Affine-invariant curvature estimators for implicit surfaces

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1. Affine curvature 2. Implicit surface
3. Isosurface 4. Affine invariants
5. Gaussian curvature 6. Mean curvature

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Geometria e Visão Computacionais

Estimativas de curvatura

Outras Informações Relevantes:

Breve Descrição do Projeto (em caso de projeto novo):

Abstract. Affine Differential Geometry provides a set of measures invariant under a larger set of transformations compared to rigid motions. This leads to several applications using robust shape descriptors. Although affine invariant operations are already used for surfaces, they do not intend to approximate the definitions of Affine Differential Geometry, which are the basis for further differential invariants. In this work we propose estimators for the local affine structure of an implicit surface, i.e. the affine metric, the co-normal and normal vectors, and the affine Gaussian and mean curvatures. The direct derivation of the formulae from the implicit function theorem lead to very intensive computations and numerical instabilities. This work further proposes a geometrical reduction allowing a much simpler and more stable formulae, and compares the results by incorporating the proposed estimators in Marching Cubes based algorithms.

Keywords: Affine curvature. Implicit surface. Isosurface. Affine invariants. Gaussian curvature. Mean curvature.

Figure 1: Three dimensional implicit model of a banana through affine transformations $p \mapsto A \cdot p$, with the affine Gaussian (left) and mean (right) curvatures, colored from dark to bright. Although those shapes are clearly similar, Euclidean curvatures does not match, but affine ones do.
1 Introduction

Implicit representation of geometric models are widely used, in particular for their design versatility, such as the efficient representation for deformation, Boolean operations and offsets [7]. The computation of the topological and geometrical structure from such representations may be delicate, although it is well known for parametric representations [5]. In particular, the curvature formulae for implicit surfaces were not gathered until recently [12], and their topological structure is tricky to compute [20]. In this paper we provide formulae with geometrical reductions to compute the affine structure of implicit surfaces and numerical tools to incorporate them into Marching Cubes based algorithms [15].

Affine invariant measures have received a lot of attention recently in the computer vision community for improving matching and registration [17] [18] [23] [26]. Indeed, curves on an object seen on two distinct photos have different Euclidean measures, i.e. distances or curvatures [16]. However, if the curve is contained in a plane parallel to the projection plane, the affine measures would correspond across photos. For three-dimensional data, the quantification of shape similarities have already a large number of applications in matching and recognition [1] [10] [23] and reconstruction [11]. Although some objects are clearly similar they do not match locally under rigid motion (see Figure 1). A better similarity measure for such objects may be obtained using a larger set of transformations, requiring more sophisticated measures. From this point of view, the simplest geometry beyond the Euclidean one is the affine geometry, which defines measures that are invariant under affine transformation, i.e. including non-uniform scaling and shearing. This paper introduces estimators of such measures for surfaces represented implicitly.

Related work Planar affine geometry is commonly used in image processing and computer vision, from affine curve motion [25, 9] to affine invariant descriptors [17]. Those works are motivated by the geometrical description of viewpoint changes in photographic projections, which are well approximated by affine transformations [16]. In particular, Mikolajczyk et al. [17] provide a survey of local shape descriptor, with emphasis on affine invariant ones [17] [18]. For planar curves described explicitly, Craizer et al. [6] propose a local affine-invariant construction of parabolic polygons, from which they deduce the affine length, normal and curvature of the discrete curve. Zuliane et al. [26] propose an global affine-invariant descriptor based on Helmholtz’s equation for vibrating membrane with a matching application. In this work we propose local affine normals and curvatures estimators for implicit surfaces.

For three dimensional shapes, affine invariant descriptors have been studied since 1990’s. Already Arbet et al. [1] derive global affine invariant descriptors for parametric models through normalization of the Fourier transform. Mortara and Patané [19] derive affine invariant skeleton for 3d shapes using partial barycenters to embed the Reeb graph of the shape. Betelu et al. [2] define 3d affine invariant distances to also derive a skeleton by erosion. Rothganger et al. [24] propose to use affine-invariant neighborhoods in 3d shapes to derive affine descriptors from local parameterization. In this work, we propose to approximate affine curvatures, which are the basis for further local differential affine invariants. Such estimators are derived in an implicit setting, following the approach from Goldman [12] and references therein.

Contributions In this paper we describe the affine structure of a surface, i.e. affine co-normal, normal vectors, and affine curvatures, for the case where $S$ is a graph, and deriving the formulae for the implicit case (see section 2). The direct computation of those leads to more than 23,000 flops per point of estimation, which leads to large numerical instabilities. Therefore, we propose a mapping from the local geometry via an affine transformation to get a simpler case with gradient vector $\nabla f = [0, 0, 1]$ and principal curvatures aligned with the global frame. In this case, simplified formulae can be applied, reducing to approximately 1,500 flops per point (see section 3). We describe how to incorporate such estimators in Marching Cubes [15] based algorithm (see section 4) and perform some numerical experiments with our estimators (see section 5).

Throughout this work, we study invariant structures under the equiaffine group of transformations, i.e. transformations of the form $p \mapsto Ap + t$ where $A$ is a $3 \times 3$ matrix of determinant 1 and $t$ is a vector in $\mathbb{R}^3$. We use the term affine instead of equiaffine to ease the reading. We use the notation $f_x$ for the partial derivative of $f$ along the $x$ direction. Finally, in the theoretical development we will consider only $C^4$ functions at regular points.

2 Affine structures and their formulae in the graph and implicit cases

In this section we will give some definitions of the main affine structures: the co-normal and normal vectors, and the Gaussian and mean curvatures. We will give the formulae for surfaces described as a graph, and deduce those for the implicit case using the implicit function theorem, as Goldman did for the Euclidean case. A more complete exposition of affine structures can be found in the books of Buchin [3] or Nomizu [21].

(a) Derivatives of an implicit surface

We consider here the affine structure of a surface described implicitly $S = \{ (x, y, z) \in \mathbb{R}^3 / f(x, y, z) = 0 \}$, where $f$ is at least of class $C^4$ and where $0$ is a regular value of $f$. Given a point $p \in S$ that is regular, i.e. $\nabla f(p) = [f_x, f_y, f_z] \neq 0$, we assume, without loss of generality that $f_z(p) \neq 0$. The implicit function theorem ensures the existence of a function $g : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ such that the equation $z = g(x, y)$ describes the surface $S$ in a neighborhood of $p$. Thus $S$ can be parameterized around $p$ as a graph $G = \{ (x, y, g(x, y)) \}, (x, y) \in U \}$ on an open set $U \subset \mathbb{R}^2$. Although $g$ is hard
to find in general, the derivates \( g \) at image point \( p \) are easily expressed from the derivatives of \( f \):

\[
\text{at } p = [x, y, g(x, y)] : \quad g_x(x, y) = -\frac{f_x(p)}{f_z(p)} \quad \text{and} \quad g_y(x, y) = -\frac{f_y(p)}{f_z(p)}. \tag{1}
\]

Further derivatives of \( g \) can be obtained from those expressions using the chain rule from \( p = [x, y, g(x, y)] \) \cite{12}.

Note that the formulas derived from the theorem may lead to numerical instabilities when \( |f_z(p)| \) has a small value.

(b) Tangent plane and affine metric

The tangent vectors, combinations of \( G_x = [1, 0, g_x] \) and \( G_y = [0, 1, g_y] \), are naturally contravariant under any linear transformation \( A \), in particular affine ones: \( G'(x,y) (Ap) = AG(x,y) (p) \) (a direct consequence of the chain rule).

The usual affine metric on this tangent plane is the Berwald-Blaschke metric, which corresponds to the elementary volume of a curve frame on the surface. It can be expressed by the bilinear form \( d^{-1/4} [L \ M \ N] \) with:

\[
L = \det G_x, \ G_y, \ G_{xx} = g_{xx}(x,y), \quad M = \det G_x, \ G_y, \ G_{xy} = g_{xy}(x,y), \quad N = \det G_x, \ G_y, \ G_{yy} = g_{yy}(x,y),
\]

\[
d = LN - M^2 = g_{xx}g_{yy} - g_{xy}^2.
\]

Throughout the theoretical developments, we suppose that the Berwald-Blaschke metric is non-degenerate: \( d \neq 0 \). Geometrically \( d > 0 \) means that the Euclidean Gaussian curvature does not vanish, i.e. the surface is strongly convex. The local affine surface area is then given by \( |d|^{1/4} \cdot dx \wedge dy \).

The implicit formulae for the metric can be derived using Equation \( \cite{1} \):

\[
g_{xx} = -\frac{f_x}{f_z} + 2\frac{f_xf_z}{f_z^2} - \frac{f_{xx}}{f_z^2}, \quad g_{xy} = -\frac{f_y}{f_z} + \frac{f_yf_z}{f_z^2} + \frac{f_{xy}}{f_z^2}, \quad g_{yy} = -\frac{f_y}{f_z} + 2\frac{f_yf_z}{f_z^2} - \frac{f_{yy}}{f_z^2}.
\]

In particular:

\[
d = \frac{1}{f_z} \left( (f_{yy}f_{zz} - f_{yz}^2) f_x^2 + 2(f_{xx}f_{xy} - f_{xx}f_{yz}) f_yf_z + (f_{yy}f_{zz} - f_{yz}^2) f_x^2 + 2(f_{xy}f_{yz} - f_{yy}f_{xx}) f_xf_y + (f_{xx}f_{yy} - f_{xy}^2) f_y^2 + 2(f_{xx}f_{xy} - f_{yy}f_{xx}) f_xf_y \right). \tag{2}
\]

(c) Affine co-normal

Orthonormality relationships are not preserved under an affine transformation \( A \), therefore the Euclidean normal \( N_e \) is not an affine covariant vector. However, the direction of the Euclidean normal is covariant (if \( \langle N_e, G'(x,y) \rangle = 0 \), then \( \langle A^{-T}N_e, A^{-1}G'(x,y) \rangle = 0 \)). Therefore, a covariant affine normal, called the affine co-normal \( \nu \), can thus be obtained by scaling the Euclidean normal vector \( \nu \) (see Figure \( \cite{2} \)).

\[
\nu = |K_e|^{-1/4} N_e = |g_{xx}g_{yy} - g_{xy}^2|^{-1/4} [-g_x, -g_y, 1], \tag{3}
\]

where \( K_e \) is the Euclidean Gaussian curvature: \( K_e = (1 + g_y^2 + g_z^2)^{-2}(g_{xx}g_{yy} - g_{xy}^2) \). This affine co-normal then satisfies \( \langle \nu, G'(x,y) \rangle = 0 \) and the affine metric satisfies \( d^{1/4} = \pm \det [\nu, \nu_x, \nu_y] \).

The implicit formulae for the co-normal can be derived from Equation \( \cite{3} \):

\[
\nu = \frac{1}{f_z} d^{1/4} [f_x, f_y, f_z].
\]

The implicit formulae for the affine structures described in the following subsections can be similarly derived. The next section will describe formulae for a generic case, which are more numerically stable.
The affine normal then satisfies: 

\[ \langle \nu, \xi \rangle = 1, \quad \langle \nu, \nu_{x,y} \rangle = 0. \]

The affine normal then satisfies: \( \langle \nu, \xi_{x,y} \rangle = 0 \) and \( \det | G_x, G_y, \xi | = \pm d^{3/4} \). This last relation shows that a local basis at a point \( p \) of the surface can be obtained by \( \{ G_x, G_y, \xi \} \). This allows to define affine structures from Cartan’s moving frames theory \( \{ 2 \} \).

Denote by \( \xi = [\xi_1, \xi_2, \xi_3] \) the affine normal vector, we know that \( \langle \xi, \nu \rangle = 1, \quad \langle \nu, \xi_x \rangle = \langle \nu, \xi_y \rangle = 0. \) Thus \( \xi = \lambda (\nu_x \times \nu_y) \), with \( \lambda = \det | \nu, \nu_x, \nu_y |^{-1} = d^{-3/4}. \)

We can deduce the explicit formulas for the coordinates of \( \xi \):

\[
\begin{align*}
\xi_1 &= \frac{1}{4} d^{-7/4} \left( g_{xx}g_{xy}g_{yxy} - g_{xx}g_{yy}g_{xyy} - g_{yy}^2 g_{xxx} - 2g_{xy}^2 g_{xyy} + 3g_{yy}g_{xy}g_{xyy} \right), \\
\xi_2 &= \frac{1}{4} d^{-7/4} \left( g_{yy}g_{xy}g_{xxx} - g_{xx}g_{yy}g_{xyy} - g_{xx}^2 g_{yxy} - 2g_{xy}^2 g_{xx}g_{xyy} + 3g_{xx}g_{xx}g_{yxy}g_{xyy} \right), \\
\xi_3 &= \frac{1}{4} d^{-7/4} \left( 4g_{xx}^2 + 4g_{xy}^2 g_{yy} - 8g_{xx}g_{yy}g_{xy}^2 + 3g_{xx}g_{yy}g_{xy}g_{xyy} + 3g_{yy}g_{xx}g_{xy}g_{xyy} - g_{xx}g_{yy}g_{xxx} - g_{yy}g_{xx}g_{xy}g_{yxy} - 2g_{xx}g_{xx}g_{xyy} - 2g_{yy}^2 g_{xx}g_{xyy} - 2g_{yy}g_{yy}g_{xy}g_{xyy} + g_{xx}g_{xx}g_{xx}g_{yxy}g_{xyy}g_{xy} + g_{yy}g_{yy}g_{xx}g_{xx}g_{xyy}g_{xy} \right). 
\end{align*}
\]

(e) **Affine curvatures**

As in Euclidean geometry, the curvature describes the variations of the normal (see Figure 3). Since \( \langle \nu, \xi_{x,y} \rangle = 0 \), the derivatives \( \xi_{x,y} \) of the affine normal are orthogonal to \( \nu \), which is parallel to \( \mathbf{N}_e \). Therefore, \( \xi_{x,y} \) are tangent to the surface:

\[
\begin{align*}
\xi_x &= -b_{11} G_x - b_{12} G_y, \\
\xi_y &= -b_{21} G_x - b_{22} G_y.
\end{align*}
\]

Consequently:

\[
\begin{align*}
b_{11} &= -d^{-1/4} \cdot \det | \xi_x, G_y, \xi |, \\
b_{12} &= -d^{-1/4} \cdot \det | G_x, \xi_x, \xi |, \\
b_{21} &= -d^{-1/4} \cdot \det | \xi_y, G_y, \xi |, \\
b_{22} &= -d^{-1/4} \cdot \det | G_x, \xi_y, \xi |.
\end{align*}
\]

The coefficients \( b_{ij} \) form a matrix \( B = [b_{ij}]_{1 \leq i,j \leq 2} \) whose determinant and trace are respectively the Gaussian and mean affine curvatures: \( K = \det B, \quad \mathcal{H} = \text{tr} B. \)

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Figure 3: Affine structure of a blobby surface \( x^2 + y^2 + z^2 - x^4 - y^4 - z^4 \). (from left to right) normal direction \( \nu \), Gaussian \( K \) and mean \( H \) curvatures, colored from red to blue, the central green part corresponding to a degenerate metric.

(f) Fundamental examples

The most basic shapes of Euclidean geometry are the plane, with constant normal and thus zero curvature, and the sphere, with constant curvature. In affine geometry, the equivalent shapes are the paraboloids, with constant affine normal (see Figure 4), and the ellipsoid and hyperboloids of one and two sheets, with constant curvatures, respectively. Their affine structures are reported on Table 1.

<table>
<thead>
<tr>
<th></th>
<th>Elliptic Paraboloid</th>
<th>Hyperbolic Paraboloid</th>
<th>Sphere</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f )</td>
<td>( z - \frac{x}{2} (x^2 + y^2) )</td>
<td>( z - \frac{x}{2} (x^2 - y^2) )</td>
<td>( x^2 + y^2 + z^2 - r^2 )</td>
</tr>
<tr>
<td>( N_e )</td>
<td>( \frac{1}{\sqrt{1+x^2+y^2}} \cdot [-x, -y, 1] )</td>
<td>( \frac{1}{\sqrt{1+x^2+y^2}} \cdot [-x, y, 1] )</td>
<td>( \frac{1}{r} \cdot [x, y, z] )</td>
</tr>
<tr>
<td>( K_e )</td>
<td>( \frac{1}{(1+x^2+y^2)^2} )</td>
<td>( -(1+x^2+y^2)^2 )</td>
<td>( r^2 )</td>
</tr>
<tr>
<td>( d )</td>
<td>1</td>
<td>-1</td>
<td>( r^2 z^{-4} )</td>
</tr>
<tr>
<td>( \nu )</td>
<td>([ -x, -y, 1 ] )</td>
<td>([ -x, y, 1 ] )</td>
<td>( r^{-1/2} [x, y, z] )</td>
</tr>
<tr>
<td>( \xi )</td>
<td>([ 0, 0, 1 ] )</td>
<td>([ 0, 0, 1 ] )</td>
<td>( r^{-3/2} [x, y, z] )</td>
</tr>
<tr>
<td>( K )</td>
<td>0</td>
<td>0</td>
<td>( r^{-3} )</td>
</tr>
<tr>
<td>( H )</td>
<td>0</td>
<td>0</td>
<td>( -2r^{-3/2} )</td>
</tr>
</tbody>
</table>

Table 1: Fundamental examples of affine structures.

Figure 4: Elliptic and hyperbolic paraboloids have a constant affine normal \( \xi \). Playing the role of the Euclidean plane in affine geometry.
Figure 5: The affine curvature of a paraboloid should be $K = 0$, but a direct estimation leads to numerical instabilities (left). With simplified formulae the estimation is more stable (right).

3 Geometric reductions and simplified formulae

The formulae for the affine structures derived in the previous section are already extensive for the case of a graph $G = \{(x, y, g(x, y))\}$, and increase in size dramatically when using the implicit function theorem to express those affine structures directly in term of the implicit function $f$. This leads to significant numerical instability during the computation (see Figure 5), and harms the affine invariance of the calculated quantities (see Figure 6).

However, we know how each affine quantity varies under an affine transformation: the metric and the affine curvatures are invariant, the co-normal is covariant and the normal is contravariant. Here we define an affine transformation $A$ that simplifies the above formulae and improve (or isolate) numerical instability. This section will first introduce this transformation, derive the formulae for the affine structure after the simplification, and finally show how to compute the affine structure for a general implicit surface using this simplification.

Figure 6: The non-invariant choice of an axis in eqrefimpl (left) leads to discontinuous errors in the estimation of $\nu$ and even further on the curvature with a geometric reduction on this Cassini oval. The mean affine curvature $H$ is better approximated (right).

(a) Simplifying transformation

Since all the implicit formulae are derived from the implicit function theorem (Equation (1)), most of the terms may be simplified if we can set the gradient of $f$ to a constant vector, e.g. $[0, 0, 1]$, after an affine transformation $A$. Moreover, aligning the Euclidean principal curvature directions with the $x$–axis and $y$–axis further reduces the size of our formulae.

More precisely, we look for an (equi-)affine transformation $A$ (neglecting the a translation part which has no effect on the derivatives) as the composition of a rotation $R_1$ and a scaling $S$ that maps the gradient to a vertical direction with unit norm, and a rotation $R_2$ in the $xy$–plane to align the Euclidean principal curvatures (see Figure 7). The effect of this transformation on the derivatives is described in the following proposition and the construction of $A$ is detailed in its proof.

Proposition 1 At each regular point $p$ of the implicit surface $\{p \in \mathbb{R}^3, f(p) = 0\}$, there exists an equiaffine transformation $A$ such that, at point $\tilde{p} = A(p)$ of the transformed implicit surface $\{\tilde{p} \in \mathbb{R}^3, \tilde{f}(\tilde{p}) = f(A^{-1}(\tilde{p})) = 0\}$:

– the gradient is the unit vertical vector: $\nabla \tilde{f}(\tilde{p}) = [0, 0, 1]$.
– the crossed derivative $\tilde{f}_{xy}$ vanishes: $\tilde{f}_{xy}(\tilde{p}) = 0$. 

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det restriction imposes the degrees of freedom by three: the angle or the scale factor, and an axis. The planar rotation

In terms of degrees of freedom, the transformation

Observation In terms of degrees of freedom, the transformation $A$ is a $3 \times 3$ matrix. From those 9 coefficients, the equiaffine restriction imposes $\det A = 1$, which reduces the degrees of freedom by one. The rotation $R_1$ and the scaling $S$ each reduce the degrees of freedom by three: the angle or the scale factor, and an axis. The planar rotation $R_2$ has one degree of freedom: its angle. Although there are still spare degrees of freedom for the coefficients, the second derivative has a quadratic dependency on the coefficients of $A$, and there is no guarantee that further simplification would be feasible without deciding the sign of the metric (or equivalently the signal of the Euclidean Gaussian curvature).

(b) Simplified formulae

In this subsection, we will consider an implicit surface $\{ p \in \mathbb{R}^3, f(p) = 0 \}$ around point $p$ such that $\nabla f(p) = [0, 0, 1]$ and $f_{xy}(p) = 0$. From the previous proposition, we ensure this condition for any regular point through an equiaffine map $A$. With this condition, we derive the simplified formulae for the implicit structures:
Tangent Since \( f_x = f_y = 0 \), we deduce from Equation (1):

\[
\mathcal{G}_x = [1, 0, 0] \quad \text{and} \quad \mathcal{G}_y = [0, 1, 0].
\]

Metric The metric reduces to a very simple expression: \( d = f_{xx} f_{yy} \).

Co-normal From the metric and gradient, we deduce: \( \nu = | f_{xx} f_{yy} |^{-1/4} [0, 0, 1] \).

Normal The affine normal reduces to:

\[
\xi = \frac{1}{4 | f_{xx} f_{yy} |^{1/4}} \begin{bmatrix}
  f_{yy}^2 f_{xxx} + f_{xx} f_{yy} f_{xyy} - 4 f_{xx} f_{yy}^2 f_{xx} \\
  f_{xx} f_{yy} f_{xyy} - 4 f_{xx}^2 f_{yy} f_{yy} + f_{xx} f_{yy}^2 \\
  f_{xx}^2 f_{yy}^2
\end{bmatrix}.
\]

Curvatures The curvature expressions also greatly simplify:

\[
b_{11} = \frac{1}{16 f_{xx} f_{yy}^3} \left( 8 f_{xx}^3 f_{yy}^2 f_{xx} - 8 f_{xx}^3 f_{yy} f_{zz} + 8 f_{xx}^2 f_{yy}^2 f_{zz} - 4 f_{xx}^2 f_{yy} f_{xyy} f_{xx} - 4 f_{xx} f_{yy}^2 f_{xxx} + 7 f_{yy}^3 f_{xxx}^2 + 3 f_{xx}^2 f_{zz}^2 + 12 f_{xx} f_{yy} f_{zz}^2 f_{xxx} + 2 f_{xx} f_{yy} f_{zz} f_{xyy} + 4 f_{xx}^2 f_{yy} f_{zz} f_{yy} + 24 f_{xx}^2 f_{yy} f_{zz} f_{xxx} - 24 f_{xx} f_{yy} f_{zz} f_{xx} - 24 f_{xx} f_{yy} f_{zz} f_{xyy} \\
-4 f_{xx} f_{yy} f_{xxx} + 24 f_{xx}^2 f_{yy} f_{xyy} + 24 f_{xx}^2 f_{yy} f_{yy} f_{xxy} + 7 f_{xx} f_{yy} f_{xxx} f_{xyy} + 7 f_{xx} f_{yy} f_{xx} f_{yyy} \right),
\]

The formulae for \( b_{12} \) and \( b_{22} \) are obtained by exchanging \( x \) and \( y \) in \( b_{21} \) and \( b_{11} \).

(c) From the geometric reduction to the general case

The previous reduction is the key to increase the numerical stability and improve the invariance of the estimated affine structures. Starting from the original implicit function \( f \) at a regular point \( p \), we compute the transformation \( A \), which is defined from the first and second derivatives of \( f \) at \( p \) following Proposition (1) leading to a new implicit function \( \tilde{f}(\tilde{p}) = f(A^{-1}(\tilde{p})) \) with \( \tilde{p} = A(p) \).

We first compute the derivatives of \( \tilde{f} \) from the derivatives of \( f \) using the chain rule:

\[
\nabla \tilde{f}(\tilde{p}) = \nabla f(p) \cdot A^{-1}
\]

\[
H_{\tilde{f}}(\tilde{p}) = A^{-T} \cdot H_f(p) \cdot A^{-1}
\]

\[
\forall (i, j, k) \in \{x, y, z\}^3, \quad \tilde{f}_{ijk}(\tilde{p}) = \sum_{(a, b, c) \in \{x, y, z\}^3} f_{abc}(p) A_{a,i}^{-1} A_{b,j}^{-1} A_{c,k}^{-1}
\]

(4)

where \( A_{a,i}^{-1} \) is the coefficient of the inverse matrix \( A^{-1} \) of row \( a \) and column \( i \).

From those derivatives, we use the simplified formula to compute the affine structure \( \tilde{\nu}(\tilde{p}), \tilde{\xi}(\tilde{p}), \tilde{K}(\tilde{p}) \) and \( \tilde{H}(\tilde{p}) \) of the implicit surface defined by \( \tilde{f} \).

From the invariance of those structures, we deduce the affine structure of the original implicit surface \( \{ p \in \mathbb{R}^3, f(p) = 0 \} \) at point \( p \):

\[
\tilde{\nu}(\tilde{p}) = \nu(p) \cdot A^{-1},
\]

\[
\tilde{\xi}(\tilde{p}) = \xi(p) \cdot A^T,
\]

\[
\tilde{K}(\tilde{p}) = K(p),
\]

\[
\tilde{H}(\tilde{p}) = H(p).
\]
Table 2: Number of floating point operations returned by Maple for a single point using the simplified formulae with the derivative mapping versus non-simplified formulae: the simplified formulae are more concise and comprise of mostly operations to map the derivatives (Equation (4)).

<table>
<thead>
<tr>
<th># Operations</th>
<th>Matrices $A, A^{-1}$</th>
<th>Mapping of $\partial(f)$</th>
<th>Simplified formulae</th>
<th>Total</th>
<th>Non-Simplified formulae</th>
</tr>
</thead>
<tbody>
<tr>
<td>749</td>
<td>7,335</td>
<td>1,783</td>
<td>9,867</td>
<td>23,690</td>
<td></td>
</tr>
</tbody>
</table>

(d) On numerical stability

The simplified formulae are significantly more stable than direct computation without the transformation, as seen in our experiments (see section 5). We use the Maple software to optimize both the direct and simplified formulae, aiming at reducing the number of operations. The comparison of the number of operations in the direct and simplified formulae clearly explains the stability gained by the simplification (see Table 2).

However the main tool for the derivation of the implicit formulae comes from the implicit function theorem (Equation (1)), where all the derivatives of $g$ are obtained through a division by $f_z$. Therefore any numerical implementation may suffer when the gradient is almost vanishing $\|\nabla f\| < \epsilon$. In the simplified formulae, this instability only appears in transformation $A$ (in particular in the non-uniform scaling $S$).

Moreover, the Berwald-Blaschke metric degenerates ($d = 0$) when the Euclidean Gaussian curvature is close to zero. In particular at inflection points, the affine curvatures should be infinite, even though they are regular. Such divergence is delicate to handle in a numerical context. A separate handling of such inflection points has been proposed for curves through a careful local resampling [6] and could be extended to surfaces in future work. This instability remains in the simplified formulae.

4 Affine structure computation for isosurfaces

From the previous sections, we compute the affine structure of an implicit surface \( \{ p \in \mathbb{R}^3, f(p) = 0 \} \) at one of its regular point \( p \) from the derivatives of \( f \) up to the fourth order. In this section we discuss the main tools we use to apply such calculi in a simple implicit surface extractor on a regular grid: how to approximate derivatives, how to incorporate the affine structure computation into Marching Cubes [15], and how to measure the quality of the results.

(a) Discrete derivatives approximation

The formulae for the affine curvature involve all the partial derivatives of \( f \) up to the fourth order. Those 34 derivatives may be very sensitive to numerical noise on \( f \), specially those of high order. When \( f \) is sampled on a regular grid, which is the common case for describing geometric object implicitly, a common choice to obtain such derivatives relies on discrete convolutions with splines [22]. We use in our experiments a trivariate spline function $\sigma_1(x, y, z)$ of degree 5 defined by $\sigma_1(x, y, z) = \sigma_1(x) \sigma_1(y) \sigma_1(z)$ (see Figure 8), where:

$$\sigma_1(t) = \frac{1}{60} \begin{cases} (3-t)^5 - 6(2-t)^5 + 15(1-t)^5, & 0 \leq |t| < 1 \\ (3-t)^5 - 6(2-t)^5, & 1 \leq |t| < 2 \\ (3-t)^5, & 2 \leq |t| < 3 \\ 0, & |t| \geq 3. \end{cases}$$

The derivatives are obtained by convolving \( f \) with the normalized derivative of the spline: $f \approx f * \sigma \Rightarrow \partial^\alpha f \approx f * (\frac{1}{\alpha!} \partial^\alpha \sigma)$. The constant $\epsilon$ is determined for each derivation in order to compensate for the scaling from the voxel spacing to the $[-3, 3]$ domain of $\sigma_1$ and to guarantee that derivatives of monomials of degree $\alpha$ are correctly estimated [8].
Figure 9: Incorporating the estimators inside Marching Cubes reveals the non-invariant pattern of the grid-based derivative estimation: (left) $K$ and (right) $H$ view, before (top) with a close up of the scale, and after (bottom) the affine transformation $[[0.9, 0, 0.9], [0, 2, 0], [1.1, 0, 0.6]]$.

(b) Implementation inside Marching Cubes

Marching Cubes \[15\] is the base algorithm for extracting implicit surfaces. It operates on each voxel of a regular grid, and eventually generate some triangles inside the voxels. The vertices of those triangles are computed by linear interpolation along the edges of the voxel, generating either 0 or 1 vertex per edge, depending if the values of the implicit function at the edge’s ends have equal or different signs, respectively.

Since we can directly evaluate the derivatives through discrete convolution only at the vertices of the voxel, we can either compute the affine structure $\nu$, $\xi$, $K$ and $H$ at the voxel corners (off the surface) and interpolate the structure (linearly) along the edge, or interpolate the derivatives along the edge and compute the affine structure at the Marching Cubes’ vertex from those interpolated derivatives. The former option has the disadvantage of computing the affine structure off-surface, and without restriction on $f$, this structure may differ from the on-surface one.

Since the Marching Cubes mechanics uses linear interpolation (to avoid having more than one vertex per voxel edge), the latter option is not completely consistent: interpolating a fourth-order derivative linearly means interpolating the function itself as a polynomial of degree 4! This is visible on the sphere of Figure 9, where the very small noise variations are correlated to the grid structure. However, this last option of interpolating derivatives at the Marching Cubes’ vertices and then computing the affine structure leads to better results in practice, as seen in the next section.

(c) Quality measures

The quality of a geometric estimator is generally measured along four criteria: invariance under geometric mapping, error compared to the exact geometric measure, convergence to the continuous measure when the sampling gets denser, and robustness to noise.

In the computation of affine structure for isosurfaces, the most delicate one is the invariance criterion since the sampling process of the implicit function $f$ on a regular grid is not invariant under affine mapping. Moreover, when checking the invariance by comparing the affine estimators on an isosurface before and after an affine transformation, the vertices generated by the Marching Cubes algorithm are not at the corresponding positions and not uniformly distributed. Although we try to reduce this disparity in the experiments with analytic implicit function by adapting the transformed domain to a bounding box of the image of the original domain, a global invariance measure is still hard to implement. We generate some results of the next section by comparing histograms of the curvatures before and after some affine transformations, similar histograms indicating better invariance of the estimator.
Affine-invariant curvature estimators for implicit surfaces

5 Experiments

We experiment the two affine estimators introduced above: applying the direct formulae referred as the direct method and computing the local mapping $A$ to use the simplified formulae referred as the transformation method. In the direct method, one of the axes has to be chosen at each point to serve as $z$ direction in the implicit function theorem. In order to reduce the numerical instability, we choose the axis with which the gradient is most aligned, i.e. we choose $x$ if $|f_x| > |f_y|$ and $|f_y| > |f_z|$ (see Figure 10). The transformation method follows the steps of section 3(a). We embedded those estimators in the Marching Cubes algorithm with the choices described in section 4 and use the open source of the Topological Marching Cubes [13].

Three groups of data were analyzed. First we generate analytic implicit functions (see Figures 4, 5, 10, 9 and 2) where we compute the exact differential affine structure $\nu$, $\xi$, $K$ and $H$, in order to compute the error of our estimators. Since the differential computation is extensive, we include only quadric surfaces (see Table 1). The second set of data groups other analytic implicit functions $f(p)$ (see Figures 3, 11 and 6), where we can sample exactly the affine image of the surface through $f(A(p))$ to check for the invariance of the estimator, with the restriction discussed in section 4(c). The last kind of data are isosurfaces, read from a static regular grid, generated as distance functions to a triangulated surface (see Figures 1 and 14). Such isosurfaces are generally more complex and would be a first step to a more specific application of the affine estimators.

Figure 11: Sine surface of equation $2z^2 - \sin(5x + 3y^2 - 1)$, comparing the mean curvature estimations $H$ using the direct method (left) and the transformation method (right), with the same color scales: both methods show discontinuity at the degenerated $K_e = 0$ regions, but the direct method artificially includes more discontinuities.

Numerical stability Since the definition of the affine curvature relies on a 4th-order derivative, any estimator will be very sensitive to numerical error. The geometric reduction we propose allows to reduce the numerical error linked to the gradient alignment with the axis. Figures 10 and 6 clearly show that the quality of the direct estimator decreases when the direction moves away from the axes. This is partly due to the discontinuity of the axis choice, similarly to the simpler case of Euclidean curvature of parametric curve [14], but is corrected by our reduction.

Moreover, this reduction greatly simplifies the formulae (see Table 2), which drastically improves the numerical stability. This is illustrated in Figure 5, where the paraboloid should have 0 curvature but the direct method introduces a noise of the order of $10^{-5}$ in this simple case. The reduction also allows to clearly identify degenerated regions where the metric $d$ is close to 0, as in Figure 11.
Estimation error and convergence  On models where we could derive the analytic expressions of the affine structure, essentially those of Table 1, we can compute the error at each point generated by the Marching Cubes algorithm using different grid sizes. The graphics obtained are all similar to Figure 12 which is the sphere case. In the direct method the residual error and its variance do not decrease with finer grid size, while the transformation method shows converging behavior even when applying an affine transformation, i.e. looking at $f(A^{-1}p)$, while maintaining a very low error variance.

Affine Invariance  As discussed in section 4(c) the measure of the affine invariance is delicate when experimenting with isosurfaces. We first generate the distribution histograms of the Gaussian affine curvature $K$ before and after an affine mapping. We compare the results obtained by the direct and transformation method on a torus model (see Figure 13). Once again the transformation method better preserves the geometric meaning of the estimated measures. On more complex isosurfaces, as the one depicted on Figures 1 and 14, the curvatures and the degenerate regions are still clearly mapped from the models of the same affine class.

Limitations  As mentioned in section 4(b) the sampling process is not affine invariant (see Figure 9). Although this introduces some error in the estimation process, it does not lead to significant loss on well sampled models as the ones of Figure 14. Moreover, as mentioned in section 5(d) the affine quantities are not well defined if the gradient $\nabla f$ of $f$ is close to 0 or if the Euclidean Gaussian curvature $K_e$ vanishes. While the second criterion correctly defines affine-invariant regions (see Figures 3 and 14), the first one is an intrinsic numerical problem. Finally the quality of the derivative approximations is crucial, which may harm the applicability of such estimators on noisy data.

Figure 12: Convergence on the sphere model, absolute error versus grid size, before (solid) and after (dashed) the affine transformation of Figure 9, comparing the direct method (left, in linear scale) with the transformation method (right, in log scale). The error bar represents one-fifth of the absolute error variance.

Figure 13: On the torus $z^2 + (\sqrt{x^2 + y^2} - 0.5)^2$, the distribution of the affine Gaussian curvature $K$ is better preserved under the affine transformation $[1.4, -0.2, 0], [0.1, 0.7, 0], [0, 0, 1]$ if using the transformation method (right) compared to the direct one (left).
6 Conclusion

This work proposes estimators for the local affine structure of implicit surfaces, i.e. the affine metric, co-normal and normal vectors, and affine Gaussian and mean curvatures. We introduce a geometric reduction in order to simplify the numerically intensive formulae, improving significantly the invariance of the estimators and reducing the error compared to the differential measure. We plan to extend this work by completing the affine estimators with the Pick invariant and eventually the affine signature, allowing for a complete set of estimators for differential affine invariants.

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