In fact, Theorem 7 asserts that the property of "total non-domination plus coexistence of saddles of several indices" provides another example of a viral property of a chain recurrence class. This property leads to the generic coexistence of a non-countable set of different (aperiodic) classes, extending previous results in [11]

We next define precisely the main definitions involved in this paper and state our main results.

1.2. **Basic definitions.** We will focus on two types of elementary pieces of the dynamics: homoclinic classes and chain recurrence classes.

The *homoclinic class* of a hyperbolic periodic point P, denoted by $H(P,f)$, is the closure of the transverse intersections of the stable and unstable manifolds of the orbit of P. Note that the class $H(P,f)$ coincides with the closure of the saddles Q *homoclinically related with* P: the stable manifold of the orbit of Q transversely meets the unstable manifold of the orbit of P and vice-versa.

To define a *chain recurrence class* we need some preparatory definitions. A finite sequence of points $(X_i)_{i=0}^n$ is an *ϵ-pseudo-orbit* of a diffeomorphism f if $\operatorname{dist}(f(X_i), X_{i+1}) < \epsilon$ for all $i = 0, \dots, n-1$. A point X is *chain recurrent* for f if for every $\epsilon > 0$ there is an ϵ-pseudo-orbit $(X_i)_{i=0}^n$, $n \geq 1$, starting and ending at X (i.e. $X = X_0 = X_n$). The chain recurrent points form the *chain recurrent set* of f, denoted by $R(f)$. This set splits into disjoint *chain recurrence classes* defined as follows. The class of a point $X \in R(f)$, denoted by $C(X,f)$, is the set of points $Y \in M$ such that for every $\epsilon > 0$ there are ϵ-pseudo-orbits joining X to Y and Y to X. A chain recurrence class that does not contain periodic points is called *aperiodic.*

As a remark, in general, for hyperbolic periodic points their chain recurrence classes contain their homoclinic ones. However, for C^1-generic diffeomorphisms the equality holds, [8, Remarque 1.10].

A key ingredient in this paper is the notion of *dominated splitting*:

Definition 1.1 (Dominated splitting)**.** *Consider a diffeomorphism f and a compact f-invariant set Λ. A Df-invariant splitting $T_\Lambda M = E \oplus F$ over Λ is* dominated *if the fibers E_x and F_x of E and F have constant dimensions and there exists $k \in \mathbb{N}$ such that*

$$\frac{||D_x f^k(u)||}{||D_x f^k(w)||} \leq \frac{1}{2}, \tag{1.1}$$

for every $x \in \Lambda$ and every pair of unitary vectors $u \in E_x$ and $w \in F_x$.

The index *of the dominated splitting is the dimension of E.*

When we want to stress on the role of the constant k we say that the splitting is k-dominated.

Given a periodic point P of $f \in \mathrm{Diff}^1(M)$ denote by $\pi(P)$ its period. We order the eigenvalues $\lambda_1(P), \dots, \lambda_d(P)$ of $D_P f^{\pi(P)}$ in increasing modulus and counted with multiplicity, that is, $|\lambda_i(P)| \le |\lambda_{i+1}(P)|$. We call $\lambda_i(P)$ the *i-th multiplier* of P. The *i-th Lyapunov exponent* of P is $\chi_i(P) = \frac{1}{\pi(P)} \log |\lambda_i(P)|$. If $\chi_i(P) < \chi_{i+1}(P) < 0$ then one can define the *strong stable manifold of dimension* i of the orbit of P, denoted by $W^{ss}_i(P, f)$, as the only f-invariant embedded manifold of dimension i tangent to the i-dimensional eigenspace corresponding to the multipliers $\lambda_1(P), \dots, \lambda_i(P)$. There are similar definitions for strong unstable manifolds.

Recall that if Λ is hyperbolic set of f then every diffeomorphism g close to f has a hyperbolic set Λ_g (called the *continuation of* Λ) that is close and conjugate to Λ. If the set Λ is transitive the dimension of its stable bundle is called its *stable index* or simply *s-index*.

Throughout this paper we consider diffeomorphisms defined on closed manifolds of dimension $d \ge 3$. Unless it is explicitly mentioned, we always consider C^1-diffeomorphisms, C^1-neighborhoods, and so on. We repeatedly consider perturbations of diffeomorphisms. By a *perturbation* of a diffeomorphism f we mean here a diffeomorphism g that is arbitrarily C^1-close to f. To emphasize the size of the perturbation we say that a diffeomorphism g is a *ε-perturbation* of $f \in \mathrm{Diff}^1(M)$ if the C^1-distance between f and g is less than ε.

1.3. **Heterodimensional cycles generated by homoclinic tangencies.**

Recall that the generation of homoclinic tangencies is closely related to the absence of dominated splittings over homoclinic classes. In fact, in [24] it is proved that if the stable/unstable splitting over the periodic points homoclinically related to a saddle P is not dominated then there are diffeomorphisms g arbitrarily C^1-close to f with a homoclinic tangency associated to P_g. See also previous results in [39].

Our main result about the interplay between homoclinic tangencies and heterodimensional cycles is stated in the following theorem.

Theorem 1. *Let f be a diffeomorphism and P a hyperbolic periodic saddle of f with stable index $i \ge 2$. Assume that*

(1) there is no dominated splitting over $H(P, f)$ of index i,
(2) there is no dominated splitting over $H(P, f)$ of index $i-1$, and
(3) the Lyapunov exponents of P satisfy $\chi_i(P) + \chi_{i+1}(P) \ge 0$.

Then there are diffeomorphisms $g \in \mathrm{Diff}^1(M)$ arbitrarily C^1-close to f with a heterodimensional cycle associated to P_g and a saddle $R_g \in H(P_g, g)$ of stable index $i-1$.

Moreover, the diffeomorphisms g can be chosen such that there are hyperbolic transitive sets L_g and K_g containing P_g and R_g, respectively, having simultaneously a robust heterodimensional cycle and a robust homoclinic tangency.

Remark 1.2.

(i) In fact, we prove Theorem 1 under the following slightly weaker hypothesis replacing condition (3).

(3') For every $\delta > 0$ there exists a periodic point Q_δ homoclinically related to P whose Lyapunov exponents satisfy $\chi_i(Q_\delta) + \chi_{i+1}(Q_\delta) \geq -\delta$.

(ii) Hypothesis (2) can be replaced by the following condition (see Proposition 8.2).

(2') There is a diffeomorphism g arbitrarily C^1-close to f having a periodic point R_g that is homoclinically related to P_g and that has a strong stable manifold of dimension $i-1$ intersecting the unstable manifold of the orbit of R_g.

Theorem 1 will be proved in Section 8.1. Let us observe that for three dimensional diffeomorphisms a version of this theorem was proved in [37] replacing condition (3) by a stronger one requiring existence of a saddle Q homoclinically related to P such that $\chi_1(Q)+\chi_2(Q)+\chi_3(Q) > 0$. Note that conditions (3) and (3') are related to the notion of a sectionally dissipative bundle that is also considered in [34, 36], see Section 1.4.

Condition (1) is used to get homoclinic tangencies associated to P. Conditions (2) and (2') assure that the homoclinic class is not contained in a normally hyperbolic surface (this would be an obstruction for the generation of heterodimensional cycles). Finally, condition (3) implies that these tangencies generate saddles of index $i-1$.

We would like to replace condition (3) (or (3')) by a weaker one about Lyapunov exponents of measures supported over the class, namely requiring the existence of an ergodic measure μ whose i-th and $(i+1)$-th Lyapunov exponents satisfy $\chi_i(\mu)+\chi_{i+1}(\mu) \geq 0$. This potential extension is related to the still open problem of approximation of ergodic measures supported on a homoclinic class by measures supported on periodic points of the class, see [6, Conjecture 2] and [2].

There is also the following "somewhat symmetric" version of Theorem 1 that is an immediate consequence of it.

Corollary 2. *Consider a hyperbolic saddle P of stable index i, $2 \leq i \leq d-2$, of a diffeomorphism f. Assume that there are no dominated splittings over $H(P,f)$ of indices $i-1$, i, and $i+1$. Then there is a diffeomorphism g arbitrarily C^1-close to f with a heterodimensional cycle associated to P_g and a saddle $R_g \in H(P_g,g)$ of stable index $i-1$ or $i+1$.*

Moreover, the diffeomorphism g can be chosen such that there are hyperbolic transitive sets L_g and K_g containing P_g and R_g, respectively, having simultaneously a robust heterodimensional cycle and a robust homoclinic tangency.

Theorem 1 has the following consequence for C^1-generic diffeomorphisms of three dimensional manifolds that slightly generalizes the dichotomy "domination versus infinitely many sources/sinks" in [15].

Corollary 3. *Let M be a closed manifold of dimension three. There is a residual subset $\mathcal{R}$ of $\mathrm{Diff}^1(M)$ such that for every diffeomorphism f and every saddle P of stable index 2 of f (at least) one of the following three possibilities holds:*

- *$H(P, f)$ has a dominated splitting;*
- *$H(P, f)$ is the Hausdorff limit of periodic sinks;*
- *f has a robust heterodimensional cycle associated to P and $H(P, f)$ is the Hausdorff limit of periodic sources.*

1.4. **Non-domination far from heterodimensional cycles implies sectional dissipativity.** One approach for settling Palis conjecture is to study dynamics *far from homoclinic tangencies.* In this case the diffeomorphisms necessarily have nice dominated splittings that are adapted to their index structure, see for instance [39]. In contrast, dynamics *far from heterodimensional cycles* is yet little understood. To address this point we will make the following "local version" of Conjecture 2 where a given homoclinic class is specified.

Conjecture 2'. *Let P be a hyperbolic saddle of a diffeomorphism f such that for every diffeomorphism g that is C^1-close to f there is no heterodimensional cycle associated to the continuation P_g of P. Then there exists a diffeomorphism g arbitrarily C^1-close to f such that the homoclinic class $H(P_g, g)$ is hyperbolic.*

To discuss Conjecture 2' let us first consider a simple illustrating case involving the notion of *sectional dissipativity*. Let P be a hyperbolic saddle of a diffeomorphism f of stable index 1 whose homoclinic class $H(P, f)$ satisfies the following two properties:

- $H(P, f)$ has no dominated splitting of index 1 and
- $H(P, f)$ is uniformly sectionally dissipative for f^{-1}, that is, there is $n > 0$ such that the Jacobian of f in restriction to any 2-plane is strictly larger than 1.

Under these hypotheses, the lack of domination of $H(P, f)$ corresponding to the index of P enables a homoclinic tangency associated to P after a perturbation. However, the sectional dissipativity prevents the existence of saddle points of stable index larger than 1 in a small neighborhood of the homoclinic class of P. Thus any diffeomorphism g that is C^1-close to f cannot have a heterodimensional cycle associated to P_g.

We wonder if the case above is the only possible setting where homoclinic tangencies far from heterodimensional cycles can occur. We provide a partial result to this question by considering homoclinic classes without any dominated splitting and a weaker notion of sectional dissipativity.

Consider a set of periodic points $\mathcal{P}$ of a diffeomorphism f and a Df-invariant subbundle E defined over the set $\mathcal{P}$. The bundle E is said to be *sectionally dissipative at the period* if for any point $R \in \mathcal{P}$ there is a constant $0 < \alpha_R < 1$ such that $|\lambda_k\, \lambda_{k+1}| < \alpha_R^{\pi(R)}$ for every pair of multipliers λ_k and

λ_{k+1} of R whose eigendirections are contained in E. When $E = T_{\mathcal{P}}M$ then we call the set of periodic points $\mathcal{P}$ *sectionally dissipative at the period.* In the case that the constant α_R can be chosen independently of R we call the bundle E (or the set $\mathcal{P}$) *uniformly sectionally dissipative at the period.*

Corollary 4. *Let M be a closed manifold M with $dim(M) \geq 3$ and $f\colon M \to M$ a diffeomorphism. Consider a homoclinic class $H(P, f)$ without any dominated splitting that is far from heterodimensional cycles. Then the set of periodic points of f homoclinically related to P is uniformly sectionally dissipative at the period either for f or for f^{-1}.*

1.5. **Robust non-domination.** We first recall that the existence of a dominated splitting is (in some sense) an open property. More precisely, if Λ is an f-invariant compact set with a dominated splitting $T_\Lambda M = E \oplus F$, then there are neighborhoods U of Λ in M and $\mathcal{U}$ of f in $\mathrm{Diff}^1(M)$ such that for every $g \in \mathcal{U}$ and every g-invariant set Σ contained in U there is a dominated splitting for Σ of the same index as $E \oplus F$, see for instance [16, Chapter B.1]. Observe that the next theorem implies that, in some cases, the absence of domination of a homoclinic class can, after a perturbation, be turned into a robust property.

Theorem 5. *Let $H(P, f)$ be a non trivial homoclinic class of a periodic point P of stable index i. Assume that for some $j \neq i$ there is no dominated splitting of index j. Then there exists a diffeomorphism g arbitrarily C^1-close to f having a periodic point Q that is homoclinically related to P_g and such that $\lambda_j(Q)$ and $\lambda_{j+1}(Q)$ are non-real, have the same modulus, and any k-th multiplier of Q has modulus different from $|\lambda_j(Q)|$, ($k \neq j, j+1$).*

An immediate consequence of this theorem is that for every diffeomorphism h close to g the homoclinic class $H(P_h, h)$ does not have a dominated splitting of index j.

A more detailed version of this theorem is given in Proposition 5.2. Unfortunately, it still remains to settle the hardest case in which the lack of domination of the class $H(P, f)$ corresponds to the stable index of P.

Observe that, under the hypotheses of Theorem 5, the constructions in [15] imply that there are points Q homoclinically related to P whose multipliers $\lambda_j(Q)$ and $\lambda_{j+1}(Q)$ can be made non-real by small perturbations. The difficulty in the theorem is to preserve the homoclinic relation between P and Q throughout the perturbation process.

The following result is a consequence of Theorem 5 and the fact that for C^1-generic diffeomorphisms two saddles in the same chain recurrence class robustly belong to the same chain recurrence class (see Section 5 for the proof).

Corollary 6. *There is a residual set $\mathcal{G}$ of $\mathrm{Diff}^1(M)$ such that for every $f \in \mathcal{G}$ and every homoclinic class $H(P, f)$ of f having periodic points of different stable indices the following holds:*

if the class $H(P, f)$ has no dominated splitting of index j then for any diffeomorphism g in a neighborhood of f the chain recurrence class of P_g has no dominated splitting of index j.

1.6. **Robust non-domination and self-replication.** In [11, Definition 1.1], for diffeomorphisms defined on three-dimensional manifolds, we consider the following open property for chain recurrence classes that we call *Property* $\mathfrak{U}$.

(i) The class contains two transitive hyperbolic sets L and K of different stable indices related by a robust heterodimensional cycle.
(ii) Each of these sets K, L contains a saddle with non-real multipliers.
(iii) Each of these sets contains a saddle whose Jacobian is greater than one and a saddle whose Jacobian is less than one.

A key ingredient in [11] is the notion of *universal dynamics:* Given a diffeomorphism f with Property $\mathfrak{U}$ by perturbation we can produce "any type" of dynamics in a ball isotopic to the identity (for large iterations of the diffeomorphisms). In particular, after perturbations one can re-obtain properties of any orientation preserving diffeomorphism of a closed ball, see [11, Definition 1.3]. As a consequence, chain recurrence classes satisfying Property $\mathfrak{U}$ generate new different classes satisfying also this property. Thus Property $\mathfrak{U}$ is a "self-replicant" or "viral" property. This is the main motivation behind the definition of a viral property in [6, Sections 7.3-7.5].

Definition 1.3 (Viral property)**.** *A property $\mathfrak{P}$ of chain recurrence classes of saddles is said to be C^k-*viral *if for every diffeomorphism f and every saddle P of f whose chain recurrence class $C(P, f)$ satisfies $\mathfrak{P}$ the following conditions hold:*

Robustness. *There is a C^k-neighborhood $\mathcal{U}$ of f such that $C(P_g, g)$ also satisfies $\mathfrak{P}$ for all $g \in \mathcal{U}$.*

Self-replication. *For every C^k-neighborhood $\mathcal{V}$ of f and for every neighborhood V of $C(P, f)$ there are a diffeomorphism $g \in \mathcal{V}$ and a hyperbolic periodic point $Q_g \in V$ of g such that $C(Q_g, g)$ is different from $C(P_g, g)$ and satisfies property $\mathfrak{P}$.*

As observed above, the existence of a robust homoclinic tangency (associated to a transitive hyperbolic set in the class) is an example of a C^2-viral property for chain recurrence classes in dimension two.

As a consequence of the above results we now confirm [6, Conjecture 14] claiming that the property of robust non-existence of any dominated splitting over a chain recurrence class of a saddle is viral in the case that the class contains saddles whose stable indices are different from 1 and $\dim(M)-1$. We formulate the following generalization of Property $\mathfrak{U}$.

Definition 1.4 (Property $\mathfrak{V}$)**.** *Given a saddle P of a diffeomorphism f, the chain recurrence class $C(P, f)$ of P satisfies* Property $\mathfrak{V}$ *if there is a C^1-neighborhood $\mathcal{U}$ of f such that for all $g \in \mathcal{U}$ the chain recurrence class $C(P_g, g)$ of P_g satisfies the following two conditions:*

- (non-domination) $C(P_g, g)$ *does not have any dominated splitting,*
- (index variability) $C(P_g, g)$ *contains a saddle* Q_g *whose stable index is different from the one of* P_g.

Observe that the set of C^1-diffeomorphisms satisfying Property $\mathfrak{V}$ is indeed non-empty, see Section 9.3.

Theorem 7. *Property* $\mathfrak{V}$ *is* C^1*-viral for chain recurrence classes.*

The following result is a consequence of Theorem 7 and the properties of C^1-generic diffeomorphisms extending [11]. In fact, the corollary holds for any viral property of a chain recurrence class containing a saddle.

Corollary 8. *Let* $C(P, f)$ *be a chain recurrence class satisfying Property* $\mathfrak{V}$. *Then there are a neighborhood* $\mathcal{U}$ *of* f *and a residual subset* $\mathcal{G}_{\mathcal{U}}$ *of* $\mathcal{U}$ *such that for every* $g \in \mathcal{G}_{\mathcal{U}}$

- *there are infinitely (countably) many pairwise disjoint homoclinic classes, and*
- *there are uncountably many aperiodic chain recurrence classes.*

Indeed the homoclinic classes obtained in the corollary can be chosen to also satisfy Property $\mathfrak{V}$. The proofs of Theorem 7 and Corollary 8 are in Section 9.

Let us observe that nature of the proof of Theorem 7 is quite different from the approach in [11], where universal dynamics is the key ingredient. In [11] this universal dynamics is obtained by considering saddles in the chain recurrence class whose Jacobians are larger and smaller than one, respectively. A restriction of this construction is that all Lyapunov exponents of the aperiodic classes obtained in [11] are zero. This follows from the fact that one considers maps whose "returns" are close to the identity. Here we use directly the self-replication property. This allows us to obtain aperiodic classes with regular points having Lyapunov exponents uniformly bounded away from zero. See [6, Section 7.4], specially Problem 6, for further discussion.

Finally, bearing in mind the results in [37] and Corollary 2, we introduce the following variation of Property $\mathfrak{V}$ for diffeomorphisms defined on manifolds of dimension $d \geq 4$.

Definition 1.5 (Property $\mathfrak{V}'$)**.** *Given a saddle* P *of a diffeomorphism* f *the chain recurrence class* $C(P, f)$ *of* P *has* Property $\mathfrak{V}'$ *if there is a* C^1*-neighborhood* $\mathcal{U}$ *of* f *such that for all* $g \in \mathcal{U}$ *the chain recurrence class* $C(P_g, g)$ *of* P_g *satisfies the following two conditions:*

- $C(P_g, g)$ *does not have any dominated splitting and*
- $C(P_g, g)$ *contains a saddle with stable index* $i \notin \{1, \dim(M) - 1\}$.

Corollary 2 implies that in this case, after a perturbation, the chain recurrence class $C(P_g, g)$ robustly satisfies the index variability condition. Thus, after a perturbation, Property $\mathfrak{V}'$ implies Property $\mathfrak{V}$. In fact, we will see that these two properties are "essentially equivalent", see Lemmas 9.2 and 9.3. Finally, we have the following:

Remark 1.6. *Theorem 7 and Corollary 8 hold for Property* $\mathfrak{V}'$.

Organization of the paper. We first observe that we will use systematically several C^1-perturbation results imported from [24, 25, 7]. These results allow us to realize dynamically perturbations of cocycles associated to the derivatives of diffeomorphisms along periodic orbits (see specially Section 3.2 and 4.2).

• In Section 2 we recall results about the generation of homoclinic tangencies and heterodimensional cycles associated to homoclinic classes.

• An ingredient of our paper is the notion of an *adapted perturbation* of a diffeomorphism, that is, a small perturbation of a diffeomorphism throughout the orbit of a periodic point that preserves some homoclinic relations and some prescribed dynamical properties of a given homoclinic class, (see Definition 3.1). An essential feature of adapted perturbations is that one can perform simultaneously finitely many of them preserving some prescribed properties of the homoclinic class. These perturbations are introduced in Section 3.

• Using adapted perturbations we prove in Section 4 two important technical results (Propositions 4.7 and 4.8) claiming that the lack of domination of a homoclinic class yields periodic orbits having multiple Lyapunov exponents and weak hyperbolicity.

• In Sections 5 and 6, in the non-dominated setting we get periodic orbits inside a homoclinic class having non-real multipliers and prove Theorem 5. This proof is based on Proposition 5.2 whose proof is the most difficult step of the paper.

• In Section 7 we obtain homoclinic intersections associated to strong invariant manifolds of periodic points that will allow us to get heterodimensional cycles and finally prove Theorem 1 in Section 8.

• Finally, we study viral properties of chain recurrence classes and prove Theorem 7 and Corollary 8 in Section 9 .

2. Homoclinic tangencies and heterodimensional cycles

In this section we recall some results about generation of homoclinic tangencies and robust heterodimensional cycles associated to homoclinic classes.

2.1. Homoclinic tangencies.

Next lemma states the relation between the lack of domination over a periodic orbit and the generation of homoclinic tangencies.

Lemma 2.1 ([24, Theorem 3.1])**.** *For any* $K > 1$, $\varepsilon > 0$, *and* $d \in \mathbb{N}$, *there are constants* k_0 *and* ℓ_0 *with the following property.*

- *For every* $f \in \mathrm{Diff}^1(M)$ *with* $\dim(M) = d$ *such that the norms of* Df *and* Df^{-1} *are both bounded by* K, *and*

- *for every periodic point P of f of saddle-type such that*
  - *the period of P is larger than ℓ_0 and*
  - *the stable/unstable splitting $E^s(f^i(P)) \oplus E^u(f^i(P))$ over the orbit of P is not k_0-dominated,*

there is an ε-perturbation g of f whose support is contained in an arbitrarily small neighborhood of the orbit of P and such that the stable and unstable manifolds $W^s(P,g)$ and $W^u(P,g)$ of P have a homoclinic tangency.

Moreover, if Q is homoclinically related to P for f then the perturbation g can be chosen such that Q_g and P are homoclinically related (for g).

Remark 2.2. *Lemma 2.1 implies that the perturbation g of f can be chosen such that the saddle P has a homoclinic tangency and its homoclinic class $H(P,g)$ is non-trivial. Moreover, the orbit of tangency can be chosen inside the homoclinic class $H(P,g)$.*

2.2. **Robust heterodimensional cycles.** Let us observe that a homoclinic class $H(P,f)$ may contain saddles of different indices. But, in principle, it is not guaranteed that such a property still holds for perturbations of f. We next collect some results from [14] that will allow us to get such a property in a robust way.

We say that a heterodimensional cycle associated to a pair of transitive hyperbolic sets has *coindex one* if the s-indices of theses sets differ by one.

Lemma 2.3 ([14])**.** *Let $f \in \mathrm{Diff}^1(M)$ be a diffeomorphism having a coindex one heterodimensional cycle associated to a pair of hyperbolic periodic points P and Q such that the homoclinic class $H(P,f)$ is non trivial. Then there is a diffeomorphism g arbitrarily C^1-close to f with a pair of hyperbolic transitive sets L_g and K_g having a robust heterodimensional cycle and containing the continuations P_g and Q_g of P and Q, respectively.*

There is the following consequence of this lemma for C^1-generic systems:

Corollary 2.4 ([14])**.** *There is a residual subset $\mathcal{G}$ of $\mathrm{Diff}^1(M)$ such that for every diffeomorphism $f \in \mathcal{G}$ and every pair of periodic points P and Q of stable indices $i < j$ in the same homoclinic class there is a (finite) sequence of transitive hyperbolic sets $K_i, K_{i+1} \dots, K_j$ such that*

- *$P \in K_i$, $Q \in K_j$,*
- *the stable index of K_n is n, $n = i, i+1, \dots, j$, and*
- *the sets K_k and K_{k+1} have a robust heterodimensional cycle for all $k = i, \dots, j-1$.*

3. Adapted perturbations and generalized Franks' lemma

In this section, we collect some results about C^1-perturbations of diffeomorphisms. Observe that if $g_1, \dots, g_n$ are ε-perturbations of f with disjoint supports $V_1, \dots, V_n$ then there is an ε-perturbation g of f supported in the union of the sets V_i such that g coincides with g_i over the set V_i.

3.1. **Adapted perturbations.** We next introduce a kind of perturbation of diffeomorphisms along periodic orbits that preserves homoclinic relations. Moreover, these perturbations can be performed simultaneously and independently along different periodic orbits.

In what follows, given $\varrho > 0$, we denote by $W^{s,u}_\varrho(P,f)$ the stable/unstable manifolds of size ϱ of the orbit of P.

Definition 3.1 (Adapted perturbations)**.** *Consider a property $\mathfrak{P}$ about periodic points. Given $f \in \mathrm{Diff}^1(M)$, a pair of hyperbolic periodic points P and Q of f that are homoclinically related, and a neighborhood $\mathcal{U} \subset \mathrm{Diff}^1(M)$ of f we say that there is a* perturbation of f in $\mathcal{U}$ along the orbit of Q that is adapted to $H(P,f)$ and property $\mathfrak{P}$ *if*

- *for every neighborhood V of the orbit of Q and*
- *for every $\varrho > 0$ and every pair of compact sets $K^s \subset W^s_\varrho(Q,f)$ and $K^u \subset W^u_\varrho(Q,f)$ disjoint from V*

there is a diffeomorphism $g \in \mathcal{U}$ such that:

- *g coincides with f outside V and along the f-orbit of Q,*
- *the points P_g and Q_g are homoclinically related for g,*
- *the sets K^s, K^u are contained in $W^s_\varrho(Q,g)$ and $W^u_\varrho(Q,g)$, respectively, and*
- *the saddle Q satisfies property $\mathfrak{P}$.*

When the neighborhood $\mathcal{U}$ of f is the set of diffeomorphisms that are ε-C^1-close to f we say that g is an ε-perturbation of f along the orbit of Q that is adapted to $H(P,f)$ and property $\mathfrak{P}$.

Examples of property $\mathfrak{P}$ for periodic points are the existence of non-real multipliers and negative Lyapunov exponents.

3.2. **Generalized Franks' lemma.** We need the following extension of the so-called Franks Lemma [22] about dynamical realizations of perturbations of cocycles along periodic orbits. The novelty of this extension is that besides the dynamical realization of the cocycle throughout a periodic orbit the perturbations also preserve some homoclinic/heteroclinic intersections. Next lemma is a particular case of [25, Theorem 1] and is a key tool for constructing adapted perturbations. Recall that a linear map $B \in GL(d,\mathbb{R})$ is *hyperbolic* if every eigenvalue λ of B satisfies $|\lambda| \neq 1$.

Lemma 3.2 (Generalized Franks' Lemma, [25])**.** *Consider $\varepsilon > 0$, a diffeomorphism $f \in \mathrm{Diff}^1(M)$ and a hyperbolic periodic point Q of period $\ell = \pi(Q)$ of f. Then*

- *for any one-parameter family of linear maps $(A_{n,t})_{n=0,\dots,\ell-1,\, t\in[0,1]}$, $A_{n,t} \in \mathrm{GL}(d,\mathbb{R})$, $d = \dim(M)$, such that*
  (1) $A_{n,0} = Df(f^n(Q))$,
  (2) for all $n = 0,\dots,\ell-1$ and all $t \in [0,1]$ it holds
  $$\max\left\{\|Df(f^n(Q)) - A_{n,t}\|,\, \|Df^{-1}(f^n(Q)) - A^{-1}_{n,t}\|\right\} < \varepsilon,$$

(3) $B_t = A_{\ell-1,t} \circ \cdots \circ A_{0,t}$ *is hyperbolic for all* $t \in [0,1]$,

- *for every neighborhood* V *of the orbit of* Q*, every* $\varrho > 0$*, and every pair of compact sets* $K^s \subset W^s_\varrho(Q,f)$ *and* $K^u \subset W^u_\varrho(Q,f)$ *disjoint from* V*,*

there is an ε*-perturbation* g *of* f *such that*

(a) g *and* f *coincide throughout the orbit of* Q *and outside* V*,*
(b) $K^s \subset W^s_\varrho(Q,g)$ *and* $K^u \subset W^u_\varrho(Q,g)$*, and*
(c) $Dg(g^n(Q)) = Dg(f^n(Q)) = A_{n,1}$ *for all* $n = 0, \dots, \ell-1$.

4. Lyapunov exponents of periodic orbits

In this section we see that the lack of domination of a homoclinic class yields perturbations such that there are periodic points of the class whose Lyapunov exponents are multiple or close to zero, see Propositions 4.7 and 4.8. We first state some preparatory results and prove these propositions in Section 4.3.

4.1. **Lyapunov exponents and homoclinic relations.** We will use repeatedly throughout the paper the following result.

Lemma 4.1. *There is a residual subset* $\mathcal{G}$ *of* $\mathrm{Diff}^1(M)$ *such that for every* $f \in \mathcal{G}$*, every saddle* P *of* f*, every non-trivial and locally maximal transitive hyperbolic set* Λ *of* f *containing* P*, and every* $\varepsilon > 0$ *there is a saddle* $Q \in \Lambda$ *such that*

- $|\chi_j(Q) - \chi_j(P)| < \varepsilon$ *for all* $j \in \{1, \dots, d\}$*, and*
- *the orbit of* Q *is* ε*-dense in* Λ*.*

In particular, the saddle Q *can be chosen with arbitrarily large period.*

This results follows from the arguments in the proofs of [3, Corollary 2] and [2, Theorem 3.10] using standard constructions that allow us to distribute these orbits throughout the "whole" transitive hyperbolic set while keeping the control of the exponents.

4.2. **Dominated splittings and cocycles over periodic orbits.** We next study the lack of domination of homoclinic classes. For that we consider periodic orbits (of large period) in the class and their associated cocycles. Next result is a standard fact about dominated splittings (see for instance [16, Appendix B]).

Lemma 4.2 (Extension of a dominated splitting to a closure)**.** *Consider an* f*-invariant set* Λ *having a* k*-dominated splitting of index* i*. Then the closure of* Λ *also has a* k*-dominated splitting of index* i *that coincides with the one over* Λ*.*

As in the case of periodic points of diffeomorphisms, given a family of linear maps $A_1, \dots, A_\ell \in GL(d,\mathbb{R})$ we consider the product $B = A_\ell \circ \cdots \circ A_1$ and the eigenvalues $\lambda_1(B), \dots, \lambda_d(B)$ of B ordered in increasing modulus and counted with multiplicity. We define the *i-th Lyapunov exponent* of B by

$$\chi_i(B) = \frac{1}{\ell} \log |\lambda_i(B)|.$$

The family of linear maps above is *bounded by* K if $\|A_n\|$ and $\|A_n^{-1}\|$ are both less than or equal to K for all $n = 1, \ldots, \ell$.

Note that Definition 1.1 of a dominated splitting over an invariant set of a diffeomorphism can be restated for sequences of linear maps.

Next lemma relates the lack of domination of a cocycle and the generation of sinks or sources.

Lemma 4.3 ([17, Corollary 2.19 and Remark 2.20])**.** *For every $K > 1$, $\varepsilon > 0$, and $d \in \mathbb{N}$, there are constants k_0 and ℓ_0 with the following property.*

- *For every $f \in \mathrm{Diff}^1(M)$ with $\dim(M) = d$ such that the norms of Df and Df^{-1} are both bounded by K, and*
- *for every periodic point P of f of period larger than ℓ_0 such that there is no any k_0-dominated splitting over the orbit of P,*

there is an ε-perturbation g of f whose support is contained in an arbitrarily small neighborhood of the orbit of P and such that P is either a sink or a source of g.

Next result is a finer version of the previous lemma that allows us to modify only two consecutive Lyapunov exponents of a cocycle.

Lemma 4.4 ([7, Theorem 4.1 and Proposition 3.1])**.** *For every $K > 1$, $\varepsilon > 0$, and $d \geq 2$, there are constants k_0 and ℓ_0 with the following property.*

Consider $\ell \geq \ell_0$ and linear maps $A_1, \ldots, A_\ell$ in $GL(d, \mathbb{R})$, such that:

- *every A_n is bounded by K,*
- *for any $i \in \{1, \ldots, d-1\}$, the linear map $B = A_\ell \circ \cdots \circ A_1$ has no any k_0-dominated splitting of index i.*

Then for every $j \in \{1, \ldots, d-1\}$, there exist one parameter families of linear maps $(A_{n,t})_{t \in [0,1]}$ in $GL(d, \mathbb{R})$, $n = 1, \ldots, \ell$, such that

(1) *$A_{n,0} = A_n$ for all $n = 1, \ldots, \ell$, and*
(2) *$A_{n,t} - A_n$ and $A_{n,t}^{-1} - A_n^{-1}$ are bounded by ε for all $t \in [0,1]$ and all $n = 1, \ldots, \ell$.*

Consider the linear map

$$B_t = A_{\ell,t} \circ \cdots \circ A_{1,t}.$$

Then, for any $t \in [0,1]$, the Lyapunov exponents of the map B_t satisfies

(3) *$\chi_m(B_t) = \chi_m(B)$ if $m \neq j, j+1$,*
(4) *$\chi_j(B_t) + \chi_{j+1}(B_t) = \chi_j(B) + \chi_{j+1}(B)$,*
(5) *$\chi_j(B_{t'})$ is non-decreasing and $\chi_{j+1}(B_{t'})$ is non-increasing, that is*

$$\chi_j(B_{t'}) \leq \chi_j(B_t) \leq \chi_{j+1}(B_t) \leq \chi_{j+1}(B_{t'}), \quad \textit{for all } t' < t,$$

(6) *$\chi_{j+1}(B_1) = \chi_j(B_1)$, and*

(7) the eigenvalues of B_1 are all real.

Remark 4.5. *Note that if $A \in GL(d,\mathbb{R})$ has real eigenvalues and if its Lyapunov exponents $\chi_j(A)$ and $\chi_{j+1}(A)$ are equal then there is $\bar{A} \in GL(d,\mathbb{R})$ arbitrarily close to A whose eigenvalues are real and whose Lyapunov exponents satisfy $\chi_m(\bar{A}) \neq \chi_j(\bar{A}) = \chi_{j+1}(\bar{A})$ for all $m \neq j, j+1$. Moreover, there is a "small path of cocycles" joining A and $\bar{A}$ that preserves the j and $j+1$ Lyapunov exponents. Thus in the conclusions of Lemma 4.4 we can replace item (3) by*

(3') $\chi_m(B_t)$ is close to $\chi_m(B)$ for all $m \neq j, j+1$ and all $t \in [0,1]$ and $\chi_m(B_1) \neq \chi_j(B_1) = \chi_{j+1}(B_1)$.

In order to get cocycles with real eigenvalues we also use the following result (see also previous results in [8, Lemme 6.6] and [17, Lemma 3.8]).

Proposition 4.6 ([7, Proposition 4.1])**.** *For every $K > 1$, $\varepsilon > 0$, and $d \geq 2$, there is a constant ℓ_0 with the following property.*

Consider $\ell \geq \ell_0$ and linear maps $A_1, \dots, A_\ell$ in $GL(d,\mathbb{R})$, such that:

For every family of linear maps $(A_n)_{n=1}^{\ell}$ in $GL(d,\mathbb{R})$ such that $\ell \geq \ell_0$ and A_n and A_n^{-1} are bounded by K for every n, there are one parameter families of linear maps $(A_{n,t})_{n=1,t\in[0,1]}^{\ell}$, in $GL(d,\mathbb{R})$, such that

- *$A_{n,0} = A_n$,*
- *$A_{n,t} - A_n$ and $A_{n,t}^{-1} - A_n^{-1}$ are bounded by ε for every n,*
- *let $B_t = A_{\ell,t} \circ \cdots \circ A_{1,t}$, then for every $j \in \{1, \dots, d\}$ the Lyapunov exponent $\chi_j(B_t)$ is constant for $t \in [0,1]$, and*
- *all the multipliers of B_1 are real.*

4.3. **Multiple Lyapunov exponents and weak hyperbolicity.** In Propositions 4.7 and 4.8 we combine Lemmas 3.2 and 4.4 to prove that the lack of domination of a homoclinic class yields periodic orbits whose Lyapunov exponents are multiple or close to zero.

Proposition 4.7. *For every $K > 1$, $\varepsilon > 0$, and $d \in \mathbb{N}$, there is a constant k_0 with the following property.*

Consider a diffeomorphism $f \in \mathrm{Diff}^1(M)$, $\dim(M) = d$, such that the norms of Df and Df^{-1} are bounded by K, a hyperbolic periodic point P of s-index i whose homoclinic class $H(P,f)$ is non-trivial, and an integer $j \in \{1, \dots, d\}$ with $j \neq i$ such that the homoclinic class $H(P,f)$ has no any k_0-dominated splitting of index j.

Then there is a periodic point $Q \in H(P,f)$ homoclinically related with P and an ε-perturbation g of f along the orbit of Q that is adapted to $H(P,f)$ and to the following property $\mathfrak{P}_{j,j+1}$:

$$\mathfrak{P}_{j,j+1} \stackrel{\text{def}}{=} \begin{cases} \chi_j(Q_g) = \chi_{j+1}(Q_g), \\ \chi_m(Q_g) \neq \chi_j(Q_g) \text{ for all } m \neq j, j+1, \\ \lambda_m(Q_g) \in \mathbb{R} \text{ for all } m. \end{cases}$$

Proof. Consider the constants $d \in \mathbb{N}$, $K > 1$, and $\varepsilon > 0$. Applying Lemma 4.4 to these constants we obtain the constants k_0 and ℓ_0.

Since the homoclinic class $H(P,f)$ is non-trivial, the set Σ_{ℓ_0} of all saddles Q of period larger than ℓ_0 that are homoclinically related to P is dense in $H(P,f)$. Observe that there is a saddle $Q \in \Sigma_{\ell_0}$ such that there is no k_0-dominated splitting of index j over the orbit of Q. Otherwise, by Lemma 4.2, the closure of the set Σ_{ℓ_0} (that is the whole class $H(P,f)$) would have a k_0-dominated splitting of index j, which is a contradiction.

Thus we can apply Lemma 4.4 to the linear maps $Df(Q), \dots, Df^{\ell-1}(Q)$, $\ell = \pi(Q) \geq \ell_0$, obtaining one-parameter families of linear maps $(A_{i,t})_{t\in[0,1]}$, $i = 0, \dots, \ell-1$, satisfying the conclusions of Lemma 4.4.

We now fix a neighborhood V of the orbit Q and compact sets $K^s \subset W^s(Q,f)$ and $K^u \subset W^u(Q,f)$ disjoint from V as in Definition 3.1. Since Q is homoclinically related to P there are transverse intersections $Y^s \in W^s(Q,f) \pitchfork W^u(P,f)$ and $Y^u \in W^u(Q,f) \pitchfork W^s(P,f)$ and (small) compact disks $\Delta^s \subset W^s(Q,f)$ and $\Delta^u \subset W^u(Q,f)$ (of the same dimension as $W^s(Q,f)$ and $W^u(Q,f)$) containing the points Y^s and Y^u. We consider the compact sets

$$\tilde{K}^s = K^s \cup \Delta^s \quad \text{and} \quad \tilde{K}^u = K^u \cup \Delta^u.$$

We now apply Lemma 3.2 to ε, f, the small path of cocycles $(A_{n,t})$ above, and the compact sets $\tilde{K}^s$ and $\tilde{K}^u$ to get an ε-perturbation g of f along the orbit of Q adapted to $H(P,f)$ and Property $\mathfrak{P}_{j,j+1}$:

- adapted to $H(P,f)$: By the choice of Δ^s and Δ^u the saddle Q_g is homoclinically related to P_g.
- adapted to Property $\mathfrak{P}_{j,j+1}$: By item (6) in Lemma 4.4 it holds $\chi_j(B_1) = \chi_{j+1}(B_1)$, by Remark 4.5 we have $\chi_m(B_1) \neq \chi_j(B_i)$ if $m \neq j, j+1$, and by item (7) all the eigenvalues of B_1 all are real.

This concludes the proof of the proposition. □

Proposition 4.8. *For every $K > 1$, $\varepsilon > 0$, and $d \in \mathbb{N}$, there is a constant k_0 with the following property.*

Consider $\delta > 0$, a diffeomorphism $f \in \mathrm{Diff}^1(M)$, $\dim(M) = d$, and a hyperbolic periodic point P of f of s-index i such that:

- *the norms of Df and Df^{-1} are bounded by K,*
- *$\chi_i(P) + \chi_{i+1}(P) > -\delta$,*
- *the homoclinic class $H(P,f)$ is non-trivial and has no k_0-dominated splitting of index i.*

Then there is a periodic point $Q \in H(P,f)$ homoclinically related with P and an ε-perturbation g of f along the orbit of Q that is adapted to $H(P,f)$ and to the following property

$$\mathfrak{P}_{i,\delta} \stackrel{\text{def}}{=} \textit{The i-th Lyapunov exponent of Q satisfies } \chi_i(Q) \in (-\delta, 0). \tag{4.2}$$

Proof. The strategy of the proof is analogous to the one of Proposition 4.7, so we will skip some repetitions. As in the proof of Proposition 4.7 we consider constants k_0 and ℓ_0 associated to K, ε, and d.

Since the homoclinic class $H(P,f)$ has no dominated splitting of index i, there is a locally maximal transitive hyperbolic subset L of $H(P,f)$ containing P and having no k_0-dominated splitting of index i. We can also assume that for every f' close to f the continuation $L_{f'}$ of L has no such a k_0-dominated splitting.

We choose f' in the residual subset $\mathcal{G}$ of $\mathrm{Diff}^1(M)$ in Lemma 4.1. Then there is a periodic point $Q_{f'} \in L_{f'}$ such that $\chi_i(Q_{f'}) + \chi_{i+1}(Q_{f'}) > -\delta$ and whose orbit has no k_0-dominated splitting of index i. Otherwise, by Lemma 4.2, the set $L_{f'}$ has a k_0-dominated splitting.

Consider the point $Q = Q_f$. We take a first small path of hyperbolic cocycles $(\bar{A}_{n,t})_{t\in[0,1]}$, $n = 0,\dots,\ell-1$, $\ell = \pi(Q)$, over the orbit of Q joining the derivatives Df and Df'. Note that, by definition, the cocycle $(\bar{A}_{n,1})$ does not have a k_0-dominated splitting and the Lyapunov exponents of $\bar{B}_1 = \bar{A}_{\ell-1,1} \circ \cdots \circ \bar{A}_{0,1}$ satisfy $\chi_i(\bar{B}_1) + \chi_{i+1}(\bar{B}_1) > -\delta$.

Observe that if $\chi_i(\bar{B}_1) > -\delta$ we are done. Otherwise we apply Lemma 4.4 to the cocycle $\bar{A}_{n,1}$, $n = 0,\dots,\ell-1$, and $j = i$. This provides new families of linear maps $(\tilde{A}_{n,t})_{t\in[0,1]}$, $n = 0,\dots,\ell-1$, satisfying the conclusions of the lemma. Define the composition $\tilde{B}_t$ as above. Let

$$\tau \stackrel{\text{def}}{=} \chi_i(\tilde{B}_t) + \chi_{i+1}(\tilde{B}_t) > -\delta.$$

Note that by item (4) of Lemma 4.4 this number does not depend on t.

By item (6) in Lemma 4.4, there is some t_0 such that

$$\chi_i(\tilde{B}_{t_0}) = \min\left(\frac{\tau-\delta}{2}, \frac{-\delta}{2}\right).$$

As the map $\chi_i(\tilde{B}_t)$ is non-decreasing (recall item (5) in Lemma 4.4) we have $\chi_i(\tilde{B}_t) \le \frac{-\delta}{2} < 0$ for all $t \in [0,t_0]$. Also

$$\chi_{i+1}(\tilde{B}_t) \ge \tau - \min\left(\frac{\tau-\delta}{2}, \frac{-\delta}{2}\right) \ge \frac{\tau+\delta}{2} + \frac{\max(0,\tau)}{2} > 0.$$

Therefore $(\tilde{A}_{n,t})_{n,t\in[0,t_0]}$ is a path of hyperbolic cocycles.

We next consider the concatenation of the paths of hyperbolic cocycles $(\bar{A}_{n,t})_{t\in[0,1]}$ and $(\tilde{A}_{n,t})_{t\in[0,t_0]}$. The end of the proof is the same as the one of Proposition 4.7 and involves the definition of the sets $\tilde{K}^s$ and $\tilde{K}^u$. We apply Lemma 3.2 to get an ε-perturbation g of f along the orbit of Q that is adapted to $H(P,f)$ and to property $\mathfrak{P}_{i,\delta}$, since by construction

$$\chi_i(Q_g) = \chi_i(\tilde{B}_{t_0}) = -\delta + \min\left(\frac{\tau+\delta}{2}, \frac{\delta}{2}\right) > -\delta.$$

This ends the proof of the proposition. □

5. "Robustizing" lack of domination

In this section we analyze the existence of dominated splittings for homoclinic classes. In some cases these splittings will have several bundles.

Definition 5.1 (Dominated splittings II)**.** *Let Λ be an invariant set of a diffeomorphism f. A Df-invariant splitting $E_1 \oplus \cdots \oplus E_s$, $s \geq 2$, over the set Λ is dominated if for all $j \in \{1, \ldots, s-1\}$ the splitting $E_1^j \oplus E_{j+1}^s$ is dominated, where $E_1^j = E_1 \oplus \cdots \oplus E_j$ and $E_{j+1}^s = E_{j+1} \oplus \cdots \oplus E_s$.*

*As in the case of two bundles, the splitting is k-*dominated *if the splittings $E_1^j \oplus E_{j+1}^k$ are k-dominated for all j.*

There are analogous definitions for cocycles.

Note that if there is a saddle Q homoclinically related to P such that $\chi_j(Q) = \chi_{j+1}(Q)$ then the class has no dominated splitting of index j. Moreover, if

$$\chi_{j-1}(Q) < \chi_j(Q) = \chi_{j+1}(Q) < \chi_{j+2}(Q) \quad \text{and} \quad \lambda_j(Q), \lambda_{j+1}(Q) \in (\mathbb{C} \setminus \mathbb{R})$$

then the lack of domination of the homoclinic class is C^1-robust. In this section we study when the converse holds (up to perturbations).

A saddle Q of a diffeomorphism f satisfies property $\mathfrak{P}_{j,j+1,\mathbb{C}}$ if

$$\mathfrak{P}_{j,j+1,\mathbb{C}} \stackrel{\text{def}}{=} \begin{cases} \textbf{(i)}\ \chi_j(Q) = \chi_{j+1}(Q), \\ \textbf{(ii)}\ \chi_m(Q) \neq \chi_j(Q) \text{ for all } m \neq j, j+1, \\ \textbf{(iii)}\ \lambda_j(Q) \text{ and } \lambda_{j+1}(Q) \text{ are non-real.} \end{cases} \tag{5.3}$$

The main technical step of our constructions is the next proposition whose proof is postponed to the next section. It immediately implies Theorem 5.

Proposition 5.2. *For any $K > 1$, $\varepsilon > 0$, and $d \in \mathbb{N}$, there is a constant k_0 with the following property.*

Consider a diffeomorphism $f \in \mathrm{Diff}^1(M)$, $\dim M = d$, such that the norms of Df and Df^{-1} are bounded by K, a hyperbolic periodic point P of s-index i, and an integer $j \in \{1, \ldots, d-1\}$, $j \neq i$. Assume that the homoclinic class $H(P, f)$ is non trivial and has no k_0-dominated splitting of index j.

Then there is a periodic point Q that is homoclinically related with P and an ε-perturbation of f along the orbit of Q that is adapted to $H(P, f)$ and property $\mathfrak{P}_{j,j+1,\mathbb{C}}$.

Remark 5.3. *The proof of the proposition provides a point Q with arbitrarily large period. In particular, there exist infinitely many periodic points Q satisfying the conclusion of the proposition.*

We postpone the proof of this proposition to Section 6. We now deduce from it Corollaries 5.4 and 5.5 below.

Corollary 5.4. *For any $K > 1$, $\varepsilon > 0$, and $d \in \mathbb{N}$, there is a constant k_0 with the following property.*

Consider a diffeomorphism $f \in \mathrm{Diff}^1(M)$, $\dim M = d$, such that the norms of Df and Df^{-1} are bounded by K, a homoclinic class $H(P, f)$ of f, and integers $0 < j_1 < \cdots < j_\ell < d$ that are different from the s-index of

P and such that there is no k_0-dominated splitting of index j_k over $H(P,f)$ for every $k \in \{1,\dots,\ell\}$.

Then there exists an ε-perturbation g of f supported in a small neighborhood of $H(P,f)$ such that for each $k \in \{1,\dots,\ell\}$ there exists a periodic point $Q_{k,g}$ of g homoclinically related to P_g satisfying property $\mathfrak{P}_{j_k,j_k+1,\mathbb{C}}$ in equation (5.3).

In particular, for every diffeomorphism $\bar g$ close to g and for every $k \in \{1,\dots,\ell\}$ there is no dominated splitting of index j_k over $H(P_{\bar g},\bar g)$.

Proof. By Proposition 5.2, for each index j_k there is a periodic point Q_k homoclinically related to P and ε-perturbations of f along the orbit of Q_k that are adapted to $H(P,f)$ and to property $\mathfrak{P}_{j_k,j_k+1,\mathbb{C}}$. For each saddle Q_k consider a pair of transverse heteroclinic points

$$Y_k^s \in W^s(Q_k,f) \pitchfork W^u(P,f) \quad \text{and} \quad Y_k^u \in W^u(Q_k,f) \pitchfork W^s(P,f).$$

For each k we also fix compact disks

$$K_k^s \subset W^s(Q_k,f) \quad \text{and} \quad K_k^u \subset W^u(Q_k,f)$$

of the same dimensions as $W^s(Q_k,f)$ and $W^u(Q_k,f)$ containing Y_k^s and Y_k^u in their interiors. By Remark 5.3, we can assume that the orbits of the saddles Q_k are different. Thus there are small neighborhoods $V_1,\dots,V_\ell$ of these orbits whose closures are pairwise disjoint and such that for each $k \neq k'$ the orbits of Y_k^s and Y_k^u do not intersect $V_{k'}$. Thus taking the disks K_k^s and K_k^u small enough, we can assume that this also holds for the forward orbit of K_k^s and the backward orbit of K_k^u.

For each k we get an adapted ε-perturbation g_k supported in V_k (and associated to the compact sets K_k^s and K_k^u). Since the supports of these perturbations are disjoint, we can perform all them simultaneously obtaining a diffeomorphism g that is ε-close to f and has saddles $Q_{k,g}$ satisfying $\mathfrak{P}_{j_k,j_k+1,\mathbb{C}}$, $j = 1,\dots,\ell$.

It remains to check that these saddles are homoclinically related to P_g. Observe that for each k the points Y_k^s and Y_k^u are transverse heteroclinic points (associated to Q_k and P) for g_k. The choices of the orbits of these heteroclinic points and of the sets V_j imply that Y_k^s and Y_k^u are also transverse heteroclinic points (associated to Q_k and P) for g (in fact, the orbits of the points Y_k^s and Y_k^u are the same for g_k and g). This completes the proof of the corollary. □

We also get the following genericity result.

Corollary 5.5. *There exists a residual subset $\mathcal{G}$ of $\mathrm{Diff}^1(M)$ such that every diffeomorphism $f \in \mathcal{G}$ satisfies the following property:*

For every $i,j \in \{1,\dots,d-1\}$, $i \neq j$, and for every periodic point P of s-index i of f such that there is no dominated splitting of index j over $H(P,f)$ there exists a periodic point Q homoclinically related to P satisfying property $\mathfrak{P}_{j,j+1,\mathbb{C}}$.

The corollary follows from standard genericity arguments after noting that for a homoclinic class $H(P,f)$ to have a saddle Q homoclinically related to P satisfying property $\mathfrak{P}_{j,j+1,\mathbb{C}}$ is an open condition.

We are now ready to prove Corollary 6.

Proof of Corollary 6. The residual subset $\mathcal{G}$ in Corollary 5.5 can be chosen with the following additional property, see [8]. For every $f \in \mathcal{G}$ and for every pair of hyperbolic periodic points P and Q of $f \in \mathcal{G}$ that are in the same chain recurrent class the following holds

- the homoclinic classes of P and Q are equal and
- there is a neighborhood $\mathcal{U}$ of f such that for all $g \in \mathcal{U}$ the chain recurrence classes of P_g and Q_g are equal.

Now it is enough to consider a point $Q \in H(P,f)$ of s-index different from the one of P and to apply Corollary 5.5 to P (if j is different to the index of P) or to Q (otherwise). □

Comment. We wonder if in the conclusion of Corollary 6 it is possible to consider homoclinic classes instead of chain recurrence classes. One difficulty is that in general one may have two hyperbolic periodic points with different stable index that are robustly in the same chain recurrence class but whose homoclinic classes do not coincide robustly. More precisely:

Question 1. *Consider an open set $\mathcal{U}$ of $\mathrm{Diff}^1(M)$ and two hyperbolic saddles P_f and Q_f whose continuations are defined for all $f \in \mathcal{U}$, have different stable indices, and whose chain recurrence classes coincide for all $f \in \mathcal{U}$.*

Does there exist an open and dense subset $\mathcal{V}$ of $\mathcal{U}$ such that for any $f \in \mathcal{V}$ one has $Q_f \in H(P_f,f)$? Or even more, $H(P_f,f) = H(Q_f,f)$?

By [8] the answer to this question is affirmative when the saddles have the same index. It is also true when the chain recurrence class is partially hyperbolic with a central direction that splits into one-dimensional central directions. This follows using quite standard arguments and we will provide the details of this construction in a forthcoming note.

6. Obtaining non-real multipliers: Proof of Proposition 5.2

In this section we prove Proposition 5.2. This proposition follows from the next lemma:

Lemma 6.1. *Consider a homoclinic class $H(P,f)$ and $j \in \mathbb{N}$ satisfying the hypothesis of Proposition 5.2. Then there are a hyperbolic periodic point Q homoclinically related to P and path of cocycles $(A_{i,t})_{t\in[0,1]}$, $0 \le i < \ell$ and $\ell = \pi(Q)$, over the orbit of Q that are ε-perturbations of $Df(f^i(Q))$ and satisfy the following properties:*

- **(A)** *the composition $B_t = A_{\ell-1,t} \circ \cdots \circ A_{0,t}$ is hyperbolic for all $t \in [0,1]$,*
- **(B)** *$A_{i,0} = Df(f^i(Q))$ for all $i = 0, \dots \ell-1$, and*

(C) *the multipliers λ_m and the exponents of χ_m of the composition B_1 satisfy the conclusions in Proposition 5.2:*

$$\chi_j = \chi_{j+1}, \qquad \chi_m \neq \chi_j \quad \text{if } m \neq j, j+1, \qquad \lambda_j, \lambda_{j+1} \notin \mathbb{R}.$$

We briefly introduce some formalism that we will use only in this section. Consider a set Σ and a bijection $g\colon \Sigma \to \Sigma$. Let E be a vector bundle over the base Σ such that it fibers E_x, $x \in \Sigma$, are endowed with an Euclidean metric. A *linear cocycle* on E over g is a map $\mathcal{A}\colon E \to E$ that sends each fiber E_x to a fiber $E_{g(x)}$ by a linear isomorphism $\mathcal{A}_x$. The map g is called the *base transformation* of the cocycle $\mathcal{A}$.

The *distance* between two linear cocycles $\mathcal{A}$ and $\mathcal{B}$ above the same base transformation g is

$$\operatorname{dist}(\mathcal{A}, \mathcal{B}) = \sup_{x \in \Sigma} \{\|\mathcal{A}_x - \mathcal{B}_x\|, \|(\mathcal{A}_x)^{-1} - (\mathcal{B}_x)^{-1}\|\}.$$

A *path of cocycles* defined on the bundle E is a one-parameter family of cocycles $(\mathcal{A}_t)_{t\in[0,1]}$ above the same base transformation g such that the map $t \mapsto \mathcal{A}_t$ is continuous for the metric above. The *radius* of the path $(\mathcal{A}_t)_{t\in[0,1]}$ is defined by

$$\max_{t\in[0,1]} \operatorname{dist}(\mathcal{A}_0, \mathcal{A}_t).$$

Here we only deal with continuous cocycles (for the ambient topology of E) whose base transformations are diffeomorphisms or restrictions of diffeomorphisms to invariant subsets of the ambient.

Finally, hyperbolicity and domination of cocycles are defined in the natural way, see for example Definition 5.1.

We will deduce Lemma 6.1 from the following result:

Lemma 6.2. *Consider a homoclinic class $H(P, f)$ and $j \in \mathbb{N}$ satisfying the hypothesis of Proposition 5.2. Then there is an arbitrarily small path of continuous cocycles $(\mathcal{A}_t)_{t\in[0,1]}$ on TM above the diffeomorphism f, a point $\bar{Q}$ homoclinically related to P, and a horseshoe K containing $\bar{Q}$ such that:*

- *$\mathcal{A}_0$ coincides with Df,*
- *the cocycle $\mathcal{A}_t$ restricted to $T_K M$ is hyperbolic, for all $t \in [0,1]$,*
- *the cocycle $\mathcal{A}_1$ restricted to $T_K M$ has a dominated splitting*

$$T_K M = E \oplus E^{j,j+1} \oplus F$$

  such that E has dimension $j-1$ and $E^{j,j+1}$ has dimension 2,
- *the cocycle $\mathcal{A}_1$ restricted to the (periodic) orbit of $\bar{Q}$ does not admit any dominated splitting over $E^{j,j+1}$.*

Here a small path, means path of small radius.

Proof of Lemma 6.2. Observe first that arguing as in the previous propositions we just get a periodic point $\bar{Q}$ homoclinically related to P and a small path of hyperbolic cocycles $(\bar{A}_{i,t})_{t\in[0,1]}$, $0 \le i < \pi(\bar{Q})$, defined over the orbit of $\bar{Q}$ such that the Lyapunov exponents of the final composition $\bar{B}_1$ are real and $\chi_j(\bar{B}_1)$ and $\chi_{j+1}(\bar{B}_1)$ are equal, see Proposition 4.7. Moreover, by Remark 4.5, we can assume that, for all $m \neq j, j+1$, the m-th exponent $\chi_m(\bar{B}_1)$ is different from $\chi_j(\bar{B}_1) = \chi_{j+1}(\bar{B}_1)$ for all $m \neq j, j+1$. Note that if the multipliers $\lambda_j(\bar{B}_1)$ and $\lambda_{j+1}(\bar{B}_1)$ are equal then one can make them non-real and conjugate by a small perturbation. However they might have opposite signs, which is why Lemma 6.1 is not obvious.

We now go to the details of the proof of the lemma. Fix a transverse homoclinic point X for $\bar{Q}$ and let

$$\Lambda = \{f^n(\bar{Q}), 0 \leq n < \pi(\bar{Q})\} \cup \{f^n(X), n \in \mathbb{Z}\}.$$

The compact invariant set Λ is hyperbolic for the cocycle Df. The path $\mathcal{A}_t$ of cocycles is obtained as a concatenation of the following three paths:

- The first path $\mathcal{A}_t^{[1]}$ "linearizes" the dynamics around $\bar{Q}$.
- The second path $\mathcal{A}_t^{[2]}$ is a path of cocycles on TM that extends the path $(\bar{A}_{i,t})_{t\in[0,1]}$ of cocycles over the orbit of $\bar{Q}$ introduced above in such a way that the set Λ is a hyperbolic set for all t.
- The third path $\mathcal{A}_t^{[3]}$ provides a cocycle having the required dominated splitting over a horseshoe containing the set Λ.

For simplicity of notations, we will assume that $\bar{Q}$ is a fixed point for f (the argument is identical in the general case), thus we write $(\bar{A}_t)_{t\in[0,1]}$ instead of $(\bar{A}_{i,t})_{t\in[0,1]}$. Finally, in what follows, the path of cocycles $(\bar{A}_t)_{t\in[0,1]}$ becomes a path of matrices of $GL(d, \mathbb{R})$.

(I) The first path of cocycles $\mathcal{A}_t^{[1]}$. Fix a chart around the point $\bar{Q}$ so that for any x in a neighborhood V of the orbit of $\bar{Q}$, we can identify the derivative Df (or any neighboring cocycle) at x to a matrix of $GL(d, \mathbb{R})$.

Claim 6.3. *There is an arbitrarily small path of continuous cocycles $(\mathcal{A}_t^{[1]})_{t\in[0,1]}$ on TM above f, starting at $\mathcal{A}_0^{[1]} = Df$, and a neighborhood $W \subset V$ of $\bar{Q}$, such that*

- *by considering the restriction to the fiber of each point $x \in W$, the cocycle $\mathcal{A}_1^{[1]}$ is identified to the derivative of f at $\bar{Q}$.*
- *the set Λ is hyperbolic for all the cocycles $\mathcal{A}_t^{[1]}$.*

Proof. Build a candidate cocycle $\mathcal{A}_1^{[1]}$ arbitrarily close to Df, by a unit partition on a (small enough) neighborhood of $\bar{Q}$. On each fibre of TM, take for the matrix of $\mathcal{A}_t^{[1]}$ the $(1-t,t)$-barycenter of the matrices of Df and $\mathcal{A}_1^{[1]}$. Since the set Λ is hyperbolic for Df, it will also be for all the cocycles $\mathcal{A}_t^{[1]}$, provided we chose $\mathcal{A}_1^{[1]}$ close enough to Df. □

(II) The second path of cocycles $\mathcal{A}_t^{[2]}$. Fix a neighborhood W of $\bar{Q}$ and a path $(\mathcal{A}_t^{[1]})_{t\in[0,1]}$, as given by Claim 6.3.

Claim 6.4. *There is a path $(\mathcal{A}_t^{[2]})_{t\in[0,1]}$ of continuous cocycles on TM above f, starting at $\mathcal{A}_0^{[2]} = \mathcal{A}_1^{[1]}$, such that:*

- *its radius is arbitrarily close to that of $\left(\bar{A}_t\right)_{t\in[0,1]}$,*
- *$\mathcal{A}_1^{[2]}$ coincides with $\bar{A}_1$ over $\bar{Q}$,*
- *for all $t \in [0,1]$, the set Λ is hyperbolic for the cocycle $\mathcal{A}_t^{[2]}$.*

Proof. For all $t \in [0,1]$, denote by E_t^u and E_t^s the stable and unstable directions of the hyperbolic point $\bar{Q}$ for the cocycle $\bar{A}_t$. These directions vary continuously with t. Hence given any $\epsilon > 0$ there exists a sequence $0 = t_0 < .... < t_N = 1$ of times such that, for all $0 \le n < N$, there is a path of linear maps $\theta_{n,t} \in GL(d,\mathbb{R})$, with $\theta_{n,t_n} = Id$, and for all $t_n \le t \le t_{n+1}$:

- $\theta_{n,t}$ is ϵ-close to identity,
- $\theta_{n,t}(E_{t_n}^u) = E_t^u$ and $\theta_{n,t}(E_{t_n}^s) = E_t^s$.

Assume that the neighborhood W of $\bar{Q}$ is small enough and consider $n_0 \in \mathbb{N}$ such that $f^{\pm n}(X) \in W$, for all $n \ge n_0$. First, we define the cocycle $\mathcal{A}_t^{[2]}$ over the segment of orbit $\{f^n(X)\}_{n\ge 0}$ and for all $t \in [0,1]$. We denote by $\mathcal{B}_{n,t}$ the linear map corresponding to the cocycle $\mathcal{A}_t^{[2]}$ over the point $f^n(X)$.

For all $t_n \le t \le t_{n+1}$, define $\mathcal{B}_{n,t}$ as follows:

- $\mathcal{B}_{k,t}$ coincides with $\mathcal{A}_1^{[1]}$ at $f^n(X)$, if $0 \le k < n_0$,
- $\mathcal{B}_{n_0+k,t} = \bar{A}_{t_k} \circ \theta_{k,t_{k+1}}$, if $k < n$,
- $\mathcal{B}_{n_0+n,t} = \bar{A}_t \circ \theta_{n,t}$,
- $\mathcal{B}_{n_0+k,t} = \bar{A}_t$, if $k > n$.

Recall that the set Λ is hyperbolic for $\mathcal{A}_1^{[1]}$. Let E_X^s and E_X^u be the stable and unstable directions at X for the cocycle $\mathcal{A}_1^{[1]}$.

By construction, the iterations of the cocycle $(\mathcal{B}_{n,t})_{n\in\mathbb{N}}$ maps E_X^s and E_X^u into the stable and unstable directions of $\bar{Q}$ for the map $\bar{A}_t$, respectively. Hence, the bundles E_X^s and E_X^u are uniformly contracted and uniformly expanded, respectively, by positive iterations of $(\mathcal{B}_{n,t})_{n\in\mathbb{N}}$.

We define $\mathcal{B}_{n,t}$ symmetrically for the backward orbit $\{f^n(X)\}_{n\le 0}$ of X.

Let $\mathcal{A}_{t,\Lambda}^{[2]}$ be the linear cocycle on $T_\Lambda M$ given by the linear maps $\mathcal{B}_{n,t}$ over the orbit of X and by the matrix $\bar{A}_t$ over the point $\bar{Q}$. Then the orbits of the bundles E_X^s and E_X^u is a hyperbolic splitting for $\mathcal{A}_{t,\Lambda}^{[2]}$. By construction, the family $(\mathcal{A}_{t,\Lambda}^{[2]})_{t\in[0,1]}$ is a path of continuous linear cocycles starting at the restriction of $\mathcal{A}_1^{[1]}$ to the set Λ. The radius of this path can be found arbitrarily close to the radius of $\left(\bar{A}_t\right)_{t\in[0,1]}$: just take both $\epsilon > 0$ and the neighborhood W of $\bar{Q}$ small enough. Now, all we need to do is to extend the path $\mathcal{A}_{t,\Lambda}^{[2]}$ of cocycles above the restriction of f to Λ to a small path $(\mathcal{A}_t^{[2]})_{t\in[0,1]}$ of continuous cocycles above f starting at $\mathcal{A}_1^{[1]}$.

Note that, for all $n > n_0 + N$, the matrix of $\mathcal{A}^{[2]}_{t,\Lambda}$ is A_t at the iterate $f^{\pm n}(X)$. So is it also at $\bar{Q}$. Fix a small neighborhood $U_{\bar{Q}} \subset M$ of the set formed by these points and $\bar{Q}$. Fix a small neighborhood U_n for each other iterate $f^n(X)$. Do this such that we have a disjoint union

$$U = U_{\bar{Q}} \cup \bigcup_{-n_0-N}^{n_0+N} U_n.$$

Let $1 = \phi + \psi$ be a unit partition on M such that $\phi = 1$ on Λ and $\phi = 0$ outside of U. Let $\mathcal{A}^{[2]}_t$ be the cocycle above f whose matrix on the fiber T_xM is the $(\phi(x), \psi(x))$-barycenter of the following two matrices:

- the matrix of $\mathcal{A}^{[1]}_1$ at x,
- $\begin{cases} \text{the matrix of } \mathcal{A}^{[2]}_{t,\Lambda} \text{ at } f^n(X), \text{ if } x \in U_n, \\ \text{the matrix } \bar{A}_t, \text{ if } x \in U_{\bar{Q}}. \end{cases}$

Choosing the neighborhood U of Λ small enough, one finds the radius of $(\mathcal{A}^{[2]}_t)_{t\in[0,1]}$ as close as wished to the radius of $(\mathcal{A}^{[2]}_{t,\Lambda})_{t\in[0,1]}$, hence as close as wished to the radius of $\left(\bar{A}_t\right)_{t\in[0,1]}$. This ends the proof of the claim. □

(III) The third path of cocycles $\mathcal{A}^{[3]}_t$. We fix a path $\mathcal{A}^{[1]}_t$ and a path $\mathcal{A}^{[2]}_t$, as given by Claims 6.3 and 6.4.

Claim 6.5. *There is an arbitrarily small path of cocycles $(\mathcal{A}^{[3]}_t)_{t\in[0,1]}$ defined on TM above the diffeomorphism f, starting at $\mathcal{A}^{[3]}_0 = \mathcal{A}^{[2]}_1$, such that:*

- *$\mathcal{A}^{[3]}_1$ coincides with A_1 at $\bar{Q}$,*
- *$\mathcal{A}^{[3]}_1$ admits, over Λ, a dominated splitting of the form*

$$T_\Lambda M = E \oplus E^{j,j+1} \oplus F$$

  such that E has dimension $j-1$, and $E^{j,j+1}$ has dimension 2,
- *for all $t \in [0,1]$, the cocycle $\mathcal{A}^{[3]}_t$ is hyperbolic over the set Λ.*

Proof. Since $\mathcal{A}^{[2]}_1$ is equal to $\bar{A}_1$ at $\bar{Q}$, recalling the properties of the exponents of $\bar{A}_1$, we have that there is a dominated splitting $T_{\bar{Q}}M = E \oplus E^{j,j+1} \oplus F$ with the required dimensions and such that $E^{j,j+1}$ is either uniformly contracted or uniformly expanded by $\mathcal{A}^{[2]}_1$. We need to extend this splitting to the whole orbit of X.

Observe that there are $(j-1)$ and $(j+1)$-dimensional spaces E_X and $\tilde{E}_X$ at the point X such that their positive iterations by $\mathcal{A}^{[2]}_1$ converge to E and $\tilde{E} = E \oplus E^{j,j+1}$, respectively. Symmetrically, there are $(j-1)$ and $(j+1)$-codimensional spaces $\tilde{F}_X$ and F_X whose negative iterations by $\mathcal{A}^{[2]}_1$ converge to $\tilde{F} = E^{j,j+1} \oplus F$ and F, respectively.

One can perturb slightly $\mathcal{A}^{[2]}_1$ at the point X in order to make $\tilde{E}_X$ transverse to F_X and E_X transverse to $\tilde{F}_X$. Then the iterates of $\tilde{E}_X$ and F_X

by the perturbed cocycle along the orbit of X extend the dominated splitting $\tilde{E} \oplus F$ to the whole set Λ. Symmetrically, we get an extension of the dominated splitting $E \oplus \tilde{F}$ to the set Λ. Taking $E^{j,j+1} = \tilde{E} \cap \tilde{F}$ we get the dominated splitting $E \oplus E^{j,j+1} \oplus F$ over Λ for that perturbed cocycle.

That perturbation of $\mathcal{A}_1^{[2]}$ may be reached by an arbitrarily small path of cocycles $\mathcal{A}_t^{[3]}$ on TM such that $\mathcal{A}_1^{[3]}$ coincides with $\bar{A}_1$ at $\bar{Q}$. In particular, it can be chosen so that $\mathcal{A}_t^{[3]}$ is hyperbolic over Λ for all t. □

End of the proof of Lemma 6.2. Define the path $(\mathcal{A}_t)_{t\in[0,1]}$ as the concatenation of the paths $(\mathcal{A}_t^{[1]})_{t\in[0,1]}$, $(\mathcal{A}_t^{[2]})_{t\in[0,1]}$, and $(\mathcal{A}_t^{[3]})_{t\in[0,1]}$ given by the three previous claims. By construction, the path $(\mathcal{A}_t)_{t\in[0,1]}$ can be found having radius arbitrarily close to the radius of $(\bar{A}_t)_{t\in[0,1]}$. Choosing $\bar{Q}$ conveniently, this last radius can be taken arbitrarily small.

Note that the diffeomorphism f has horseshoes K containing the set Λ that are arbitrarily close to Λ for the Hausdorff distance. Choosing the horseshoe K Hausdorff-close enough to Λ, we have the following:

- for all $t \in [0,1]$, the cocycles $\mathcal{A}_t$ are continuous on TM and hyperbolic over Λ. Thus, by a compactness argument on $t \in [0,1]$, the cocycles $\mathcal{A}_t$ are also hyperbolic over K for all $t \in [0,1]$.
- The dominated splitting $T_\Lambda M = E \oplus E^{j,j+1} \oplus F$ for $\mathcal{A}_1 = \mathcal{A}_1^{[3]}$ extends to K, see [16, Appendix B]).

All the conclusions of Lemma 6.2 are then satisfied. This ends its proof. □

Proof of Lemma 6.1. Let $\mathcal{A}_t$, $\bar{Q}$, K, and $T_K M = E \oplus E^{j,j+1} \oplus F$ be as in Lemma 6.2. Consider a transverse homoclinic point X of $\bar{Q}$, $X \in W^u_{\mathrm{loc}}(\bar{Q}, f) \cap K$, and an iterate of it $f^r(X) \in W^s_{\mathrm{loc}}(\bar{Q}, f) \cap K$. These two points can be chosen arbitrarily close to $\bar{Q}$.

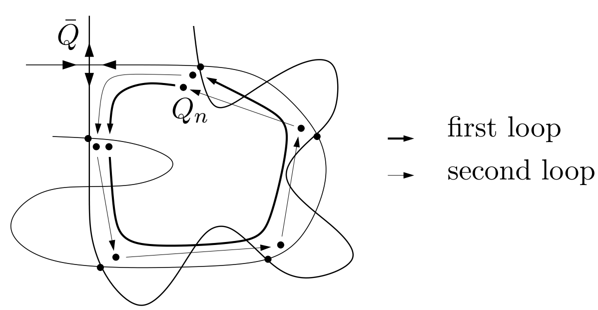

FIGURE 1. Two-loops orbits Q_n

We next consider periodic points Q_n passing close to X and having orbits with "two loops". For every large n there is a periodic point $Q_n \in K$ of period $2\,n+2+2\,r$ as follows (see Figure 1): Let $Q_n = Q_n^0$ and $Q_n^i = f^i(Q_n)$, where

$$\begin{aligned}
&\bullet\ Q_n^0 \text{ is close to } f^r(X) \text{ and } Q_n^0, \dots, Q_n^n \text{ are close to } \bar{Q},\\
&\bullet\ Q_n^{n+i} \text{ is close to } f^i(X) \text{ for all } i = 0, \dots, r,\\
&\bullet\ Q_n^{n+r}, \dots, Q_n^{n+r+n+2} \text{ are close to } \bar{Q}, \text{ and}\\
&\bullet\ Q_n^{2\,n+r+2+i} \text{ is close to } f^i(X) \text{ for all } i = 0, \dots, r.
\end{aligned} \tag{6.4}$$

Claim 6.6. *For n large enough, the linear cocycle $\mathcal{A}_1$ preserves the orientation of the central bundle $E^{j,j+1}$ at the periodic orbit of Q_n.*

Proof. Let $\mathcal{A}_1^c$ be the restriction of $\mathcal{A}_1$ to the central bundle $E^{j,j+1}$. Since the base K of the 2-dimensional bundle $E^{j,j+1}$ is a Cantor set, there is a continuous identification between $E^{j,j+1}$ and $K \times \mathbb{R}^2$. Thus, for any $x \in K$, the restriction $\mathcal{A}_{1,x}^c$ of $\mathcal{A}_1^c$ to the fiber T_xM identifies to a 2×2 matrix.

By continuity, if the distance between a pair of points $x, y \in K$ is less than some $\eta > 0$, then the determinants of the matrices $\mathcal{A}_{1,x}^c$ and $\mathcal{A}_{1,y}^c$ have the same sign. One then easily checks that for n great enough (when "close" in (6.4) means distance less than $\eta/2$), the composition of the matrices $\mathcal{A}_{1,x}^c$ along the (finite) entire orbit of Q_n has positive determinant. □

If the multipliers λ_j and λ_{j+1} of the first return map of $\mathcal{A}_1$ at some Q_n are complex, then all the conclusions of Lemma 6.1 are satisfied by $Q = Q_n$ and the restriction $(A_{i,t})_{t\in[0,1]}$ of the path $\mathcal{A}_t$ to the orbit of Q.

Otherwise, by Claim 6.6, these multipliers are real and have the same sign. Recall that the linear cocyle $\mathcal{A}_1^c$ admits no dominated splitting at the point $\bar{Q}$. Since the orbits of Q_n accumulate on $\bar{Q}$, then with increasing n the strength of domination (if any) of the splitting of the bundle $E^{i,j}$ along the orbit of Q_n for the cocycle $\mathcal{A}_1$ will decrease. We can now apply [15, Proposition 3.1]. This result claims that, for n great enough, the cocycle $\mathcal{A}_1$ can be perturbed along the two-dimensional bundle $E_{j,j+1}$ and along the orbit of Q_n to get a pair of non-real and conjugate eigenvalues.

For n great enough, that perturbation can be reached through a small path $(\mathcal{B}_{t,n})_{t\in[0,1]}$ of cocycles over the orbit of Q_n. If the perturbation is small enough then, for all t, the hyperbolicity and the domination of the splitting $E \oplus E^{i,j} \oplus F$ of $\mathcal{A}_1$ over the horseshoe K are preserved. Thus the conclusions of Lemma 6.1 are all satisfied for $Q = Q_n$ and the cocycle $(A_{i,t})_{t\in[0,1]}$ defined as the concatenation of

- the restriction of the path $\mathcal{A}_t$ to the orbit of $Q = Q_n$ and
- the path $\mathcal{B}_{t,n}$.

This concludes the proof of the Lemma 6.1. □

7. Formation of strong homoclinic connections

We say that a saddle P has a *strong homoclinic intersection* if there is a strong stable manifold of the orbit of P that intersects the unstable manifold of the orbit of P or vice-versa. That is, let i be the s-index of P, then either $W^{ss}_k(P) \cap W^u(P) \neq \emptyset$ for some $k < i$ or $W^{uu}_j(P) \cap W^s(P) \neq \emptyset$ for some $j < \dim(M) - i$ (recall the definitions of $W^{ss}_k(P)$ and $W^{uu}_j(P)$ in Section 1.2). In this section, we see how the lack of domination of a homoclinic class yields strong homoclinic intersections.

Proposition 7.1. *For every $K > 1$, $\varepsilon > 0$ and $d \geq 2$, there exists a constant k_0 with the following property.*

Consider $f \in \mathrm{Diff}^1(M)$, $\dim(M) = d$, and a hyperbolic periodic point P of s-index i, $i \in \{2, \dots, d-1\}$, such that $H(P, f)$ is non-trivial and has no k_0-dominated splitting of index $i - 1$. Then there is a periodic point Q homoclinically related to P and an ε-perturbation of f along the orbit of Q that is adapted to $H(P, f)$ and to property $\mathfrak{P}_{ss}$ defined as follows

$$\mathfrak{P}_{ss} \stackrel{\text{def}}{=} \begin{cases} \textbf{\textit{(i)}}\ \chi_{i-1}(Q) < \chi_i(Q), \\ \textbf{\textit{(ii)}}\ W^{ss}_{i-1}(Q) \cap W^u(Q) \neq \emptyset. \end{cases} \tag{7.5}$$

Proof. By Proposition 5.2 there is a hyperbolic periodic point Q that is homoclinically related to P and an $\frac{\varepsilon}{2}$-perturbation f' of f along the orbit of Q that is adapted to $H(P, f)$ and to property $\mathfrak{P}_{i-1,i,\mathbb{C}}$ (see equation (5.3)). This means that fixed small $\varrho > 0$, a neighborhood V of the orbit of Q, and compact sets $K^s \subset W^s_\varrho(Q)$ and $K^u \subset W^u_\varrho(Q)$ disjoint from V, there is a diffeomorphism f' that is $\frac{\varepsilon}{2}$-close to f such that

(1) $f' = f$ outside V and along the f-orbit of Q,
(2) the points P and Q are homoclinically related for f',
(3) $K^s \subset W^s_\varrho(Q, f')$ and $K^u \subset W^u_\varrho(Q, f')$, and
(4) the saddle $Q_{f'} = Q$ satisfies property $\mathfrak{P}_{i-1,i,\mathbb{C}}$.

By Remark 5.3, the period of Q can be chosen arbitrarily large. Hence Proposition 4.6 provides a small path of hyperbolic cocycles joining the restriction of Df' over the orbit of Q and a cocycle with real multipliers. Applying Lemma 3.2 to this cocycle and to f' we get an $\frac{\varepsilon}{2}$-perturbation f'' of f', such that $Q = Q_{f''}$ has a pair of real multipliers $\lambda_{i-1}(Q)$ and $\lambda_i(Q)$ such that $|\lambda_{i-1}(Q)| = |\lambda_i(Q)|$ and $|\lambda_i(Q)| \neq |\lambda_j(Q)|$ for all $j \neq i, i-1$, and such that conditions (1)–(3) also hold for f''. Note that f'' is ε-close to f.

Consider now local coordinates around Q such that

$$W^s_{\mathrm{loc}}(Q, f'') = [-1, 1]^i \times \{0^{d-i}\} \quad \text{and} \quad W^u_{\mathrm{loc}}(Q, f'') = \{0^i\} \times [-1, 1]^{d-i}.$$

To conclude the proof of the proposition it is enough to get a diffeomorphism g arbitrarily C^1-close to f'' and a small neighborhood $V_0 \subset V$ of the orbit of Q such that

(a) $g = f''$ outside V_0 and along the g-orbit of $Q_g = Q$,
(b) $W^s_{\mathrm{loc}}(Q, g) = [-1, 1]^i \times \{0^{d-i}\}$ and $W^u_{\mathrm{loc}}(Q, g) = \{0^i\} \times [-1, 1]^{d-i}$, and
(c) Q_g satisfies property $\mathfrak{P}_{ss}$.

This will be done in several steps. To simplify the presentation, let us assume in the remainder steps of the proof that the period of Q is one.

Claim 7.2. *There is an arbitrarily C^1-small perturbation g' of f'' satisfying (a) and (b) and such that the restriction of g' to a small neighborhood of Q in $W^s_{\mathrm{loc}}(Q,g')$ is linear. Moreover, one has that $Dg'(Q) = Df''(Q)$.*

This claim allows us to define a two dimensional locally invariant center-stable manifold $W^{cs}_\tau(Q,g')$ of Q tangent to the space corresponding to the $(i-1)$-th and i-th multipliers of Q. Up to a linear change of coordinates, we have

$$W^{cs}_\tau(Q,g') = \{0^{i-2}\} \times [-\tau,\tau]^2 \times \{0^{d-i}\} \text{ and } Df''(Q)(x^s,x^u) = (A^s,A^u).$$

Proof of Claim 7.2. Using the coordinates (x^s,x^u) corresponding to the stable and unstable bundles, in a neighborhood of Q, we write

$$f''(x^s,x^u) = (f^s(x^s,x^u), f^u(x^s,x^u)).$$

By the invariance of the local stable and unstable manifolds we have that $f^u(x^s,0) = 0^u$ and $f^s(0^s,x^u) = 0^s$.

Next step is to linearize the restriction of f^s to the local stable manifold. Consider a local perturbation $\tilde f^s$ of $x^s \mapsto f^s(x^s,0^u)$ supported in a small neighborhood of 0^s such that $\tilde f^s(x^s) = A^s(x^s)$ for small x^s. Note that

$$\tilde f^s(x^s) = f^s(x^s,0^u) + h^s(x^s),$$

where h^s is C^1-close to the zero map and has support in a small neighborhood of 0^s.

Finally, we choose a bump-function $\psi(x^u)$ such that $\psi = 1$ in a neighborhood of 0^u, it is equal to 0 outside another small neighborhood of 0^u, and it has small derivative. We define g' in a neighborhood of $(0^s,0^u)$ by

$$g'(x^s,x^u) = \big(f^s(x^s,x^u) + h^s(x^s)\,\psi(x^u), f^u(x^s,x^u)\big).$$

By construction, the restriction of g' to a small neighborhood in the local stable manifold of Q coincides with $\tilde f^s = A^s$. Moreover, the local unstable manifold of Q is also preserved and $Dg'(Q) = Df''(Q)$. This completes the proof of the claim. □

Claim 7.3. *There is an arbitrarily small C^1-perturbation g'' of g' satisfying Claim 7.2 (in particular, (a) and (b)) and such that there is a transverse homoclinic point of Q in F_{i-1}, where F_{i-1} is a $Dg''(Q)$-invariant one-dimensional linear space corresponding to the Lyapunov exponent $\chi_{i-1}(Q)$.*

Proof. Note first that as the homoclinic class of Q is non-trivial there is a transverse homoclinic point Y of Q that belongs to the local stable manifold of Q where the dynamics is linear. Next two steps are quite standard. First, by a perturbation we can assume that $Y \notin W^{ss}_{i-2}(Q)$.

Second, after replacing the point Y by some forward iterate of it and after a new perturbation, we can assume that Y belongs to the $Dg'(Q)$-invariant (central) two-dimensional linear space F corresponding to the Lyapunov exponents $\chi_{i-1}(Q)$ and $\chi_i(Q)$. This follows noting that any stable non-zero vector of Q that is not in the linear space corresponding to the first $(i-2)$ Lyapunov exponents has normalized iterations which approximate to F.

There are two cases according to the restriction of g' to the two dimensional space F.

Case 1: the restriction of g' to F is a homothety. In this case, the point Y belongs to a one-dimensional $Dg'(Q)$-invariant space and we are done.

Case 2: the restriction of g' to F is parabolic. In this case, the restriction of $Dg'(Q)$ to F is conjugate to a matrix of the form

$$\begin{pmatrix} \lambda_i & 1 \\ 0 & \lambda_i \end{pmatrix}, \quad 0 < |\lambda_i| < 1.$$

Then the normalized iterations of any non-zero vector in F accumulate to the unique one-dimensional invariant sub-space F_{i-1} of $Dg'(Q)$ in F. As above, after a new perturbation we can assume that there is some iterate of Y in F_{i-1} ending the proof of the claim. □

To conclude the proof of the proposition it is now enough to make the Lyapunov exponent $\chi_{i-1}(Q)$ smaller than $\chi_i(Q)$ so that the space F_{i-1} is now locally contained in the strong stable manifold of Q of dimension $i-1$. To perform this final perturbation we argue as in Claim 7.2. □

8. Homoclinic tangencies yielding heterodimensional cycles

In this section we prove Theorem 1 and its alternative version in item (ii) of Remark 1.2. For that we need the following two propositions.

Proposition 8.1. *Consider $f \in \mathrm{Diff}^1(M)$, $\dim(M) = d$, and a hyperbolic periodic point P of f of s-index $i \in \{2, \dots, d-1\}$ such that:*

- *(i) for any C^1-neighborhood $\mathcal{U}$ of f there exist a hyperbolic periodic point R homoclinically related to P and perturbations of f in $\mathcal{U}$ along the orbit of R that are adapted to $H(P,f)$ and to property $\mathfrak{P}_{ss}$,*
- *(ii) for any C^1-neighborhood $\mathcal{U}$ of f and any $\delta > 0$ there exist a hyperbolic periodic point Q homoclinically related to P and perturbations of f in $\mathcal{U}$ along the orbit of Q that are adapted to $H(P,f)$ and to property $\mathfrak{P}_{i,\delta}$,*

Then there exists a diffeomorphism $g \in \mathcal{U}$ arbitrarily C^1-close to f having a heterodimensional cycle associated to P_g and a saddle S_g of s-index $i-1$.

Recall that properties $\mathfrak{P}_{ss}$ and $\mathfrak{P}_{i,\delta}$, see (7.5) and (4.2), mean that the saddles R and Q satisfy

$$\chi_{i-1}(R) < \chi_i(R) \quad \text{and} \quad W^{ss}_{i-1}(R) \cap W^u(R) \neq \emptyset,$$
$$\chi_i(Q) \in (-\delta, 0).$$

Proposition 8.2. *Consider $f \in \mathrm{Diff}^1(M)$, $\dim(M) = d$, having a hyperbolic periodic point P of s-index $i \in \{2, \dots, d-1\}$ such that:*

- *(1) $H(P,f)$ is non trivial and has no dominated splitting of index i,* *(2') there is a diffeomorphism g arbitrarily C^1-close to f with a hyperbolic periodic point R_g homoclinically related to P_g satisfying property $\mathfrak{P}_{ss}$, and*

(3') for every $\delta > 0$ there exists a hyperbolic periodic point Q_δ homoclinically related to P such that $\chi_i(Q_\delta) + \chi_{i+1}(Q_\delta) \geq -\delta$.

Then, there exists a diffeomorphism g arbitrarily C^1-close to f with a heterodimensional cycle associated to a P and to a saddle of s-index $i-1$.

Note that item (1) in Proposition 8.2 corresponds exactly to the same item in Theorem 1, items (2') and (3') are exactly items (2') and (3') in Remark 1.2. Therefore Proposition 8.2 implies the conclusions in Remark 1.2.

We postpone the proof of these propositions to Sections 8.4 and 8.5. Assuming these propositions we now prove Theorem 1 and Corollary 3

8.1. **Proof of Theorem 1.** Proposition 7.1 and assumption (2) in the theorem imply that condition (i) in Proposition 8.1 is satisfied.

Proposition 4.8 and assumptions (1) and (3) in the theorem imply that condition (ii) in Proposition 8.1 is satisfied.

Proposition 8.1 now provides a diffeomorphism g with a heterodimensional cycle associated to P_g and a saddle S_g of s-index $i-1$. By Lemma 2.3, we can assume that the diffeomorphism g has a pair of transitive hyperbolic sets L_g and K_g having a robust heterodimensional cycle, where L_g contains P_g and K_g contains a periodic point R_g of stable index $i-1$.

We now explain how to improve the previous arguments to obtain robust homoclinic tangencies.

Fix $\varepsilon > 0$ and consider the integer k_0 associated to ε in Proposition 5.2. Since $H(P,f)$ has no dominated splittings of indices $i-1$ and i, there are $r > 0$ and a neighborhood $\mathcal{U}$ of f such that for any $f' \in \mathcal{U}$ and any f'-invariant set K having an r-neighborhood containing $H(P,f)$ there is no k_0-dominated splitting over K.

We perform a first perturbation g_0 of f, $g_0 \in \mathcal{U}$, as above, obtaining a robust heterodimensional cycle between two transitive hyperbolic sets containing the saddles P_{g_0} and R_{g_0}. By [8], taking g_0 in a residual subset of $\mathrm{Diff}^1(M)$, we can assume that $H(P,g_0)$ and $H(R_{g_0},g_0)$ coincide. In particular, these homoclinic classes are non-trivial and their r-neighborhoods contain $H(P,g)$. Thus for every diffeomorphism h close to g_0, the homoclinic classes $H(P_h,h)$ and $H(R_h,h)$ have no k_0-dominated splittings of indices $i-1$ and i.

We now consider another small perturbation $g_1 \in \mathcal{U}$ of g_0 such that the saddles P_{g_1} and R_{g_1} have a heterodimensional cycle.

Since the classes $H(P_{g_1},g_1)$ and $H(R_{g_1},g_1)$ have no k_0-dominated splittings of indices $i-1$ and i, Proposition 5.2 provides a pair of hyperbolic periodic points Q_{g_1} and T_{g_1} homoclinically related to P_{g_1} and R_{g_1}, respectively, and two "independent" local ε-perturbations g_Q and g_T of g_1 such that

- the supports of g_Q and g_T are disjoint and contained in arbitrarily small neighborhoods of the orbits of Q_{g_1} and T_{g_1}, respectively,
- these perturbations preserve the heterodimensional cycle associated to P_{g_1} and T_{g_1},
- the i-th and $(i-1)$-th multipliers of Q_{g_1} for g_Q and of T_{g_1} for g_T are non-real.

As the supports of the perturbations g_Q and g_T are disjoint, combining these perturbations one gets a diffeomorphism g_2 such that P_{g_2} and T_{g_2} have a heterodimensional cycle and the classes $H(P_{g_2}, g_2)$ and $H(T_{g_2}, g_2)$ robustly have no dominated splittings of indices i and $i-1$, respectively.

By Lemma 2.3, one can perform a last perturbation g so that $P_g \in K_g$ and $T_g \in L_g$ where K_g and L_g are transitive hyperbolic sets having a robust heterodimensional cycle. Finally, we choose g in the residual subset of $\mathrm{Diff}^1(M)$ in [13, Theorem 1], this choice implies that the sets K_g and L_g have robust homoclinic tangencies. $\square$

8.2. **Proof of Corollary 3.** We first recall that there is a residual subset $\mathcal{R}$ of $\mathrm{Diff}^1(M)$ such that every homoclinic class $H(P,f)$ of $f \in \mathcal{R}$ that does not have any dominated splitting is the Hausdorff limit of sinks or sources, see [15, Corollary 0.3]. More precisely, if there is a saddle Q homoclinically related to P whose Jacobian is less (resp. greater) than one then the class $H(P,f)$ is the Hausdorff limit of sinks (resp. sources), see the proof of [15, Proposition 2.6]. Thus to prove the corollary it is enough to consider a saddle P of s-index two whose homoclinic class $H(P,f)$ does not have any dominated splitting and such that every saddle Q homoclinically related to P has Jacobian greater than one. By the previous comments, the class $H(P,f)$ is limit of sources.

Observe that the assumption on the Jacobians implies that $\chi_2(Q) + \chi_3(Q) > 0$. Thus the homoclinic class satisfies all hypotheses in Theorem 1. Hence there is a perturbation g of f with a robust heterodimensional cycle associated to a hyperbolic set containing Q_g and P_g. The corollary now follows from standard genericity arguments. $\square$

8.3. **Sectional dissipativiness. Corollary 4.** Let P be a hyperbolic saddle of a diffeomorphism f such that:

- for every diffeomorphism g that is C^1-close to f there is no heterodimensional cycle associated to P_g, and
- let i the stable index of P, then the homoclinic class $H(P,f)$ has no dominated splitting of index i.

Under these hypotheses we consider a dominated splitting with three bundles (see Definition 5.1)

$$T_{H(P)}M = E_1 \oplus E^c \oplus E_3$$

such that $\dim(E_1) < i < \dim(E_1 \oplus E^c)$ and E^c does not admit any dominated splitting. Note that the bundles E_1 and E_3 may be empty and that $\dim(E^c) \geq 2$.

We now see some properties of the homoclinic class $H(P,f)$ that follow from Theorem 1 and will imply the corollary. There are the following cases:

- $\dim(E^c)=2$: Assume that E^c is sectionally dissipative. Then, by Theorem 1 and Remark 1.2, for every diffeomorphism g C^1-close to f and every saddle R_g homoclinically related to P_g the unstable and strong stable manifolds of R_g have empty intersection. There is similar statement when E^c is sectionally dissipative for f^{-1}.
- $\dim(E^c)\geq 3$. Since the diffeomorphisms close to f cannot have heterodimensional cycles, Corollary 2 implies that

  $$\textbf{(I)} \quad i=\dim(E_1\oplus E^c)-1 \qquad \text{or} \qquad \textbf{(II)} \quad i=\dim(E_1)+1.$$

  In case (I), by Theorem 1, the bundle E^c is uniformly sectionally dissipative. Moreover, by Remark 1.2, for every diffeomorphism g C^1-close to f and every saddle R_g homoclinically related to P_g the unstable and strong stable manifolds of R_g have empty intersection. There is similar statement for case (II) considering f^{-1}.

The previous discussion implies Corollary 4. □

8.4. **Proof of Proposition 8.1.** We fix a small neighborhood $\mathcal{U}$ of f and small $\delta>0$. Conditions (i) and (ii) in the proposition provide saddles R and Q having different orbits and local perturbations g_R and g_Q throughout these orbits as follows. Consider small neighborhoods V_R and V_Q of the orbits of R and Q having disjoint closures. Then there are perturbations g_R and g_Q of f in $\mathcal{U}$ whose supports are contained in V_R and V_Q such that R satisfies $\mathfrak{P}_{ss}$ for g_R and Q satisfies $\mathfrak{P}_{i,\delta}$ for g_Q.

As the supports of these perturbations are disjoint, we can consider a perturbation g_0 of f which coincides with g_R in V_R, with g_Q in V_Q, and with f outside these neighborhoods. Note that if $\mathcal{U}$ is small then the diffeomorphism g_0 can be chosen arbitrarily close to f. Moreover, since we are considering adapted perturbations, we have that the saddles R and Q are all homoclinically related to P (recall the proof of Corollary 5.4).

The proposition is an immediate consequence of the following two claims. We observe that there are similar results in [31] and [20, section 2.5], so we just sketch their proofs.

Claim 8.3. *There is a perturbation g_1 of g_0 having a hyperbolic periodic point S_{g_1} that is homoclinically related to P_{g_1} and that satisfies simultaneously properties $\mathfrak{P}_{ss}$ and $\mathfrak{P}_{i,\delta}$.*

Claim 8.4. *The dynamical configuration in Claim 8.3 yields diffeomorphisms g having heterodimensional cycles associated to a periodic orbit homoclinically related to P_g and to a saddle of index $i-1$. Moreover, if $\delta>0$ is small and g_1 is close enough to f then $g\in\mathcal{U}$.*

Sketch of the proof of Claim 8.3. The idea of the proof of the claim is the following. First, consider a strong homoclinic intersection X of the orbit of

R. Then there are N_1 and $N_2 > 0$ such that

$$X \in g_0^{-N_1}\big(W^{ss}_{\mathrm{loc}}(R, g_0)\big) \cap g_0^{N_2}\big(W^{u}_{\mathrm{loc}}(R, g_0)\big).$$

Observe also that, since R and Q are homoclinically related, there is a locally maximal transitive hyperbolic set L of g_0 containing R and Q. Moreover, we can assume (and we do) that L is disjoint from the orbit of the point X.

We consider a "generic" perturbation g_0' of g_0 given by Lemma 4.1 obtaining a periodic point $S_{g_0'} \in L_{g_0'}$ which satisfies $\mathfrak{P}_{i,\delta}$ and having iterates arbitrarily close to $R_{g_0'}$. This implies that

$$(g_0')^{-N_1}\big(W^{ss}_{\mathrm{loc}}(S_{g_0'}, g_0')\big) \quad \text{and} \quad (g_0')^{N_2}\big(W^{u}_{\mathrm{loc}}(S_{g_0'}, g_0')\big)$$

have points that are close to X. Since X is disjoint from the orbit of $S_{g_0'}$ we can perform a local perturbation g_1 of g_0' in a small in a neighborhood of X having a strong homoclinic intersection associated to S_{g_1}. This completes the sketch of the proof of the claim. □

Sketch of the proof of Claim 8.4. By a small local perturbation g_2 of g_1 bifurcating the point S_{g_1} we get two points $\bar{R}_{g_2}$ and $\bar{S}_{g_2}$ of indices $i-1$ and i such that

- $\bar{S}_{g_2}$ is still homoclinically related to P_{g_2},
- the manifolds $W^u(\bar{R}_{g_2}, g_2)$ and $W^s(\bar{S}_{g_2}, g_2)$ have a transverse intersection point Y, and
- the N_2-th iterate of $W^u_{\mathrm{loc}}(\bar{S}_{g_2}, g_2)$ and the N_1-th iterate by g_2^{-1} of $W^s_{\mathrm{loc}}(\bar{R}_{g_2}, g_2)$ have points that are close.

As above, there is a small local perturbation g of g_2 such that the intersection $W^u(\bar{S}_g, g) \cap W^s(\bar{R}_g, g)$ is non-empty. The support of this perturbation is disjoint from the orbits of the saddles $\bar{S}_{g_2}$ and $\bar{R}_{g_2}$, the transverse intersection point Y, and a pair of transverse heteroclinic points between $\bar{S}_{g_2}$ and P_{g_2}. As a consequence, the diffeomorphism g has a heterodimensional cycle associated to $\bar{S}_g$ and $\bar{R}_g$ and $\bar{S}_g$ is homoclinically related to P_g. This completes the proof of the claim. □

This completes the proof of Proposition 8.1. □

8.5. **Proof of Proposition 8.2.** Consider any small $\varepsilon, \delta > 0$. The proof of this proposition follows exactly as the one of Proposition 8.1 after finding an ε-perturbation g_0 of f and two saddles R and Q of g_0 that are homoclinically related to P_{g_0} and satisfy properties $\mathfrak{P}_{ss}$ and $\mathfrak{P}_{i,\delta}$, respectively.

Let $k_0 \geq 1$ be an integer associated to ε given by Proposition 4.8. Fix a point $Q = Q_\delta$ as in item (3) in the proposition. For an arbitrarily small perturbation g' given by item (2') consider the point $R_{g'}$ homoclinically related to $P_{g'}$ and satisfying $\mathfrak{P}_{ss}$. Note that $Q_{g'}$ also satisfies item (3). Moreover, the homoclinic class $H(P_{g'}, g')$ does not have any dominated splitting of index i.

We now apply Proposition 4.8 to get a perturbation g_0 of g' supported on an arbitrarily small neighborhood of the orbit of $Q_{g'}$ and such that property $\mathfrak{P}_{i,\delta}$ holds for Q_{g_0} and g_0. Therefore all conditions in the proposition are satisfied. □

9. Viral classes

In this section we prove Theorem 7. We begin with a definition.

Definition 9.1 (Property $\mathfrak{V}''$). *The chain recurrence class $C(P,f)$ of a saddle P of a diffeomorphism $f \in \mathrm{Diff}^1(M)$, $\dim(M) = d$, satisfies* Property $\mathfrak{V}''$ *if the following conditions hold:*

(1) *for every $j \in \{1,\dots,d-1\}$ there exists a periodic point Q_j whose multipliers $\lambda_j(Q)$ and $\lambda_{j+1}(Q)$ are non-real and whose Lyapunov exponents satisfy $\chi_k(Q) \neq \chi_j(Q)$ for all $k \neq j, j+1$,*
(2) *let i be the s-index of P, if j is different from i then the points P and Q_j are homoclinically related,*
(3) *if $j = i$ then Q_i has s-index $i+1$ or $i-1$ and there are two hyperbolic transitive sets L and K containing P and Q_i and having a robust heterodimensional cycle, and*
(4) *there are saddles Q^+ and Q^- homoclinically related to P such that*

$$\chi_1(Q^-) + \chi_2(Q^-) < 0 \quad \textit{and} \quad \chi_{d-1}(Q^+) + \chi_d(Q^+) > 0. \tag{9.6}$$

Note that the points Q_j in the definition belong to the chain recurrence class $C(P,f)$. This is obvious for the saddles Q_j, $j \neq i$, that are homoclinically related to P. For the saddle Q_i this follows from the existence of the hyperbolic transitive sets L and K containing P and Q_i and related by a heterodimensional cycle.

Note also that properties $\mathfrak{V}$, $\mathfrak{V}'$, and $\mathfrak{V}''$ (recall Definitions 1.4 and 1.5) are open by definition. The next two lemmas imply that these three properties are equivalent "open and densely".

Lemma 9.2. *Consider a saddle P and its chain recurrence class $C(P,f)$. If Property $\mathfrak{V}''$ holds for $C(P,f)$ then Property $\mathfrak{V}$ holds for $C(P,f)$. Moreover, if the dimension $d \geq 4$, then property $\mathfrak{V}'$ also holds for $C(P,f)$.*

Proof. Let i be the s-index of P and denote by Q_j the saddles in Property $\mathfrak{V}''$. Condition (1) and the fact that Q_j belongs to $C(P,f)$ robustly implies that there is a neighborhood $\mathcal{V}_j$ of f such that, for all $h \in \mathcal{V}_j$, the class $C(P_h,h)$ cannot have a dominated splitting $E \oplus F$ of index j. Since this holds for all $j = 1,\dots,d-1$, the non-domination condition follows for the class $C(P_h,h)$ for every diffeomorphism $h \in \mathcal{V} = \cap_{j=1}^{d-1} \mathcal{V}_j$.

The fact that $C(P_h,h)$ contains a saddle of s-index different from i for all $h \in \mathcal{V}$ follows from condition (3) after recalling that $Q_{i,h} \in C(P_h,h)$ and that its s-index is $i \pm 1$. In dimension $d \geq 4$, either P or Q_i has s-index different from 1 and $d-1$. □

Lemma 9.3. *Consider a saddle P_f and its chain recurrence class $C(P_f, f)$. Let $\mathcal{V}$ be an neighborhood of f such that Property $\mathfrak{V}$ holds for $C(P_g, g)$ for all $g \in \mathcal{V}$. Then there is an open and dense subset $\mathcal{W}$ of $\mathcal{V}$ such that $C(P_g, g)$ satisfies $\mathfrak{V}''$ for all $g \in \mathcal{W}$. In dimension $d \geq 4$, the same holds when $\mathfrak{V}$ is replaced by $\mathfrak{V}'$.*

Proof. Assume that $C(P_g, g)$ satisfies property $\mathfrak{V}$ for all $g \in \mathcal{V}$. Let i be the s-index of P_f. Proposition 5.2 implies that there is an open and dense subset $\mathcal{W}'$ of $\mathcal{V}$ such that for all $j \neq i$ and all $g \in \mathcal{W}'$ there is a saddle $Q_{j,g}$ of s-index i homoclinically related to P_g whose j-th multipliers and exponents satisfy condition (1). This implies items (1) and (2) in Property $\mathfrak{V}''$ for $j \neq i$.

In what follows we use some properties of C^1-generic diffeomorphisms. Given two hyperbolic saddles P_f and Q_f of a generic diffeomorphism f then $C(P_f, f) = H(P_f, f)$. Moreover, if $Q \in C(P, f)$ then there is a neighborhood $\mathcal{U}$ of f such that $Q_g \in C(P_g, g)$ for all $g \in \mathcal{U}$, see [8]. Furthermore, if $H(P, f)$ contains saddles of s-indices $i < j$ then it contains a saddle of s-index k for all $k \in (i, j) \cap \mathbb{N}$, see [3].

By the comments above, after a perturbation, we can assume that the saddle Q_g in Property $\mathfrak{V}$ has s-index $i \pm 1$ for all $g \in \mathcal{W}'$. Let us assume, for instance, that this index is $i + 1$. Note that $C(P_g, g) = C(Q_g, g)$ and that, by hypothesis, this class has no dominated splitting. Arguing as above, but now considering the saddle Q_g of s-index $i + 1$, we get saddles Q'_g homoclinically related to Q_g whose multipliers and exponents satisfy condition (1) for $j = i + 1$. By construction, these saddles are robustly in the same chain recurrence class of Q_g and therefore in $C(P_g, g)$.

By Corollary 2.4, there exists two hyperbolic transitive sets L and K containing P_g and Q'_g with a robust heterodimensional cycle. Taking $Q_{i,g} = Q'_g$ we get condition (1) for $j = i$ and condition (3).

Observe that condition (4) is trivial if the s-index of P_f is $i \neq 1, d - 1$. Suppose that the index is 1 (the case $d - 1$ is analogous). In this case every saddle Q^+ homoclinically related to P_f satisfies $\chi_{d-1}(Q^+) + \chi_d(Q^+) > 0$. Note that, after a perturbation if necessary, we can assume that the homoclinic class of P_f contains saddles Q_f of stable index 2. After a new perturbation, one gets a diffeomorphism h with a heterodimensional cycle associated to Q_h and P_h. By the arguments in [3] (see Corollary 2) the unfolding of these cycle provides diffeomorphisms g with a saddle Q_g^- homoclinically related to P_g whose Lyapunov exponent $\chi_2(Q_g^-)$ is arbitrarily close to 0^+ while $\chi_1(Q_g^-)$ is negative and uniformly away from 0. In particular, one has $\chi_1(Q_g) + \chi_2(Q_g) < 0$. This proves that property $\mathfrak{V}''$ holds for g.

When $d \geq 4$, let us now assume that $C(p_g, g)$ satisfies property $\mathfrak{V}'$ for all $g \in \mathcal{V}$. Corollary 2 implies that there is a dense and open subset of $\mathcal{V}$ consisting of diffeomorphisms g such that there exists a hyperbolic periodic point Q_g in $C(P_g, g)$ with s-index different from the s-index of P. In particular Property $\mathfrak{V}$ holds and for a smaller dense and open subset $\mathfrak{V}''$ holds. $\square$

Theorem 7 is now a consequence of the two lemmas above and the following proposition.

Proposition 9.4 (Viral contamination)**.** *Consider $f \in \mathrm{Diff}^1(M)$ and a saddle P of f. Assume that the chain recurrence class $C(P, f)$ of P satisfies Property $\mathfrak{V}''$. Then for every neighborhood V of $H(P, f)$ there exist a diffeomorphism g arbitrarily C^1-close to f and a hyperbolic periodic point Q_g of g such that:*

1. *(1) the orbit of Q_g is arbitrarily close to $H(P_f, f)$ for the Hausdorff distance,*
2. *(2) there is an open neighborhood $U \subset V$ of the orbit of Q_g such that $P \notin U$ and either $f(\overline{U}) \subset U$ or $f^{-1}(\overline{U}) \subset U$, and*
3. *(3) $C(Q_g, g)$ satisfies Property $\mathfrak{V}''$.*

Note that item (2) implies that $C(Q_g, g)$ is disjoint from the chain-recurrence class of P_g (that contains $H(P_g, g)$). Recall that property $\mathfrak{V}$ is robust. Thus this proposition implies that Property $\mathfrak{V}''$ satisfies the self-replication condition in Definition 1.3.

9.1. **Proof of Proposition 9.4.** We consider small $\varepsilon > 0$ and an upper bound K of the norms of Df and Df^{-1}. Let k_0 and ℓ_0 be the constants associated to ε and K in Lemmas 2.1 and 4.3. Let i be the s-index of P. For clearness, we split the proof of the proposition into six steps.

Step I: Construction of the saddle Q. Consider periodic points Q^+ and Q^- as in equation (9.6) in Definition 9.1, i.e. $\chi_1(Q^-) + \chi_2(Q^-) < 0$ and $\chi_{d-1}(Q^+) + \chi_d(Q^+) > 0$. Note that there exists a locally maximal transitive hyperbolic set Λ_f such that

- Λ_f contains P, Q^+, and Q^-,
- $\Lambda_f \subset H(P, f)$, and
- Λ_f is arbitrarily close to $H(P, f)$ for the Hausdorff metric.

In particular, the set Λ has no k_0-dominated splitting.

Claim 9.5. *There is a perturbation g_0 of f such that the continuation Λ_{g_0} of Λ is the Hausdorff limit of the orbits of periodic points $Q_{g_0} \in \Lambda_{g_0}$ such that*

$$\chi_1(Q_{g_0}) + \chi_2(Q_{g_0}) < 0 \quad \textit{and} \quad \chi_{d-1}(Q_{g_0}) + \chi_d(Q_{g_0}) > 0. \tag{9.7}$$

Moreover, the set Λ_{g_0} has no k_0-dominated splitting of any index.

Proof. If the s-index i of P belongs to $\{2, \dots, d-2\}$ then the condition on the Lyapunov exponents holds for any saddle homoclinically related to P. Thus it is enough to consider the cases $i = 1$ and $i = d$.

Let assume that $i = 1$ (the case $i = d - 1$ is similar). In this case, $\chi_{d-1}(Q) + \chi_d(Q) > 0$ for every saddle Q that is homoclinically related to P. Consider the saddle $Q^- \in \Lambda$. Taking a perturbation g of f in the residual set $\mathcal{G}$ in Lemma 4.1, we can to "spread" the property $\chi_1(Q) + \chi_2(Q) < 0$

over the hyperbolic set Λ_{g_0}, obtaining the point Q_{g_0}. This completes the first part of the claim.

Since g_0 is close to g and Λ_{g_0} is close to Λ, there is no k_0-dominated splitting over Λ_{g_0}. This ends the proof of the claim. □

By Lemma 4.2, we can take the point Q_{g_0} in Claim 9.5 such that its orbit does not have any k_0-dominated splitting, has period larger than ℓ_0, and its distance to the homoclinic class $H(P_{g_0}, g_0)$ is arbitrarily small. This completes the choice of the point $Q = Q_{g_0}$.

Step II: Separation of homoclinic classes. By Lemma 4.3 there is an ε-perturbation g_1 of g_0 supported on an arbitrarily small neighborhood of the orbit of Q_{g_0} such that the orbit of Q_{g_1} is a sink or a source for g_1. In what follows, let us assume that Q_{g_1} is a sink. Thus there is an open set $U \subset V$ containing the orbit of Q_{g_1} such that $g_1(\overline{U}) \subset U$ and U is disjoint from the homoclinic class of P_{g_1}. Note that these properties hold for any diffeomorphism g that is C^0-close to g_1. This implies item (2) in the proposition.

Recall that the choice of Q and the neighborhood U imply that, for any perturbation g of g_1, the homoclinic class $H(Q_g, g)$ is close to $H(P_f, f)$. This gives item (1) of the proposition.

Step III: Non-trivial homoclinic class of Q. Note that after an ε-perturbation we can "recover" the original cocycle given by the derivative Dg_0 over the orbit of Q_{g_0}, now defined over the orbit of Q_{g_1}. In particular, there is no k_0-dominated splitting over the orbit of Q_{g_1}, conditions in equation (9.7) hold, and the saddle Q_{g_1} has s-index i. In what follows all perturbations g we consider will preserve the cocycle over the orbit of Q_{g_1}. Hence the homoclinic class of Q_g will satisfy item (4) in Property $\mathfrak{V}''$.

Finally, by Lemma 2.1 and Remark 2.2, there is an ε-perturbation g_2 of g_1 supported on an arbitrarily small neighborhood of the orbit of Q_{g_1} such that the homoclinic class of Q_{g_2} is not-trivial.

Step IV: No domination for the homoclinic class of Q. Since there is no k_0-dominated splitting over the orbit of Q_{g_1}, by Corollary 5.4, there is a ε-perturbation g_3 of g_2 such that for any $j \neq i$, $j \in \{1, \dots, d-1\}$, there is a periodic point Q_{j,g_3} homoclinically related to Q_{g_3} that satisfies Property $\mathfrak{P}_{j,j+1,\mathbb{C}}$. In what follows, all perturbations that we will perform will preserve these properties. This implies that the homoclinic class will satisfy items (1) and (2) in the definition of Property $\mathfrak{V}''$ for every $j \neq i$.

Finally, for $j = i$, as the class $H(Q_{g_3}, g_3)$ is not k_0-dominated, using Lemma 2.1 and Remark 2.2 we can generate a homoclinic tangency inside the class after an ε-perturbation g_4 of g_3. This prevents the existence of a dominated splitting of index i for g_4.

Note that to complete the proof of the proposition it remains to get items (1) for $j = i$ and (3) of Property $\mathfrak{V}''$.

Step V: Generation of a robust heterodimensional cycle. Recall that the homoclinic class $H(Q_{g_4}, g_4)$ has no any dominated splitting. There are three possibilities for the s-index i of P. If $i \in \{2, \dots, d-2\}$ we can apply Corollary 2 to get g_5 close to g_4 with a robust heterodimensional cycle associated to a hyperbolic set L_{g_5} containing P_{g_5} and a hyperbolic set K_{g_5} of stable index $i+1$ or $i-1$.

Assume now that $i = d-1$. Recall that $\chi_{d-1}(Q_{g_4}) + \chi_d(Q_{g_4}) > 0$. Thus the hypotheses in Theorem 1 are satisfied by g_4 and we get a diffeomorphism g_5 having a robust heterodimensional cycle as before.

Finally, the case $i = 1$ is analogous to the case $i = d-1$. Hence we obtain item (3) in Property $\mathfrak{V}''$.

Step VI: And finally Property $\mathfrak{V}''$ holds. Note that since the sets L_{g_5} and K_{g_5} have a robust heterodimensional cycle, for all g close to g_5 they are contained in the same chain recurrence class. Thus by [8, Remarque 1.10] there is a residual subset $\mathcal{G}'$ of $\mathrm{Diff}^1(M)$ such that for every $f \in \mathcal{G}'$, every periodic point of f is hyperbolic and its homoclinic and chain recurrence classes coincide. In particular, for diffeomorphisms in $\mathcal{G}'$ the homoclinic classes of two periodic points either coincide or are disjoint.

Therefore for any $g_6 \in \mathcal{G}'$ close to g_5 there is a periodic point $R_{g_6} \in K_{g_6}$ such that the homoclinic classes $H(R_{g_6}, g_6)$ and $H(Q_{g_6}, g_6)$ coincide. Hence the homoclinic class $H(R_{g_6}, g_6)$ does not have any k_0-dominated splitting of index i.

By Proposition 5.2, there is a saddle Q_{i,g_6} homoclinically related to R_{g_6} such that there is an ε-perturbation of g along the orbit of Q_{i,g_6} that is adapted to $H(R_{g_6}, g_6)$ and to property $\mathfrak{P}_{i,i+1,\mathbb{C}}$. Since the perturbation is adapted, there is a transitive hyperbolic set K'_g containing $Q_{i,g}$ and K_g. Thus the diffeomorphism g has a robust heterodimensional cycle associated to L_g and K'_g. This ends the proof of the proposition. □

9.2. **Proof of Corollary 8.** Recall that the residual subset $\mathcal{G}'$ of $\mathrm{Diff}^1(M)$ introduced in Step VI consists of diffeomorphisms whose periodic points are all hyperbolic. In particular, these diffeomorphisms have at most countably many periodic points and hence countably many homoclinic classes which are either disjoint or coincide.

By Lemma 9.3, there exists a dense open subset $\mathcal{W} \subset \mathcal{V}$ such that $C(P_g, g)$ satisfies $\mathfrak{V}''$ for all $g \in \mathcal{U}$.

Recall that a filtrating neighborhood is an open set U such that $U = U_+ \cap U_-$ where U_+ and U_- are open sets such that $f(\overline{U_+}) \subset U_+$ and $f^{-1}(\overline{U_-}) \subset U_-$. Observe that there is filtrating neighborhood for the chain recurrence class of Q_g separating this class and the class of P_g. In particular, these two recurrence classes are disjoint. Thus Theorem 7 allows to repeat this process, generating new classes satisfying Property $\mathfrak{V}''$. Inductively, for each $n \in \mathbb{N}$ we get an open and dense subset $\mathcal{U}_n$ of $\mathcal{U}$ consisting of diffeomorphisms having (at least) n disjoint homoclinic classes. Therefore, the set

$$\mathcal{G}_{\mathcal{U}} = \mathcal{G}' \cap \bigcap_{n\in\mathbb{N}} \mathcal{U}_n$$

is a residual subset of $\mathcal{U}$ consisting of diffeomorphisms with infinitely (countably) many homoclinic classes. This implies the first part of the corollary.

To see that there are uncountably many chain recurrence classes note that the first step of the construction provides two disjoint filtrating open sets, the set $V_0 = U$ containing the chain recurrence class of $P_g = Q^0$ and the set V_1 containing the chain recurrence class of $Q_g = Q^1$.

Repeating this process n times, we can assume that for each map $g \in \mathcal{G}_{\mathcal{U}}$ at each step we get 2^n open filtrating sets $V_{i_1,\dots,i_n}$, $i_k = 0, 1$, that are pairwise disjoint and nested (i.e. $V_{i_1,\dots,i_n} \subset V_{i_1,\dots,i_{n-1}}$), and each set contains a chain recurrence class with property $\mathfrak{V}''$. Note that these classes are different and pairwise disjoint.

Arguing inductively, we can repeat the construction of the first step for every finite sequence $i_1, \dots, i_n$, getting a new pair of filtrating neighborhoods $V_{i_1,\dots,i_n,0}$ and $V_{i_1,\dots,i_n,1}$ contained in $V_{i_1,\dots,i_n}$ and each of them containing a chain recurrence class satisfying Property $\mathfrak{V}''$.

Finally, for each infinite sequence $\iota = (i_k)$ consider the set

$$K_\iota = \bigcap_{k=1}^{\infty} \overline{V_{i_1,\dots,i_k}}.$$

By construction, each set K_ι contains some recurrent point X_ι and given two different sequences ι and ι' the chain recurrence classes of X_ι and $X_{\iota'}$ are different. Thus for $g \in \mathcal{G}_U$ to each sequence ι we associate a chain recurrent class $C(X_\iota, g)$ and this map is injective.

We have shown that every $g \in \mathcal{G}_U$ has uncountably many chain recurrence classes. Since, by the definition of $\mathcal{G}'$, the diffeomorphism g has only countably many periodic points, there are uncountably many aperiodic classes. This completes the proof of the corollary. $\square$

9.3. **Examples.** We close this paper by providing examples of diffeomorphisms satisfying viral properties that do not exhibit universal dynamics.

Proposition 9.6. *Given any closed manifold M of dimension $d \geq 3$ there is a non-empty open set of diffeomorphisms having homoclinic classes satisfying Property $\mathfrak{V}$. Moreover, the open set can be chosen such that the Jacobians of the diffeomorphisms are strictly less than one over these homoclinic classes.*

The construction follows arguing exactly as in [11, Appendix 6]. Just note that in this case we do not assume the existence of a pair of points P' and Q' with Jacobians less and larger than one as in [11]. A different approach is to consider perturbations of systems having *heterodimensional tangencies* as in [21].

References

[1] F. Abdenur, *Generic robustness of spectral decompositions,* Ann. Sci. École Norm. Sup., **36** (2003), 213–224.

[2] F. Abdenur, Ch. Bonatti, and S. Crovisier, *Nonuniform hyperbolicity for C^1-generic diffeomorphisms,* arXiv:0809.3309v1 and to appear in Israel Jour. of Math..

[3] F. Abdenur, Ch. Bonatti, S. Crovisier, L. J. Díaz, and L. Wen, *Periodic points and homoclinic classes,* Ergod. Th. and Dynam. Syst., **27** (2007), 1–22.

[4] R. Abraham and S. Smale, *Nongenericity of Ω-stability,* Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968), 5–8 Amer. Math. Soc., Providence, R.I, (1970).

[5] M. Asaoka, *Hyperbolic sets exhibiting C^1-persistent homoclinic tangency for higher dimensions,* Proc. Amer. Math. Soc. **136** (2008), 677–686.

[6] Ch. Bonatti, *Towards a global view of dynamical systems, for the C^1-topology,* prépublication Institut de Mathématiques de Bourgogne (2010).

[7] J. Bochi and Ch. Bonatti, *Perturbation of the Lyapunov spectra of periodic orbits,* arXiv:1004.5029.

[8] Ch. Bonatti and S. Crovisier, *Récurrence et généricité,* Inventiones Math., **158** (2004), 33–104.

[9] Ch. Bonatti and L.J. Díaz, *Persistence of transitive diffeomorphims,* Ann. Math., **143** (1995), 367–396.

[10] Ch. Bonatti and L.J. Díaz, *Connexions hétéroclines et généricité d'une infinité de puits ou de sources,* Ann. Scient. Éc. Norm. Sup., **32**, 135-150, (1999).

[11] Ch. Bonatti and L.J. Díaz, *On maximal transitive sets of generic diffeomorphisms,* Publ. Math. Inst. Hautes Études Sci., **96** (2002), 171–197.

[12] Ch. Bonatti and L.J. Díaz, *Robust heterodimensional cycles and C^1-generic dynamics,* Journal of the Inst. of Math. Jussieu, **7** (2008), 469–525

[13] Ch. Bonatti and L.J. Díaz, *Abundance of C^1-robust homoclinic tangencies,* to appear in Trans. A. M. S and arXiv:0909.4062.

[14] Ch. Bonatti, L.J. Díaz, and S. Kiriki, *Robust heterodimensional cycles and hyperbolic continuations,* in preparation.

[15] Ch. Bonatti, L.J. Díaz, and E.R. Pujals, *A $\mathcal{C}^1$-generic dichotomy for diffeomorphisms: Weak forms of hyperbolicity or infinitely many sinks or sources,* Ann. of Math., **158** (2003), 355–418.

[16] Ch. Bonatti, L.J. Díaz, and M. Viana, *Dynamics beyond uniform hyperbolicity*, Encyclopaedia of Mathematical Sciences (Mathematical Physics), **102**, Springer Verlag, (2004).

[17] Ch. Bonatti, N. Gourmelon, and T. Vivier, *Perturbations of the derivative along periodic orbits,* Ergodic Th. and Dynam. Syst., **26** (2006), 1307–1337.

[18] E. Colli, *Infinitely many coexisting strange attractors,* Ann. Inst. H. Poincaré Anal. Non Linéaire, **15** (1998), 539–579.

[19] S. Crovisier, *Birth of homoclinic intersections: a model for the central dynamics of partially hyperbolic systems,* arXiv:math/0605387 and to appear in Ann. Math..

[20] S. Crovisier and E. R. Pujals, *Essential hyperbolicity and homoclinic bifurcations: a dichotomy phenomenon/mechanism for diffeomorphisms*, in preparation.

[21] L. J. Díaz, A. Nogueria, and E. R. Pujals, *Heterodimensional tangencies,* Nonlinearity, **19** (2006), 2543–2566.

[22] J. Franks, *Necessary conditions for stability of diffeomorphisms,* Trans. Amer. Math. Soc., **158** (1971), 301–308.

[23] S. Gan and L. Wen, *Heteroclinic cycles and homoclinic closures for generic diffeomorphisms,* J. Dynam. Differential Equations, **15** (2003), 451–471.

[24] N. Gourmelon, *Generation of homoclinic tangencies by C^1-perturbations,* Discrete Contin. Dyn. Syst. **26** (2010), 1–42.

[25] N. Gourmelon, *A Franks' lemma that preserves invariant manifolds,* arXiv:0912.1121v2.
[26] R. Mañé, *Contributions to the stability conjecture,* Topology, **17** (1978), 383–396.
[27] C. G. Moreira, *There are no C^1-stable intersections of regular Cantor sets,* Pre-print IMPA 2008, `http://www.preprint.impa.br/cgi-bin/MMMsearch.cgi`
[28] S. Newhouse, *Diffeomorphisms with infinitely many sinks,* Topology, **13** (1974), 9–18.
[29] S. Newhouse, *The abundance of wild hyperbolic sets and nonsmooth stable sets for diffeomorphisms,* Publ. Math. I. H.E.S, **50** (1979), 101–151.
[30] S. Newhouse, *New phenomena associated with homoclinic tangencies,* Ergodic Theory Dynam. Systems, **24** (2004), 1725–1738.
[31] M. J. Pacifico, E. R. Pujals, and J. L. Vietez, *Robustly expansive homoclinic classes,* Ergod. Th. and Dynam. Syst., **25** (2005), 271–300.
[32] J. Palis, *A global view of dynamics and a conjecture on the denseness of finitude of attractors,* Astérisque, **261** (2000), 335–347.
[33] J. Palis and F. Takens, *Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations. Fractal dimensions and infinitely many attractors,* Cambridge Studies in Advanced Mathematics, **35**, Cambridge University Press, Cambridge, (1993).
[34] J. Palis and M. Viana, *High dimension diffeomorphisms displaying infinitely many periodic attractors,* Ann. of Math., **140** (1994), 207–250.
[35] E. R. Pujals and M. Sambarino, *Homoclinic tangencies and hyperbolicity for surface diffeomorphisms,* Ann. of Math., **151** (2000), 961–1023.
[36] N. Romero, *Persistence of homoclinic tangencies in higher dimensions,* Ergodic Theory Dynam. Systems, **15** (1995), 735–757.
[37] K. Shinohara, *On the indices of periodic points in C^1-generic wild homoclinic classes in dimension three,* arXiv:1006.5571.
[38] C.P. Simon, *Instability in Diff(T^3) and the nongenericity of rational zeta function,* Trans. A.M.S., **174** (1972), 217–242.
[39] L. Wen, *Generic diffeomorphisms away from homoclinic tangencies and heterodimensional cycles,* Bull. Braz. Math. Soc. (N.S.), **35** (2004), 419–452.

Institut de Mathématiques de Bourgogne, BP 47 870, 1078 Dijon Cedex, France

E-mail address: `bonatti@u-bourgogne.fr`

Institut Galilée, Université Paris 13, Avenue J.-B. Clément, 93430 Villetaneuse, France

E-mail address: `crovisie@math.univ-paris13.fr`

Depto. Matemática, PUC-Rio, Marquês de S. Vicente 225, 22453-900 Rio de Janeiro RJ, Brazil

E-mail address: `lodiaz@mat.puc-rio.br`

Institut de Mathématiques de Bordeaux, Université de Bordeaux, 351, cours de la Libération, F33405 Talance Cedex France

E-mail address: `nicholas.gourmelon@math.u-bordeaux1.fr`