Intersections of Hugoniot curves with Sonic Surfaces in the Wave Manifold

C.S.Eschenazi C.F.B.Palmeira

Abstract

Local 1-shock curve arcs and local reverse 2-shock curve arcs were constructed, in the geometric context of the wave manifold, in a previous paper. In that geometric context, sonic and sonic’ surfaces are the boundaries of admissible shock curve arcs; in order to introduce the concept of non-local 1-shock curve arcs and non-local reverse 2-shock curve arcs it is necessary to understand how a hugoniot curve intersects the sonic and sonic’ surfaces; intersections of such curves with sonic’ surface are well known. In this paper we present a complete study on how Hugoniot curves intersect the sonic surface for a quadratic system of two conservation laws.

1 Introduction

The classical construction of Riemann solutions for certain systems of conservation laws is unable to establish their global existence or stability. The main obstacle is the fact that Hugoniot curves do not foliate state space, so they can only be part of a local coordinate system. Structural stability in the classical context was considered in [3] and [7]. In [1] a new construction of Riemann solutions was presented: the state space was replaced by a 3-dimensional manifold, the wave manifold, which is foliated (possibly with singularities) by rarefaction and Hugoniot curves. Lax’s and Liu’s construction of Riemann solutions for systems of two conservation laws were generalized to the geometric framework by introducing the concept of intermediate surfaces, which replace wave curves. As transversality is frequent in the wave manifold, structural stability of Riemann solutions follows naturally. This is the main advantage of the new construction over the classical construction. Results in that paper extend Liu’s admissibility criterion, [5],
to the wave manifold and introduce local 1-shock curves and local reverse 2-shock curves for case IV in the classification of Schaeffer and Shearer, [8]. It was also shown that Liu’s admissibility criterion implies Lax’s inequalities. Since Lax’s inequalities are reversed on sonic and sonic’ surfaces, [4], these two surfaces are the boundaries of admissible shock curve arcs.

In order to solve a Riemann problem it is necessary to characterize non-local 1-shock curves and non-local reverse 2-shock curves. So, it is necessary to study intersections of Hugoniot curves with sonic and sonic’ surfaces. Intersections of Hugoniot curves with the sonic’ surface were studied in [2]. In this paper we will study their intersections with the sonic surface in the symmetric case.

Generically, a Hugoniot curve intersects, transversally, the sonic surface in 0, 2 or 4 points. Tangencies occur along a curve, called sonic fold, which splits the sonic surface into regions. The sonic fold curve is the hysteresis locus in [4]. We characterize precisely the projection of the sonic fold curve in state space, for each of the four cases in the classification of Scheaffer and Shearer.

In a forthcoming paper, we will combine these results with the ones in [2], to split the wave manifold into regions according to the existence of non-local admissible arcs, in the manner of [1].

This paper is organized as follows: In Section 2 we recall the definition of the wave manifold, extend to this manifold the concepts of Hugoniot and Hugoniot’ curves, describe the characteristic, sonic and sonic’ surfaces, [4], and discuss their relationship. In Section 3 we characterize the intersections of Hugoniot curves with sonic surface in each of the four cases in the classification of Scheaffer and Shearer. In the Appendix we present the proof of a technical lemma stated in Section 3.3.

2 The wave manifold and Hugoniot curves

We refer the reader to section 2 of [2] for basic definitions. For sake of completeness we will briefly present what is need for this paper. We consider the equation

\[ W_t + F(W)_x = 0, \] (1)
with $F$ given by
\[
\begin{align*}
  f(u, v) &= \frac{v^2}{2} + (b_1 + 1)\frac{u^2}{2} + a_1 u + a_2 v \\
  g(u, v) &= uv - b_2 v^2/2 + a_3 u + a_4 v
\end{align*}
\] (2)

where, $b_1 \notin \{0, \pm 1, 1 + b_2^2/4\}, b_2^2 + 4/(b_1 + 1) \neq 0$ and $a_3 - a_2 \neq 0$. We will work with $b_2 = 0$ (symmetric case).

After a suitable change of coordinates (explicitly done in [2]) we get the manifold $M^3$ given by
\[
(1 - Z^2)\tilde{V} - Z\tilde{U} + c = 0
\]

and
\[
Y = ZX.
\]

Since $Z$ is a direction, $M^3$ is actually a submanifold of $\mathbb{R}^4 \times \mathbb{RP}^1$, or equivalently, of $\mathbb{R}^4 \times S^1$. In previous work [6], [2] the authors have taken advantage of the fact that there are no special features at $Z = \infty$, to use only $\tilde{U}, \tilde{V}, Z, X$ coordinates and consider $M^3$ as given by $G = 0$ in $(\tilde{U}, \tilde{V}, Z, X)$-space, where $G = (1 - Z^2)\tilde{V} - Z\tilde{U} + c$.

We also have the following expression for the speed of a shock
\[
s = Z\tilde{V} + m + (b_1 + 1)(\tilde{U} - b_2\tilde{V})/b_1,
\] (3)

where $m = a_1 + (b_1 + 1)(a_4 + b_2a_2 - a_4)/b_1$.

In state space ($(u, v)$-plane) we introduce new coordinates $k = b_1 u + 2a_1 + 2b_2a_2 - 2a_4$ and $l = 2v + 2a_2$. Given a point $(k, l)$ we define the Hugoniot curve in $M^3$ as the set of points $(\tilde{U}, \tilde{V}, Z, X)$ satisfying the system
\[
\begin{align*}
  (1 - Z^2)\tilde{V} - Z\tilde{U} + c &= 0 \\
b_1 X + 2\tilde{U} - 2b_2\tilde{V} &= k \\
ZX + 2\tilde{V} &= l.
\end{align*}
\] (4)

The projection of this curve in state space are the classical Hugoniot curves, as defined in [1]. Hugoniot’ curves are obtained by changing $X$ by $-X$ in system (4).

System (4) is a linear system in $\tilde{U}, \tilde{V}$, and $X$ with determinant $2Zp(Z)$, where $p(Z) = Z^2 + b_2Z + b_1 - 1$. So, if $Zp(Z) \neq 0$, Hugoniot curves are parametrized by $Z$. Solving the above system when $Zp(Z) \neq 0$, one gets
\[
\begin{aligned}
\tilde{U} &= (kZ^3 - b_1lZ^2 + (2cb_2 - k)Z + b_1(l + 2c))/[2Zp(Z)] \\
\tilde{V} &= (-kZ + b_1l + 2c)/[2p(Z)] \\
X &= (lZ^2 + (k + b_2l)Z - (l + 2c))/[Zp(Z)].
\end{aligned}
\] (5)

Since we assume that the discriminant \(b_2^2 - 4(b_1 - 1)\) of \(p\) is nonzero, \(p\) has distinct roots; if these roots are real, denote them by \(Z_1 < Z_2\). Also, let \(Z_0 = 0\). Hugoniot curves are singular along one or three straight lines \(B_i, i = 0, 1, 2\), defined by:

\[
\begin{aligned}
Z &= Z_i, \quad \tilde{V} = \frac{Z_i\tilde{U} - c}{1 - Z_i^2}, \quad X = -2[(Z_i^2 + 1)\tilde{U} - 2cZ_i] \frac{b_1 + b_2Z_i}{b_1(1 - Z_i^2)^2}.
\end{aligned}
\]

In the same way, Hugoniot' curves are singular along the straight lines \(B'_i\), obtained from \(B_i\) by changing \(X\) into \(-X\). Let \(B = \cup_{i=0}^2 B_i\), and \(B' = \cup_{i=0}^2 B'_i\).

The local structure of Hugoniot curves near \(B_i\) is the same as the curve \(x^2 - y^2 = c, Z = c'\) near the \(z\)-axis in \(\mathbb{R}^3\).

From (5) it is clear that the number of connected components in a generic Hugoniot curve is given by the number of real roots of \(Zp(Z)\).

We define the sonic surface \(Son\) in \(M^3\) as the set of points where the speed \(s\) is extremal along a Hugoniot curve. In [6] it is shown that \(Son\) is given by the equation

\[
A_U\tilde{U} + A_V\tilde{V} + A_XX = 0,
\] (6)

where

\[
\begin{aligned}
A_U &= 2(Z^2 + b_1 + 1) \\
A_V &= 2Z(Z^2 - b_2Z + b_1 + 3) \\
A_X &= -Z^2 - b_2(b_1 + 1)Z + b_1 + 1.
\end{aligned}
\]

Similarly, we define the sonic' surface \(Son'\) as the set of points where the speed \(s\) is extremal along a Hugoniot' curve. The equation for \(Son'\) is obtained by changing \(X\) by \(-X\) in equation 6.

\[
A_U\tilde{U} + A_V\tilde{V} - A_XX = 0.
\] (7)

Both \(Son\) and \(Son'\) are Moebius band embedded in \(M^3\), and not cylinders as stated in [6].

We remark that what we are calling here Hugoniot curves following [1] were called shock curves in [6], [4], [2].
3  Hugoniot curves and the Sonic Surface $Son$

3.1 Preliminaries

In this section we will show that a Hugoniot curve intersects $Son$ in 0, 2 or 4 distinct points depending on $(k, l)$. Substituting $\tilde{U}$, $\tilde{V}$ and $X$ in equation (5) we get the following polynomial:

$$S(Z) = (b_2k + 2c - l)Z^4 - (2b_2l(b_1 + 1) + 4k)Z^3$$
$$+ (2l + 4b_1c - b_2k(b_1 + 1) - b_2^2l(b_1 + 1) + 4(b_1l + 2c))Z^2$$
$$+ 2b_2(b_1 + 1)(l + 2c)Z + (b_1^2 - 1)(l + 2c).$$

Real roots of $S(Z)$ correspond to intersections points of Hugoniot curves and $Son$. Double roots of $S(Z)$ correspond to tangency points of Hugoniot curves and $Son$. So tangency points are obtained solving system:

$$\begin{cases}
S(Z) = 0 \\
\dot{S}(Z) = 0,
\end{cases}$$

where we use a dot to indicate differentiation with respect to $Z$.

By noting that $S(Z)$ can be written as $S(Z) = \alpha(Z)k + \beta(Z)l + \gamma(Z)$, we can regard the last system as a linear system in $k$ and $l$:

$$\begin{cases}
\alpha(Z)k + \beta(Z)l = -\gamma(Z) \\
\dot{\alpha}(Z)k + \dot{\beta}(Z)l = -\dot{\gamma}(Z),
\end{cases} \quad (8)$$

Assuming, $\alpha\dot{\beta} - \beta\dot{\alpha} \neq 0$ we can solve this system, to obtain:

$$\begin{cases}
k = \frac{-2c(b_1 + 1)(b_2Z^3 - 6Z^2 - 3b_2(b_1 + 1)Z - b_2^2(b_1 + 1) + 2(b_1 - 1))}{(2 + b_2^2(b_1 + 1))Z^3 - 3b_2(b_1 + 1)Z^2 + 6(b_1 + 1)Z + b_2(b_1 + 1)} \\
l = \frac{2c(2Z^3 + 3b_2(b_1 + 1)Z^2 - 6(b_1 + 1)Z - b_2(b_1 + 1)^2)}{(2 + b_2^2(b_1 + 1))Z^3 - 3b_2(b_1 + 1)Z^2 + 6(b_1 + 1)Z + b_2(b_1 + 1)}.
\end{cases} \quad (9)$$

Hugoniot curves through points $(k, l)$ satisfying equations (9) are tangent to the sonic surface. These Hugoniot curves define a regular curve in $Son$, called sonic fold. Equations (9) can be interpreted as the parametric equations of the projection, by Hugoniot curves, in $(k, l)$-space of the sonic fold. We will call this curve $(k, l)$-sonic fold.

Other tangency points are obtained if $\alpha\dot{\beta} - \beta\dot{\alpha}$ and $\beta\dot{\gamma} - \dot{\beta}\gamma$ have common real roots. If $Z_1$ is a common real root of $\alpha\dot{\beta} - \beta\dot{\alpha}$ and $\beta\dot{\gamma} - \dot{\beta}\gamma$, then we have
a straight line in the \((k,l)\)-plane given by \(\alpha(Z_1)k + \beta(Z_1)l + \gamma(Z_1) = 0\), as part of solution of the system (8). Straightforward computations show that \(\alpha\dot{\beta} - \beta\dot{\alpha}\) and \(\beta\dot{\gamma} - \beta\gamma\) have \(2Z(Z^2 + b_2Z + b_1 - 1)\) as their greatest common divisor. In this way, we have the straight line \(l = -2c\), \(r_0\), corresponding to \(Z = 0\) and 2 others if \(b_2^2 - 4(b_1 - 1) > 0\). We remark that, for each \(i = 0, 1, 2\), \(r_i\) is the projection onto \((k,l)\)-plane of the straight lines \(B_i\), \(i = 0, 1, 2\), defined in section 2. Classically these straight lines are called \textit{secondary bifurcations}.

From now on we will consider \(b_2 = 0\) and \(c > 0\). Substituting \(b_2 = 0\) and using \(Z = 1/z\) in equations (9), the \((k,l)\)-sonic fold is written as

\[
\begin{align*}
k &= \frac{-2cz((b_1^2 - 1)z^2 - 3(b_1 + 1))}{3(b_1 + 1)z^2 + 1} \\
l &= \frac{-2c(3(b_1 + 1)z^2 - 1)}{3(b_1 + 1)z^2 + 1}.
\end{align*}
\]

It is easy to see that this curve has the line \(l = -2c\) as horizontal asymptote as \(z\) goes to infinity. There are two more asymptotes if \(\frac{1}{3(b_1 + 1)} < 0\) or, equivalently, \(b_1 < -1\), corresponding to case I, in the Schaeffer and Shearer classification (D3.2 case in [2]).

Since the \((k,l)\)-sonic fold is symmetric with respect to the \(l\)-axis, self-intersection points, if they exist, are on the \(l\)-axis, i.e., \(k = 0\). The \(k\) coordinates equals zero for \(z = 0\) or \((b_1 - 1)z^2 - 3 = 0\). So there is a self-intersection point if and only if \(b_1 > 1\), corresponding to case IV. Straightforward computations show that for \(b_1 > 1\), the \(l\)-coordinate of the self-intersection point is \(l_{si} = \frac{-2c(4b_1 + 5)}{5b_1 + 4}\). Sumarizing:

i- In case I, there are three asymptotes;

ii- In cases II, III and IV the \((k,l)\)-sonic fold has only the line \(l = -2c\) as asymptote;

iii- Only in case IV there is a self-intersection point. Its coordinates are \(k = 0\) and \(l = l_{si}\).

### 3.2 Intersections in case IV

In case IV, i.e., \(b_1 > 1\), the \((k,l)\)-sonic fold splits the \((k,l)\)-space in 4 regions: \(R_1\) above the curve; \(R_2\) inside the loop; \(R_3\) between the curve and the
asymptote and $R_1$ below the asymptote, as illustrated in figure 1, here as in all other cases, these are not the $R_1, R_2, R_3,$ and $R_4$ regions considered in [2]. The following result describes intersections of Hugoniot curves with the sonic surface in case IV.

**Proposition 1**: Suppose $b_1 > 1$, then

i - Hugoniot curves through points in $R_1$ and $R_4$ intersect the sonic surface in two distinct points;

ii - Hugoniot curves through points in $R_2$ do not intersect the sonic surface;

iii - Hugoniot curves through points in $R_3$ intersect the sonic surface in four distinct points.

![Figure 1: Decomposition of $(k,l)$-space in case IV](image)

**Proof**: Since the $(k,l)$-sonic fold and secondary bifurcations are the loci of tangency, it is sufficient to consider one point in each region. As we said before real roots of $S(Z)$ correspond to intersection points between Hugoniot curves and $Son$. For $b_2 = 0$, $S(Z)$ is written as

$$S_1(Z) = (2c-l)Z^4-4Z^3k+(2l(2b_1+1)+4c(b_1+2))Z^2+(2c+l)(b_1^2-1).$$

(11)

It is easy to see that we can choose in each region a point with $k = 0$. Choosing $k = 0$ and $l = 3c$, $l = 0$, $l = \frac{l_{ii}-2c}{2}$ and $l = -3c$, respectively, $S_1(Z)$ becomes a biquadratic polynomial, i.e., a polynomial of the form $aZ^4+bZ^2+c$ and the analysis is simple. The conclusion is
$R_1$, 2 intersection points;
$R_2$, 0 intersection points;
$R_3$, 4 intersection points;
$R_4$, 2 intersection points.

### 3.3 Intersections in cases I, II and III

For cases I, II and III, we will need the following lemma, which will be proved in the Appendix:

**Lemma 1**: Let $p(z) = az^4 + bz^3 + cz^2 + d$ be a polynomial with real coefficients and $ad \neq 0$. Let $\text{cond}_1 = 9b^2 - 32ac$ and $\text{cond}_2 = a(-4c^3b^2 - 27b^4d + 144acb^2d - 128a^2c^2d + 16ac + 256a^3d^2)$. Let $m_1$ and $m_2$ be the real zeros of $p'(z)$, if they exist. We have:

i- If $ad < 0$ and $\text{cond}_1 > 0$ and $\text{cond}_2 < 0$, then $p(z)$ has 4 real roots;

ii- If $ad < 0$ and $\text{cond}_1 > 0$ and $\text{cond}_2 > 0$, then $p(z)$ has 2 real roots;

iii- If $ad < 0$ and $\text{cond}_1 < 0$, then $p(z)$ has 2 real roots;

iv- If $ad > 0$ and $ac < 0$ and $\text{cond}_2 < 0$, then $p(z)$ has 2 real roots;

v- If $ad > 0$ and $ac < 0$ and $\text{cond}_2 > 0$ and $ap(m_1) > 0$ then $p(z)$ has 0 real roots;

vi- If $ad > 0$ and $ac < 0$ and $\text{cond}_2 > 0$ and $ap(m_1) < 0$ then $p(z)$ has 4 real roots;

vii- If $ad > 0$ and $ac > 0$ and $\text{cond}_1 < 0$, then $p(z)$ has 0 real roots;

viii- If $ad > 0$ and $ac > 0$ and $\text{cond}_1 > 0$ and $\text{cond}_2 < 0$, then $p(z)$ has 2 real roots;

ix- If $ad > 0$ and $ac > 0$ and $\text{cond}_1 > 0$ and $\text{cond}_2 > 0$, then $p(z)$ has 0 real roots;
Applying Lemma 1 to $S_1(Z)$, we get $a = 2c - l$, $b = -4k$, $c = 2l(2b_1 + 1) + 4c(b_1 + 2)$, $d = (2c + l)(b_1^2 - 1)$. A straightforward computation shows that $\text{cond}_2 = -(l - 2c)((2c + b_1 l)^2 - k^2(1-b_1))((\alpha k^2 + \beta)$, where $\alpha = 27(2c + l)(b_1 + 1)$ and $\beta = (l - 2c)(4l + 10c + 5b_1 l + 8b_1 c)^2$. It can be shown that $(\alpha k^2 + \beta) = 0$ is the implicit equation of the $(k, l)$-sonic fold and $(2c + b_1 l)^2 - k^2(1-b_1) = (b_1 l + 2c - \sqrt{1 - b_1 k})(b_1 l + 2c + \sqrt{1 - b_1 k})$ is the equation of both $r_1$ and $r_2$, as defined in Section 3.3.1. So, Hugoniot curves through points $(k, l)$ satisfying $\text{cond}_2 = 0$ are tangent to the sonic surface.

3.3.1 Intersections in case III

In case III, $0 < b_1 < 1$, tangency points are in the union of the $(k, l)$-sonic fold with the secondary bifurcations, $l = 2c$, $r_1$, given by $k = \frac{2c + b_1 l}{\sqrt{1 - b_1}}$, and $r_2$ given by $k = -\frac{2c + b_1 l}{\sqrt{1 - b_1}}$.

Straightforward computations show that $r_1$ and $r_2$ intersect the $(k, l)$-sonic fold tangentialy( in the same way as $y = x^3$ intersects $y = 0$), at points with $l_{bc} = \frac{2c(2b_1 + 1)}{b_1 + 2}$. It is easy to see that $r_1$ intersects $r_2$ at $(0, \frac{-2c}{b_1})$.

![Figure 2: Decomposition of $(k, l)$-space in case III](image)

The union of the $(k, l)$-sonic fold and the secondary bifurcations splits the $(k, l)$-space in ten regions, illustrated in figure 2: $R_1$, the region above the curve and between $r_1$ and $r_2$; $R_2$, the region below the curve, above $l = -2c$ and between $r_1$ and $r_2$; $R_3$, the triangle defined by $r_1$, $r_2$ and $l = -2c$; $R_4$, ...
the region between \( r_1 \) and \( r_2 \), below their intersection point; \( R_5 \), the region below \( l = -2c \), and to the right of \( r_1 \) and \( r_2 \); \( R_6 \), the small region above \( l = -2c \), below the curve and to the right of \( r_1 \); \( R_7 \), the region above the curve and to the right of \( r_1 \); \( R_8 \), \( R_9 \) and \( R_{10} \), the symmetric regions of \( R_7 \), \( R_6 \) and \( R_5 \), respectively.

Next result states how Hugoniot curves intersect \( Son \) in case III.

**Proposition 2**: Suppose \( 0 < b_1 < 1 \), then

i - Hugoniot curves through points in \( R_3 \) do not intersect the sonic surface;

ii - Hugoniot curves through points in \( R_2 \), \( R_5 \), \( R_7 \), \( R_8 \) and \( R_{10} \) intersect the sonic surface in two distinct points;

iii - Hugoniot curves through points in \( R_1 \), \( R_4 \), \( R_6 \) and \( R_9 \) intersect the sonic surface in four distinct points.

**Proof**: As before for regions \( R_1 \), \( R_2 \), \( R_3 \) and \( R_4 \) we can take points \( k = 0 \), which makes \( S_1(Z) \) biquadratic and again the analysis is simple.

A representative point of \( R_5 \) is \( (k_5, l = -\frac{2c}{b_1}) \), where \( k_5 \) is the \( k \) value at the intersection point of \( r_1 \) and \( k \)-axis. The fourth degree polynomial obtained by substituting \( k = k_5 \) and \( l = -\frac{2c}{b_1} \) in equation (11), has two real roots from Lemma 1-viii. Taking \( (-k_5, l = -\frac{2c}{b_1}) \) as a representative point of \( R_{10} \), from Lemma 1-viii we get the same conclusion as in \( R_5 \).

A point to the right of \( (k_5, 0) \) is a representative point of \( R_7 \). Such a point is \( (\frac{4c}{\sqrt{1-b_1}}, l = 0) \). Substituting this point in \( S_1(Z) \), from Lemma 1-iii we get that Hugoniot curves through points in \( R_7 \) intersect \( Son \) in two distinct points. In the same way, taking \( (-\frac{4c}{\sqrt{1-b_1}}, l = 0) \) as a representative point of \( R_8 \), applying Lemma 1-iii we get the same conclusion as in \( R_7 \).

As we said before, the \( l \) value at the intersection point of \( r_1 \) and the \((k, l)\)-sonic fold is \( l_{bc} = -\frac{2c(2b_1 + 1)}{b_1} \). In this case \( l_{bc} < 0 \). The horizontal line \( l = l_{\mu} \), where \( l_{\mu} \) is the midpoint between \(-2c\) and \( l_{bc} \), intersects \( R_6 \) and \( R_9 \). Let us consider points in \( R_6 \). Straightforward computations show that \( (k_{sf6}, l_{\mu}) \),
the intersection of \( l = l_\mu \) and the \((k, l)\)-sonic fold, is to the right of \((k_{sb6}, l_\mu)\), the intersection of \( r_1 \) and \( l = l_\mu \). We take the point \((k_\mu, l_\mu)\), where \(k_\mu\) is the midpoint between \(k_{sf6}\) and \(k_{sb6}\), as a representative point of \(R_6\). Substituting this point in equation (11), we get a fourth degree polynomial satisfying Lemma 1-i. In the same way, using the point \((-k_\mu, l_\mu)\) as a representative point of \(R_9\), from Lemma 1-i we get the same conclusion as in \(R_6\).

### 3.3.2 Intersections in case II

In case II, \(-1 < b_1 < 0\), the subdivision of the \((k, l)\)-plane is similar to the one in case III. The main difference is that \(r_1 \cap r_2\) is above of the \(k\)-axis and not below. So the union of \((k, l)\)-sonic fold and \(r_1 \) and \(r_2\) splits the \((k, l)\)-plane in ten regions illustrated in figure 3: \(R_1\) the region between \(r_1\) and \(r_2\) above their intersection point; \(R_2\) the region above the curve, between \(r_1\) and \(r_2\) and below their intersection point; \(R_3\) the region between \(r_1\) and \(r_2\), below the curve and above \(l = -2c\); \(R_4\) the unlimited region between \(r_1\) and \(r_2\) below \(l = -2c\); \(R_5\) the region to the right of \(r_1\) and below \(l = -2c\); \(R_6\) the small region to the right of \(r_1\), below the curve and above \(l = -2c\); \(R_7\) the region above the curve and to the right of \(r_1\) and \(r_2\); \(R_8\), \(R_9\) and \(R_{10}\) the symmetric regions of \(R_7\), \(R_6\) and \(R_5\), respectively.

![Figure 3: Decomposition of \((k, l)\)-space in case II](image)

**Proposition 3**: Suppose \(-1 < b_1 < 0\), then
i - Hugoniot curves through points in $R_1$ and $R_4$ do not intersect the sonic surface;

ii - Hugoniot curves through points in $R_3$, $R_5$, $R_7$, $R_8$ and $R_{10}$ intersect the sonic surface in two distinct points;

iii - Hugoniot curves through points in $R_2$, $R_6$ and $R_9$ intersect the sonic surface in four distinct points.

**Proof:** Regions $R_1$, $R_2$, $R_3$ and $R_4$ are analyzed as in Proposition 1 (using points with $k = 0$).

The line $l = -3c$ intersects the straight line $r_1$ at $k = \frac{c(2 - 3b_1)}{\sqrt{1 - b_1}}$. So a representative point of $R_5$ is $(\frac{c(4 - 3b_1)}{\sqrt{1 - b_1}}, -3c)$. After substituting these values of $k$ and $l$ in equation (11), it follows from Lemma 1-viii that the fourth degree polynomial has two real roots. Taking the point $(\frac{-c(4 - 3b_1)}{\sqrt{1 - b_1}}, -3c)$ as a representative point of $R_{10}$, and using Lemma 1-viii we get the same conclusion as in $R_5$.

A representative point of $R_7$ is $(k = \frac{-2c}{b_1}, l = \frac{-2c}{b_1})$. Substituting this point in equation (11) we get a fourth degree polynomial satisfying hypothesis of item vi from Lemma 1. So it has two real roots. Taking $(k = \frac{2c}{b_1}, l = \frac{-2c}{b_1})$ as representative point for $R_8$, the same conclusion extends to points in that region.

To study how Hugoniot curves through points in $R_6$ intersect the sonic surface, we consider two cases:

i) $b_1 < -1/2$, in this case the line $l = l_{bc}$ is in $l > 0$ half-plane. So the horizontal line $l = 0$ crosses the regions $R_6$ and $R_7$, as $k$ varies. It is easy to see that $(k_{sf}, 0)$, the intersection of $l = 0$ and the $(k, l)$-sonic fold is to the right of $(k_{bs}, 0)$, the intersection of $l = 0$ and $r_1$. In this way representative points of $R_6$ and $R_9$ are, respectively, $(k_{m}, 0)$ and $(-k_{m}, 0)$, where $k_{m}$ is the midpoint between $k_{sf}$ and $k_{bs}$. From Lemma 1-i Hugoniot curves through points in $R_6$ and $R_9$ intersect $Son$ in four distinct points.

ii) $b_1 > -1/2$, in this case the line $l = l_{bc}$ is in $l < 0$ half-plane. So, as in the proof of Proposition 2, representative points of $R_6$ and $R_9$ are $(k_{\mu}, l_{\mu})$
and \((-k\mu, l\mu)\), respectively. From Lemma 1-i the fourth degree polynomials obtained after substituting these points in \(S_1(Z)\), has four real roots.

### 3.3.3 Intersections in case I

As we said before, in case I, \(b_1 < -1\), and the \((k,l)\)-sonic fold has three asymptotes, \(l = -2c\) and two others, \(as_1\) and \(as_2\), given by
\[
3\sqrt{-3(b_1 + 1)k + (5b_1 + 4)l - 2c(b_1 - 1)} = 0 \quad \text{and} \quad -3\sqrt{-3(b_1 + 1)k + (5b_1 + 4)l - 2c(b_1 - 1)} = 0,
\]
respectively. Their intersection point is \((k = 0, l = \frac{2c(b_1 - 1)}{5b_1 + 4})\). Except for \(b_1 = -2\), the asymptotes \(as_1\) and \(as_2\) do not coincide with the secondary bifurcations \(r_1\) and \(r_2\). It is easy to see that \(as_1 \cap as_2\) is above \(r_1 \cap r_2\). Straightforward computations show that at the intersection of \(as_i, i = 1, 2\), and the \((k,l)\)-sonic fold, \(l = \frac{-2c(17b_1^2 + 38b_1 + 26)}{3(5b_1^2 + 14b_1 + 8)}\). It follows that the intersection points are in \(l > 0\) half-plane or \(l < 0\) half-plane, according to \(-2 < b_1 < -1\) or \(b_1 < -2\). So, case I subdivides in two subcases: case Ia, \(b_1 < -2\) and case Ib, \(-2 < b_1 < -1\).

In case Ia, the \((k,l)\)-plane is subdivided in 12 regions, as illustrated in figure 4(a): \(R_1\) the unlimited region above the upper branch of the curve; \(R_2\), the region between \(r_1\) and \(r_2\) below the upper branch; \(R_3\) the triangular region defined by \(r_1, r_2\) and \(l = -2c\). The intersections of the \((k,l)\)-sonic fold with the secondary bifurcations, \(r_1\) and \(r_2\) are in the \(l < -2c\) region. These intersections split \(l < -2c\) in 7 regions: A central region \(R_4\); three regions on the right side: \(R_5\) the region to right of \(r_1\) below the curve; \(R_6\), the small region to the right of \(r_1\) above the curve and \(R_8\), the small region to the left of \(r_1\) below the curve; three regions on the left side: \(R_{12}, R_{11}\) and \(R_9\) symmetric to \(R_3, R_6\) and \(R_8\), respectively. The other regions are \(R_7\), the region to the right of \(r_1\) and \(r_2\) above \(l = -2c\) and its symmetric region \(R_{10}\).

In case Ib, the \((k,l)\)-plane is also subdivided in 12 regions, as illustrated in figure 4(b). In this case the secondary bifurcations \(r_1\) and \(r_2\) intersect the upper branch of the \((k,l)\)-sonic fold. These intersection points split the region above the upper branch in three regions: a central region, \(R_1\) and two small regions: \(R_8\), above the curve below \(r_2\) and \(R_9\), symmetric to \(R_8\). The other regions are: \(R_2\), the region between \(r_1\) and \(r_2\) below the upper branch; \(R_3\) the triangular region defined by \(r_1, r_2\) and \(l = -2c\); \(R_4\), the region \(l < -2c\)
between $r_1$ and $r_2$; $R_6$, the small region below $l = -2c$ to the right of $r_1$ above the right branch of the curve; $R_5$, the region below the right branch of the curve; $R_7$, the region above $l = -2c$ to the right of $r_1$ and $r_2$; $R_{10}$, $R_{11}$ and $R_{12}$ the symmetric regions of $R_7$; $R_6$ and $R_5$, respectively.

Next proposition states how Hugoniot curves intersect $Son$ in case I:

**Proposition 4**: Suppose $b_1 < -1$, then

i - Hugoniot curves through points in $R_3$ do not intersect or intersect the sonic surface in four distinct points, according to $b_1 < -5/4$ or $-5/4 < b_1$, respectively;

ii - Hugoniot curves through points in $R_1$, $R_4$, $R_5$, $R_7$, $R_{10}$ and $R_{12}$ intersect the sonic surface in two distinct points;

iii - Hugoniot curves through points in $R_2$, $R_6$, $R_8$, $R_9$ and $R_{11}$ intersect the sonic surface in four distinct points.

**Proof**: Regions $R_1$, $R_2$ and $R_4$ are analyzed as in Proposition 1. The conclusions for $R_3$ follows using $k = 0$ and $l = 0$ in $S_1(Z)$. We get a biquadratic polynomial such that its discriminant equals zero for $b_1 = -5/4$.

The horizontal line $l = -4c$ intersects regions $R_5$, $R_6$ and their symmetric regions. So we can take points on this line as representative points of these regions. Let us consider points in $R_5$ and $R_6$. The line $l = -4c$ intersects

Figure 4: Decomposition of $(k,l)$-space in case Ia and case Ib, respectively
the \((k, l)\)-sonic fold at \(k_{ds} = -\frac{2c(2b_1 + 1)}{\sqrt{-b_1 - 1}}\). In this way, the point \((k_5 = -\frac{4c(2b_1 + 1)}{\sqrt{-b_1 - 1}}, l = -4c)\) is a point in \(R_5\). Substituting \(k = k_5\) and \(l = -4c\) in \(S_1(Z)\), it follows from Lemma 1-ii that the fourth degree polynomial has two real roots. Using \((k = -k_5, l = -4c)\) in \(S_1(Z)\), similar computations give the same conclusion for points in \(R_{12}\).

To get a point in \(R_6\), we follow the same strategy as in Proposition 3, i.e., we take the \(k\) value, \(k_{m6}\), as the midpoint between \(k_{ds}\) and \(k_6\), the intersection of \(l = -4c\) and \(r_1\). Using \(k = k_{m6}\) and \(l = -4c\) in equation (11) and Lemma 1-i we have that Hugoniot curves through points is \(R_6\) intersect \(Son\) in four distinct points. Similar computations with \(k = -k_{m6}\) and \(l = -4c\) give the same conclusion for points in \(R_{10}\).

Any point on \(k\)-axis to the right of \((\frac{2c}{\sqrt{1 - b_1}}, l = 0)\), the intersection of \(r_1\) and \(k\)-axis, is a representative point of \(R_7\). Using \(k = \frac{4c}{\sqrt{1 - b_1}}\), and \(l = 0\) in equation (11) and Lemma 1 we see that Hugoniot curves through points in \(R_7\) intersect \(son\) in two distinct points. The same conclusions extend to points in \(R_{10}\).

In case Ia, regions \(R_8\) and \(R_9\) are in \(l < 0\) half-plane. Since \(l_{int} = -\frac{4c(2b_1 + 1)}{b_1 + 2} < l_{bc}\), the line \(l = l_{int}\) crosses regions \(R_8\) and \(R_9\). Let us consider its intersection with \(R_8\). Let \(k_{sf8}\) and \(k_{sb8}\) be the intersections of \(l_{int}\) with the \((k, l)\)-sonic fold and \(r_1\), respectively. We take the point \((k_{m8}, l_{int})\), where \(k_{m8}\) is the midpoint between \(k_{sf8}\) and \(k_{sb8}\), as a representative point of \(R_8\). From Lemma 1-i the fourth degree polynomial obtained from equation (11) has four distinct real roots. Taking the point \((-k_{m8}, l_{int})\) as a representative points of \(R_9\), we get the same conclusion as in \(R_8\). In case Ib, the horizontal line \(l = l_{int}\) is above \(l_{bc}\). As the calculations in case Ib are the same as in case Ia, results for case Ia extend to case Ib.  \(\square\)
4 Appendix: Proof of Lemma 1, a Polynomial Lemma

We are considering a fourth degree polynomial \( p(z) = az^4 + bz^3 + cz^2 + d \). Lemma 1 gives the number of real roots of \( p(z) \) in terms of its coefficients and extremum values.

As \( p'(x) = z(4az^2 + 3bz + 2c) \), the graph of \( p(z) \) has an extremum value at \((0, d)\). According to the sign of \( \text{cond}_1 = 9b^2 - 32ac \) it may have two others.

If \( \text{cond}_1 > 0 \), let \( m_1, m_2 \) be the extreme values of \( p(z) \), and consider the product \( \text{cond}_2 = p(m_1)p(m_2) \), i.e., \( \text{cond}_2 = a(-4c^3b^2 - 27b^4d + 144acb^2d - 128a^2c^2d + 16ac^4 + 256a^3d^2) \).

The number of real roots of \( p(z) \) depends on the sign of \( ad \):

\[ a) \; ad < 0. \] Since \( a \) and \( d \) have opposite signs, in this case \( p(z) \) has at least two real roots.

- If \( \text{cond}_1 < 0 \) or \( \text{cond}_1 > 0 \) and \( \text{cond}_2 > 0 \), \( p(z) \) has two real roots, proving items iii and ii, respectively.
- If \( \text{cond}_1 > 0 \) and \( \text{cond}_2 < 0 \), \( p(z) \) has four real roots, proving item i.

\[ b) \; ad > 0, \] we have two subcases: \( ac < 0 \) or \( ac > 0 \).

We remark that \( p(z) \) reaches a local maximum or local minimum at \( z = 0 \) according to \( ac < 0 \) or \( ac > 0 \).

\[ b1) \; ac < 0 \] implies \( \text{cond}_1 > 0 \).
• If $\text{cond}_2 < 0$ then $p(z)$ has two real roots, proving item iv;

• If $\text{cond}_2 > 0$, $p(z)$ may has 0 or 4 real roots according to $ap(m_1) > 0$ or $ap(m_1) < 0$, respectively, proving itens v and vi.

b2) $ac > 0$ and $\text{cond}_1 < 0$, $p(z)$ has no real roots, proving item vii;

b3) $ac > 0$ and $\text{cond}_1 > 0$

• $\text{cond}_2 > 0$, $p(z)$ has no real roots, proving item ix;

• $\text{cond}_2 < 0$, $p(z)$ has two real roots, proving item viii.
Figure 8: A possible graph of $p(z)$ satisfying conditions in item viii.

References


Cesar Eschenazi
Departamento de Matemática
UFMG
30123-970 Belo Horizonte, MG
cesar@mat.ufmg.br

Carlos Frederico Palmeira
Departamento de Matemática
PUC-RIO
22453-900 Rio de Janeiro, RJ
fredpalm@mat.puc-rio.br