The geometry of finite difference discretizations of semilinear elliptic operators

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Abstract. Discretizations by finite differences of some semilinear elliptic equations lead to maps $F(u) = Au - f(u), u \in \mathbb{R}^n$, given by nonlinear convex diagonal perturbations of symmetric matrices A. For natural nonlinearity classes, we consider the equation $F(u) = y - tp$, where t is a large positive number, and p is a vector with negative coordinates. As the range of the derivative f_i' of the coordinates of f encloses more eigenvalues of A, the number of solutions increases geometrically, eventually reaching 2^n. This phenomenon, somewhat in contrast with behavior associated to the Lazer-McKenna conjecture, has a very simple geometric explanation: a perturbation of a multiple fold gives rise a function which sends connected components of its critical set to hypersurfaces with large rotation numbers with respect to vectors with very negative coordinates.

Strictly speaking, the results have nothing to do with elliptic equations: they are properties of the interaction of a (self-adjoint) linear map with increasingly stronger nonlinear convex diagonal interactions.

Keywords: Stieltjes matrices, semilinear elliptic operators, Lazer-McKenna conjecture, discretizations.

AMS classification scheme numbers: 34G20, 39A12, 47Hxx, 65M22.

1. Statement of main results

In [3], Costa, Figueiredo and Srikanth proved the so called one-dimensional Lazer-McKenna conjecture ([8], [9]), completing a series of interesting previous results ([10], [12], [13]). Briefly, let $H^2_D([0,\pi])$ be the Sobolev space of functions satisfying Dirichlet boundary conditions with square integrable second derivatives. Recall that $u \mapsto -u''$ acting on $H^2_D([0,\pi])$ has eigenvalues k^2, $k = 1, 2, \ldots$ with corresponding eigenfunctions $\sin(kx)$. A strictly convex smooth function f is *asymptotically linear* with parameters a and b if the range of its derivative f' is the interval (a, b).

Theorem 1.1 *[3] Let f be asymptotically linear with parameters a and b satisfying*

$$a < 1, \quad k^2 < b < (k+1)^2.$$

Then the equation $-u'' - f(u) = -t \sin x$, $u(0) = u(\pi) = 0$, has exactly $2k$ solutions for $t > 0$ sufficiently large.

In higher dimensions, an active area of research is the related equation, $\mathcal{F}(u) = -\Delta_D u - f(u) = -t\phi_1$ for u satisfying Dirichlet conditions on a bounded set $\Omega \subset \mathbb{R}^n$ with smooth boundary, where $\phi_1 \geq 0$ is the normalized eigenfunction associated to the smallest eigenvalue λ_1 of $-\Delta_D$. Among the basic results, Clark [2] proved that, for odd nonlinearities which essentially interact only with the first k smallest eigenvalues of $-\Delta_D$, the equation has at least $2k$ solutions for large values of $t > 0$. Dancer ([5] and references therein) showed that, for arbitrary k, there are nonlinearities interacting with the first k eigenvalues for which the equation, instead, has at most 4 solutions for large $t > 0$. For $f(x) = x^p$, Dancer and Yan [6] showed that the number of solutions goes to infinity as the asymptotic parameter b goes to ∞ ([6] contains also a good description of the history of the problem).

In this paper we consider finite difference discretizations of the differential operator $\mathcal{F}(u)$, which arise naturally in numerical analysis. More precisely, we study *admissible maps*

$$\begin{array}{rccc} F: & \mathbb{R}^n & \to & \mathbb{R}^n \\ & u & \mapsto & Au - f(u) \end{array}$$

where A is a real $n \times n$ symmetric matrix and the nonlinearity f

$$f(u) = (f_1(u_1), f_2(u_2), \ldots, f_n(u_n))^T,$$

has coordinate functions f_i with properties to be specified later. The index i appears from the discretization of a nonlinearity of the form $f(x, u)$. Strictly speaking, our results have nothing to do with elliptic equations: they are properties of the interaction of a (self-adjoint) linear map with increasingly stronger nonlinear convex diagonal interactions.

An *orthant* of $\mathbb{R}^n$ is one of the 2^n (open) connected components consisting of vectors with nonzero coordinates. We denote by $u > 0$ a vector u in the positive orthant. We are interested in *Property (Sol)*:

$$\begin{array}{c}\text{For any } y,\, p \in \mathbb{R}^n,\, p > 0, \text{ there exists } t_0 \text{ so that, for } t > t_0, \\ F(u) = y - tp \ \text{ has exactly one solution in each orthant.}\end{array} \tag{Sol}$$

As we shall see, in the finite dimensional case, property (Sol) is a consequence of some robust topological facts. We begin with a simple example. For an *$s\ell$-admissible map* $F(u) = Au - f(u)$, each coordinate $f_i : \mathbb{R} \to \mathbb{R}$ is *superlinear*: it is smooth, strictly convex and the range of each f_i' is the whole line $\mathbb{R}$.

Theorem 1.2 *An $s\ell$-admissible map $F : \mathbb{R}^n \to \mathbb{R}^n$ satisfies property (Sol).*

Indeed, for large t, the nonlinear term prevails and its coordinates roughly look like $x \mapsto x^2$. But the same holds for milder hypotheses. For an *$a\ell$-admissible map* $F(u) = Au - f(u)$, each coordinate $f_i : \mathbb{R} \to \mathbb{R}$ is asymptotically linear with common asymptotic parameters a and b. Recall that a symmetric positive definite matrix A is a *Stieltjes matrix* if its off-diagonal entries are non-positive. A symmetric matrix M is *reducible* if there is a permutation matrix P such that PMP^{-1} splits in two diagonal blocks. Let λ_1 be the smallest eigenvalue of A.

Theorem 1.3 *Let A be a symmetric matrix and a and b satisfy one of the two hypotheses below:*

(i) $a < 0$, $b > 0$, where $|a|$ and b are sufficiently large;

(ii) A is an irreducible Stieltjes matrix, $a < \lambda_1$ and $b > 0$ is sufficiently large.

Then any $a\ell$-admissible map $F(u) = Au - f(u)$ with asymptotic parameters a and b satisfies property (Sol).

The estimates for a and b in the statement above only depend on A.

We also consider *$p\ell$-admissible maps* $F_{p\ell}(u) = Au - f_{p\ell}(u)$, where $f_{p\ell}$ is *piecewise linear*: all coordinates $f_{p\ell,i}$ are equal, given by ax for $x < 0$ and bx for $x > 0$. As is well known ([3], [4], [14]), results about $F_{p\ell}(u) = -p$ frequently convert into statements about solutions of $F_{a\ell}(u) = y - tp$ for large $t > 0$.

Numerical experiments for the special choice $p = \phi_1 > 0$, the eigenvector related to the smallest eigenvalue λ_1 of A, display an interesting pattern. Typically, when only the first k eigenvalues of A lie in the interval (a, b), the number of solutions of $F_{p\ell}(u) = -p$ is roughly $2k$, in the spirit of the Lazer-McKenna conjecture. As k gets large, the number of solutions essentially *doubles* whenever a new eigenvalue interacts with the nonlinearity.

More concretely, let A be the symmetric tridiagonal matrix with diagonal entries equal to 2 and adjacent subdiagonal entries equal to -1. The eigenvalues of A lie between $\lambda_1 \approx .03843$ and $\lambda_{15} \approx 3.96157$. We consider the $p\ell$-admissible maps $F^k_{p\ell}(u) = Au - f^k_{p\ell}(u)$ for nonlinearities $f^k_{p\ell}$ given by parameters

$$a = a_k = \frac{\lambda_1}{2}, \quad b_k = \frac{\lambda_k + \lambda_{k+1}}{2}, \ k = 1, \ldots, 14 \text{ and } b_{15} = \frac{\lambda_{15} + \lambda_1}{2}.$$

Thus, whenever k increases by 1, a new eigenvalue interacts with $f^k_{p\ell}$. The number of solutions of $F^k_{p\ell}(u) = -t\phi_1$ is given in the table below. From homogeneity, it is independent of $t > 0$.

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
# sols	2	4	6	8	12	12	22	24	26	100	286	634	972	1320	2058

For $b = 10 + \lambda_{15}$, there are 32.224 solutions, and the last four of $2^{15} = 32768$ come up when b changes from $44.274 + \lambda_{15}$ to $44.587 + \lambda_{15}$.

The equation in Theorem 1.1 admits a variational formulation: as shown in [3], there is a pair of solutions to each different value of the Morse index of critical points of the related functional. The variational formulation for our case is introduced in Theorems 1.2 and 1.3. Now, when property (Sol) holds, critical points with the same index are counted by binomial numbers.

We outline the proof of Theorems 1.2 and 1.3. Two homotopies deform the functions F into simpler models. For $\tau \in [0, 1]$, set $H_{s\ell}(\tau, u) = \tau Au - f_{s\ell}(u)$ in the superlinear case. In the asymptotically linear case, set $H_{a\ell}(\tau, u) = Au - f_\tau(u)$ where $f_\tau(u) = \tau f_{a\ell} + (1 - \tau) f_{p\ell}$ for nonlinearities $f_{a\ell}$ and $f_{p\ell}$ with the same asymptotic parameters.

We start by showing that the appropriate versions of Theorems 1.2 and 1.3 hold for $\tau = 0$ and then transfer these results to $\tau = 1$. This requires checking a few properties. First, the homotopies should be proper. In the $a\ell$-case, this requires *non-resonance*, which we now define. Let a and b be the asymptotic parameters of $F_{a\ell}$ and let D be any of the 2^n diagonal matrices with diagonal entries equal to either a or b. The map $F_{a\ell}$ is *non-resonant* if all the matrices $A-D$ are invertible; in this case a and b are *non-resonant* asymptotic parameters.

We then bound the critical sets $\mathcal{C}(\tau)$ of the restrictions $H(\tau, \cdot)$ in both cases: assuming non-resonance, all such sets lie in a *cross of width* α,

$$\mathcal{X}_\alpha = \{(u_1, u_2, \ldots, u_n) \in \mathbb{R}^n \mid \exists i, |u_i| < \alpha\}.$$

Nonsmooth points, as some in $p\ell$-admissible maps, are automatically included in the critical set. The properties of critical sets are presented in Section 2.

A second necessary bound is more geometric: half-lines of the form $y - tp$, $p > 0$ leave images of crosses for large $t > 0$. Thus, rather imprecisely, vectors with very negative coordinates are regular values of the homotopies $H_{s\ell}$ and $H_{a\ell}$. We are then left with checking that the solutions for $\tau = 0$ are deformed in a standard way to solutions of $\tau = 1$ along a regular region of the domain of both homotopies. In Section 3 we handle the homotopy related to the $s\ell$ case, which is simpler, and then proceed to the $a\ell$ case in Section 4.

In Section 5 we interpret geometrically the dramatic increase of preimages as the parameters a and b give rise to larger spectral interaction. In a nutshell, critical components are taken by F to surfaces with high turning number around points of the form $y - tp$, for $t > 0$ large, giving rise to what we would find appropriate to call *topological turbulence*.

Theorems 1.2 and 1.3 were presented in the thesis of the first author [15].

2. The critical set of an admissible function

2.1. Smooth admissible maps: the $s\ell$ and $a\ell$ cases

Let $\mathrm{diag}(v) = \mathrm{diag}(v_1, v_2, \ldots, v_n)$ be the diagonal matrix with $(\mathrm{diag}(v))_{ii} = v_i$ and $f'(u)$ be the vector with entries $f_i'(u)$. In some occasions, we also write $\mathrm{diag}(x_i)$ to denote the diagonal matrix with (i,i)-entry equal to x_i. For an admissible map $F \in C^1$, $F(u) = Au - f(u)$, its Jacobian at u is $DF(u) = A - \mathrm{diag}(f'(u))$. Label the eigenvalues $\tilde{\lambda}_k$ of $DF(u)$ in nondecreasing order, $\tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \ldots \leq \tilde{\lambda}_n$. The *critical set* $\mathcal{C}(F)$, or simply $\mathcal{C}$, is

$$\mathcal{C}(F) = \{u \in \mathbb{R}^n \,|\, 0 \in \sigma(DF(u))\} = \cup_k \, \mathcal{C}_k(F),$$

where

$$\mathcal{C}_k(F) = \{u \in \mathbb{R}^n \,|\, \tilde{\lambda}_k = 0\}.$$

Let $p \in \mathbb{R}^n$ be a vector with strictly positive coordinates, which we denote by $p > 0$. We first show that the subsets $\mathcal{C}_k(F)$ are graphs of continuous functions $\gamma_k : p^\perp \to \mathbb{R}$. The arguments follow those used for the continuous case in [1].

Lemma 2.1 *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be an admissible C^1 map and $u, w \in \mathbb{R}^n$. Let $\tilde{\lambda}(u)$ be a simple eigenvalue of $DF(u)$ with normalized eigenvector $v(u)$. Then*

$$\frac{\partial \tilde{\lambda}}{\partial w}(u) = \frac{\mathrm{d}\tilde{\lambda}}{\mathrm{d}t}(u + tw)\bigg|_{t=0} = -\sum_i f_i''(u_i) w_i v_i^2(u).$$

Proof: As is well known, along a curve $S(t)$ of symmetric matrices, a simple eigenvalue λ is differentiable and $\dot{\lambda}(t) = \langle \dot{S}(t) v(t), v(t)\rangle$ for normalized eigenvectors $v(t)$. We are interested in $S(t) = DF(u+tw) = A - \mathrm{diag}(f'(u+tw))$:

$$\frac{\mathrm{d}\tilde{\lambda}}{\mathrm{d}t}(u + tw)\bigg|_{t=0} = -\langle \mathrm{diag}(f_i''(u_i) w_i) v(u), v(u)\rangle = -\sum_i f_i''(u_i) w_i v_i^2(u).$$

□

For convenience, we say that the asymptotic parameters of an $s\ell$-admissible nonlinearity are $a = -\infty$ and $b = \infty$.

Proposition 2.2 *Let $F = A - f$ be either $s\ell$ or $a\ell$-admissible, with smooth convex coordinates f_i, so that $f_i'' = 0$ at isolated points and asymptotic parameters* *a and b. Then each eigenvalue $\tilde{\lambda}_k(t)$ of DF is strictly decreasing along lines $u+tp$, $p>0$ and*

$$\lim_{t\to-\infty} \tilde{\lambda}_k(u+tp) = \lambda_k - a \quad \text{and} \quad \lim_{t\to+\infty} \tilde{\lambda}_k(u+tp) = \lambda_k - b.$$

Each $\mathcal{C}_k(F) = \tilde{\lambda}_k^{-1}(0)$ is the graph of a continuous function $\gamma_k : p^\perp \to \mathbb{R}$. Near critical points u in which $\tilde{\lambda}_k(u)$ is simple, $\mathcal{C}(F)$ is a smooth hypersurface.

Thus, the sets $\mathcal{C}_k(F)$ intersect exactly at multiple eigenvalues, where they cease to be smooth manifolds. Moreover, $\mathcal{C}_k(F)$ is nonempty if and only if the free eigenvalue λ_k of A in the range of f_i'. For $p>0$, each line $u+tp$, trespasses a nonempty $\mathcal{C}_k(F)$ exactly once.

Proof: As in the lemma above, at points $u+t^*p$ for which $\tilde{\lambda}_k$ is simple,

$$\frac{\mathrm{d}\tilde{\lambda}}{\mathrm{d}t}(u+tp)\bigg|_{t=t^*} = -\sum_i f_i''(u_i+t^*p_i)p_i v_i^2(u+t^*p) \le 0,$$

since $f_i'' \ge 0$ and $p>0$. Equality holds for isolated points on each line $u+tp$. Hence each $\tilde{\lambda}_k$ is strictly decreasing as t increases along such lines. In particular, there exists at most a point at which $\tilde{\lambda}_k(t)=0$.

The diagonal entries of $DF(u+tp)$ are $A_{ii} - f_i'(u_i+tp_i)$ and, by admissibility,

$$\lim_{t\to-\infty}[f_i'(u_i+tp_i)] = a \quad \text{and} \quad \lim_{t\to+\infty}[f_i'(u_i+tp_i)] = b.$$

As $t\to-\infty$ (resp. $t\to\infty$), $DF(u+tp)$ converges to $A-aI$ (resp. $A-bI$) and $\lim_{t\to-\infty}\tilde{\lambda}_k(t) = \lambda_k - a$ (resp.$\lim_{t\to+\infty}\tilde{\lambda}_k(t) = \lambda_k - b$). Thus $\lambda_k - b < \tilde{\lambda}_k < \lambda_k - a$ and $\tilde{\lambda}_k$ becomes zero (i.e., $\mathcal{C}_k(F)$ is nonempty) exactly when $\lambda_k \in (a,b)$. This requirement is independent of u: for such an index k, each straight line $u+tp$ meets $\mathcal{C}_k(F)$ exactly once. Now, for $u \in p^\perp$, define $\gamma_k(u) = t_k$, where $u+t_kp$ is the point in the line $u+tp$ in $\mathcal{C}_k(F)$ — continuity of γ_k is clear.

Consider now a critical point $u+t_kp$ for which $\tilde{\lambda}_k(u+t_kp)=0$. By the implicit function theorem, if $\frac{\mathrm{d}\lambda}{\mathrm{d}t}(u+tp)\big|_{t=t_k} < 0$, then $\mathcal{C}_k(F)$ is a graph of a smooth function $\gamma_k : p^\perp \cap \mathcal{V}(u) \to \mathbb{R}$ for a neighborhood $\mathcal{V}(u)$ of $u \in p^\perp$. □

As is well known, all eigenvalues of a Jacobi matrix, or more generally, of a tridiagonal symmetric matrix with nonzero off-diagonal entries $(i,i+1)$, are simple. This is the case of discretizations of second order ODE's. Let

$$A_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad \sigma(A_3) = \{\lambda_1 \approx 0.58, \lambda_2 = 2, \lambda_3 \approx 3.41\}$$

and nonlinearities $f_{s\ell} = (g,g,g)$ and $f_{a\ell} = (h,h,h)$, where

$$g(x) = x^2 \quad \text{and} \quad h' = 1.4(\frac{2}{\pi}\arctan x + 1), \quad h(0) = 0.$$

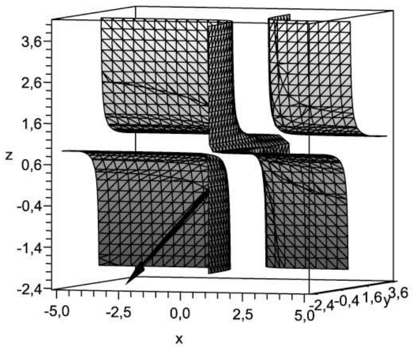

Figure 1. $\mathcal{C}(F_{s\ell}) = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$

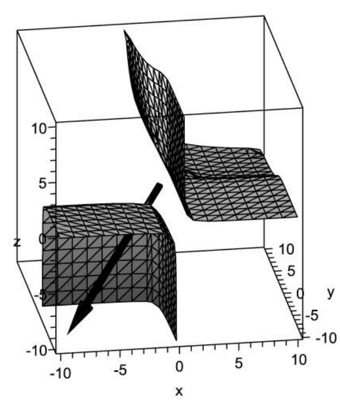

Figure 2. $\mathcal{C}(F_{a\ell}) = \mathcal{C}_1 \cup \mathcal{C}_2$

Clearly, $F_{s\ell} = A_3 - f_{s\ell}$ is $s\ell$-admissible and $F_{a\ell} = A_3 - f_{a\ell}$ is $a\ell$-admissible. The range of h', the open interval $(0.0, 2.8)$, contains only λ_1 and λ_2. Critical sets are displayed in Figures 1 and 2, together with a vector $-p$, $p > 0$.

Now set

$$A = \begin{bmatrix} 7 & 1 & -2 \\ 1 & 7 & -2 \\ -2 & -2 & 10 \end{bmatrix}, \quad \text{with } \lambda_1 = \lambda_2 = 6, \ \lambda_3 = 12$$

and $F_{s\ell} = A - f_{s\ell}$, with $f_i(x) = x^2$. The critical set $\mathcal{C}(F_{s\ell})$ ceases to be a manifold at $u_0 = (3, 3, 3)$, where $\mathcal{C}_1$ and $\mathcal{C}_2$ meet.

2.2. The critical set of $p\ell$-admissible maps

As discussed in the introduction, we must handle critical sets of $p\ell$-admissible maps $F_{p\ell}(u) = Au - f_{p\ell}(u)$. All vectors $u \in \mathcal{O}$ in the (open) orthant $\mathcal{O}$ have entries with a fixed *sign pattern* $s_1 s_2 \ldots s_n$. There, $F_{p\ell}(u) = (A - D^{\mathcal{O}})u$, where $D^{\mathcal{O}}_{ii} = a$ or b depending if s_i is negative or positive. We require $F_{p\ell}$ to be *non-resonant* — the restriction to each orthant is invertible, so that $\det(A - D^{\mathcal{O}})$ has a well defined sign. It is clear then that $F_{p\ell}$ is continuous in $\mathbb{R}^n$ but not smooth at the boundaries of orthants, the coordinate hyperplanes. We say that two orthants are *adjacent* if their sign patterns differ exactly at one location.

The *critical set* $\mathcal{C}(F_{p\ell})$ is then defined as the points in coordinate hyperplanes having neighboring points in adjacent orthants for which the restrictions $\det(A - D^{\mathcal{O}})$ have opposite signs. Thus, $\mathcal{C}(F_{p\ell})$ consists of unions of the closures of *tiles*, which are subsets of the coordinates hyperplanes which are common to the closure of exactly two adjacent orthants.

At a critical point u in a tile, clearly the generic case, $F_{p\ell}$ is a (topological) fold. Said differently, after composition with local homeomorphisms, $F_{p\ell}$ near u takes the normal form $(x, y_1, \ldots, y_{n-1}) \mapsto (x^2, y_1, \ldots, y_{n-1})$.

For

$$A = \begin{bmatrix} 4 & -2 \\ -2 & 5 \end{bmatrix}$$

with eigenvalues $\{2.44, 6.56\}$ take $f_{p\ell}$ with parameters $a = 0$ and $b = 3$. Figures 3 and 4 show the critical set, consisting of two semi-axes, and its image. We also indicate the determinant of $DF_{p\ell}$ in each quadrant: $F_{p\ell}$ is non-resonant. Adding up, the image of f' contains only the smallest eigenvalue of A, and the critical set, as we shall see in the next proposition, is the smallest possible. As the asymptotic parameters a and b enclose more eigenvalues, the critical set increases.

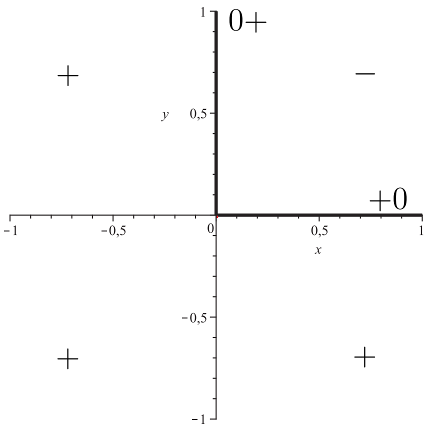

Figure 3. $\mathcal{C}(F_{p\ell})$: $a = 0$ and $b = 3$

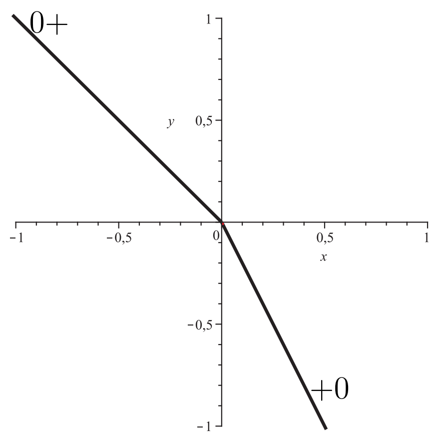

Figure 4. $F_{p\ell}(\mathcal{C}(F_{p\ell}))$

Proposition 2.3 *Let $F_{p\ell}$ be a $p\ell$-admissible map whose asymptotic parameters a, b are not eigenvalues of A. Suppose that the interval (a, b) contains the eigenvalues $\lambda_\ell \leq \ldots \leq \lambda_r$ of A. Then a generic straight line $r(t) = u + tp$, $p > 0$, hits the critical set $\mathcal{C}(F_{p\ell})$ at $r - \ell + 1$ points. In particular, if $\ell = 1$ and $r = n$, then $\mathcal{C}(F_{p\ell})$ is the union of all coordinate hyperplanes.*

Proof: Replace f_i by a strictly convex, smooth function $\tilde{f}_i$, with the same asymptotic parameters than f_i. Then, from Proposition 2.2, the claim above is true for the critical set of the $a\ell$-admissible map $\tilde{F}_{a\ell}(u) = Au - \tilde{f}_{a\ell}(u)$. For large positive t, the signs of $\det DF_{p\ell}(u)$ and $\det D\tilde{F}_{a\ell}(u)$ are the same for points of the form $t(\pm 1, \pm 1, \ldots, \pm 1)$. The result is now obvious. □

For A as above and parameters $a = 0$ and $b = 7$, the critical set of $F_{p\ell}$ is at its maximum: it is shown in Figure 7 of Section 4.

3. Superlinear admissible maps

In this section we prove Theorem 1.2, following the sketch of proof given in the introduction. Let $f = f_{s\ell} : \mathbb{R} \to \mathbb{R}$ be $s\ell$-admissible. We consider the homotopy $H_{s\ell}(\tau, u) = \tau Au - f(u)$, from $[0, 1] \times \mathbb{R}^n$ to $\mathbb{R}^n$.

3.1. The result for $\tau = 0$

When $\tau = 0$, $H_{s\ell}(0, u) = -f(u)$ and $DH_{s\ell}(0, u) = -\operatorname{diag}(f_1'(u_1), \ldots, f_n'(u_n))$. This matrix ceases to be invertible exactly when a diagonal entry becomes zero, and for each i this occurs exactly for one real number, by the strict convexity of f_i. Hence, the critical set of $H_{s\ell}(0, \cdot)$ consists of a union of n affine hyperplanes, parallel to the coordinate hyperplanes.

Again due to the convexity and superlinearity of each f_i, for $t > 0$ sufficiently large, each equation $f_i(x) = -(y_i - tp_i) > 0$ has exactly one positive and one negative solution, so that $H_{s\ell}(0, u) = y - tp$, $p > 0$ satisfies property (Sol).

3.2. Properness and bounds for the critical set

Proposition 3.1 *The homotopy $H_{s\ell}(\tau, u)$ is a proper map.*

Proof: Suppose by contradiction that the bounded sequence $z^k = H_{s\ell}(\tau^k, u^k)$ is the image of pairs (τ^k, u^k) going to infinity. Since $\tau^k \in [0, 1]$, we must have $u^k \to \infty$. Then

$$\tau^k A \frac{u^k}{||u^k||} - \frac{f(u^k)}{||u^k||} = \frac{z^k}{||u^k||}$$

and thus $f(u^k)/||u^k||$ is bounded, contradicting the superlinearity of f. □

For $\tau \in [0, 1]$, the critical sets $\mathcal{C}_{s\ell}(\tau)$ of $H_{s\ell}(\tau, \cdot)$ should not deviate much from the critical set of $H_{s\ell}(0, \cdot)$, described in the subsection above. This is indeed suggested by Figures 1 and 2 in the previous chapter. More precisely, we define the *cross* $\mathcal{X}_\alpha = \cup_i \mathcal{S}^i_\alpha$, a union of *strips of width* $\alpha > 0$,

$$\mathcal{S}^i_\alpha = \{(u_1, u_2, \ldots, u_n) \in \mathbb{R}^n \,|\, |u_i| < \alpha\}.$$

We now show that the critical sets $\mathcal{C}_{s\ell}(\tau), \tau \in [0, 1]$ lie in a common cross.

Proposition 3.2 *There is $\alpha > 0$ such that, for all $\tau \in [0, 1]$, $\mathcal{C}_{s\ell}(\tau) \subset \mathcal{X}_\alpha$.*

Proof: For a fixed $\tau \in [0, 1]$, $DH_{s\ell}(\tau, u) = \tau A - \operatorname{diag}(f_i'(u_i))$. The diagonal entries of $DH_{s\ell}(\tau, u)$ are $DH_{s\ell}(\tau, u)_{ii} = \tau A_{ii} - f_i'(u_i)$. By Gerschgorin theorem, the eigenvalues $\tilde{\lambda}_k$ of $DH_{s\ell}(\tau, u)$ lie in the union of the disks

$$|z - (\tau A_{ii} - f_i'(u_i))| \leq R_i, \quad \text{where} \quad R_i(\tau) = \tau \sum_{j \neq i} |A_{ij}|.$$

Set $R^* = \max_{i, \tau \in [0,1]} (R_i(\tau))$. If $u \in \mathcal{C}_{s\ell}(\tau)$, the eigenvalue $z = 0$ satisfies

$$|\tau A_{ii} - f_i'(u_i)| \leq R^* \quad \text{for some } i,$$

and $|f_i'(u_i)| \leq \max_i |A_{ii}| + R^*$. Since $|f_i'| \to \infty$, we have $|u_i| < \alpha$ for some α. □

3.3. An estimate for the image of the critical set

As an example, take $F_{s\ell} = A_2 - f_{s\ell}$ for $f_{s\ell} = (g, g)$, $g(x) = x^2$ and

$$A_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

Figures 5 and 6 show the sets $\mathcal{C}(F_{s\ell})$ and $\mathcal{X}_3$, their images and a half line $-tv$, $tv > 0$. One critical curve consists only of folds, the other has a cusp. A line $y - tp$ trespasses the image of the critical set twice, going from a region with $t << 0$, where points do not belong to the range of $F_{s\ell}$ to points with $t >> 0$ which have four preimages. Most of the quadrant consisting of vectors with negative coordinates consists of regular values — this fact is made precise by the next proposition.

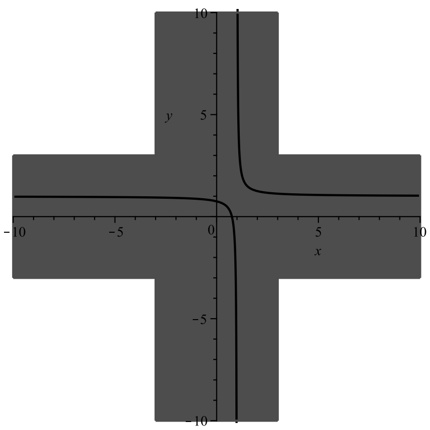

Figure 5. $\mathcal{C}(F_{s\ell}) \subset \mathcal{X}_3$ for A_2

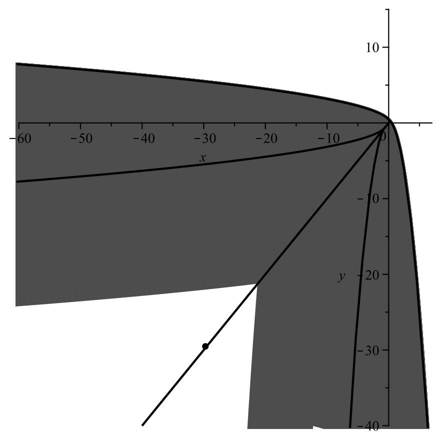

Figure 6. $F_{s\ell}(\mathcal{C}(F_{s\ell})) \subset F_{s\ell}(\mathcal{X}_3)$

Proposition 3.3 *Let $F_{s\ell}$ be an $s\ell$-admissible map and $H_{s\ell}(\tau, u) = \tau Au - f_{s\ell}(u)$ be the associated homotopy. Let $\mathcal{X}_\alpha \subset \mathbb{R}^n$ be the cross of width α and $r(t) = y - tp$, for fixed $p > 0$, $||p|| = 1$. Then, for $t > 0$ sufficiently large, $r(t) \notin H_{s\ell}([0,1] \times \mathcal{X}_\alpha)$.*

Proof: Write $f = f_{s\ell}$. Take sequences $\tau^k \in [0,1]$, $u^k \in \mathcal{X}_\alpha$ and $t^k > 0$ such that

$$\tau^k Au^k - f(u^k) = y - t^k p, \qquad (*)$$

and suppose by contradiction that $t^k = ||\tau^k Au^k - f(u^k) - y|| \to \infty$. Since $H_{s\ell}$ is proper (Proposition 3.1), $u^k \to \infty$. Thus $t^k/||u^k|| \to \infty$, by the superlinearity of f. Taking a subsequence if necessary, suppose $|u_i^k| \le \alpha$ for a fixed index i. Dividing equation $(*)$ by t^k and equating the i-th coordinate,

$$\frac{(\langle \tau^k Au^k, e_i \rangle - f_i(u_i^k) - y_i)/||u^k||}{t^k/||u^k||} = p_i.$$

The three terms in the numerator are bounded, the denominator goes to $k \to \infty$: we must then have $p_i = 0$, a contradiction. □

3.4. From $\tau = 0$ to $\tau = 1$: Theorem 1.2

We prove Theorem 1.2. Write $f = f_{s\ell}$, $F = F_{s\ell}$, $H = H_{s\ell}$: we want to show that $H(1, u) = F(u) = Au - f(u)$ satisfies property (Sol). By Proposition 3.2, all critical sets $\mathcal{C}(\tau)$ of the homotopy $H(\tau, \cdot)$ lie in a cross $\mathcal{X}_\alpha$ and by Proposition 3.3, for $t > 0$ sufficiently large, $w = y - tp$ is a regular value of H, since $H(\tau, \mathcal{X}_\alpha)$, $\tau \in [0, 1]$, contains all the critical values.

Take such w: by properness of H (Proposition 3.1), $K_w = H^{-1}(w)$ is a compact one-dimensional manifold with regular boundary points. As we shall see, each of its 2^n components is diffeomorphically parameterized by $\tau \in [0, 1]$. Indeed, for $u \in K_w$, $DH(\tau, u(\tau))$ is invertible. Take derivatives in τ of $H_{s\ell}(\tau, u(\tau)) = w$ to obtain $Au(\tau) + \tau Au'(\tau) - f'(u(\tau))u'(\tau) = 0$, so that

$$u'(\tau) = -\left[\tau A - f'(u(\tau))\right]^{-1} Au(\tau) = -[DH(\tau, u(\tau))]^{-1} Au(\tau). \qquad (1)$$

The differential equation is well defined on K_w. By Section 3.1, $H(0, u) = w$ has 2^n solutions $u^{\mathcal{O}}$, one in each orthant $\mathcal{O} \subset \mathbb{R}^n$. Solve for $u(\tau)$ with initial conditions $u(\tau = 0) = u^{\mathcal{O}}$ to obtain 2^n arcs $(\tau, u^{\mathcal{O}}(\tau)) \subset K_w$ for which $H_{s\ell}(\tau, u^{\mathcal{O}}(\tau)) = w$ which extend to $\tau = 1$, by compactness of K_w. Invert the sense of the vector field to show that $(0, u(0)) \mapsto (1, u(1))$ is a bijection between solutions at $\tau = 0$ and $\tau = 1$. Finally, since each $(\tau, u^{\mathcal{O}}(\tau))$ does not leave the cross, it stays in $\mathcal{O}$. Thus, solutions for $\tau = 1$ are distributed among orthants, as the solutions for $\tau = 0$. $\square$

4. Asymptotically linear admissible maps

We now consider $a\ell$-admissible maps $F_{a\ell}$ and the homotopy

$$H_{a\ell}(\tau, u) = Au - (\tau f_{a\ell}(u) + (1 - \tau) f_{p\ell}(u)),$$

where $f_{a\ell}$ and $f_{p\ell}$ have the same asymptotic parameters. We require that $F_{p\ell}$ be non-resonant, as defined in Section 2.

We follow the same sequence of steps as in the $s\ell$ case, but the arguments are subtler. We start with $p\ell$-admissible maps.

4.1. The homotopy at $\tau = 0$: $p\ell$-admissible maps

We start with an example. Consider a non-resonant map $F_{p\ell}$, with A given in Section 2.2 and $f_{p\ell}$ with asymptotic parameters $a = 0$ and $b = 7$, whose critical set $\mathcal{C}$ and image $F_{p\ell}(\mathcal{C})$ are shown in the figures below. We indicate in each quadrant the sign of the determinant of $DF_{p\ell}$. Two half lines $-tp$, $p > 0$, consist of critical values. Other half-lines $-tp$ do not reach the component of the image with maximal number of preimages, the sector between the images of $+0$ and $0+$. As we shall see, as the asymptotic parameters a and b get large, a larger portion of the negative orthant consists of points with 2^n preimages.

We prove the analogous of Theorem 1.3 for $p\ell$-admissible maps: each proposition handles one possible hypothesis — the proofs are quite different.

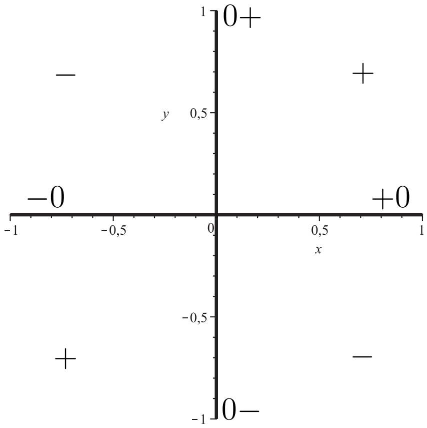

Figure 7. $\mathcal{C}(F_{p\ell})$: $a = 0$ e $b = 7$

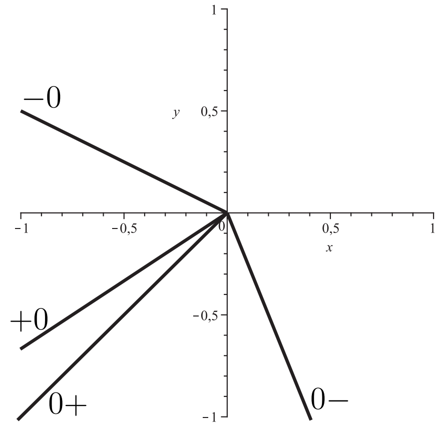

Figure 8. $F_{p\ell}(\mathcal{C}(F_{p\ell}))$

Proposition 4.1 *Let A be a real, symmetric matrix. Then for $a < 0$, $b > 0$ and $|a|, b$ sufficiently large, the map $F_{p\ell}(u) = Au - f_{p\ell}(u)$ is non-resonant. Also, $F_{p\ell}$ satisfies property (Sol).*

Proof: Let $s_1 s_2 \ldots s_n$ be the sign pattern of the vectors in the orthant $\mathcal{O}$. In $\mathcal{O}$, the restriction $F_{p\ell}^{\mathcal{O}} v = (A - f_{p\ell})v$ is the linear map $F_{p\ell}^{\mathcal{O}} v = (A - D^{\mathcal{O}})v$, where $D_{jj}^{\mathcal{O}} = a$ (resp. b) if and only if $s_j < 0$ (resp. $s_j > 0$). By Gerschgorin theorem, for $a < 0$ and $b > 0$ sufficiently large, this matrix is invertible. For parameters a and b such that $A - D^{\mathcal{O}}$ is invertible for each $\mathcal{O}$, $F_{p\ell}$ is non-resonant.

We now show that, for large $t > 0$, the solution $x^{\mathcal{O}}$ of $(A - D^{\mathcal{O}})x^{\mathcal{O}} = y - tp$, $p > 0$, belongs to $\mathcal{O}$, i.e., $\operatorname{sgn} x_j^{\mathcal{O}} = s_j$. Set $-w = -(y - tp) > 0$. From the homogeneity of $F_{p\ell}$, we may suppose that $\|w\| = 1$.

Let $(A - D^{\mathcal{O}})_j$ be the matrix obtained by replacing column j of $A - D^{\mathcal{O}}$ by the vector w. By Cramer's rule,

$$x_j^{\mathcal{O}} = \frac{\det(A - D^{\mathcal{O}})_j}{\det(A - D^{\mathcal{O}})}$$

For large $|a|$ and b, the monomials consisting of the product of the diagonal entries in the expansions of $\det(A - D^{\mathcal{O}})$ and $\det(A - D^{\mathcal{O}})_j$ dominate the other terms. Their values differ only at entry (j, j): for $s_j = -1$, the positive entry $A_{jj} - a$ is replaced by $w_j < 0$ and thus $\operatorname{sgn} x_j^{\mathcal{O}} = -1 = s_j$; for $s_j = 1$, $A_{jj} - b$ and w_j have the same sign and again $\operatorname{sgn} x_j^{\mathcal{O}} = 1 = s_j$. □

For completeness, we present a proof of a basic property of Stieltjes matrices.

Lemma 4.2 *The inverse of a Stieltjes matrix A is a positive definite matrix with non-negative entries. The smallest eigenvalue λ_1 of A has an eigenvector $\phi \geq 0$. If A is irreducible, λ_1 is simple and one may take $\phi > 0$.*

Proof: Let $p \in \mathbb{R}^n$, $p \geq 0$. Solve $Ax = p$ by Gauss-Seidel iteration ([7]): split $A = L + U$, for L lower triangular and U strictly upper triangular, and set

$$x_0 = (1, 1, \ldots, 1), \quad x_{k+1} = L^{-1}(p - Ux_k).$$

As is well known, the iteration converges because A is positive definite. Also, if $x_k \geq 0$ then $x_{k+1} \geq 0$. Indeed, U has non-positive entries, so $p - Ux_k \geq 0$ and the inversion under L may be performed by computing sequentially the coordinates of x_{k+1} starting from the first one, yielding non-negative entries for x_{k+1}.

In particular, the i-th column of A^{-1} is the solution of the system for $p = e_i$, and hence the entries of A^{-1} are non-negative. By the Perron-Frobenius theorem, we must have a non-negative eigenvector $\phi \geq 0$ associated to the largest positive eigenvalue λ_1^{-1} of A^{-1}. There are no other eigenvalues with the same absolute value, due to the positivity of A^{-1}, but λ_1^{-1} is not necessarily simple. This is the case, however, when A is irreducible, and then we may take $\phi > 0$. □

Proposition 4.3 *Let $\lambda_1 \leq \lambda_n$ be the extreme eigenvalues of a Stieltjes matrix A. Then, for $a_0 < \lambda_1$ and b sufficiently above λ_n, the map $F_{p\ell}(u) = Au - f_{p\ell}(u)$ with asymptotic parameters a and b is non-resonant. Also, $F_{p\ell}$ satisfies property (Sol).*

Proof: Fix an orthant $\mathcal{O}$ with sign pattern $s_1 \ldots s_n$ and, without loss, permute coordinates so that the first ν signs are negative and the remaining ones positive: now $D^{\mathcal{O}} = \operatorname{diag}(a, \ldots, a, b, \ldots, b)$. Again, set $w = y - tp$, $p > 0$, for $t > 0$ so large that $-w > 0$. We want to show that the first ν coordinates of the solution $x^{\mathcal{O}}$ of $(A - D^{\mathcal{O}})x^{\mathcal{O}} = w$ are negative, and the remaining ones, positive. Write in block form $x^{\mathcal{O}} = (x_1, x_2)^T$, $w = (w_1, w_2)^T$ and

$$\begin{bmatrix} A_1 - aI_\nu & N \\ N^T & A_2 - bI_{n-\nu} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix},$$

where I_j is the $j \times j$ identity matrix and the matrix N has non-positive entries, since A is a Stieltjes matrix. The block $A_1 - aI_\nu$ is Stieltjes, since the eigenvalues of A_1 interlace those of A and $a < \lambda_1$. The solution $(x_1, x_2)^T$ depends on b: write

$$\begin{bmatrix} A_1 - aI_\nu & N \\ N^T/b & A_2/b - I_{n-\nu} \end{bmatrix} \cdot \begin{bmatrix} x_1(b) \\ x_2(b) \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2/b \end{bmatrix},$$

and take the limit $b \to \infty$, to get

$$\begin{bmatrix} A_1 - aI_\nu & N \\ 0 & -I_{n-\nu} \end{bmatrix} \cdot \begin{bmatrix} x_1(\infty) \\ x_2(\infty) \end{bmatrix} = \begin{bmatrix} w_1 \\ 0 \end{bmatrix}.$$

Thus $x_2(\infty) = 0$ and $-x_1(\infty) = -(A_1 - aI_\nu)^{-1}(w_1) > 0$, by Lemma 4.2. By continuity, for b sufficiently large, $-x_1(b) > 0$. From the second equation,

$$(I_{n-\nu} - A_2/b)x_2(b) = (N^T/b)\ x_1(b) - w_2/b > 0.$$

By the Neumann expansion, for large b,

$$x_2(b) = (I_{n-\nu} + A_2/b + O(1/b^2))\ ((N^T/b)\ x_1(b) - w_2/b)\ >\ 0.$$

In particular, for such b, $(A - D^{\mathcal{O}})$ is invertible, and, choosing b large enough so that this holds for all orthants, one obtains the non-resonance of $F_{p\ell}$. □

4.2. Properness and bounds for the critical set

The proposition below is a simple exercise.

Proposition 4.4 *For a smooth, strictly convex $f : \mathbb{R} \to \mathbb{R}$,*

(i) $\lim_{x\to-\infty} f'(x) = a \iff \lim_{x\to-\infty} f(x)/x = a$,

(ii) $\lim_{x\to\infty} f'(x) = b \iff \lim_{x\to\infty} f(x)/x = b$.

Proposition 4.5 *For non-resonant asymptotic parameters $a < b$, the homotopy $H_{a\ell} : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ given by $H_{a\ell}(\tau, u) = \tau F_{a\ell}(u) + (1-\tau)F_{p\ell}(u)$ is proper.*

Proof: By contradiction, take an unbounded sequence (τ^k, u^k) with bounded image $z^k = H_{a\ell}(\tau^k, u^k)$, so $||u^k|| \to \infty$. Without loss, $u^k/||u^k||$ converges to a unit vector u^* and all vectors u^k lie in the closure of a common orthant $\mathcal{O}$. Consider the i-th coordinate:

$$(Au^k)_i - \left(\tau^k f_{a\ell,i}(u_i^k) + (1-\tau^k) f_{p\ell,i}(u_i^k)\right) = z_i^k.$$

By Proposition 4.4, if u_i^* is negative, zero or positive,

$$\lim_k \left(\tau^k \frac{f_{a\ell,i}\left(\frac{u_i^k}{||u^k||}||u^k||\right)}{||u^k||} + (1-\tau^k) \frac{f_{p\ell,i}\left(\frac{u_i^k}{||u^k||}||u^k||\right)}{||u^k||} \right) \ = \ a u_i^*,\ 0 \ \text{ or } \ b u_i^*.$$

As $||u^k|| \to \infty$, $\frac{z^k}{||u^k||} \to 0$. Taking limits,

$$Au^* - (d_1 u_1^*, \dots, d_n u_n^*)^T = (A - \operatorname{diag}(d_1, d_2, \dots, d_n))u^* = 0,$$

where each entry d_i is equal to a or b and thus $\ker(A - D) \neq \{0\}$, where D is a diagonal matrix having for diagonal entries only the values a and b. This contradicts the non-resonance hypothesis. □

As described in Section 2.2, the critical set of the non-resonant $p\ell$-admissible map $H_{a\ell}(0, \cdot)$ is contained in the union of the coordinate hyperplanes, and hence lie in crosses of width zero. For each $\tau \in [0, 1)$, the function $H_{a\ell}(\tau, u) = \tau F_{a\ell}(u) + (1-\tau)F_{p\ell}(u)$ ceases to be differentiable at the coordinate hyperplanes. For τ fixed, define the critical set $\mathcal{C}_{a\ell}(\tau)$ of $H_{a\ell}(\tau, \cdot)$ to be the union of the coordinate planes with the points u for which the Jacobian is not invertible.

Proposition 4.6 *The sets $\mathcal{C}_{a\ell}(\tau)$, for $\tau \in [0, 1]$, lie in a common cross $\mathcal{X}_\alpha$.*

Proof: Clearly, any cross contains the coordinate hyperplanes. For a point u in some (open) orthant $\mathcal{O}$,

$$DH_{a\ell}(\tau, u) = A - D^{\mathcal{O}} + \tau(D^{\mathcal{O}} - \tilde{D}^{\mathcal{O}}),$$

where $D^{\mathcal{O}} = \text{diag}(c_i)$, c_i equals a or b depending on the sign pattern in $\mathcal{O}$, and $\tilde{D}^{\mathcal{O}} = \text{diag}(f'_{a\ell,1}(u_1), f'_{a\ell,2}(u_2), \ldots, f'_{a\ell,n}(u_n))$. By non-resonance, $A - D^{\mathcal{O}}$ is invertible, and so are matrices sufficiently close to it. For N sufficiently large, $\sup_i |u_i| > N$ implies that $DH_{a\ell}(\tau, u)$ is also invertible. Take, then $\alpha = N$. $\square$

In resonant situations, critical sets do not lie in crosses. Let

$$A = \begin{bmatrix} 5 & -2 \\ -2 & 6 \end{bmatrix}$$

and take $f = (g, g)$, where

$$g(x) = \frac{65x}{7} + \int_0^x \frac{38\arctan(t)}{7\pi}\,\mathrm{d}t.$$

Then $g' : \mathbb{R} \to (a = 46/7, b = 12)$ is an increasing diffeomorphism. The critical set of $F_{a\ell} = A - f$ is the graph of

$$y(x) = -\tan\left[\frac{\pi(13\pi + 23\arctan(x))}{2(15\pi + 19\arctan(x))}\right].$$

Since $\lim_{x\to+\infty} y(x) = -\infty$, $\mathcal{C}(F_{a\ell})$ does not lie in a cross.

4.3. An estimate for the image of the critical set

We begin with a lemma which states that, at infinity, the regular sets of a non-resonant map $F_{a\ell}$ and its $p\ell$-version $F_{p\ell}$ are pretty similar: half-lines $r(t) = y - tp$, $p > 0$ leave the images of crosses.

Lemma 4.7 *Let $H_{a\ell}(\tau, u) = \tau F_{a\ell} + (1-\tau)F_{p\ell}$ be the linear homotopy between non-resonant admissible maps $F_{a\ell}$ and $F_{p\ell}$ with asymptotic parameters a and b. Let $\mathcal{X}_\alpha \subset \mathbb{R}^n$ be a cross of width α and $p \in \mathbb{R}^n, p > 0$, be such that all solutions of $F_{p\ell}(u) = -p$ only have non-zero entries. Then, for $t > 0$ sufficiently large, the half-line $r(t) = y - tp$ is not in $\cup_{\tau\in[0,1]} H_{a\ell}(\tau, \mathcal{X}_\alpha)$.*

Proof: Start as in the proof of Proposition 4.5. Suppose by contradiction that there exist $\tau^k \in [0,1]$, $t^k \to \infty$ and $u^k \in \mathcal{X}_\alpha \cap \mathcal{O}$ for some orthant $\mathcal{O}$ such that

$$\frac{1}{||u^k||} H_{a\ell}(\tau^k, u^k) = \frac{r(t^k)}{||u^k||} = \frac{y - t^k p}{||u^k||}. \qquad (*)$$

From the properness of $H_{a\ell}$, $u^k \to \infty$ and we suppose that $u^k/||u^k|| \to u^*$. Since the u^k's belong to a cross, there is some coordinate i for which u_i^k is uniformly bounded, so that $u_i^* = 0$. By taking $k \to \infty$, depending if u_i^* is negative, zero or positive, the left hand side converges to

$$(A - \text{diag}(d_1, \ldots, d_n))u^*, \text{ where } d_i = a, 0 \text{ or } b.$$

Changing the d_i's equal to zero does no change $(A - \text{diag}(d_1, \ldots, d_n))u^*$:

$$(A - \text{diag}(d_1, \ldots, d_n))u^* = (A - D^{\mathcal{O}})u^* = F_{p\ell}(u^*) = -cp, \quad p > 0,$$

where the convergence of the right hand side of $(*)$ implies $t^k/||u^k|| \to c \geq 0$. If $c = 0$, then $(A - D^{\mathcal{O}})u^* = 0$ with $u^* \neq 0$, contradicting non-resonance. If $c > 0$ and u^* is a solution of $F_{p\ell}(u^*) = -cp$, then $u = u^*/c$ solves $F_{p\ell}(u) = -p$, by homogeneity. This contradicts the hypothesis: one entry of u^*/c is zero. □

Proposition 4.8 *Let $\mathcal{X}_\alpha \subset \mathbb{R}^n$ be a cross of width α and a half-line $r(t) = y - tp$, $t > 0$, where $p > 0$ and $\|p\|_\infty = 1$. Let A be real, symmetric, with extreme eigenvalues $\lambda_1 \leq \lambda_n$, $F_{a\ell}$ and $F_{p\ell}$ be admissible maps with asymptotic parameters a and b and $H_{a\ell}(\tau, u) = \tau F_{a\ell} + (1-\tau)F_{p\ell}$. Suppose also one of the two hypotheses:*

(i) a is sufficiently below λ_1 and b is sufficiently above λ_n

(ii) A is a Stieltjes matrix, $a < \lambda_1$ and b is sufficiently above λ_n

Then, for $t > 0$ sufficiently large, $r(t) \notin \cup_{\tau \in [0,1]} H_{a\ell}(\tau, \mathcal{X}_\alpha)$.

Proof: For the first (resp. second) hypothesis, Proposition 4.1 (resp. 4.3) yields non-resonant parameters a and b, for which the equation $(A - D^{\mathcal{O}})u = -cp$, $p > 0$, only has solutions with non-zero entries. This is exactly the hypothesis of the previous lemma. □

4.4. From $\tau = 0$ to $\tau = 1$: proof of Theorem 1.3

The proof of Theorem 1.3 now follows closely the argument in Section 3.4, adapted to the homotopy $H_{a\ell} = \tau F_{a\ell} + (1 - \tau)F_{p\ell}$. Each step — property (Sol) for $\tau = 0$; properness of the homotopy; a cross encloses critical sets; appropriate half-lines leave the images of crosses under the homotopy — have been proved in the subsections above for the $a\ell$ case. The deformation of the solutions along the homotopy imitates the $s\ell$ case. □

5. Some variational and topological properties

As in the differential setting, there is a variational form for the equation $F(u) = Au - f(u) = -p$ with $s\ell$ or $a\ell$-nonlinearities. It is easy to check that the appropriate functional is $\Phi : \mathbb{R}^n \to \mathbb{R}$, given by

$$\Phi(u) = \frac{1}{2}\langle Au, u\rangle - \langle \hat{f}(u), \mathbf{1}\rangle + \langle p, u\rangle,$$

where $\hat{f}' = f$ and $\mathbf{1} = (1, 1, \ldots, 1)$. From Theorems 1.2 and 1.3, for $p > 0$ sufficiently large, every solution u of $F(u) = -p$ lies in some orthant $\mathcal{O}$ and is a nondegenerate critical point of Φ, i.e., $D\Phi(u) = F(u) + p = 0$ and $\det DF(u) = \det D^2\Phi(u) \neq 0$.

Define, as usual, the Morse index $k(u)$ of a critical point u of Φ to be the number of negative eigenvalues of $D^2\Phi(u)$.

Proposition 5.1 *Let $F_{s\ell}$ and $F_{a\ell}$ satisfy the hypotheses of Theorems 1.2 and 1.3. Take $t, p > 0$ for t sufficiently large. Then the Morse index of each solution u of $F_{s\ell}(u) = y - tp$ or $F_{a\ell}(u) = y - tp$ is the number of its positive coordinates. There are exactly $\binom{n}{k}$ solutions of index k.*

Proof: In the notation of Sections 3.4 and 4.4, the index does not change along each arc of solutions $(\tau, u(\tau))$ of both homotopies $H_{s\ell}$ and $H_{a\ell}$, so that $k(u^{\mathcal{O}}(0)) = k(u^{\mathcal{O}}(1))$, for each orthant $\mathcal{O}$ (recall that the arcs of solutions stay out of a cross which contains the critical sets along the homotopies).

In the $s\ell$ case, when $\tau = 0$, $H(0, u) = -f(u)$ and the index is simply the number of negative diagonal entries of $\operatorname{diag}(-f_1'(u^{\mathcal{O}}(0)_1), \ldots, -f_n'(u^{\mathcal{O}}(0)_n))$, or, by the convexity of f, the number of positive entries of either $u^{\mathcal{O}}(0)$ or $u^{\mathcal{O}}(1)$.

In the $a\ell$ case, we must count the number of negative eigenvalues of $A - D^{\mathcal{O}}$, where, as defined in Section 2.2, $D^{\mathcal{O}}$ is a diagonal matrix with diagonal entries equal to a or b depending on the sign pattern of vectors in $\mathcal{O}$. Let $\mathcal{O}^-$ (resp. $\mathcal{O}^+$) be the orthant whose vectors only have negative (resp. positive) coordinates. All the eigenvalues of $A - D^{\mathcal{O}^-}$ (resp. $A - D^{\mathcal{O}^+}$) are positive (resp. negative). Draw a generic straight oriented line $r(x)$, $x \in \mathbb{R}$ from the $\mathcal{O}^-$ to $\mathcal{O}^+$, so that the line only crosses orthants at vectors with a single null cordinate. By non-resonance (Proposition 4.1), there are exactly $n+1$ different invertible matrices of the form $A - D^{\mathcal{O}}$ along r. By Proposition 2.3, determinants of matrices related to adjacent orthants have opposite signs. By a standard min-max argument, their eigenvalues are non-increasing. Then, necessarily, whenever $r(x)$ hits a hyperplane, one (and only one) additional eigenvalue becomes negative, exactly when one (and only one) additional coordinate of $r(x)$ becomes positive.

Since there is exactly one solution in each orthant, in both cases, the count of solutions with a fixed index is now immediate. □

We now focus on an interesting feature of the map F, first illustrated for a $p\ell$-admissible map $F : \mathbb{R}^3 \to \mathbb{R}^3$. Let the nonlinearity interact with the three eigenvalues of A: from the results in Section 2, the critical set consists of three connected components. The figure on the left represents $\mathcal{C}_2$: notice the similarity with the central component in Figure 1. Its image $F(\mathcal{C}_2)$, on the right, winds *twice* around the vector $(-1, -1, -1)$.

From Theorems 1.2 and 1.3, $s\ell$ and non-resonant $a\ell$-admissible maps F satisfy property (Sol) for $p > 0$ large. We compute the turning of the images of the critical surfaces $\mathcal{C}_k$ of F around extreme points $-p$. From Proposition 2.2, the

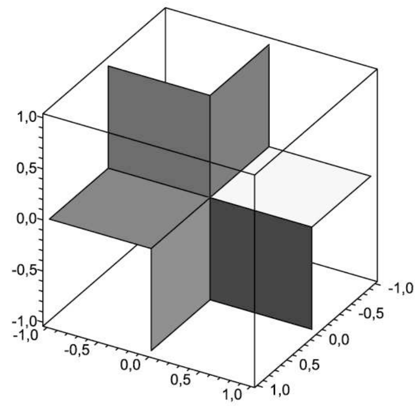

Figure 9. $\mathcal{C}(F)$

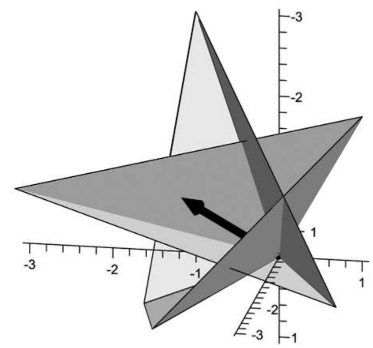

Figure 10. $F(\mathcal{C}(F))$

critical surfaces $\mathcal{C}_k$, $k = 1, \dots, n$, are graphs of continuous functions $\gamma_k : p^\perp \to \mathbb{R}^n$, for any $p > 0$. Denote by $\mathcal{R}_0, \dots, \mathcal{R}_n$ the connected component of regular points between $\mathcal{C}_1, \dots, \mathcal{C}_n$. After removing an appropriate cross, $\mathcal{R}_0$ contains only vectors with negative coordinates, $\mathcal{R}_1 - \mathcal{R}_0$ contains only vectors with a single positive coordinate, and, in general, $\mathcal{R}_\ell - \mathcal{R}_{\ell_1}$ contains only vectors with ℓ positive coordinates. Define the closed (topological) half-spaces $\mathcal{U}_\ell = \bigcup_{i=0}^{\ell} \overline{\mathcal{R}_i}$ and, for consistency, set $\mathcal{U}_n = \mathbb{R}^n$. By properness (Propositions 3.1 and 4.5), F extends continuously to a map $\tilde{F} : \mathbb{R}^n \cup \{\infty\} \to \mathbb{R}^n \cup \{\infty\}$.

Proposition 5.2 *For $p > 0$ large,* $\deg(F, \mathcal{U}_\ell, -p) = (-1)^\ell \binom{n-1}{\ell}$, $\ell = 0, \dots, n-1$ *and* $\deg(F, \mathcal{U}_n, -p) = 0$.

Proof: We want

$$\deg(\tilde{F}, \mathcal{U}_\ell \cup \{\infty\}, -p) = \deg(F, \mathcal{U}_\ell, -p) = \sum_{u \in F^{-1}(-p) \cap \mathcal{U}_\ell} \operatorname{sgn}(\det(DF(u))).$$

Since $-p$ is a regular value, there are no preimages u in $\partial \mathcal{U}_\ell = \mathcal{C}_\ell$, and the degree is indeed given by the sum above. We have $\operatorname{sgn}(\det(DF(u))) = (-1)^{k(u)}$ and, by Proposition 5.1, $k(u)$ is the number of positive coordinates in u. Now,

$$\deg(F, \mathcal{U}_\ell, -p) = \sum_{i=0}^{\ell} (-1)^i \binom{n}{i} = (-1)^\ell \binom{n-1}{\ell}$$

since the admissible maps satisfy property (Sol) — in each regular region $\mathcal{R}_\ell$, there are $\binom{n}{\ell}$ preimages u, for which $k(u) = \ell$. □

We may interpret some of the results in this paper as follows. The map $N(x_1, \dots, x_n) = (-x_1^2, \dots, -x_n^2)$ is a *multiple fold*, sending each orthant to the negative orthant of $\mathbb{R}^n$. In particular, there are 2^n preimages for each vector with negative coordinates and we may think of the coordinate hyperplanes as forming the critical set of N. As we add a linear perturbation to N, the critical set of the resulting map $F(u) = Au + N(u)$ is generically regular, splitting into n critical surfaces $\mathcal{C}_\ell$. The original folding effect gives rise to the high amount of turning of the surfaces $F(\mathcal{C}_\ell)$ around points in the negative quadrant. It is the turning of $F(\mathcal{C}_\ell)$ around points in the image that leads to a large number of preimages.

In the following related example, we may see what is happening. Consider $g(z) = z^7 + 5\overline{z}^4 + z$, for $z \in \mathbb{C}$. The map has two critical curves γ_1 and γ_2, encircling the origin as shown in Figure 11. Their images under g are shown on Figure 12: the image of $g(\gamma_1)$ has been enlarged for visual convenience. As γ_1 and γ_2 are traversed in counter-clockwise fashion, their images are traversed in opposite senses: this *topological turbulence* gives rise to points with many preimages. The number of cusps, five in γ_1 and eleven in γ_2, together with the turning numbers of the images of the critical curves, are compatible with the results in [11] ‡.

Clearly, $g : \mathbb{R}^2 \to \mathbb{R}^2$ is a proper map, and points in the same connected component of the set of regular values have the same number of preimages. In particular, points in the unbounded regular component have seven preimages, due to the z^7 type behavior at ∞. From the normal form of a map at fold points, the number of preimages of points at opposite sides of the image of a critical arc differ by two. In this case, as one gets closer to the origin, the number of preimages *increases* by two. Hence, for example, g has 17 roots: the origin and its preimages are represented in the figures.

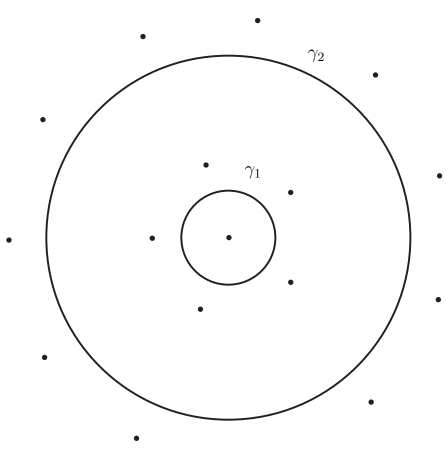

Figure 11. $\mathcal{C}(g)$

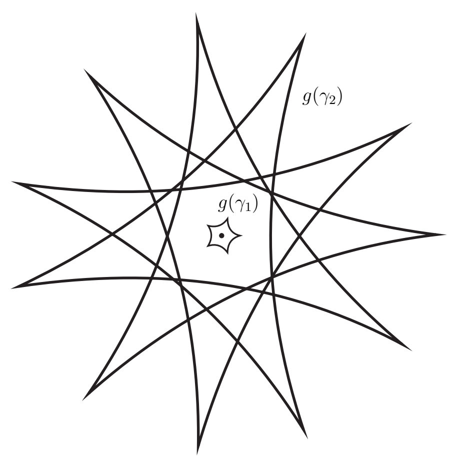

Figure 12. $g(\mathcal{C}(g))$

Acknowledgments

The second author is grateful to CNPq and FAPERJ for continuous support.

‡ For the computer program, see http://www.mat.puc-rio.br/~hjbortol/2x2/

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