TRANSITIVITY OF FINSLER GEODESIC FLOWS IN COMPACT SURFACES WITHOUT CONJUGATE POINTS AND HIGHER GENUS

JOSÉ BARBOSA GOMES AND RAFAEL O. RUGGIERO

Abstract. We show that the geodesic flow of a compact, C^4, Finsler surface without conjugate points and genus ≥ 2 is transitive. This result generalizes the work of P. Eberlein for compact Riemannian surfaces without conjugate points in the context of visibility manifolds. The core of the paper is a complete proof of the backward divergence of geodesic rays, that in the case of reversible Finsler metrics without conjugate points is straightforward from the forward divergence of rays proved by Green for Riemannian surfaces in [24], and by Barbosa-Ruggiero in [21] for Finsler surfaces. The divergence of geodesic rays in both senses allows to extend Eberlein's work [13] about visibility manifolds which leads to the proof of the transitivity of the geodesic flow in the case of compact surfaces of genus greater than one and no conjugate points.

1. Introduction

Finsler manifolds are geometric models for high energy levels of Tonelli Hamiltonians: the Hamiltonian flow of a Tonelli Hamiltonian with large enough energy is, up to parametrization, the geodesic flow of a Finsler manifold [**?**]. The word "geometric" refers to the local geometric theory of Finsler manifolds, which provides a series of shape operators generalizing Riemannian shape operators. Shape operators in Riemannian manifolds, notably the Gaussian sectional curvature, play an important role in the study of the dynamics of the geodesic flow. Sectional curvatures define the Jacobi equation that describes the linear part of the dynamics. In the Finsler realm the connection between shape operators and the dynamics of the geodesic flow is not always this clear. The flag sectional curvatures appear indeed in the second variation formula of the Finsler geometry, giving rise to a second order equation that is usually more complicated than the Riemannian Jacobi equation. But there are many other shape operators known in the Finsler literature: the Cartan scalar, the Landsberg scalar, the Berwald tensor, etc, whose connection with the geodesic flow might not be so straightforward.

Nevertheless, a dynamical approach to study Finsler manifolds has proved to be fruitful and revealing. This is the case of manifolds without conjugate points, characterized by the fact that every geodesic is a global minimizer (see Section 1 for a rigorous definition). The work of Akbar-Zadeh [**?**] applies some ideas of Anosov dynamics to show that constant negative flag curvature implies that the Finsler manifold is Riemannian. The core of this result is the relationship between the Cartan scalar and the Jacobi equation, a consequence of the Finsler Bianchi

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identities under some assumptions on the flag curvature. This result inspired some rigidity results concerning compact k-basic Finsler surfaces with higher genus [31], [20]. The work of P. Foulon (see for instance [?], [?]) is one of the first to extend the ideas of Hopf [26] and Green [24] to Finsler surfaces without conjugate points. Notably, the existence of a Riccati equation associated to the Finsler Jacobi equation and its connection to Lyapunov exponents and entropy. Recently, a natural extension of the so-called Hopf conjecture was obtained by ——, who shows for instance that the integral in the unit tangent bundle of the Ricci curvature for Finsler metrics without conjugate points is non-positive. In the case of Riemannian metrics in the two torus, this implies that the metric is flat because of Gauss-Bonnet Theorem. However, a Finsler metric might not satisfy the Gauss-Bonnet theorem, Landsberg metrics have indeed a Gauss-Bonnet theorem but there exist examples of non-flat Finsler metrics without conjugate points in the two torus due to Busemann [?] and many others. In [?], a substantial progress toward the proof of the Hopf conjecture for k-basic Finsler metrics in the two torus has been made.

We should mention the work of Egloff [18] for reversible Finsler manifolds with non-positive flag sectional curvatures, extending many geometric features of Riemannian manifolds of non-positive curvature to the Finsler category. The work of Paternain-Paternain [33] applies dynamical methods to show that the expansiveness of the Finsler geodesic flow in a compact surface implies the absence of conjugate points, with no further restrictions in the sign of the flag curvature. A Hamiltonian version for compact manifolds of a famous result du to R. Mañé [28], asserting that if the Hamiltonian flow in a regular energy level preserves a continuous, Lagrangian bundle then the flow has no conjugate points, was proved by Paternain-Paternain [?]. A celebrated characterization of Anosov Riemannian geodesic flows due to P. Eberlein [?] in terms of the transversality of the Green bundles was extended to Hamiltonian flows by Contreras-Iturriaga [11].

In a recent paper [21], a dynamical point of view is used to show that compact Landsberg surfaces without conjugate points and genus greater than one are Riemannian, a result motivated by an old conjecture about Landsberg metrics [?]. Many fundamental questions naturally raised by the theory of Riemannian manifolds without conjugate points remain still unknown in the Finsler category.

The main result of the present article is based in a well known theorem due to P. Eberlein [13]: the transitivity of the geodesic flow of a compact manifold without conjugate points whose universal covering is a visibility manifold.

Theorem 1.1. *Let (M, F) be a compact C^∞ Finsler surface without conjugate points and genus greater than one. Then the geodesic flow is transitive.*

Despite the prolific literature about Finsler manifolds without conjugate points, the transitivity of the geodesic flow was an unknown fundamental result in the category of compact Finsler surfaces of higher genus.

Let us make an outline of the proof of Theorem 1.1. The main contribution of the present paper to the subject is to show that Eberlein's ideas for Riemannian surfaces without conjugate points and higher genus can be extended to Finsler surfaces if Finsler geodesic rays diverge forwardly and backwardly. Namely, given two different unit speed geodesics $\gamma(t), \beta(t)$ in the universal covering endowed with the pullback of the Finsler metric such that $\gamma(0) = \beta(0)$, the distances $d(\gamma(t), \beta(t))$ go to $+\infty$ if $t \to +\infty$ and $t \to -\infty$. The divergence of geodesic rays is one of the most intriguing fundamental problems of the theory of manifolds without conjugate points. In the Riemannian case, it is not even known for n-dimensional compact manifolds if $n \geq 3$ (see [?]). The divergence of rays is related to regularity properties of horospheres [?], and is clear that for a reversible Finsler metric, the forward divergence of rays implies the backward divergence (by just inverting the parametrization of the rays).

The forward divergence of geodesic rays for Finsler surfaces without conjugate points was proved in [21]. So one of the main results of the paper is to show the backward divergence of geodesic rays. This requires an involved analysis of Green's proof [24] for Riemannian surfaces without conjugate points. With this result in hands, we can extend Eberlein's proof of the transitivity of the geodesic flow for Riemannian visibility manifolds using an appropriate definition of visibility for Finsler metrics. His proof is based in the interplay between the dynamics of the geodesic flow and the behavior of the extended action of fundamental group of the manifold in the compactification of the universal covering by asymptotic classes of geodesics. The compactification of the universal covering by asymptotic classes is a subtle matter: if geodesic rays are forward divergent then the compactification by forward asymptotic classes is homeomorphic to the unit disk. This was already observed in [21] for Finsler surfaces without conjugate points. If geodesic rays are backward divergent then the compactification by backward asymptotic classes is also homeomorphic to the unit disk. In the last two statements the topology involved is the so-called cone topology. We give a rigorous treatment of this problem in Section 2. Notice that in the Riemannian case, the natural involution $F(p, v) = (p, -v)$ defined in the unit tangent bundle induces an involution in the ideal boundary of the compactification which sends forward asymptotic classes to backward asymptotic classes. This might not be the case in Finsler manifolds.

The regularity of both compactifications of the universal covering allows to extend the action of covering isometries to the ideal boundary, and in Section 3 we check step by step Eberlein's proof of the transitivity of the geodesic flow for Riemannian surfaces without conjugate points and genus greater than one. Most of the steps follow straightforwardly for Finsler surfaces, so the section is sketchy but at some delicate arguments where we do point out the role of backward divergence of rays.

We also generalize Eberlein's results for visibility manifolds in any dimensions, this is the content of the last section of the paper.

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2. Preliminaries

In this section we follow [7] as main reference.

Let M be a n-dimensional, C^∞ manifold, let T_pM be the tangent space at $p \in M$, and let TM be its tangent bundle. In local coordinates, an element of T_xM can be expressed as a pair (x, y), where y is a vector tangent to x. Let $TM_0 = \{(x, y) \in TM; y \neq 0\}$ be the complement of the zero section. A C^k $(k \geq 2)$ *Finsler structure* on M is a function $F : TM \to [0, +\infty)$ with the following properties:

(i) F is C^k on TM_0;

(ii) F is positively homogeneous of degree one in y, where $(x, y) \in TM$, that is,

$$F(x, \lambda y) = \lambda F(x, y) \; \forall \, \lambda > 0$$

(iii) The Hessian matrix of $F^2 = F \cdot F$

$$g_{ij} = \frac{1}{2} \frac{\partial^2}{\partial y^i \partial y^j} F^2$$

is positive definite on TM_0.

A C^k *Finsler manifold* (or just a Finsler manifold) is a pair (M, F) consisting of a C^∞ manifold M and a C^k Finsler structure F on M.

Given a Lipschitz continuous curve $c : [a, b] \to M$ on a Finsler manifold (M, F), we define the Finsler length of c as

$$L_F(c) := \int_a^b F(c(t), \frac{dc}{dt}(t)) \, dt$$

L_F gives rise to a function $d = d_F : M \times M \longrightarrow [0, \infty)$ by

$$d_F(p, q) := \inf_c L_F(c) = \inf_c \int_a^b F(c(t), \frac{dc}{dt}(t)) \, dt$$

where the infimum is taken over all Lipschitz continuous curves $c : [0, 1] \to M$ with $c(0) = p$ and $c(1) = q$. It is clear that

$$d_F(p, q) \leq d_F(p, r) + d_F(r, q)$$

and

$$d_F(p, q) = 0 \Leftrightarrow p = q,$$

but since F is not absolutely homogeneous or reversible, $d_F(p, q)$ might not be equal to $d_F(q, p)$ and therefore, d_F might not be a distance. If M is compact, there exists $C > 0$ such that $\frac{1}{C} d_F(p, q) \leq d_F(q, p) \leq C d_F(p, q)$ for every $p, q \in M$. There are many natural ways to get a true distance from d_F, all of them equivalent when the manifold M is compact due to the above assertion ([7]).

Therefore, throughout the paper we shall use d_F as a true distance without loss of generality. One of the goals of this article is to show that indeed the lack of reversibility of the Finsler metric does not pose relevant obstructions to get Finsler versions of many well known results about the global geometry of Riemannian geodesics. The geodesic flow $\phi_t : TM \longrightarrow TM$ is the Euler-Lagrange flow associated to (M, F), a Finsler geodesic is the canonical projection of an orbit of the geodesic flow. We shall assume throughout the paper that geodesics have unit speed.

The Finsler manifold (M, F) induces naturally a Finsler structure in the universal covering $\tilde{M}$ of M, just by pulling back the Finsler structure F to the tangent space of $\tilde{M}$ by the covering map. Let us denote by $(\tilde{M}, \tilde{F})$ this Finsler manifold.

A geodesic $\sigma : [a, b] \longrightarrow \tilde{M}$ is called *forward minimising*, or simply minimising if $L_{\tilde{F}}(\sigma) \leq L_{\tilde{F}}(c)$ for all rectifiable curves $c : [a, b] \longrightarrow \tilde{M}$ such that $c(a) = \sigma(a)$, $c(b) = \sigma(b)$ (this implies that $\sigma : [s, t] \longrightarrow \tilde{M}$ is also minimising for every $a \leq s \leq t \leq b$). Notice that in general, a minimising geodesic σ might fail to be minimising if one reverses its orientation, because the Finsler metric might not be reversible. That is the meaning of the word "forward" in the term "forward minimising", the minimisation property is attached to an orientation of the geodesic.

For a non-vanishing vector $y \in T_xM$, we shall denote by $\sigma_{(x,y)}(t)$ the geodesic with initial conditions $\sigma_{(x,y)}(0) = x$ and $\sigma'_{(x,y)}(0) = y$. The *exponential map* at x, $exp_x : T_xM \to M$ is defined as usual: $exp_x(y) := \sigma_{(x,y)}(1)$.

2.1. **Chern-Rund connection (or Chern connection) and Jacobi fields.** Let T^*_xM be the cotangent space at x, and let T^*M be the cotangent bundle of M. Take local coordinates $(x_1, x_2, .., x_n)$ for M, and let $b_i = \frac{\partial}{\partial x^i}$, dx_i, $i = 1, 2, .., n$, be the corresponding basis for TM and T^*M respectively. The so-called fundamental tensor of the Finsler metric is given by

$$g_{ij(x,y)}dx^i \otimes dx^j$$

where $g_{ij(x,y)} = (\frac{1}{2}F^2)_{ij}(x, y)$, namely, g_{ij} is the ij-entry of the Hessian of $\frac{1}{2}F^2$.

The fundamental tensor is very convenient to study Finsler Jacobi fields: vector fields defined along a geodesic obtained by differentiating C^2 variations of the geodesic. The second variation formula for the Finsler length gives the Jacobi equation of the metric, whose solutions are just the Jacobi fields. This equation is more complicated than the Riemannian Jacobi equation in general (see for instance [11] for a Hamiltonian expression of the Jacobi equation). Using the fundamental tensor, it is possible to define a sort of covariant differentiation along geodesics that is "almost compatible" with the fundamental tensor (see [7] for details), such that the Jacobi equation of the geodesic acquires a Riemannian form. In the next lemma, that is suited for the applications in the present paper, we summarise some basic properties of this connection (see [7], [36] for details).

Lemma 2.1. *Let (M, F) be a C^4 Finsler manifold, let $\sigma(t)$ be a C^∞ curve, and $\sigma(t, u) : \triangle = \{(t, u); 0 \leq t \leq r, -\varepsilon < u < \varepsilon\} \longrightarrow M$ be a C^2 variation of $\sigma(t, 0) = \sigma(t)$ by C^∞ curves. Then, in the tangent space $T_{\sigma(t,u)}M$ the inner product*

$$g_T := g_{ij(\sigma(t,u),T(t,u))}dx^i \otimes dx^j$$

where $T = T(t, u) := \sigma_* \frac{\partial}{\partial t} = \frac{\partial \sigma}{\partial t}$, satisfies the following properties:

(1) *$g_T(T, T) = F^2(T)$.*
(2) *$\sigma(t)$ is a Finslerian geodesic if and only if*

$$\frac{d}{dt}g_T(V, W) = g_T(D_TV, W) + g_T(V, D_TW)$$

where V and W are two arbitrary vector fields along σ. The operator $D_T = \frac{d}{dt}$ is called covariant differentiation with reference vector T.

(3) *In particular, Finslerian geodesics satisfy*

$$D_T\left[\frac{T}{F(T)}\right] = 0.$$

The constant speed Finslerian geodesics $F(v) = c$ are the solutions of

$$D_TT = 0,$$

(4) *Assume that $\sigma(t)$ is a unit speed geodesic. Then a Jacobi field $J(t) = \partial\delta\delta u\sigma(0, u)$ along $\sigma(t)$ satisfies*

$$D_TD_TJ + R(J, T)T = 0,$$

where R is the Jacobi tensor of the Finsler metric (We shall denote as usual $J'' = D_TD_TJ$, $J' = D_TJ$). When $dim(M) = 2$,

$$R(y, u)u = K(y)[g_y(y, y)u - g_y(y, u)y], \ y, u \in T_xM \setminus \{0\}$$

where $K(y)$ is the flag curvature (see [7] for the definition).

(5) *Let $\sigma(t)$ be a unit speed geodesic. Then, if $J(t)$ is a Jacobi field along $\sigma(t)$, the component $J_\perp(t)$ of $J(t)$ that is perpendicular to $\sigma'(t)$ with respect to g_T satisfies the scalar Jacobi equation*

$$J''_\perp + KJ_\perp = 0.$$

Moreover, if $g_T(T, J(t_0)) = g_T(T, J'(t_0)) = 0$ at some point t_0, then $g_T(T, J) = 0$ at every point.

Throughout the paper, all covariant differentiations will be carried out with reference vector T. Lemma 2.1 reduces many Finsler problems concerning Jacobi fields to Riemannian ones. We shall often call the inner product g_T the *adapted Riemannian metric.*

2.2. **Conjugate points.** We say that q is *conjugate* to p along a geodesic σ if there exists a nonzero Jacobi field J along σ which vanishes at p and q. We say that (M, F) *has no conjugate points* if no geodesic has conjugate points. The following result taken from [7] (Proposition 7.1.1) has a similar, well known counterpart in Riemannian geometry.

Proposition 2.2. *Let $\sigma(t) = exp_p(tv)$, $0 \leq t \leq r$, be a unit speed geodesic. Then the following statements are mutually equivalent:*

(1) *The point $q = \sigma(r)$ is not conjugate to $p = \sigma(0)$ along σ;*
(2) *Any Jacobi field defined along σ that vanishes at p and q must be identically zero;*
(3) *Given any $V \in T_pM$ and $W \in T_qM$, there exists a unique Jacobi field $J : [0, r] \longrightarrow TM$ defined along σ such that $J(0) = V$, $J(r) = W$;*
(4) *The derivative $(exp_p)_*$ of the exponential map exp_p is nonsingular at rv.*
(5) *Each geodesic $\gamma : \mathbb{R} \longrightarrow \tilde{M}$ is minimising.*

In T_xM, we define the *tangent spheres*

$$\mathcal{S}_x(r) := \{y \in T_xM;\ F(x, y) = r\}$$

of radii r. For r small enough, $S_x(r) = exp_x[\mathcal{S}_x(r)]$ is diffeomorphic to $\mathcal{S}_x(r)$ and will be called a *geodesic sphere* in M centred at x. A natural generalisation of the Gauss Lemma is available for Finsler spheres (see for instance, [7]).

2.3. **Forward Divergence of geodesic rays.** For an arbitrary fixed point $p \in M$, the Finsler metric induces a Riemannian metric $\hat{g}$ on the punctered plane $T_pM - \{0\}$ by

$$\hat{g} = g_{ij}(y)dy^i \otimes dy^j$$

where $y = (y^1, y^2)$ are the global coordinates in T_pM. Let $d_{\mathcal{S}_p}$ be the distance induced by the Riemannian metric on the indicatrix $\mathcal{S}_p$.

Definition 2.3. The *Landsberg angle* between two directions $v, w \in T_pM - \{0\}$ is defined as

$$\measuredangle_p(v, w) = d_{\mathcal{S}_p}\left(\frac{v}{F(v)}, \frac{w}{F(w)}\right)$$

The Landsberg angle is a natural generalization of the notion of radian, which measures angles in Riemannian manifolds. There are many natural possibilities of angle notions in Finsler geometry, all of them equivalent. We shall adopt the Landsberg angle just to follow the classical literature about the subject.

We stress that the domain of the Landsberg angle is not necessarily equal to $[0, 2\pi]$. The next result is proved in [**?**].

Proposition 2.4. *[Divergence of geodesic rays] Let (M, F) be a C^4 closed Finsler surface without conjugate points, $(\tilde{M}, \tilde{F})$ the lift of F to the universal covering. Then geodesic rays diverge uniformly in $(\tilde{M}, \tilde{F})$. Namely, given $\epsilon > 0$, $R > 0$, there exists $r > 0$ such that for every pair of geodesic rays $\gamma_\theta : [0, +\infty) \longrightarrow \tilde{M}$, $\gamma_\eta : [0, +\infty) \longrightarrow \tilde{M}$, with $\gamma(0) = \beta(0)$, parametrized by arc length, and $\angle(\theta, \eta) \geq \epsilon$, we have*

$$inf\{d_{\tilde{F}}(\gamma_\theta(t), \gamma_\eta(t)), d_{\tilde{F}}(\gamma_\eta(t), \gamma_\theta(t))\} \geq R$$

for every $t \geq r$.

Since two different geodesic rays subtend a nonzero Landsberg angle at their starting point, the previous proposition shows that the distance between two geodesic rays increases uniformly with positive time at a rate depending on the initial Landsberg angle.

2.4. **Morse's shadowing of geodesics.** In this subsection we recall briefly the work of Morse [30] about quasi-geodesics in surfaces.

We say that a Lipschitz continuous curve $c : [a, b] \to \tilde{M}$ is a *forward* (A, B)-*quasi-geodesic* or just quasi-geodesic if

$$\frac{1}{A} \cdot d_{\tilde{F}}(c(s), c(t)) - B \leq L_{\tilde{F}}(c|_{[s,t]}) \leq A \cdot d_{\tilde{F}}(c(s), c(t)) + B \;\; \forall \; s \leq t \in [a, b]$$

Observe that a geodesic $\gamma : I \longrightarrow M$ is forward minimising if it is a $(1, 0)$-forward quasi-geodesic. From the work of Morse [30] about minimising geodesics in surfaces we have,

Theorem 2.5 (Morse)**.** *Let (M, g) be a Riemannian metric in a compact surface M of genus greater than 1. Let $(\tilde{M}, \tilde{g})$ be the pullback of a Riemannian metric g in M to the universal covering $\tilde{M}$. Given $A \geq 1, B \geq 0$, there exists $D > 0$ such that every forward (A, B)-quasi-geodesic $c : [t_1, t_2] \longrightarrow \tilde{M}$ of $(\tilde{M}, \tilde{g})$ is within a hyperbolic distance D from the hyperbolic geodesic joining $c(t_1)$ to $c(t_2)$ in the hyperbolic plane $(\tilde{M}, g_0)$.*

Let us define

$$\bar{d} : \tilde{M} \times \tilde{M} \to \mathbb{R}, \; \bar{d}(p, q) = \frac{1}{2}[d_{\tilde{F}}(p, q) + d_{\tilde{F}}(q, p)]$$

that is a distance in $\tilde{M}$.

The compactness of M implies

Corollary 2.6. *Let (M, F) be a compact C^2 Finsler surface. There exists $\lambda \geq 1$ such that every forward minimising geodesic $c : I \longrightarrow \tilde{M}$ is a $(\lambda, 0)$-quasi-geodesic of $(\tilde{M}, \bar{d})$.*

So combining with Morse's Theorem we get

Proposition 2.7. *Given a C^2 Finsler metric (M, F) in a compact surface of genus greater than 1 there exists $Q > 0$ such that every minimising geodesic $c : [a, b] \longrightarrow \tilde{M}$ is in a Q-tubular neighbourhood of a hyperbolic geodesic.*

Notice that Proposition 2.7 applies to the geodesics $c_{p,q}$ (from p to q) and $c_{q,p}$ (from q to p) for every $p, q \in \tilde{M}$. In particular, both geodesics (if different from each other) are in the Q tubular neighbourhood of the hyperbolic geodesic joining p to q.

2.5. **Asymptotic classes and central stable foliation.**

Proposition 2.8. *Let (M, F) be a C^4 Finsler compact surface of genus greater than 1 without conjugate points. There exists $Q > 0$ such that given $(p, v) \in T_1\tilde{M}$, $x \in \tilde{M}$, there exists a unique minimising geodesic $\gamma_{(x,w(x,v))} : [0, \infty) \to \tilde{M}$ such that*

(1) $\gamma_{(x,w(x,v))}(0) = x$.
(2) *If $\gamma^0_{(p,v)}$ is the hyperbolic geodesic in $\tilde{M}$ that satisfies $\gamma^0_{(p,v)}(0) = p$ and ${\gamma^0_{(p,v)}}'(0) = v$, then*
$$d_0(\gamma_{(x,w(x,v))}(t), \gamma^0_{(p,v)}) \leq Q$$
for every $t \geq 0$, where d_0 is the hyperbolic distance and $d_0(\gamma(t), \beta)$ is the hyperbolic distance from $\gamma(t)$ to the geodesic β.
(3) *The map $x \mapsto w(x, v)$ is continuous for every $(p, v) \in T_1\tilde{M}$.*
(4) *The geodesic $\gamma_{(x,w(x,v))}$ extends to a unique geodesic $\tilde{\gamma}_{(x,w(x,v))} : (-\infty, \infty) \to \tilde{M}$ (i.e., $\gamma_{(x,w(x,v))}(t) = \tilde{\gamma}_{(x,w(x,v))}(t)$ for every $t \geq 0$) that is forward minimising).*

We list in the next lemma some of the most important basic properties of the centre stable sets, the next result is taken from [21].

Proposition 2.9. *Let (M, F) be a C^4 closed, oriented C^4 Finsler surface without conjugate points. Then the following assertions hold:*

(1) *The family of sets $\tilde{\mathscr{F}}^{cs} = \cup_{\theta \in T_1\tilde{M}} \tilde{\mathscr{F}}^{cs}(\theta)$ is a collection of C^0 submanifolds which are either disjoint or coincide.*
(2) *The sets $\tilde{\mathscr{F}}^{cs}(\theta)$, $\theta \in T_1\tilde{M}$, depend continuously on θ, uniformly on compact subsets of $T_1\tilde{M}$, and hence the collection $\mathscr{F}^{cs} = \cup_{\theta \in T_1 M} \mathscr{F}^{cs}(\theta)$ is a continuous foliation by C^0 leaves of $T_1\tilde{M}$.*
(3) *Given $p \in \tilde{M}$, there exists a homeomorphism*
$$\Psi_p : \tilde{M} \times \tilde{V}_p \longrightarrow T_1\tilde{M}$$
such that
$$\Psi_p(\tilde{M} \times \{(p, v)\}) = \tilde{\mathscr{F}}^{cs}(p, v).$$
In particular, the collections $\mathscr{F}^{cs}$, $\tilde{\mathscr{F}}^{cs}$, are continuous foliations, and the space of leaves of $\tilde{\mathscr{F}}^{cs}(\theta)$ is homeomorphic to the vertical fibre $\tilde{V}_p$ for any $p \in \tilde{M}$.

2.6. **Compactification of the universal covering: "forward" cone topology.** The cone topology in the context of Riemannian manifolds without conjugate points goes back to Eberlein [13], many of the results of the subsection will be just natural generalizations to the Finsler context. So in most of the proofs we shall skip details and just give the main ideas. In the subsection we shall suppose that M is a compact surface of genus greater than one.

Let us start with the definition of an equivalent relation in the set of geodesics: γ, β are related if there exists a constant $C > 0$ depending on γ, β such that

$d(\gamma(t), \beta(t)) \leq C$ for every $t \geq 0$. We shall call each class a *forward asymptotic class* or the ω-*limit* of a geodesic in the class, and the collection of such classes will be denoted by Ω. By the divergence of geodesic rays, given $p \in \tilde{M}$, and $v \in \mathcal{S}_p(1)$, the geodesic $\gamma_{(p,v)}$ is unique in its equivalence class. Moreover, the classes of the geodesics $\gamma_{p,w)}$, $w \in \mathcal{S}_p(1)$, are all the equivalence classes. This is because of Lemma 2.9, the geodesics in the central set of $\gamma_{(p,w)}$ are all in the same asymptotic class, and the central sets form a continuous foliation. The standard notation for the space $\tilde{M} \cup \Omega$ is $\tilde{M}(\infty)$.

The natural topology to endow this space is the *cone topology*: a basis for the topology of a point in $\tilde{M}$ is just a basis of the topology of $\tilde{M}$; a basis for an asymptotic class $[[\gamma_{(p,v)}]]$ is the collection of complements with respect to balls centered at p of cones of geodesic rays

$$C_{(p,v)}(\epsilon) = \{exp_p(tw),\ F(w,w) = 1,\ t \geq 0,\ , d_S(v,w) < \epsilon\},$$

where $d_S(v,w)$ is the Sasaki-like distance in T_1M (see [7] for the definition) in the unit sphere $\mathcal{S}_p$, and the class of $\gamma_{(p,v)}$ is $[[\gamma]]$. Notice that we made a choice of a base point p for the construction of the basis, but it is not difficult to show, using the divergence of geodesic rays, that the different topologies obtained in this way are all equivalent. The space $\tilde{M}(\infty)$ endowed with the cone topology becomes a compact space homeomorphic to the unit disk, whose boundary is exactly Ω, the usual notation for the classes is then $\partial\tilde{M}(\infty)$.

The next result proved in [21] tells us that the center stable foliation is somehow parametrized by $\partial\tilde{M}(\infty)$.

Proposition 2.10. *There exists a homeomorphism*

$$\Psi_\infty : T_1\tilde{M} \longrightarrow \tilde{M} \times \partial\tilde{M}(\infty)$$

such that

$$\Psi_\infty(\tilde{\mathscr{F}}^{cs}_\omega) = \tilde{M} \times \{\omega\},$$

where $\tilde{\mathscr{F}}^{cs}_\omega$ *is the centre stable leaf such that all the orbits in the leaf project into geodesics of* $\tilde{M}$ *whose* ω*-limit is* $\omega \in \partial\tilde{M}(\infty)$.

We can define as well center unstable sets and *backward asymptotic* classes: the center unstable set of $(p,v) \in T_1\tilde{M}$ is the set of orbits of the geodesic flow such that

$$d(\phi_t(q,w), \phi_t(p,v)) \leq C(p,q)$$

for every $t \leq 0$ and some $C(p,q) > 0$ which depends on (p,q). The backward asymptotic class of $\gamma_{(p,v)}(t) = \pi(\phi_t(p,v))$ would be the equivalence class of geodesics satisfying the above inequality. We can consider the union of the universal covering and the backward asymptotic classes. However, if geodesic rays are not backward divergent we would get a backward asymptotic class that is an open set !! So the collection of α-limits would not be homeomorphic to the circle, the unit circle tangent to a point would not represent backward asymptotic classes. Moreover, the center unstable sets might not form a foliation.

The reversibility of the Finsler metric would allow to extend the cone topology to the union of $\tilde{M}$ and the collection of α-limits. This space would be homeomorphic to the compactification by the ω-limits, and the central unstable sets would form a continuous foliation.

This sort of "reversibility" of the cone topology plays a crucial role in the proof of the transitivity of the geodesic flow of compact Riemannian surfaces without conjugate points and genus greater than one. Of course, in the Riemannian case this is a consequence of the reversibility of the metric. This subtle problem will be the subject of the forthcoming sections.

3. Backward divergence of geodesic rays

Proposition 3.1. *Let (M, F) be a compact Finsler surface without conjugate points and genus greater than 1. Given $\epsilon > 0$, $R > 0$, there exists $T = T(\epsilon, R) < 0$ such that if $\gamma_\theta(t)$, $\gamma_\eta(t)$ are two geodesics in $\tilde{M}$ with $\gamma_\theta(0) = \gamma_\eta(0)$ then*

$$d(\gamma_\theta(t), \gamma_\eta(t)) > R$$

for every $t \leq T$.

3.1. **Backward "polar" coordinates.** Let us start with an elementary but important remark involving non-reversibility of the Finsler metric.

Lemma 3.2. *Let γ, β be two different (forward) minimizing geodesics in $\tilde{M}$ parametrized by arc length, such that $\gamma(0) = \beta(0)$. Then, any point of intersection $\gamma(t_1)$ with $\beta(-\infty, 0]$ must satisfy $t_1 > 0$.*

Proof. Otherwise, let $p = \gamma(t_1) = \beta(t_2)$ where $t_1, t_2 < 0$. Since $\gamma : [t_1, 0] \longrightarrow \tilde{M}$, $\beta : [t_2, 0] \longrightarrow \tilde{M}$ minimize the length of curves joining p and $\gamma(0) = \beta(0) = q$, we have that $t_1 = t_2$. Moreover, $\gamma'(t_1) = \beta'(t_1)$, otherwise we could apply a shortcut argument to get that there exists $\epsilon < 0$ such that $\gamma[t_1 + \epsilon, 0]$ is not a length minimizing curve. Therefore, by uniqueness of geodesics with respect to initial conditions γ must be equal to β contradicting the assumptions. □

Lemma 3.3. *Let $p \in \tilde{M}$, and let $S_p \in T_p\tilde{M}$ be the set of unit vectors tangent to $\tilde{M}$ at p. Then the map $F : T_p\tilde{M} \longrightarrow \tilde{M}$ given by $F(tv) = exp_p(tv)$, $t \leq 0$, is a homeomorphism. Moreover, the map F restricted to $T_p\tilde{M} - \{0\}$ is a diffeomorphism.*

Proof. The fact that F is a homeomorphism is straightforward from Lemma 3.2. To see that F is a diffeomorphism outside 0, observe that the differential of F applied to a unit vector W tangent to tS_p at tv is just a Jacobi field $J_{tv}(t)$ tangent to a variation by geodesics of $\gamma_{(p,v)} : [t, 0] \longrightarrow \tilde{M}$, all of them meeting at $t_0 = 0$. So $J_{tv}(0) = 0$, and the absence of conjugate points implies that $J_{tv}(t) \neq 0$ for every $t \neq 0$, in particular $DF(W) \neq 0$. By the Gauss Lemma for the adapted metric we get that the differential of F at every point is nonsingular.

□

3.2. **Backward divergence of radial Jacobi fields.** The next Lemma complements the asymptotic description of radial Jacobi fields: we already knew that the norm of radial Jacobi fields $J(t)$ tends to $+\infty$ as $t \to +\infty$, we shall show that this holds if $t \to -\infty$ too.

Lemma 3.4. *Let (M, F) be a Finsler compact surface without conjugate points, and let γ_θ be a unit speed geodesic, where $\theta \in T_1M$. Let $J(t)$ a non-vanishing Jacobi field such that $J(0) = 0$. Given $c > 0$, $a > 0$, there exists $T = T(c, a) < 0$ such that if $\| J'(0) \| \geq a$, then $\| J(t) \| \geq c$ for every $t \leq T$.*

Proof. The proof follows the same line of reasoning due to Green [24] to study forward divergence of Riemannian, radial Jacobi fields, and Finsler radial Jacobi fields [21]. So in many steps of the proof we shall just refer to [24], [21] to avoid repetition.

By the Chern-Rund connection, we can reduce the study of the Jacobi equation along $\gamma_\theta(t)$ to the study of the scalar equation $f''(t) + K(t)f(t) = 0$, where $K(t) = K((\gamma_\theta(t), \gamma'_\theta(t)))$ is the flag curvature, and the derivatives are the covariant derivatives with respect to the adapted metric. Since (M, F) has no conjugate points, the existence of Green Jacobi fields implies that there exists a solution $Y(t)$ which never vanishes. Let us suppose that $Y(t) > 0$ for every $t \in \mathbb{R}$.

Since the set of solutions $J(t)$ of the Jacobi equation which vanish at $t = 0$ is a one dimensional subspace, we can assume that $J(0) = 0$, $J'(0) = -1$ to show the lemma. The solutions $J(t)$, $Y(t)$ are linearly independent, so their Wronskian is a non-zero constant,

$$W(J(t), Y(t)) = J'(0)Y(0) - J(0)Y'(0) = J'(0)Y(0) = -Y(0) < 0.$$

Let us consider the function $h(t) = \frac{J(t)}{Y(t)}$. Its derivative is

$$h'(t) = \frac{1}{Y^2(t)}(J'(t)Y(t) - J(t)Y'(t)) = \frac{1}{Y^2(t)}W(J, Y) < 0,$$

and thus $h(t)$ strictly decreasing.

Let us suppose by contradiction that the lemma is false. Then there exists a sequence $x_n \to -\infty$, $x_{n+1} < x_n$ for every $n \in \mathbb{N}$, such that $\lim_{n\to+\infty} J(x_n) = c \in \mathbb{R}$. Let a_n the sequence given by $Y(x_n) = a_n J(x_n)$, which is positive since $J(t) > 0$ for every $t < 0$, and decreasing with n since $a_n = \frac{1}{h(x_n)}$ and $h(t)$ is decreasing. In this way we get that $\lim_{\to+\infty} a_n = a \geq 0$, so let us consider the solution of the Jacobi equation given by

$$Z(t) = Y(t) - aJ(t).$$

The above solution is linearly independent of $J(t)$, so their Wronskian $W(Z, J) = d$ is a non-vanishing constant. Moreover, we have

$$\lim_{n\to+\infty} Z(x_n) = 0.$$

Claim: $Z(t) \neq 0$ for every $t < 0$.

In fact, we have $Z(0) = Y(0) > 0$, and $Z(x_n) = Y(x_n) - aJ(x_n)$, which implies that

$$\frac{Z(x_n)}{J(x_n)} = a_n - a > 0,$$

since a_n is decreasing, so $Z(x_n) > 0$ for every $n \in \mathbb{N}$. Thus, if we had some number $\sigma < 0$ such that $Z(\sigma) < 0$, the solution $Z(t)$ would have at least two zeroes which is impossible by the no conjugate points assumption.

So we get that $Z(t) \geq 0$ for every $t < 0$, and if $Z(s) = 0$ for some $s > 0$, we would have $Z'(s) = 0 = Y'(s) - aJ'(s)$. This implies that $Y(t)$, $J(t)$ satisfy $Y(s) = aJ(s)$, $Y'(s) = aJ'(s)$, which yields $Y(t) = aJ(t)$ for every $t \in \mathbb{R}$ by the uniqueness of solutions of the Jacobi equation. Since $Y(0) > 0$ and $J(0) = 0$ this latter identity is a contradiction, so we get that $Z(t) > 0$ for every $t < 0$ as we claimed.

By the comparison theory of the Riccati equation associated to the Jacobi equation, the solutions $u_1(t) = \frac{Z'(t)}{Z(t)}$, $u_2(t) = \frac{Y'(t)}{Y(t)}$ of the Riccati equation $u' + u^2 + K = 0$ are bounded by a constant L for every $\mid t \mid \geq 1$. If we divide $W(Z,Y)$ by $Z(t)$ we get

$$\frac{Z'(t)}{Z(t)} Y(t) - Y'(t) = \frac{d}{Z(t)},$$

and taking $t = x_n$ we have that the left hand side of the equation is bounded, while the right hand side tends to $+\infty$, which is absurd. The contradiction arose from assuming that $J(t)$ is bounded for every $t < 0$, so $J(t)$ tends to $+\infty$ as $t \to -\infty$ as claimed.

The independence of T with respect to the geodesic γ_θ follows from Green's original proof [24]. □

Now, the backward divergence of geodesic rays (Proposition 3.1) follows from the same argument applied in the case of forward divergence (i.e., estimating the spherical distance between two backward geodesic rays at time $t < 0$) (see [21] Barbosa-Ruggiero Ergodic Theory and dyn syst 2012).

3.3. **Some consequences of the divergence of geodesics: "backward" compactification.** Once we know that geodesic rays are backward divergent, we can define a "backward" compactification $\tilde{M}(\infty)^* = \tilde{M} \cup \Lambda$ of $\tilde{M}$. The topology of this space is given by a basis in each point: if $p \in \tilde{M}$ the basis is the family of open sets of $\tilde{M}$ containing p; if $[[\gamma_{(p,v)}]]^* \in \Lambda$ then a basis is the set of complements of backward geodesic cones

$$C_{(p,v)}(\epsilon)^- = \{exp_p(tw),\ F(w,w) = 1,\ t \leq 0,\ , d_S(v,w) < \epsilon\},$$

with respect to balls centered at p. Then it is not difficult to show,

Lemma 3.5. *The space $\tilde{M}(\infty)^*$ is homeomorphic to $\tilde{M}(\infty)$. Its boundary is the set Λ of backward asymptotic classes of geodesics; that is homeomorphic to the circle.*

A trivial homeomorphism between Λ and Ω is obtained just by regarding each asymptotic class either as a forward or as a backward asymptotic class. Indeed, let $\gamma(t)$ be a Finsler geodesic in $(\tilde{M}, \tilde{F})$, whose α and ω limits are respectively $[[\gamma]]^*$ and $[[\gamma]]$. Let $\gamma^H(t)$ be the hyperbolic geodesic shadowing γ having the same α and ω limits. Now, consider the hyperbolic geodesic $\beta^H(t) = \gamma^H(-t)$, whose α-limit is $[[\gamma]]$ and whose ω-limit is $[[\gamma]]^*$. Using Morse's shadowing Lemma, we can construct a Finsler geodesic β in $\tilde{M}$ in a tubular neighborhood of β^H with the same α and ω limits of β^H. Therefore, $[[\gamma]]$ is the backward asymptotic class of the Finsler geodesic β, which implies that $[[\gamma]]$ is an element of Λ too.

Notice that in the Riemannian case, the involution $\Psi : T_1\tilde{M} \longrightarrow T_1\tilde{M}$, $\Psi(p,v) = (p,-v)$, induces an involution in $\partial\tilde{M}(\infty)$ where we identify Λ and Ω through the above homeomorphism. This might not be the case in Finsler geometry because the metric might not be reversible: the Finsler geodesic $\gamma_{(p,-v)}(t)$ might be different from $\gamma_{(p,v)}(t)$, with different limit points in the ideal boundary. Another application of Proposition 3.1 is the regularity of the collection of center unstable sets.

Lemma 3.6. *The collection of center unstable sets is a continuous foliation of $T_1\tilde{M}$. It gives rise to a continuous foliation of $T_1 M$ that is invariant by the geodesic* *flow. Moreover, Proposition 2.10 extends to the center unstable foliation, replacing forward asymptotic classes by backward asymptotic classes.*

For the proof we just follow step by step the proofs of Proposition 2.9, 3.1, replacing center stable sets by center unstable sets, and ω-limits by α-limits.

4. Eberlein's work about transitivity of visibility Riemannian manifolds

Now we are in shape to show the main Theorem of the article, that is inspired in the following result due to Eberlein [13]:

Theorem 4.1. *Let (M, g) be a compact Riemannian manifold without conjugate points such that $\tilde{M}$ endowed with the pullback of g by the covering map is a visibility manifold. Then the geodesic flow is topologically transitive. In particular, the geodesic flow of a compact Riemannian surface without conjugate points and genus greater than one is transitive.*

We are interested in the two dimensional case of the theorem, but we think that our methods can be extended to any dimension. The notion of visibility is strongly related to hyperbolic geometry in the large, and it involves the notion of angle between vectors. There are many alternative, more general definitions of visibility which do not depend on an angle notion, we can find a wide literature on the subject in the theory of Gromov-hyperbolic spaces. However, these definitions are well suited to study coarse geometry, the issue of transitivity of the geodesic flow is not only a global problem.

4.1. **Finsler Visibility.**

Definition 4.2. Let (M, F) be a complete Finsler metric without conjugate points. We say that (M, F) is a visibility manifold if given $\epsilon > 0$, $p \in M$, there exists $T = T(\epsilon, p) > 0$ such that

(1) For every two unit speed geodesic rays γ, β with $\gamma(0) = p = \beta(0)$, if the distance from p to every point of the geodesic joining $\gamma(r)$ to $\beta(s)$, $0 \geq s \geq r$ is larger than T then the Landsberg angle formed by $\gamma'(0)$ and $\beta'(0)$ is less than ϵ.

(2) For every two unit speed geodesics σ, η such that $\sigma(0) = p = \eta(0)$, if the distance from every point of the geodesic joining $\sigma(r)$ to $\eta(s)$, $s \leq r \leq 0$, to p is larger than T, then the Landsberg angle formed by $\sigma'(0)$ and $\eta'(0)$ is less than ϵ.

When T does not depend on p we say that (M, F) is a uniform visibility manifold.

Our visibility notion has two parts: the first one refers to angles formed by geodesics at their common starting point, we shall refer to this first item as **forward visibility**; the second one refers to angles formed by geodesics ending at the same point, we shall refer to this part as **backward visibility**. In the Riemannian case both items are equivalent of course. We focus our study on the universal covering $(\tilde{M}, \tilde{F})$ endowed with the pullback of F by the covering map. If M is compact, $(\tilde{M}, \tilde{F})$ is a visibility manifold if and only if it is a uniform visibility manifold. The first fundamental result of the section is

Theorem 4.3. *Let (M, F) be a compact Finsler surface without conjugate points and genus greater than 1. Then $(\tilde{M}, \tilde{F})$ is a uniform visibility manifold.*

Proof. The main ideas of the proof are already in Eberlein's work [13] for Riemannian manifolds. It follows essentially from a combination of Morse's shadowing Lemma and the divergence of geodesic rays, both forward and backward. We shall sketch the proof of backward visibility of geodesic triangles that is actually our new contribution to visibility theory. The proof of forward visibility is just as in Eberlein's paper.

Let us suppose by contradiction that backward visibility does not hold for $(\tilde{M}, \tilde{F})$. Then there exists $\epsilon > 0$ and a sequence of geodesic triangles Δ_n in $\tilde{M}$ with vertices a_n, b_n, c_n such that

(1) The distance from every point of the geodesic $[b_n, c_n]$ to a_n is greater than n,
(2) The angle subtended by $[b_n, a_n]$ and $[c_n, a_n]$ at the point a_n is greater or equal to ϵ.

Up to covering isometries, we can suppose that a_n belongs to a compact subset $K \subset \tilde{M}$. So there is a subsequence of such points which converges to a point p. Let us assume without loss of generality that the whole sequence a_n converges to p. Let $[x, y]_H$ be the hyperbolic geodesic joining $x, y \in \tilde{M}$, and consider the geodesics $[b_n, c_n]_H$, $[b_n, a_n]_H$ and $[c_n, a_n]_H$. Let Q be the shadowing constant defined in Proposition 2.7. Then we have

(1) The $\tilde{F}$-distance from every point in $[b_n, c_n]_H$ to a_n is greater than $n - Q$, so the hyperbolic distance from such points to a_n goes to ∞,
(2) The hyperbolic angle subtended by $[b_n, a_n]_H$ and $[c_n, a_n]_H$ at a_n goes to zero as $n \to +\infty$, by the visibility property of hyperbolic space,
(3) The geodesics $[b_n, a_n]$ and $[c_n, a_n]$ are respectively, in Q-tubular neighborhoods of the geodesics $[b_n, a_n]_H$, $[c_n, a_n]_H$.

Therefore, a subsequence of the geodesics $[b_n, a_n]_H$, $[c_n, a_n]_H$ converges to a single hyperbolic geodesic $\gamma(t)$ with $\gamma(0) = p$, whose image $\gamma(-\infty, 0]$ shadows within a distance Q two different $\tilde{F}$-geodesics

$$\alpha : (-\infty, 0] \longrightarrow \tilde{M},$$

$$\beta : (-\infty, 0] \longrightarrow \tilde{M},$$

with $\alpha(0) = \beta(0)$, which are limits of subsequences of, respectively, the geodesics $[b_n, a_n]$ and $[c_n, a_n]$. The geodesics α and β are different because the geodesics $[b_n, a_n]$ and $[c_n, a_n]$ form an angle of at least ϵ at a_n. This clearly contradicts the backward divergence of geodesic rays (Proposition 3.1). □

4.2. **Local and global geometry of Visibility manifolds.** Here we list a series of geometric consequences of the visibility property, based on Eberlein's article [13]. First of all, let us adopt a more suggestive notation for points at infinity in the compactification of $\tilde{M}$. Given a geodesic $\gamma \in \tilde{M}$, let $\gamma(\infty) \in \partial\tilde{M}(\infty)$ be its forward asymptotic class, and $\gamma(-\infty)$ be its backward asymptotic class.

Let $V^+ : \tilde{M} \times \tilde{M} \longrightarrow T_1\tilde{M}$ be the unit vector tangent at p to the geodesic joining p to q. Let $V^- : \tilde{M} \times \tilde{M} \longrightarrow T_1\tilde{M}$ be the unit vector tangent at q to the geodesic joining p to q. By the divergence of geodesic rays we can extend the functions V^+, V^- to

$$V^+ : \tilde{M} \times \tilde{M}(\infty) \longrightarrow T_1\tilde{M},$$

$$V^- : \tilde{M}(\infty) \times \tilde{M} \longrightarrow T_1\tilde{M}.$$

Indeed, if $p \in \tilde{M}$ and $q \in \partial\tilde{M}(\infty)$, there exists a unique geodesic γ with $\gamma(0) = p$ and $\gamma(\infty) = q$. Analogously, if $p \in \partial\tilde{M}(\infty)$, $q \in \tilde{M}$, there exists a unique geodesic β such that $\beta(0) = q$, $\beta(-\infty) = p$.

Let us say that $x, y \in \partial\tilde{M}(\infty)$ are *heteroclinically related* if there exists a geodesic $\gamma \subset \tilde{M}$ such that $\gamma(-\infty) = x$, $\gamma(\infty) = y$. In this case we shall say that γ *starts* at x and *ends* at y, and that x, y are the *end points* of γ. Given two points $p, q \in \tilde{M}(\infty)$, let us denote by $[p, q]$ the geodesic starting at p and ending at q. The next lemma gathers some of the basic properties of the above functions, their proofs follows from a combination of Morse's shadowing lemma, the divergence (backward and forward) of geodesic rays and the ideas in Eberlein's paper.

Proposition 4.4. *Let (M, F) be a compact Finsler surface without conjugate points and genus greater than one. Then the following statements hold:*

(1) *The functions $V^+ : \tilde{M} \times \tilde{M}(\infty) \longrightarrow T_1\tilde{M}$, $V^- : \tilde{M}(\infty) \times \tilde{M} \longrightarrow T_1\tilde{M}$ are continuous in the product cone topology.*
(2) *The function $\omega : T_1\tilde{M} \longrightarrow \partial\tilde{M}(\infty)$ which associates to each (p, v) the asymptotic class $[[\gamma_{(p,v)}]]$ is continuous.*
(3) *The function $\alpha : T_1\tilde{M} \longrightarrow \partial\tilde{M}(\infty)$ which associates to each (p, v) the backward asymptotic class $[[\gamma_{(p,v)}]]^*$ is continuous.*
(4) *Every pair of different points in $\partial\tilde{M}(\infty) \times \partial\tilde{M}(\infty)$ is heteroclinically related.*
(5) *There exists $L > 0$ such that given a pair of different points $x, y \in \partial\tilde{M}(\infty)$, the geodesics starting at x and ending at y lie in a strip $\Sigma(x, y)$ homeomorphic to $[0, 1] \times (-\infty, \infty)$ foliated by such geodesics and whose width is at most L.*
(6) *If a sequence of pairs $(x_n, y_n) \in \tilde{M}(\infty) \times \tilde{M}(\infty)$ converges in the product cone topology to (a, b), where $a \neq b$, then*
    - *If either a or b is in $\tilde{M}$ then the geodesics $[x_n, y_n]$ converge uniformly on compact sets to the unique geodesic $[a, b]$.*
    - *If $a, b \in \partial\tilde{M}(\infty)$ then the geodesics $[x_n, y_n]$ converge uniformly on compact sets to one of the geodesics in the strip $\Sigma(a, b)$.*

4.3. **The action of the fundamental group at infinity.** With the results of the previous subsection it is not difficult to get a Finsler version of all the standard properties of the extended action of $\pi_1(M)$ in $\tilde{M}(\infty)$. We follow the main ideas of Eberlein [13]. We can extend continuously in the cone topology the action of π_M to $\tilde{M}(\infty)$, as done with hyperbolic surfaces. In particular, if the surface is compact with genus greater than one the covering isometries have two fixed points; their fixed points either coincide or are disjoint; the extended action in $\tilde{M}(\infty)$ is not properly discontinuous and the limit set of this action is the whole set of boundary points $\partial\tilde{M}(\infty)$. The proofs of these statements for Finsler surfaces can be done following step by step Eberlein's arguments using the notion of Finsler visibility. We would like to focus on a notion that is crucial to prove the transitivity of the geodesic flow, called *duality* by Eberlein.

Definition 4.5. We say that two points $a, b \in \partial\tilde{M}(\infty)$ are dual if given an open neighborhood V of a and an open neighborhood W of b there exists a covering isometry ψ such that $\psi(\tilde{M}(\infty) - V) \subset W$.

Applying the Finsler visibility notion we can extend to Finsler surfaces with no major problems the next result, which gathers Proposition 2.5 and Proposition 3.6 in [13].

Proposition 4.6. *Let (M, F) be a compact surface without conjugate points of genus greater than one. Then*

(1) *Two points $a, b \in \partial\tilde{M}(\infty)$ are dual if and only if there exists a sequence of covering isometries ψ_n such that for every $p \in \tilde{M}$ we have $\psi_n(p) \to a$ and $\psi_n^{-1}(p) \to b$ if $n \to +\infty$.*

(2) *Duality for every pair of different points in $\partial\tilde{M}(\infty)$ is equivalent to*
   - *The limit set of the extended action of $\pi_1(M)$ to $\partial\tilde{M}(\infty)$ is $\partial\tilde{M}(\infty)$.*
   - *$P^+(\theta) = T_1M$ for every $\theta \in T_1M$.*

4.4. **Transitivity of the geodesic flow.** Let $\theta \in T_1M$, let $P^+(\theta)$ be the set of elements $\eta \in T_1M$ such that if O, U are any open sets containing θ, η respectively then $\varphi_t(O)$ meets U for arbitrarily large positive values of t. Equivalently $\eta \in P^+(\theta)$ if and only if there exist a sequence $\theta_n \to \theta$ and a sequence of real numbers $t_n \to +\infty$ such that $\varphi_{t_n}(\theta_n) \to \eta$. The following key remark made by Eberlein in [13] (Proposition 3.4) leads to the proof of the transitivity. We shall state it for Finsler surfaces since the proof in this case is straightforward.

Proposition 4.7. *Let (M, F) be a compact Finsler surface without conjugate points. If $P^+(\theta) = T_1M$ for every $\theta \in T_1M$ then the geodesic flow is transitive.*

So from Proposition 4.6 and Proposition 4.7, the geodesic flow is transitive if every pair of different points in the ideal boundary of $\tilde{M}(\infty)$ are dual. The last step of the proof of the transitivity is therefore based on Lemma 3.5 in Eberlein's paper [13].

Proposition 4.8. *Let (M, F) be a compact Finsler surface without conjugate points and genus greater than 1. If geodesic rays are backward divergent then the following assertion holds: let $\theta, \eta \in T_1M$, let $\tilde{\theta}, \tilde{\eta} \in T_1\tilde{M}$ be lifts of θ and η respectively by the natural projection $\pi : T_1\tilde{M} \longrightarrow T_1M$. Then $\eta \in P^+(\theta)$ if and only if $\gamma_{\tilde{\theta}}(\infty)$ and $\gamma_{\tilde{\eta}}(-\infty)$ are dual.*

Proof. We reproduce Eberlein's proof of Lemma 3.5, pointing out where the backward divergence of geodesic rays is relevant. This is perhaps the most important and subtle issue of the present article. Suppose that $\eta \in P^+(\theta)$. Let $\theta_n \to \theta$ such that $\phi_{t_n}(\theta_n) \to \eta$. Let $\tilde{\theta}, \tilde{\eta} \in T_1\tilde{M}$ be lifts of θ, η respectively, and choose a sequence of lifts $\tilde{\theta}_n$ of the points θ_n such that $\tilde{\theta}_n \to \tilde{\theta}$. Then there exists a sequence $\psi_n \in \pi_1(M)$ such that $(\psi_n \circ \gamma_{\tilde{\theta}_n})'(t_n) \to \tilde{\eta}$. Since geodesic rays are forward divergent, it is easy to show that $\gamma'_{\tilde{\theta}_n}(t_n) \to \gamma_{\tilde{\theta}}(\infty)$ in the cone topology. If $\tilde{\eta} = (q, w)$ then by the choice of $\tilde{\theta}_n$, t_n we have that $d(\gamma_{\tilde{\theta}_n}(t_n), \psi_n^{-1}(q)) \to 0$. Hence, $\psi_n^{-1}(q) \to \gamma_{\tilde{\theta}}(\infty)$ in the cone topology.

Claim: $\psi_n(q) \to \gamma_{\tilde{\eta}}(-\infty)$.

Based on the proof of Lemma 3.5 in [13], we consider the geodesics $\alpha_n(t) = (\psi_n \circ \gamma_{\tilde{\theta}_n})(t + t_n)$. Observe that $\alpha'_n(0) = D\psi_n(\gamma_{\tilde{\theta}_n})'(t_n)) \to \tilde{\eta}$. Moreover, $\alpha_n(-t_n) = (\psi_n \circ \gamma_{\tilde{\theta}_n})(0)$. Let $\tilde{\theta}_n = (p_n, v_n)$, $\tilde{\theta} = (p, v)$. Then we have

$$\alpha_n(-t_n) = (\psi_n \circ \gamma_{\tilde{\theta}_n})(0) = \psi_n(p_n).$$

Since geodesic rays diverge backwards and $\psi_n^{-1}(\alpha_n(-t_n) = p_n \to p$, by Proposition 4.4 item (3) the sequence $\alpha_n(-\infty)$ converges in the cone topology to $\gamma_{\tilde{\eta}}(-\infty)$. By the uniform backward divergence of geodesic rays, we conclude that the sequence $\psi_n(p)$ converges to $\gamma_{\tilde{\eta}}(-\infty)$ in the cone topology. By uniform Finsler visibility, we get that $\psi_n(q)$ converges to $\gamma_{\tilde{\eta}}(-\infty)$ as well.

Conversely, suppose that $\gamma_{\tilde{\theta}}(\infty)$ and $\gamma_{\tilde{\eta}}(-\infty)$ are dual. Consider a sequence $\psi_n \in \pi_1(M)$ such that for any point $x \in \tilde{M}$, $\psi_n^{-1}(x) \to \gamma_{\tilde{\theta}}(\infty)$ and $\psi_n(x) \to \gamma_{\tilde{\eta}}(-\infty)$. Let $\tilde{\theta} = (\tilde{p}, \tilde{v}), \tilde{\eta} = (\tilde{q}, \tilde{w})$. Let $t_n = d(\psi_n(\tilde{p}), \tilde{q})$, $v_n = V^+(\psi_n(\tilde{p}), \tilde{q})$. Since $\psi_n(\tilde{p}) \to \gamma_{\tilde{\eta}}(-\infty)$ we have by Proposition 4.4 (1), (3) that

$$\tilde{\phi}_{t_n}(v_n) = V^-(\tilde{q}, \psi_n(\tilde{p})) \to w$$

if $n \to +\infty$. Here, $\tilde{\phi}_t$ is the geodesic flow in $T_1\tilde{M}$. Notice that Proposition 4.4 item (3) holds because geodesic rays are backward divergent. Hence, there exists $\delta_n \to 0$ such that $d_S(\tilde{\phi}_{t_n}(v_n), w) \leq \delta_n$ for every n large enough, where d_S is the distance in the Sasaki-like metric.

Moreover, since the image by ψ_n^{-1} of the geodesic joining $\psi_n(\tilde{p})$ to $\tilde{q}$ is the geodesic joining $\tilde{p}$ to $\psi_n^{-1}(\tilde{q})$, and $\psi_n^{-1}(\tilde{q}) \to \gamma_{\tilde{\theta}}(\infty)$, we have by Proposition 4.4 items (1), (2) that

$$\psi_n^{-1}(v_n) = V^+(\tilde{p}, \psi_n^{-1}(\tilde{q})) \to v$$

if $n \to \infty$. Notice that Proposition 4.4 (2) follows from the forward divergence of geodesic rays. This clearly implies that $d_S(v_n, \psi_n(v)) = d_S(\psi_n^{-1}(v_n), v) \to 0$ as $n \to \infty$, where d_S is the Sasaki distance. So in any open set of T_1M containing $\theta = (p, v)$ we find a point $\pi(\psi_n^{-1}(v_n))$ whose positive orbit by the geodesic flow ϕ_t meets an open ball around $\eta = (q, w)$ of radius δ_n. This implies that $\eta \in P^+(\theta)$ as we wished to show. □

Definition 4.9. Let $L(\pi_1(M))$ be the set of accumulation points in $\tilde{M} \cup \partial\tilde{M}$ of an orbit of a point $p \in \tilde{M}$. By the uniform visibility axiom, $L(\pi_1(M))$ does not depend on the particular point $p \in \tilde{M}$.

Definition 4.10. We say that $\theta \in T_1M$ is nonwandering if $\theta \in P^+(\theta)$. Let Ω denote the set of nonwandering points in T_1M.

Proposition 4.11. *Let (M, F) be a compact Finsler surface without conjugate points. Then $P^+(\theta) = T_1M$ for every $\theta \in T_1M$.*

Proof. The fact that M is compact implies that $\Omega = T_1M$. Now, we prove that $\Omega = T_1M$ implies $P^+(\theta) = T_1M$ for every $\theta \in T_1M$. Let $x \in \partial\tilde{M}(\infty)$ be given and let γ_θ be any geodesic such that $\gamma_{\tilde{\theta}}(\infty) = x$. Then $\theta \in P^+(\theta)$ since $\Omega = T_1M$. By Proposition 4.8, $\gamma_{\tilde{\theta}}(\infty)$ and $\gamma_{\tilde{\theta}}(-\infty)$ are dual. By Proposition 4.6, (1), dual points must lie in $L(\pi_1(M))$. Then $x \in L(\pi_1(M))$. Therefore, if $\tilde{\theta}, \tilde{\eta} \in T_1\tilde{M}$ are lifts of $\theta, \eta \in T_1M$ respectively by the natural projection, $\gamma_{\tilde{\theta}}(\infty)$ and $\gamma_{\tilde{\eta}}(-\infty)$ are in $L(\pi_1(M))$. It follows, by Proposition 4.6, (2), that $\gamma_{\tilde{\theta}}(\infty)$ and $\gamma_{\tilde{\eta}}(-\infty)$ are dual. By Proposition 4.8, $\eta \in P^+(\theta)$. As θ, η are arbitrary, we conclude that $P^+(\theta) = T_1M$ for every $\theta \in T_1M$. □

Observe that $P^+(\theta) = T_1M$ for every $\theta \in T_1M$ implies at once that $\Omega = T_1M$. From Proposition 4.7 and Proposition 4.11 it follows Theorem 1.1.

5. Further consequences of the forward and backward divergence of geodesic rays

5.1. **Regularity of horocycles.**

Lemma 5.1. *Let (M,F) be a compact Finsler surface without conjugate points. Then stable and unstable horocycles are $C^{1,L}$ codimension 1 submanifolds, as well as Busemann functions b^{θ}. The central stable set of θ is the collection of orbits of the geodesic flow of $T_1\tilde{M}$ such that their canonical projections give rise to geodesics which are everywhere tangent to the vector field $-\nabla b^{\theta}$.*

Proof. We sketch the main ideas of the proof since it goes as in [21] Barbosa-Ruggiero ETDS (Lemma ??): By the theory of the Riccati equation, there exists $L > 0$ such that for every $r > 1$ the modulus of the curvature of the sets $S^{\sharp}_{p}(r) = \{x \in \tilde{M}, d(x,p) = r\}$ is bounded by L. The stable horocycle $H_{\theta}(0)$ of the geodesic γ_{θ} at $t = 0$ is the limit, as $T \to +\infty$, in the compact-open topology, of the sets $S^{\sharp}_{T}(\gamma_{\theta}(T))$. So by Arzela-Ascoli theorem, stable horocycles are $C^{1,L}$ submanifolds. The same regularity for Busemann functions follows from this fact, as in the case of Riemannian manifolds.

The geodesics starting at points in $S^{\sharp}_{T}(\gamma_{\theta}(T))$ and ending at $\gamma_{\theta}(T)$ have as limits, as $T \to +\infty$, geodesics which are forwardly asymptotic to $\gamma_{\theta}(t)$. This follows from Morse's shadowing lemma. Such geodesics generate a flow whose tangent vector field is $-\nabla b^{\theta}$. By the (forward) divergence of geodesic rays, such geodesics must coincide with the geodesics defining the central stable set of θ. Analogous results hold for the unstable horocycles, replacing in the above argument the sets $S^{\sharp}_{T}(\gamma_{\theta}(T))$ by true spheres centered at $\gamma_{\theta}(T))$, with $T \to -\infty$. This finishes the proof of the lemma. □

5.2. **Results in dimension n.**

What about higher dimensions? The divergence of geodesic rays in the universal covering of manifolds without conjugate points is not known. However, Eberlein in [13] shows that any Riemannian metric without conjugate points in a compact manifold having one visibility structure, is a visibility manifold too provided that geodesic rays diverge. We next sketch the proof of a version of this statement for Finsler manifolds, in the light of Gromov hyperbolic spaces. The theory of Gromov hyperbolic spaces was developed later than the theory of visibility manifolds. In many respects, the former generalizes the latter and better captures the impact of topology in the course geometry of the universal covering. In order to state the main results of the section we shall need to extend some definitions concerning Gromov hyperbolicity.

Definition 5.2. Given a set X, a function $d : X \times X \longrightarrow \mathbb{R}$ will be called a Finsler distance if

(1) $d(p,q) = 0$ if and only if $p = q$.
(2) There exists $L > 0$ such that $d(q,p) \leq Ld(p,q)$ for every $p,q \in X$.
(3) $d(p,q) \leq d(p,x) + d(x,q)$ for every $p,x,q \in X$.

The pair (X,F) will be called a Finsler metric space.

A Finsler metric space might not be a metric space, however it inherits a natural distance $\bar{d}(p,q) = \frac{1}{2}(d(p,q) + d(q,p)$ from the Finsler distance. It is straightforward to see that d is continuous in $X \times X$, where we endow X with the $\bar{d}$-topology and

$X \times X$ with the product $\bar{d}$-topology. We shall say that (X, d) is complete if $(X, \bar{d})$ is complete as a metric space.

Definition 5.3. Let (X, d) be a complete, geodesic Finsler metric space. (X, d) is a *Gromov hyperbolic space* if there exists $\delta > 0$ such that every geodesic triangle formed by the union of three geodesic segments $[x_0, x_1], [x_1, x_2], [x_2, x_0]$ satisfies the following property: the distance from any $p \in [x_i, x_{i+1}]$ to $[x_{i+1}, x_{i+2}] \cup [x_{i+2}, x_i]$ is bounded above by δ (the indices are taken mod. 3).

A complete Finsler metric space (X, d) is called *geodesic* if for every pair of points $p, q \in X$ there exists a continuous embedded curve $\gamma_{pq} : [0, a] \longrightarrow X$ such that $\gamma_{pq}(0) = p$, $\gamma_{pq}(a) = q$, and $d(\gamma(t), \gamma(s)) =\mid t - s \mid$ for every $0 \leq s \leq t \leq a$. The curve γ_{pq} is a sort of "oriented" isometry from the inteval $[0, a]$ endowed with the Euclidean metric and its image endowed with the restriction of the distance d. The usual notion of geodesic metric space is made for true distances, in particular an usual geodesic metric space is a "reversible" Finsler geodesic , in the sense that the constant L in the definition is 1, and the curves γ_{pq} are actually isometries: $d(\gamma(t), \gamma(s)) =\mid t - s \mid$ for every $0 \leq s, t \leq a$.

The notion of geodesic space is quite general, it allows to metricize the fundamental group $\pi_1(M)$ through the so-called Cayley graph: given a finite set of symmetric generators of $\pi_1(M)$ the Cayley graph is a graph whose vertices are the elements of the group where an edge joins any two elements which differ by the addition of just one more generator to the left. The elements of the group are viewed as words in the generators, and we assign length 1 to each edge, we get a left invariant distance in the Cayley graph called the word metric d_W: if $a, b \in \pi_1(M)$ then $d_W(a, b)$ is the connected union of edges connecting the identity and $a^{-1}b$ with the smallest number of edges. For details see [**?**].

The following result settles the relationship between Gromov hyperbolicity and visibility manifolds, it is proved for Riemannian manifolds in [**?**] and its proof for Finsler is just about the same with the definition of visibility given in Section 1.

Theorem 5.4. *Let (M, F) be a compact C^∞ Finsler manifold without conjugate points. Then $(\tilde{M}, \tilde{F})$ is a visibility manifold if and only if $(\tilde{M}, \tilde{F})$ is a Gromov hyperbolic space and geodesic rays diverge.*

The next statement is a fundamental result of the theory of Gromov hyperbolic spaces.

Theorem 5.5. *Let (M, F) be a compact, complete geodesic Finsler metric space. Then $\pi_1(M)$ is Gromov hyperbolic if and only if $(\tilde{M}, \tilde{F})$ is Gromov hyperbolic.*

The proof for the Finsler case is a straightforward extension of the Riemannian case and is a consequence of two basic facts:

(1) The existence of a quasi-isometry between the Cayley graph of $\pi_1(M)$ endowed with the word metric and $(\tilde{M}, \tilde{F})$,

(2) The invariance of Gromov hyperbolicity under quasi-isometric maps.

To fulfill the details we refer to [**?**].

Theorem 5.6. *Let (M, F) be a compact C^∞ Finsler manifold without conjugate points such that $(\tilde{M}, \tilde{F})$ is a visibility manifold. Then any Finsler structure (M, G) without conjugate points such that geodesic rays diverge in $(\tilde{M}, \tilde{G})$ satisfies the visibility property.*

The proof is a combination of Theorems 5.4 and 5.5: since $(\tilde{M},\tilde{F})$ is a visibility manifold, by Theorem 5.4 the fundamental group of M is Gromov hyperbolic, which implies that $(\tilde{M},\tilde{G})$ is Gromov hyperbolic too by Theorem 5.5. Since geodesic rays diverge in $(\tilde{M},\tilde{G})$ we get by Theorem 5.4 that it is a visibility manifold.

Finally, Theorem 1.1, Proposition **??**, Lemma 5.1, have natural extensions to higher dimensions.

Theorem 5.7. *Let (M,F) be a compact Finsler manifold without conjugate points such that $(\tilde{M},\tilde{F})$ is a visibility manifold. Then the geodesic flow of (M,F) is topologically transitive, center stable (unstable) sets define a center stable (unstable) continuous foliation of T_1M, and stable (unstable) horospheres are $C^{1,L}$ submanifolds.*

5.3. **Density of horocycle flows.**

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Lemma 5.8. *Let p_n, q_n, r_n be sequences in $\tilde{M}$ such that $p_n \to p$, $q_n \to q$ and $r_n \to +\infty$. Then:*
a) If β, γ are geodesics with

$$\beta'(0) = v, \gamma'(0) = w, V^+(p_n, r_n) \to v, \text{ and } V^+(q_n, r_n) \to w$$

then β and γ are forward asymptotic;
b) If β, γ are geodesics with $\beta'(0) = v$, $\gamma'(0) = w$, $V^-(p_n, r_n) \to v$, and $V^-(q_n, r_n) \to w$ then β and γ are backward asymptotic.

Proof. The proof follows from forward and bacward visibility and the triangular inequality, as in [13]. For a proof of triangular inequality in Finsler metrics, a reference is [7]. $\square$

Lemma 5.9. *Let β be a geodesic in $\tilde{M}$ and p any point of $\tilde{M}$. Then there exists a unique geodesic γ such that $\gamma(0) = p$ and γ is forward asymptotic to β. Analogously, there exists a unique geodesic γ such that $\gamma(0) = p$ and γ is backward asymptotic to β.*

Proof. The uniqueness follows from divergence of geodesic rays forward, and backward respectively. The existence follows from Lemma 5.8. $\square$

Corollary 5.10. *Let $p \in \tilde{M}$, $x \in \tilde{M}(\infty)$ be given. If σ is a geodesic of $\tilde{M}$ such that $\sigma(\infty) = x$ then $V^+(p, \sigma(t)) \to V^+(p, x)$ as $t \to +\infty$. If σ is a geodesic of $\tilde{M}$ such that $\sigma(-\infty) = y$ then $V^-(\sigma(t), p) \to V^-(y, p)$ as $t \to -\infty$.*

Proposition 5.11. *Every pair of different points $x, y \in \partial\tilde{M}(\infty)$ is heteroclinically related, that is, there exists a geodesic $\sigma \subset \tilde{M}$ such that $\sigma(-\infty) = x, \sigma(\infty) = y$.*

Proof. Let $p \in \tilde{M}$ fixed and let β, γ be the unique geodesics such that

$$\beta(0) = \gamma(0) = p, \ \beta(-\infty) = x, \gamma(\infty) = y,$$

given by Lemma 5.9. For a sequence (t_n) with $t_n \to +\infty$, let c_n be the unique geodesic segment from $\beta(t_n)$ to $\gamma(t_n)$. Let p_n be a point on c_n closest (using the metric $\overline{d}$) to p and parametrize c_n so that $c_n(0) = p_n$. By backward and forward visibility, $d(p, p_n)$ and $d(p_n, p)$ is bounded. Therefore the sequence $(c_n'(0))$ belongs to a compact subset, and passing to a subsequence if necessary, $(c_n'(0))$ is convergent, $c_n'(0) \to v$. Let σ be the geodesic with $\sigma'(0) = v$. By Lemma 5.9 $\sigma(-\infty) = x$ and $\sigma(\infty) = y$. $\square$

Proposition 5.12. *The functions V^+ and V^- are continuous.*

Proposition 5.13. *If v_n is a sequence of unit vectors in $\tilde{M}$ that converge to a vector v then*

$$\gamma_{v_n}(t_n) \to \gamma_v(\infty) \text{ as } t_n \to +\infty$$

and

$$\gamma_{v_n}(t_n) \to \gamma_v(-\infty) \text{ as } t_n \to -\infty$$

Proof. Let p_n the point of tangency of v_n, and p the point of tangency of v. By Corollary 5.10 $V^+(p_n, \gamma_{v_n}(t_n)) = v_n \to v = V^+(p, \gamma_v(\infty))$. By Proposition 5.12 it follows that $\gamma_{v_n}(t_n) \to \gamma_v(\infty)$ The case $t_n \to -\infty$ is analogous. $\square$

Departamento de Matemática, Universidade Federal de Juiz de Fora, Juiz de Fora, MG, Brazil, 36036-900
E-mail address: barbosa.gomes@ufjf.edu.br

Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro, Rio de Janeiro, RJ, Brazil, 22453-900
E-mail address: rorr@mat.puc-rio.br