Intersections of Hugoniot curves with Sonic Surfaces in the Wave Manifold

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Abstract

Local 1-shock curve arcs and local reverse 2-shock curve arcs were constructed, in the geometric context of the wave manifold, in a previous paper. In that geometric context, sonic and sonic' surfaces are the boundaries of admissible shock curve arcs; in order to introduce the concept of non-local 1-shock curve arcs and non-local reverse 2-shock curve arcs it is necessary to understand how a hugoniot curve intersects the sonic and sonic' surfaces; intersections of such curves with sonic' surface are well known. In this paper we present a complete study on how Hugoniot curves intersect the sonic surface for a quadratic system of two conservation laws.

1 Introduction

The classical construction of Riemann solutions for certain systems of conservations laws is unable to establish their global existence or stability. The main obstacle is the fact that Hugoniot curves do not foliate state space, so they can only be part of a local coordinate system. Structural stability in the classical context was considered in [3] and [7]. In [1] a new construction of Riemann solutions was presented: the state space was replaced by a 3-dimensional manifold, the wave manifold, which is foliated (possibly with singularities) by rarefaction and Hugoniot curves. Lax's and Liu's construction of Riemann solutions for systems of two conservation laws were generalized to the geometric framework by introducing the concept of intermediate surfaces, which replace wave curves. As transversality is frequent in the wave manifold, structural stability of Riemann solutions follows naturally. This is the main advantage of the new construction over the classical construction. Results in that paper extend Liu's admissibility criterion, [5], to the wave manifold and introduce local 1-shock curves and local reverse 2-shock curves for case IV in the classification of Schaeffer and Shearer, [8]. It was also shown that Liu's admissibility criterion implies Lax's inequalities. Since Lax's inequalities are reversed on sonic and sonic' surfaces, [4], these two surfaces are the boundaries of admissible shock curve arcs.

In order to solve a Riemann problem it is necessary to characterize non-local 1-shock curves and non-local reverse 2-shock curves. So, it is necessary to study intersections of Hugoniot curves with sonic and sonic' surfaces. Intersections of Hugoniot curves with the sonic' surface were studied in [2]. In this paper we will study their intersections with the sonic surface in the symmetric case.

Generically, a Hugoniot curve intersects, transversally, the sonic surface in 0, 2 or 4 points. Tangencies occur along a curve, called *sonic fold*, which splits the sonic surface into regions. The sonic fold curve is the hysteresis locus in [4]. We characterize precisely the projection of the sonic fold curve in state space, for each of the four cases in the classification of Scheaffer and Shearer.

In a forthcoming paper, we will combine these results with the ones in [2], to split the wave manifold into regions according to the existence of non local admissible arcs, in the manner of [1].

This paper is organized as follows: In Section 2 we recall the definition of the wave manifold, extend to this manifold the concepts of Hugoniot and Hugoniot' curves, describe the characteristic, sonic and sonic' surfaces, [4], and discuss their relationship. In Section 3 we characterize the intersections of Hugoniot curves with sonic surface in each of the four cases in the classification of Scheaffer and Shearer. In the Appendix we present the proof of a technical lemma stated in Section 3.3.

2 The wave manifold and Hugoniot curves

We refer the reader to section 2 of [2] for basic definitions. For sake of completeness we will briefly present what is need for this paper. We consider the equation

$$W_t + F(W)_x = 0, \tag{1}$$

with F given by

$$\begin{cases} f(u,v) = v^2/2 + (b_1+1)u^2/2 + a_1 u + a_2 v \\ \\ g(u,v) = uv - b_2 v^2/2 + a_3 u + a_4 v \end{cases} \tag{2}$$

where, $b_1 \notin \{0, \pm 1, 1 + b_2^2/4\}, b_2^2 + 4/(b_1+1) \neq 0$ and $a_3 - a_2 \neq 0$. We will work with $b_2 = 0$ (symmetric case).

After a suitable change of coordinates (explicitly done in [2]) we get the manifold M^3 given by

$$(1 - Z^2)\tilde{V} - Z\tilde{U} + c = 0$$

and

$$Y = ZX.$$

Since Z is a direction, M^3 is actually a submanifold of $\Re^4 \times \Re P^1$, or equivalently, of $\Re^4 \times S^1$. In previous work [6], [2] the authors have taken advantage of the fact that there are no special features at $Z = \infty$, to use only $\tilde{U}, \tilde{V}, Z, X$ coordinates and consider M^3 as given by $G = 0$ in $(\tilde{U}, \tilde{V}, Z, X)$-space, where $G = (1 - Z^2)\tilde{V} - Z\tilde{U} + c$.

We also have the following expression for the speed of a shock

$$s = Z\tilde{V} + m + (b_1 + 1)(\tilde{U} - b_2\tilde{V})/b_1, \tag{3}$$

where $m = a_1 + (b_1 + 1)(a_4 + b_2 a_2 - a_1)/b_1$.

In state space ((u,v)-plane) we introduce new coordinates $k = b_1 u + 2a_1 + 2b_2 a_2 - 2a_4$ and $l = 2v + 2a_2$. Given a point (k,l) we define the Hugoniot curve in M^3 as the set of points $(\tilde{U}, \tilde{V}, Z, X)$ satisfying the system

$$\begin{cases} (1 - Z^2)\tilde{V} - Z\tilde{U} + c = 0 \\ b_1 X + 2\tilde{U} - 2b_2\tilde{V} = k \\ ZX + 2\tilde{V} = l. \end{cases} \tag{4}$$

The projection of this curve in state space are the classical Hugoniot curves, as defined in [1]. Hugoniot' curves are obtained by changing X by $-X$ in system (4).

System (4) is a linear system in $\tilde{U}$, $\tilde{V}$, and X with determinant $2Zp(Z)$, where $p(Z) = Z^2 + b_2 Z + b_1 - 1$. So, if $Zp(Z) \neq 0$, Hugoniot curves are parametrized by Z. Solving the above system when $Zp(Z) \neq 0$, one gets

$$\begin{cases} \tilde{U} = (kZ^3 - b_1 l Z^2 + (2cb_2 - k)Z + b_1(l+2c))/[2Zp(Z)] \\ \tilde{V} = (-kZ + b_1 l + 2c)/[2p(Z)] \\ X = (lZ^2 + (k + b_2 l)Z - (l+2c))/[Zp(Z)]. \end{cases} \quad (5)$$

Since we assume that the discriminant $b_2^2 - 4(b_1 - 1)$ of p is nonzero, p has distinct roots; if these roots are real, denote them by $Z_1 < Z_2$. Also, let $Z_0 = 0$. Hugoniot curves are singular along one or three straight lines B_i, $i = 0, 1, 2$, defined by:

$$Z = Z_i, \ \tilde{V} = \frac{Z_i \tilde{U} - c}{1 - {Z_i}^2}, \ X = -2[({Z_i}^2 + 1)\tilde{U} - 2cZ_i] \frac{b_1 + b_2 Z_i}{b_1(1 - {Z_i}^2)^2}.$$

In the same way, Hugoniot$'$ curves are singular along the straight lines B_i', obtained from B_i by changing X into $-X$. Let $B = \cup_{i=0}^{2} B_i$, and $B' = \cup_{i=0}^{2} B_i'$. The local structure of Hugoniot curves near B_i is the same as the curve $x^2 - y^2 = c$, $Z = c'$ near the z-axis in $\Re^3$.

From (5) it is clear that the number of connected components in a generic Hugoniot curve is given by the number of real roots of $Zp(Z)$.

We define the *sonic surface Son* in M^3 as the set of points where the speed s is extremal along a Hugoniot curve. In [6] it is shown that Son is given by the equation

$$A_U \tilde{U} + A_V \tilde{V} + A_X X = 0, \quad (6)$$

where

$$\begin{aligned} A_U &= 2(Z^2 + b_1 + 1) \\ A_V &= 2Z(Z^2 - b_2 Z + b_1 + 3) \\ A_X &= -Z^2 - b_2(b_1 + 1)Z + b_1 + 1. \end{aligned}$$

Similarly, we define the *sonic' surface* Son' as the set of points where the speed s is extremal along a Hugoniot' curve. The equation for Son' is obtained by changing X by $-X$ in equation 6.

$$A_U \tilde{U} + A_V \tilde{V} - A_X X = 0. \quad (7)$$

Both Son and Son' are Moebius band embedded in M^3, and not cylinders as staded in [6].

We remark that what we are calling here Hugoniot curves following [1] were called shock curves in [6], [4], [2].

3 Hugoniot curves and the Sonic Surface Son

3.1 Preliminaries

In this section we will show that a Hugoniot curve intersects Son in 0,2 or 4 distinct points depending on (k,l). Substituting $\tilde{U}$, $\tilde{V}$ and X in equations (6) by their expressions in equation (5) we get the following polynomial:

$$\begin{aligned} S(Z) = {} & (b_2k + 2c - l)Z^4 - (2b_2l(b_1+1) + 4k)Z^3 \\ & +(2l + 4b_1c - b_2k(b_1+1) - b_2^2 l(b_1+1) + 4(b_1l + 2c))Z^2 \\ & +2b_2(b_1+1)(l+2c)Z + (b_1^2 - 1)(l+2c). \end{aligned}$$

Real roots of $S(Z)$ correspond to intersections points of Hugoniot curves and Son. Double roots of $S(Z)$ correspond to tangency points of Hugoniot curves and Son. So tangency points are obtained solving system:

$$\begin{cases} S(Z) = 0 \\ \dot{S}(Z) = 0, \end{cases}$$

where we use a dot to indicate differentiation with respect to Z.

By noting that $S(Z)$ can be written as $S(Z) = \alpha(Z)k + \beta(Z)l + \gamma(Z)$, we can regard the last system as a linear system in k and l:

$$\begin{cases} \alpha(Z)k + \beta(Z)l = -\gamma(Z) \\ \dot{\alpha}(Z)k + \dot{\beta}(Z)l = -\dot{\gamma}(Z), \end{cases} \tag{8}$$

Assuming, $\alpha\dot{\beta} - \beta\dot{\alpha} \neq 0$ we can solve this system, to obtain:

$$\begin{cases} k = \dfrac{-2c(b_1+1)(b_2Z^3 - 6Z^2 - 3b_2(b_1+1)Z - b_2^2(b_1+1) + 2(b_1-1))}{(2 + b_2^2(b_1+1))Z^3 - 3b_2(b_1+1)Z^2 + 6(b_1+1)Z + b_2(b_1+1)} \\[2ex] l = \dfrac{2c(2Z^3 + 3b_2(b_1+1)Z^2 - 6(b_1+1)Z - b_2(b_1+1)^2)}{(2 + b_2^2(b_1+1))Z^3 - 3b_2(b_1+1)Z^2 + 6(b_1+1)Z + b_2(b_1+1)}. \end{cases} \tag{9}$$

Hugoniot curves through points (k,l) satisfying equations (9) are tangent to the sonic surface. These Hugoniot curves define a regular curve in Son, called *sonic fold*. Equations (9) can be interpreted as the parametric equations of the projection, by Hugoniot curves, in (k,l)-space of the sonic fold. We will call this curve (k,l)-*sonic fold*

Other tangency points are obtained if $\alpha\dot{\beta} - \beta\dot{\alpha}$ and $\beta\dot{\gamma} - \dot{\beta}\gamma$ have commom real roots. If Z_1 is a commom real root of $\alpha\dot{\beta} - \beta\dot{\alpha}$ and $\beta\dot{\gamma} - \dot{\beta}\gamma$, then we have a straight line in the (k,l)-plane given by $\alpha(Z_1)k + \beta(Z_1)l + \gamma(Z_1) = 0$, as part of solution of the system (8). Straightforward computations show that $\alpha\dot{\beta} - \beta\dot{\alpha}$ and $\beta\dot{\gamma} - \dot{\beta}\gamma$ have $2Z(Z^2 + b_2 Z + b_1 - 1)$ as their greatest commom divisor. In this way, we have the straight line $l = -2c$, r_0, corresponding to $Z = 0$ and 2 others if $b_2^2 - 4(b_1 - 1) > 0$. We remark that, for each $i = 0, 1, 2$, r_i is the projection onto (k,l)-plane of the straight lines B_i, $i = 0, 1, 2$, defined in section 2. Classically these straight lines are called *secondary bifurcations.*

From now on we will consider $b_2 = 0$ and $c > 0$. Substituting $b_2 = 0$ and using $Z = 1/z$ in equations (9), the (k,l)-sonic fold is written as

$$\begin{cases} k = \dfrac{-2cz((b_1^2 - 1)z^2 - 3(b_1 + 1))}{3(b_1 + 1)z^2 + 1} \\ \\ l = -\dfrac{2c(3(b_1 + 1)z^2 - 1)}{3(b_1 + 1)z^2 + 1}. \end{cases} \tag{10}$$

It is easy to see that this curve has the line $l = -2c$ as horizontal asymptote as z goes to infinity. There are two more asymptotes if $\dfrac{1}{3(b_1 + 1)} < 0$ or, equivalently, $b_1 < -1$, corresponding to case I, in the Schaeffer and Shearer classification (D3.2 case in [2]).

Since the (k,l)-sonic fold is symmetric with respect to the l-axis, self-intersection points, if they exist, are on the l-axis, i. e., $k = 0$. The k coordinates equals zero for $z = 0$ or $(b_1 - 1)z^2 - 3 = 0$. So there is a self-intersection point if and only if $b_1 > 1$, correponding to case IV. Straightforward computations show that for $b_1 > 1$, the l-coordinate of the self-intersection point is $l_{si} = \dfrac{-2c(4b_1 + 5)}{5b_1 + 4}$. Sumarizing:

- i- In case I, there are three asymptotes;
- ii- In cases II, III and IV the (k,l)-sonic fold has only the line $l = -2c$ as asymptote;
- iii- Only in case IV there is a self-intersection point. Its coordinates are $k = 0$ and $l = l_{si}$.

3.2 Intersections in case IV

In case IV, i.e., $b_1 > 1$, the (k,l)-sonic fold splits the (k,l)-space in 4 regions: R_1 above the curve; R_2 inside the loop; R_3 between the curve and the asymptote and R_4 below the asymptote, as illustrated in figure 1, here as in all other cases, these are not the $R1$, $R2$, $R3$, and $R4$ regions considered in [2]. The following result describes intersections of Hugoniot curves with the sonic surface in case IV.

Proposition 1 *: Suppose $b_1 > 1$, then*

- **i** - *Hugoniot curves through points in R_1 and R_4 intersect the sonic surface in two distinct points;*
- **ii** - *Hugoniot curves through points in R_2 do not intersect the sonic surface;*
- **iii** - *Hugoniot curves through points in R_3 intersect the sonic surface in four distinct points.*

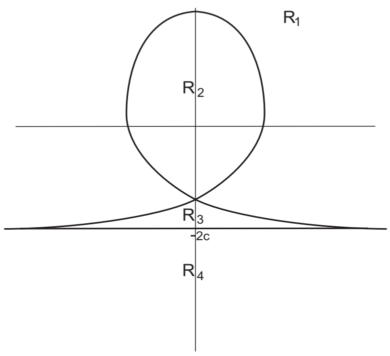

Figure 1: Decomposition of (k, l)-space in case IV

Proof: Since the (k, l)-sonic fold and secondary bifurcations are the loci of tangency, it is sufficient to consider one point in each region. As we said before real roots of $S(Z)$ correspond to intersection points between Hugoniot curves and Son. For $b_2 = 0$, $S(Z)$ is written as

$$S_1(Z) = (2c-l)Z^4-4Z^3k+(2l(2b_1+1)+4c(b_1+2))Z^2+(2c+l)(b_1^2-1). \quad (11)$$

It is easy to see that we can choose in each region a point with $k = 0$. Choosing $k = 0$ and $l = 3c$, $l = 0$, $l = \dfrac{l_{si} - 2c}{2}$ and $l = -3c$, respectively, $S_1(Z)$ becomes a biquadratic polynomial, i.e., a polynomial of the form $aZ^4 + bZ^2 + c$ and the analysis is simple. The conclusion is

R_1, 2 intersection points;

R_2, 0 intersection points;

R_3, 4 intersection points;

R_4, 2 intersection points.

□

3.3 Intersections in cases I, II and III

For cases I, II and III, we will need the following lemma, which will be proved in the Appendix:

Lemma 1 *: Let $p(z) = az^4 + bz^3 + cz^2 + d$ be a polynomial with real coefficients and $ad \neq 0$. Let $cond1 = 9b^2 - 32ac$ and $cond2 = a(-4c^3b^2 - 27b^4d + 144acb^2d - 128a^2c^2d + 16ac^4 + 256a^3d^2)$. Let m_1 and m_2 be the real zeros of $p'(z)$, if they exist. We have:*

i- If $ad < 0$ and $cond1 > 0$ and $cond2 < 0$, then $p(z)$ has 4 real roots;

ii- If $ad < 0$ and $cond1 > 0$ and $cond2 > 0$, then $p(z)$ has 2 real roots;

iii- If $ad < 0$ and $cond1 < 0$, then $p(z)$ has 2 real roots;

iv- If $ad > 0$ and $ac < 0$ and $cond2 < 0$, then $p(z)$ has 2 real roots;

v- If $ad > 0$ and $ac < 0$ and $cond2 > 0$ and $ap(m_1) > 0$ then $p(z)$ has 0 real roots;

vi- If $ad > 0$ and $ac < 0$ and $cond2 > 0$ and $ap(m_1) < 0$ then $p(z)$ has 4 real roots;

vii- If $ad > 0$ and $ac > 0$ and $cond1 < 0$, then $p(z)$ has 0 real roots;

viii- If $ad > 0$ and $ac > 0$ and $cond1 > 0$ and $cond2 < 0$, then $p(z)$ has 2 real roots;

ix- If $ad > 0$ and $ac > 0$ and $cond1 > 0$ and $cond2 > 0$, then $p(z)$ has 0 real roots;

Applying Lemma 1 to $S_1(Z)$, we get $a = 2c - l$, $b = -4k$, $c = 2l(2b_1 + 1) + 4c(b_1 + 2)$, $d = (2c + l)(b_1^2 - 1)$. A straightforward computation shows that $cond2 = (l - 2c)((2c + b_1 l)^2 - k^2(1 - b_1))(\alpha k^2 + \beta)$, where $\alpha = 27(2c + l)(b_1 + 1)$ and $\beta = (l - 2c)(4l + 10c + 5b_1 l + 8b_1 c)^2$. It can be shown that $(\alpha k^2 + \beta) = 0$ is the implicit equation of the (k, l)-sonic fold and $(2c + b_1 l)^2 - k^2(1 - b_1) = (b_1 l + 2c - \sqrt{1 - b_1}k)(b_1 l + 2c + \sqrt{1 - b_1}k)$ is the equation of both r_1 and r_2, as defined in Section 3.3.1. So, Hugoniot curves through points (k, l) satisfying $cond2 = 0$ are tangent to the sonic surface.

3.3.1 Intersections in case III

In case III, $0 < b_1 < 1$, tangency points are in the union of the (k, l)-sonic fold with the secondary bifurcations, $l = -2c$, r_1, given by $k = \frac{2c + b_1 l}{\sqrt{1 - b_1}}$, and r_2 given by $k = -\frac{2c + b_1 l}{\sqrt{1 - b_1}}$.

Straightforward computations show that r_1 and r_2 intersect the (k, l)-sonic fold tangentialy(in the same way as $y = x^3$ intersects $y = 0$), at points with $l_{bc} = -\frac{2c(2b_1 + 1)}{b_1 + 2}$. It is easy to see that r_1 intersects r_2 at $(0, \frac{-2c}{b_1})$.

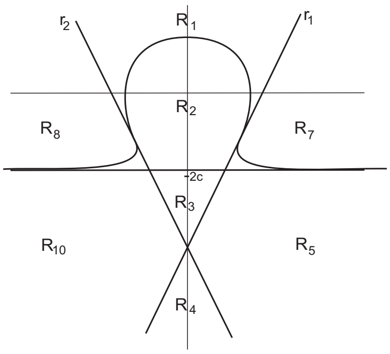

Figure 2: Decomposition of (k, l)-space in case III

The union of the (k, l)-sonic fold and the secondary bifurcations splits the (k, l)-space in ten regions, illustrated in figure 2: R_1, the region above the curve and between r_1 and r_2; R_2, the region below the curve, above $l = -2c$ and between r_1 and r_2; R_3, the triangle defined by r_1, r_2 and $l = -2c$; R_4, the region between r_1 and r_2, below their intersection point; R_5, the region below $l = -2c$, and to the right of r_1 and r_2 ; R_6, the small region above $l = -2c$, below the curve and to the right of r_1; R_7, the region above the curve and to the right of r_1; R_8, R_9 and R_{10}, the symmetric regions of R_7, R_6 and R_5, respectively.

Next result states how Hugoniot curves intersect *Son* in case III.

Proposition 2 *: Suppose* $0 < b_1 < 1$*, then*

- **i** - *Hugoniot curves through points in* R_3 *do not intersect the sonic surface;*
- **ii** - *Hugoniot curves through points in* R_2*,* R_5*,* R_7*,* R_8 *and* R_{10} *intersect the sonic surface in two distinct points;*
- **iii** - *Hugoniot curves through points in* R_1*,* R_4*,* R_6 *and* R_9 *intersect the sonic surface in four distinct points.*

Proof: As before for regions R_1, R_2, R_3 and R_4 we can take points $k = 0$, which makes $S_1(Z)$ biquadratic and again the analysis is simple.

A representative point of R_5 is $(k_5, l = -\frac{2c}{b_1})$, where k_5 is the k value at the intersection point of r_1 and k-axis. The fourth degree polynomial obtained by substituting $k = k_5$ and $l = -\frac{2c}{b_1}$ in equation (11), has two real roots from Lemma 1-*viii*. Taking $(-k_5, l = -\frac{2c}{b_1})$ as a representative point of R_{10}, from Lemma 1-*viii*, we get the same conclusion as in R_5.

A point to the right of $(k_5, 0)$ is a representative point of R_7. Such a point is $(\frac{4c}{\sqrt{1-b_1}}, l = 0)$. Substituting this point in $S_1(Z)$, from Lemma 1-*iii* we get that Hugoniot curves through points in R_7 intersect *Son* in two distinct points. In the same way, taking $(-\frac{4c}{\sqrt{1-b_1}}, l = 0)$ as a representative point of R_8, applying Lemma 1-*iii* we get the same conclusion as in R_7.

As we said before, the l value at the intersection point of r_1 and the (k,l)-sonic fold is $l_{bc} = -\frac{2c(2b_1+1)}{b_1+2}$. In this case $l_{bc} < 0$. The horizontal line $l = l_\mu$, where l_μ is the midpoint between $-2c$ and l_{bc}, intersects R_6 and R_9. Let us consider points in R_6. Straightforward computations show that (k_{sf6}, l_μ), the intersection of $l = l_\mu$ and the (k, l)-sonic fold, is to the right of (k_{sb6}, l_μ), the intersection of r_1 and $l = l_\mu$. We take the point (k_μ, l_μ), where k_μ is the midpoint between k_{sf6} and k_{sb6}, as a representative point of R_6. Substituting this point in equation (11), we get a fourth degree polynomial satisfying Lemma 1-i. In the same way, using the point $(-k\mu, l_\mu)$ as a representative point of R_9, from Lemma 1-i we get the same conclusion as in R_6.

□

3.3.2 Intersections in case II

In case II, $-1 < b_1 < 0$, the subdivision of the (k, l)-plane is similar to the one in case III. The main difference is that $r_1 \cap r_2$ is above of the k-axis and not below. So the union of (k, l)-sonic fold and r_1 and r_2 splits the (k, l)-plane in ten regions illustrated in figure 3: R_1 the region between r_1 and r_2 above their intersection point; R_2 the region above the curve, between r_1 and r_2 and below their intersection point; R_3 the region between r_1 and r_2, below the curve and above $l = -2c$; R_4 the unlimited region between r_1 and r_2 below $l = -2c$; R_5 the region to the right of r_1 and below $l = -2c$; R_6 the small region to the right of r_1, below the curve and above $l = -2c$; R_7 the region above the curve and to the right of r_1 and r_2; R_8, R_9 and R_{10} the symmetric regions of R_7, R_6 and R_5, respectively.

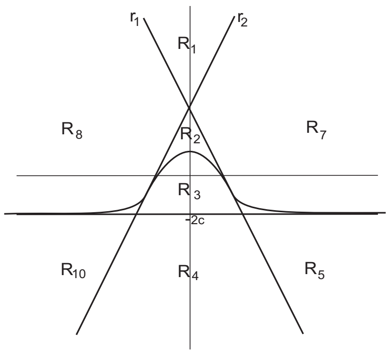

Figure 3: Decomposition of (k, l)-space in case II

Proposition 3 : *Suppose* $-1 < b_1 < 0$, *then*

i - *Hugoniot curves through points in R_1 and R_4 do not intersect the sonic surface;*

ii - *Hugoniot curves through points in R_3, R_5, R_7, R_8 and R_{10} intersect the sonic surface in two distinct points;*

iii - *Hugoniot curves through points in R_2, R_6 and R_9 intersect the sonic surface in four distinct points.*

Proof: Regions R_1, R_2, R_3 and R_4 are analyzed as in Proposition 1(using points with $k = 0$).

The line $l = -3c$ intersects the straight line r_1 at $k = \dfrac{c(2-3b_1)}{\sqrt{1-b_1}}$. So a representative point of R_5 is $(\dfrac{c(4-3b_1)}{\sqrt{1-b_1}}, -3c)$. After substituting these values of k and l in equation (11), it follows from Lemma 1-*viii* that the fourth degree polynomial has two real roots. Taking the point $(-\dfrac{c(4-3b_1)}{\sqrt{1-b_1}}, -3c)$ as a representative point of R_{10}, and using Lemma 1-*viii* we get the same conclusion as in R_5.

A representative point of R_7 is $(k = \dfrac{-2c}{b_1}, l = \dfrac{-2c}{b_1})$. Substituting this point in equation (11) we get a fourth degree polynomial satisfying hypoyhesis of item *vi* from Lemma 1. So it has two real roots. Taking $(k = \dfrac{2c}{b_1}, l = \dfrac{-2c}{b_1})$ as representative point for R_8, the same conclusion extends to points in that region.

To study how Hugoniot curves through points in R_6 intersect the sonic surface, we consider two cases:

i) $b_1 < -1/2$, in this case the line $l = l_{bc}$ is in $l > 0$ half-plane. So the horizontal line $l = 0$ crosses the regions R_6 and R_7, as k varies. It is easy to see that $(k_{sf}, 0)$, the intersection of $l = 0$ and the (k, l)-sonic fold is to the right of $(k_{bs}, 0)$, the intersection of $l = 0$ and r_1. In this way representative points of R_6 and R_9 are, respectively, $(k_m, 0)$ and $(-k_m, 0)$, where k_m is the midpoint between k_{sf} and k_{bs}. From Lemma 1-*i* Hugoniot curves through points in R_6 and R_9 intersect *Son* in four distinct points.

ii) $b_1 > -1/2$, in this case the line $l = l_{bc}$ is in $l < 0$ half-plane. So, as in the proof of Proposition 2, representative points of R_6 and R_9 are (k_μ, l_μ)

and $(-k_\mu, l_\mu)$, respectively. From Lemma 1-i the fourth degree polynomials obtained after substituting these points in $S_1(Z)$, has four real roots. $\square$

3.3.3 Intersections in case I

As we said before, in case I, $b_1 < -1$, and the (k,l)-sonic fold has three asymptotes, $l = -2c$ and two others, as_1 and as_2, given by $3\sqrt{-3(b_1+1)}k + (5b_1+4)l - 2c(b_1-1) = 0$ and $-3\sqrt{-3(b_1+1)}k + (5b_1+4)l - 2c(b_1-1) = 0$, respectively. Their intersection point is $(k = 0, l = \dfrac{2c(b_1-1)}{5b_1+4})$. Except for $b_1 = -2$, the asymptotes as_1 and as_2 do not coincide with the secondary bifurcations r_1 and r_2. It is easy to see that $as_1 \cap as_2$ is above $r_1 \cap r_2$. Straightforward computations show that at the intersection of as_i, $i = 1, 2$, and the (k,l)-sonic fold, $l = \dfrac{-2c(17b_1^2+38b_1+26)}{3(5b_1^2+14b_1+8)}$. It follows that the intersection points are in $l > 0$ half-plane or $l < 0$ half-plane, according to $-2 < b_1 < -1$ or $b_1 < -2$. So, case I subdivides in two subcases: case Ia, $b_1 < -2$ and case Ib, $-2 < b_1 < -1$.

In case Ia, the (k,l)-plane is subdivided in 12 regions, as illustrated in figure 4(a): R_1 the unlimited region above the upper branch of the curve; R_2, the region between r_1 and r_2 below the upper branch; R_3 the triangular region defined by r_1, r_2 and $l = -2c$. The intersections of the (k,l)-sonic fold with the secondary bifurcations, r_1 and r_2 are in the $l < -2c$ region. These interserctions split $l < -2c$ in 7 regions: A central region R_4; three regions on the right side: R_5 the region to right of r_1 below the curve; R_6, the small region to the right of r_1 above the curve and R_8, the small region to the left of r_1 below the curve; three regions on the left side: R_{12}, R_{11} and R_9 symmetric to R_5, R_6 and R_8, respectively. The other regions are R_7, the region to the right of r_1 and r_2 above $l = -2c$ and its symmetric region R_{10}.

In case Ib, the (k,l)-plane is also subdivided in 12 regions, as illustrated in figure 4(b). In this case the secondary bifurcations r_1 and r_2 intersect the upper branch of the (k,l)-sonic fold. These intersection points split the region above the upper branch in three regions: a central region, R_1 and two small regions: R_8, above the curve below r_2 and R_9, symmetric to R_8. The other regions are: R_2, the region between r_1 and r_2 below the upper branch; R_3 the triangular region defined by r_1, r_2 and $l = -2c$; R_4, the region $l < -2c$

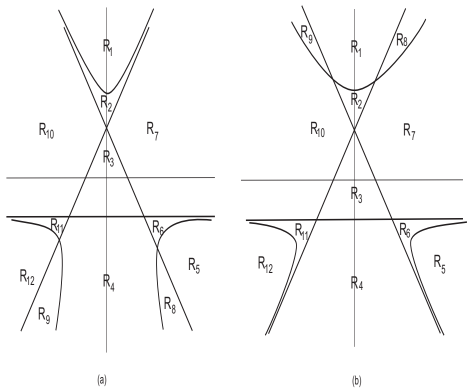

Figure 4: Decomposition of (k, l)-space in case Ia and case Ib, respectively

between r_1 and r_2; R_6, the small region below $l = -2c$ to the right of r_1 above the right branch of the curve; R_5, the region below the right branch of the curve; R_7, the region above $l = -2c$ to the right of r_1 and r_2; R_{10}, R_{11} and R_{12} the symmetric regions of R_7; R_6 and R_5, respectively.

Next proposition states how Hugoniot curves intersect *Son* in case I:

Proposition 4 *: Suppose $b_1 < -1$, then*

- **i** - *Hugoniot curves through points in R_3 do not intersect or intersect the sonic surface in four distinct points, according to $b_1 < -5/4$ or $-5/4 < b_1$, respectively;*
- **ii** - *Hugoniot curves through points in R_1, R_4, R_5, R_7, R_{10} and R_{12} intersect the sonic surface in two distinct points;*
- **iii** - *Hugoniot curves through points in R_2, R_6, R_8, R_9 and R_{11} intersect the sonic surface in four distinct points.*

Proof: Regions R_1, R_2 and R_4 are analyzed as in Proposition 1. The conclusions for R_3 follows using $k = 0$ and $l = 0$ in $S_1(Z)$. We get a biquadratic polynomial such that its discriminant equals zero for $b_1 = -5/4$.

The horizontal line $l = -4c$ intersects regions R_5, R_6 and their symmetric regions. So we can take points on this line as representative points of these regions. Let us consider points in R_5 and R_6. The line $l = -4c$ intersects the (k,l)-sonic fold at $k_{ds} = -\dfrac{2c(2b_1+1)}{\sqrt{-b_1-1}}$. In this way, the point ($k_5 = -\dfrac{4c(2b_1+1)}{\sqrt{-b_1-1}}, l = -4c$) is a point in R_5. Substituting $k = k_5$ and $l = -4c$ in $S_1(Z)$, it follows from Lemma 1-*ii* that the fourth degree polynomial has two real roots. Using $(k = -k_5, l = -4c)$ in $S_1(Z)$, similar computations give the same conclusion for points in R_{12}.

To get a point in R_6, we follow the same strategy as in Proposition 3, i. e., we take the k value, k_{m6}, as the midpoint between k_{ds} and k_6, the intersection of $l = -4c$ and r_1. Using $k = k_{m6}$ and $l = -4c$ in equation (11) and Lemma 1-*i* we have that Hugoniot curves through points is R_6 intersect *Son* in four distinct points. Similar computations with $k = -k_{m6}$ and $l = -4c$ give the same conclusion for points in R_{11}.

Any point on k-axis to the right of $(\dfrac{2c}{\sqrt{1-b_1}}, l = 0)$, the intersection of r_1 and k-axis, is a representative point of R_7. Using $k = \dfrac{4c}{\sqrt{1-b_1}}$, and $l = 0$ in equation (11) and Lemma 1 we see that Hugoniot curves through points in R_7 intersect *son* in two distinct points. The same conclusions extend to points in R_{10}.

In case Ia, regions R_8 and R_9 are in $l < 0$ half-plane. Since $l_{int} = -\dfrac{4c(2b_1+1)}{b_1+2} < l_{bc}$, the line $l = l_{int}$ crosses regions R_8 and R_9. Let us consider its intersection with R_8. Let k_{sf8} and k_{sb8} be the intersections of l_{int} with the (k,l)-sonic fold and r_1, respectively. We take the point (k_{m8}, l_{int}), where k_{m8} is the midpoint between k_{sf8} and k_{sb8}, as a representative point of R_8. From Lemma 1-*i* the fourth degree polynomial obtained from equation (11) has four distinct real roots. Taking the point $(-k_{m8}, l_{int})$ as a representative points of R_9, we get the same conclusion as in R_8. In case Ib, the horizontal line $l = l_{int}$ is above l_{bc}. As the calculations in case Ib are the same as in case Ia, results for case Ia extend to case Ib. □

4 Appendix: Proof of Lemma 1, a Polynomial Lemma

We are considering a fourth degree polynomial $p(z) = az^4 + bz^3 + cz^2 + d$. Lemma 1 gives the number of real roots of $p(z)$ in terms of its coefficients and extremum values.
As $p'(x) = z(4az^2 + 3bz + 2c)$, the graph of $p(z)$ has an extremum value at $(0, d)$. According to the sign of $cond1 = 9b^2 - 32ac$ it may have two others. If $cond1 > 0$, let m_1, m_2 be the extreme values of $p(z)$, and consider the product $cond2 = p(m_1)p(m_2)$, i.e., $cond2 = a(-4c^3b^2 - 27b^4d + 144acb^2d - 128a^2c^2d + 16ac^4 + 256a^3d^2)$

The number of real roots of $p(z)$ depends on the sign of ad:

a) $ad < 0$. Since a and d have opposite signs, in this case $p(z)$ has at least two real roots.

- If $cond1 < 0$ or $cond1 > 0$ and $cond2 > 0$, $p(z)$ has two real roots, proving itens *iii* and *ii*, respectively.
- If $cond1 > 0$ and $cond2 < 0$, $p(z)$ has four real roots, proving item *i*.

b) $ad > 0$, we have two subcases: $ac < 0$ or $ac > 0$.
We remark that $p(z)$ reachs a local maximum or local minimum at $z = 0$ according to $ac < 0$ or $ac > 0$.

b1) $ac < 0$ implies $cond1 > 0$.

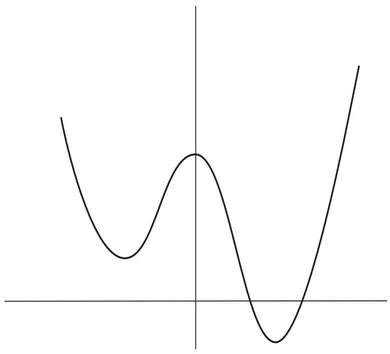

Figure 5: A possible graph of $p(z)$ satisfying conditions in item *iv*

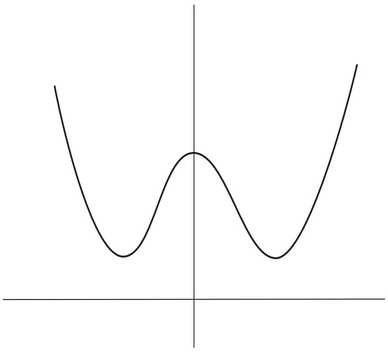

Figure 6: A possible graph of $p(z)$ satisfying conditions in item *v*.

- If $cond2 < 0$ then $p(z)$ has two real roots, proving item *iv*;
- If $cond2 > 0$, $p(z)$ may has 0 or 4 real roots according to $ap(m_1) > 0$ or $ap(m_1) < 0$, respectively, proving itens *v* and *vi*.

b2) $ac > 0$ and $cond1 < 0$, $p(z)$ has no real roots, proving item *vii*;

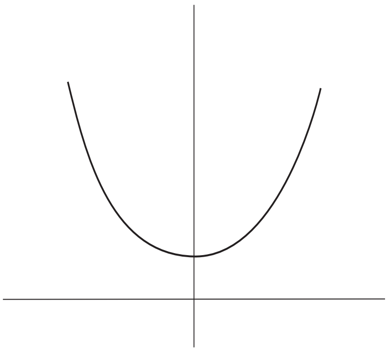

Figure 7: A possible graph of $p(z)$ satisfying conditions in item *vii*.

b3) $ac > 0$ and $cond1 > 0$

- $cond2 > 0$, $p(z)$ has no real roots, proving item *ix*;
- $cond2 < 0$, $p(z)$ has two real roots, proving item *viii*.

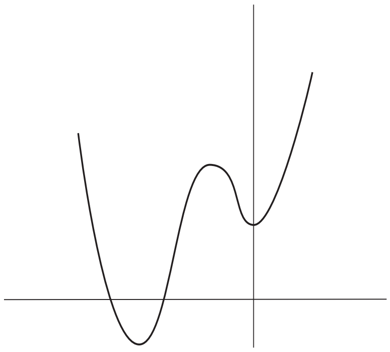

Figure 8: A possible graph of $p(z)$ satisfying conditions in item *viii*.

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