BOUNDARY OF ANOSOV DYNAMICS AND EVOLUTION EQUATIONS FOR SURFACES

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Abstract. We show that a C^∞ compact surface of genus greater than one, without focal points and a finite number of bubbles ("good" shaped regions of positive curvature) is in the closure of Anosov metrics. Compact surfaces of nonpositive curvature and genus greater than one are in the closure of Anosov metrics, by Hamilton's work about the Ricci flow. We generalize this fact to the above surfaces without focal points admitting regions of positive curvature using a "magnetic" version of the Ricci flow, the so-called Ricci-Yang-Mills flow.

1. Introduction

Stable dynamics, and in particular Anosov dynamics, are very well understood nowadays. Since the 1980s, a great deal of effort has been devoted to the study of unstable or chaotic systems. One of the most appealing subjects of the area is the study of the boundary of stable systems since it is closely related to bifurcation theory and applications. In this paper we deal with the boundary of stable geodesic flows, a classical family of systems whose generic behavior cannot be studied using common tools of generic dynamics because of its conservative nature. A celebrated theorem of Klingenberg [9] states that Anosov geodesic flows on compact manifolds have no conjugate points, and since Anosov systems are C^1 structurally stable, they are in the C^2 interior of the set of metrics without conjugate points. Recall that a C^∞ Riemannian manifold has no conjugate points if the exponential map at every point is non-singular. A result due to Ruggiero [17] shows that the interior of the set of metrics without conjugate points coincides with the set of Anosov metrics in the C^2 topology. So it is natural to ask whether, for compact manifolds, the set of metrics without conjugate points is the closure of the set of Anosov metrics (whenever this set is nonempty). This is a very difficult problem whose complete solution seems far from reach with the current tools.

We start to study the problem in the set of surfaces without conjugate points, based on recent results of the theory of evolution equations. The celebrated work of Hamilton [8] about the Ricci flow yields that compact surfaces with nonpositive curvature are in the boundary of surfaces with negative curvature. The main tool of the proof is the maximum principle for parabolic equations (see Section 1 for details), which works very well when the curvature of the surface does not change sign. Indeed, in the presence of regions with curvature of different signs, the short time evolution of the curvature is much more complicated to understand. Although

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long time evolution of the normalized Ricci equation of a compact surface of genus greater than one is hyperbolic geometry, our problem is rather connected to short time evolution of metrics. The effect of the Ricci flow on the curvature is determined by a diffusion-reaction equation and hence it is quite difficult in general to describe how the curvature evolves in the surface. Besides, it is very hard to estimate how long the Ricci flow takes to eliminate regions of positive curvature in the case of genus greater than one. This places great technical limitations on the use of the Ricci flow in the study of the boundary of Anosov surfaces: one has to consider surfaces with variable curvature sign and without conjugate points.

In this article, we deal with a subset of the surfaces without conjugate points: surfaces without focal points admitting "nicely shaped" regions of positive curvature. Our main result is the following.

Theorem 1.1. *Let M be a compact C^∞ surface of genus greater than one. Suppose that (M,g) has no focal points and that the closure of the set of points of positive curvature is contained in a finite union of non-degenerate bubbles. Then there exists a C^∞ family of metrics g_ρ, $\rho \in [0,\rho_0]$, $g_0 = g$, in the same conformal class of g, such that g_ρ is an Anosov metric for every $\rho \in (0,\rho_0]$.*

Basically, a non-degenerate bubble is a simply connected region of positive curvature, bounded by a smooth strictly convex closed curve where the Gaussian curvature vanishes, and having an open annulus of negative curvature surrounding the boundary of the bubble (see Section 4 for details). We would like to point out that almost all known examples of surfaces without conjugate points are constructed by gluing non-degenerate bubbles into surfaces of non-positive curvature. (The only exception being the Donnay-Pugh examples of compact surfaces in $\mathbb{R}^3$ with Anosov geodesic flows [4].)

Let us give a brief account of the main ideas of the proof. First of all, we use a modified version of the Ricci flow, the so-called Ricci-Yang-Mills flow (see Section 2), to produce a conformal family of metrics whose regions of positive curvature shrink, as well as strictly reducing the curvature inside such regions.

Theorem 1.2. *Let M be a compact C^∞ surface of genus greater than one. Then there exists a family of metrics g_ρ, $\rho \in [0,\rho_0]$, $g_0 = g$, in M with the following property: let K_ρ be the curvature of g_ρ, and let P_ρ be the set of points of non-negative curvature of K_ρ, then*

(1) *$K_\rho(x) < K_s(x)$ for every $x \in P_0$ and $\rho > s$,*
(2) *$P_\rho \subset P_s$ for every $\rho > s$.*

Next, we make a careful study of the geometry of Jacobi fields of surfaces with non-degenerate bubbles and without focal points, and show that Jacobi fields of conformal metrics in the family share many of the geometric properties of the initial Jacobi fields. This is the hardest part of the proof; in particular we have to show that the conformal metrics in the family have no conjugate points. Actually, we show more than that: we show that if there are "central" Jacobi fields in the conformal metrics (namely, Jacobi fields whose norm is uniformly bounded in time), then their norm is "mostly" constant (see Section 5). Using this fact, and Eberlein's criterion to classify Anosov geodesic flows in terms of Green's bundles, we show that central Jacobi fields cannot exist under the hypotheses of the main theorem. At this point the assumption on the bubbles is crucial as it allows us to neglect considerations about the effect of the conformal perturbations on the global behavior of geodesics. Notice that perturbed geodesics might have a chaotic behavior: we are in the boundary of stable systems. Further, the Ricci-Yang-Mills flow gives us no hint whatsoever about the global geometry of perturbed geodesics. We would like to remark that the existence of conformal families of metrics preserving the absence of conjugate points has been conjectured previously, for example Eberlein [7]. As far as we know, Theorem 1.1 gives the first explicit examples of such metrics.

The Ricci flow g_t has been used to study the evolution of the entropy of geodesic flows for compact surfaces of negative curvature by Manning [12]. Moreover, Theorem 1.2 gives us some hope of answering the following question posed by Manning: does the evolution of a metric without conjugate points by Ricci flow avoid metrics with conjugate points? In the case of the Ricci-Yang-Mills flow with a particular magnetic field, Theorem 1.2 tells us that the answer to this question is likely to be affirmative, since the "bad" regions of a surface without conjugate points (namely, the regions of positive curvature) shrink under the action of the flow. Of course, this is not enough to show that the evolution metrics have no conjugate points, but Theorem 1.1 is certainly an encouraging result. Theorem 1.1 partially answers the question posed at the beginning of the Introduction: certain surfaces without focal points admitting regions of positive curvature are indeed in the closure of Anosov metrics. This result is remarkable in many senses, notably because of the great difficulty in controlling the global behavior of geodesics of perturbed metrics. We think that compact surfaces without focal points should be in the closure of Anosov metrics, as well as surfaces without conjugate points.

2. Surfaces of non-positive curvature are in the closure of Anosov metrics

The main result of the section is the following,

Proposition 2.1. *Take $k \geq 4$. Any C^k metric with non-positive curvature on a compact surface of genus greater than one is in the C^∞ closure of the set of metrics of negative curvature.*

This result is the main motivation of our article. The proof was communicated to us by M. Struwe. It is a consequence of the strong maximum principle for parabolic equations, that we state next following Protter-Weinberger [16].

Theorem 2.2. *Let M be a compact surface and let $P : M \times (-a, a) \longrightarrow \mathbb{R}$ be a C^2 function satisfying*

$$\Delta P(x,0) - \frac{\partial}{\partial \rho}\bigg|_{\rho=0} P(x,\rho) \geq 0.$$

Then if $P(x,0) \leq 0 = \sup_x P(x,0)$, and $P(x,\rho) = 0$ for some $\rho > 0$, then $P(x,\rho) = 0$ for every (x,ρ).

To show the proposition, let the metrics g_ρ be solution of the Ricci flow with $g_0 = g$, and

$$\frac{\partial g_\rho}{\partial \rho} = -2K \cdot g_\rho,$$

where $K = K(x,\rho)$ is the curvature of g_ρ. By the standard results in the field [20] this family exists and is unique, and for $\rho > 0$ the metric g_ρ is smooth. By assumption $K(x,0)$ is non-positive, and a little calculation shows that generally the curvature evolves as

$$\frac{\partial K}{\partial \rho} = \triangle K + K^2.$$

Consider $H(x,\rho) = \exp(n\rho)K(x,\rho)$ for an integer n (to be determined). Then

$$\Delta H - \frac{dH}{d\rho} = \exp(n\rho)(\Delta K - \frac{dK}{d\rho}) - n\exp(n\rho)K = -K\exp(n\rho)(2K+n).$$

If we take $n > 2\|K\|_\infty$ we see that the above inequality is greater or equal to zero. As $K(x,0) \leq 0$ we can apply the maximum principle to H to conclude that $H(x,\rho)$ cannot be zero for any $\rho > 0$: otherwise $H(x,\rho)$ would be identically zero for all time. This would contradict Gauss-Bonnet, as $H(x,0) = K(x,0)$ is the curvature of a compact surface of genus greater than one. This finishes the proof of Proposition 2.1.

3. Subharmonic functions and the Ricci Yang-Mills flow

The purpose of the section is to show Theorem 1.2. We work with a compact, smooth, oriented Riemannian 2-manifold (M,g), which we call a *surface*, and let $K : M \to \mathbb{R}$ be the associated Gaussian curvature. In the negatively curved case, where $K(x) < 0$ for all points $x \in M$, very strong dynamical properties hold for the geodesic flow of (M,g) since the flow is Anosov. From the topological point of view, geodesic flows of surfaces without conjugate points are close to Anosov flows, but much less can be said in this case about the ergodic properties of the dynamics (positive metric entropy for instance). We shall subdivide the section into two subsections. In the first one we describe the action of the Ricci-Yang-Mills flow on metrics on the regions of M with positive curvature when the "magnetic potential" m is a subharmonic function. Basic facts of the existence theory of subharmonic functions in subsets of compact surfaces will be recalled in the second subsection, where we shall finish the proof of Theorem 1.2.

3.1. **The action of RYM flows for subharmonic magnetic fields.**

Proposition 3.1. *If there exists a smooth positive function on M whose Laplacian is negative on the non-negative curvature set $P = \{x \in M : K(x) \geq 0\}$ of the surface (M,g), then there exists a positive value T and a smooth family of smooth Riemannian metrics $\{g_\rho\}_{\rho\in[0,T)}$ on M such that*

1. *$g_0 = g$.*
2. *The curvature K_ρ of g_ρ is strictly decreasing in P.*
3. *The sets P_ρ of points of non-negative K_ρ-curvature shrink with t: $P_\rho \subset int(P_s)$ for every $\rho > s$ in $[0,T]$, where $int(A)$ is the interior of a set A.*

The proof of this Proposition relies on the *Ricci Yang-Mills flow*, a modification of the Ricci flow recently introduced by J. Streets [18] and A. Young [22]. A particular case of the Ricci Yang-Mills flow can be written as a coupled evolution equation of a family of metrics g_ρ on a surface M along with a family of a functions $m_\rho : M \to \mathbb{R}$:

$$\frac{\partial g_\rho}{\partial \rho} = (m_\rho^2 - 2K_\rho)\cdot g_\rho, \tag{1a}$$

$$\frac{\partial m_\rho}{\partial \rho} = \triangle_\rho m_\rho + 2K_\rho m_\rho - m_\rho^3. \tag{1b}$$

See [5] for the explicit derivation. As with the Ricci flow on surfaces, short term existence has been established [18, 22] and in the long term, up to a volume renormalisation, the metric convergences to a metric with constant curvature and the function m_ρ converges to a constant function [19]. The evolution of curvature under the Ricci Yang-Mills flow (1) can be calculated as

$$\frac{\partial K_\rho}{\partial \rho} = \triangle_\rho K_\rho + K_\rho^2 - 2K_\rho m_\rho^2 + \tfrac{1}{2}\triangle_\rho m_\rho^2. \tag{2}$$

The last term provides the extra control that is missing in a perturbative argument for decreasing the curvature of non-negative curvature points in equation (2) in the Ricci flow setting.

Proof of Proposition 3.1. We have assumed the existence of a positive function on (M, g), which we define as $m^2 : M \to \mathbb{R}$, such that the Laplacian on the set $P := \{x \in M : K(x) \geq 0\}$ is negative. As the set P is compact, $\max\{0, \triangle K/(-\triangle m^2)\}$ restricted to the set P attains its supremum c^2. We consider the Ricci Yang-Mills family generated by the function $2\lambda \cdot m$, for a real number $\lambda > c$, coupled with the metric g. Take λ large enough in order that we also have

$$\|K\|_\infty - 2\lambda \min m < 0.$$

From the evolution of the curvature (2), for every $x \in P$ we have

$$\begin{aligned}\frac{\partial K_\rho}{\partial \rho}\Big|_{\rho=0} &= \triangle K + K(K - 2\lambda m) + \tfrac{1}{2}\triangle(4\lambda^2 \cdot m^2)\\ &\leq \triangle m^2 \cdot (-c^2 + 2\lambda^2) < 0.\end{aligned}$$

Thus on the set P the curvature will immediately decrease along the Ricci Yang-Mills family g_ρ. By the continuity of the derivatives of K_ρ with respect to ρ, there exists a positive time T_1 and an open neighbourhood U of P, such that the curvatures $K_\rho(x)$ for each $x \in U$ decrease for all metrics $\{g_\rho : \rho \in (0, T_1)\}$.

It remains to show that the curvature is negative for some positive time on the complement of U. Since $M \backslash U$ is compact the supremum of the initial curvature is attained and is strictly negative. Since the curvature evolves continuously under the Ricci Yang-Mills flow, there must exist some positive time T_2 (possibly infinite) before any points of zero curvature are created. Thus for $\rho \in (0, \min\{T_1, T_2\})$, a non-trivial interval, the regions of negative curvature of the surfaces (M, g_t) increase with t. □

3.2. On the existence of subharmonic functions in Riemann surfaces.

The existence of subharmonic functions is a classical subject of complex analysis closely related to the classification of Riemann surfaces. The following result is taken from Mok [13]:

Theorem 3.2. *Let M be a compact Riemann surface, and let $X \subset M$ be an open Riemann surface with boundary. Suppose that there exists a nontrivial, smooth harmonic function in X. Then there exists a smooth, positive, strictly subharmonic function f in X.*

So the existence of subharmonic functions in X is connected (in fact, equivalent) to the solution of the Dirichlet problem in X with initial conditions defined in the boundary of X. The theory of harmonic functions (see for instance [11]) gives us the following:

Lemma 3.3. *Let M be a compact Riemann surface of genus greater than one, and let $X \subset M$ be an open Riemann surface. If the boundary of X is smooth and the growth of balls in a complete hyperbolic structure in X is exponential, then the Dirichlet problem in X has a smooth solution.*

Given a complete Riemannian metric (X, h) in X, the growth of balls is exponential if there exist $c > 0$, $a > 0$, such that for every metric ball $B_r(p)$ of radius $r > 0$ centered at $p \in X$ we have $\mathrm{vol}(B_r(p)) \geq ce^{ar}$.

Proof of Theorem 1.2.

Let (M, g) be a compact, C^∞ surface of genus greater than one. By the Gauss-Bonnet theorem, there exists a point $q \in M$ with negative curvature. Let $B_\delta(q)$ be a closed ball around q where the curvature is strictly negative, and consider the Riemann surface $X_\delta = M - B_\delta(q)$. X_δ is conformally equivalent to a hyperbolic surface with one end, with infinite area. So the growth of balls in this conformal structure is exponential, and by Theorem 3.2, there exists a positive, strictly subharmonic function a in $X_{\frac{\delta}{2}}$. Now, let $b : M \longrightarrow \mathbb{R}^+$ be a smooth function which coincides with a in X_δ. Since X_δ contains the set of points of non-negative curvature of (M, g), Proposition 3.1 yields that the Ricci-Yang-Mills flow applied to (M, g, b) gives metrics (M, g_ρ) whose regions of positive curvature shrink and whose regions of negative curvature increase with ρ, for all small $\rho > 0$.

4. The geometry of surfaces without focal points

The purpose of this section is to introduce surfaces without focal points with non-degenerate bubbles. The special growth properties of the Jacobi fields of such surfaces will be crucial to prove Theorem 1.1.

4.1. **Surfaces without focal points.** There are many equivalent definitions of manifolds without focal points. The most geometric is perhaps the convexity of metric spheres in the universal covering endowed with the pullback of the metric of the manifold. For convenience, we shall use the following one:

Definition 4.1. Take any geodesic γ in a C^∞ Riemannian manifold (M, g), and any Jacobi field $J(t)$ along $\gamma(t)$ with $J(0) = 0$ and $J(t)$ perpendicular to $\gamma'(t)$. Then (M, g) has no focal points if the norm of $J(t)$ is strictly increasing for $t \geq 0$, for all such choices of γ and J.

Surfaces without focal points satisfy the so-called flat strip Theorem (see [15] for instance): two geodesics in $(\tilde{M}, \tilde{g})$ bounding a topological strip bound in fact a flat strip. The infinitesimal version of this result is:

Lemma 4.2. *Assume (M, g) has no focal points. Given a geodesic γ, every Jacobi field $J(t)$ such that $\| J(t) \| \leq L$ for every $t \in \mathbb{R}$ is parallel: there exists $a \geq 0$ such that $\| J(t) \| = a$ for every $t \in \mathbb{R}$. Further, if $J(t)$ is perpendicular to $\gamma'(t)$ for every t, and $a > 0$, we have that the Gaussian curvature $K(\gamma(t)) = 0$ for every $t \in \mathbb{R}$.*

We shall call any geodesic where the Gaussian curvature vanishes at every point of the geodesic a **flat geodesic**.

Definition 4.3. Let (M, g) be a C^∞ complete manifold without conjugate points. Let $\theta = (p, v) \in T_1M$, let S_p be the orthogonal complement of $\gamma'(0)$ with respect to g. Given a geodesic $\gamma_\theta(t)$ and a vector $W \in S_p$, a stable Jacobi field $J^s_W(t)$ with initial condition $J^s_W(0) = W$ is defined by

$$J^s_W(t) = \lim_{T\to+\infty} J_T(t),$$

where $J_T(t)$ is a Jacobi field perpendicular to $\gamma'(t)$ for every $t \in \mathbb{R}$ such that $J_T(0) = W$, $J_T(T) = 0$. An unstable Jacobi field $J^u_W(t)$ with initial condition $J^u_W(0) = W$ is defined by

$$J^u_W(t) = \lim_{T\to-\infty} J_T(t).$$

The following statement is a special feature of asymptotic Jacobi fields in the absence of focal points.

Lemma 4.4. *Let (M, g) be a compact manifold without focal points. Then, given $(p, v) \in T_1M$, $W \in T_pM$, $\| W \|= 1$ and $g(W, v) = 0$, the Jacobi fields $J_T(t)$ defined by $J_T(0) = W$, $J_T(T) = 0$ converge uniformly in (p, v) and t to $J^s_W(t)$ as $T \to +\infty$, Namely, given $\epsilon > 0$ there exists $T(\epsilon) > 0$ such that for every $(p, v) \in T_1M$, and W as above, we have that*

$$\| J^s_W(t) - J_T(t) \|\leq \epsilon$$

for every $t \in [-T, T]$ and every $(p, v) \in T_1M$. The same holds for unstable Jacobi fields and $J_T(t)$ as $T \to -\infty$.

Indeed, the proof of Lemma 4.4 follows from the **uniform divergence of radial Jacobi fields** (see [6] for the definition) that holds in the absence of focal points. Combining Lemma 4.4 and the uniform divergence of radial Jacobi fields we get,

Lemma 4.5. *Given $C > 0$, $\epsilon > 0$, there exists $T = T(C, \epsilon)$ such that if a perpendicular Jacobi field $J(t)$ satisfies $J(0) = W$ and*

$$\sup_{t\in[-T,0]} \| J(t) \|\leq\| J(0) \|$$

then

$$\| J^u_W(t) - J(t) \|\leq \epsilon$$

for every $t \in [-T, C]$.

Let $N_\theta \subset T_\theta T_1M$ be the orthogonal complement of the geodesic vector field with respect to the Sasaki metric. The subspace N_θ is the direct sum of $H_\theta \bigoplus V_\theta$, where H_θ is the intersection of the horizontal space of $T_\theta T_1M$ and N_θ, and V_θ is the vertical subspace.

The **Green subspaces** E^s_θ, E^u_θ are the subspaces of N_θ defined in canonical coordinates by

$$E^s_\theta = \bigcup_{W\in S_p} \{(J^s_W(0), {J^s_W}'(0))\},$$

$$E^u_\theta = \bigcup_{W\in S_p} \{(J^u_W(0), {J^u_W}'(0))\}.$$

E^s_θ is often called the stable Green subspace, E^u_θ is called the unstable Green subspace. The collection of the subspaces E^s_θ is called the **stable Green subbundle**, and the collection of the E^u_θ is called the **unstable Green subbundle**. Both are Lagrangian, invariant subspaces, and in general they are not continuous bundles.

Lemma 4.6. *Let (M, g) be a compact surface without focal points. The following assertions hold.*

(1) *The norm of stable Jacobi fields is non-increasing, the norm of unstable Jacobi fields is non-decreasing.*
(2) *A Jacobi field is parallel if and only if it is both stable and unstable.*
(3) *Stable and unstable Green subbundles are continuous.*
(4) *A geodesic is not flat if and only its stable and unstable Green subbundles are linearly independent, and generate the space of perpendicular Jacobi fields along this geodesic. In particular, the geodesic flow is Anosov if and only if every geodesic crosses a region of non-vanishing curvature.*
(5) *Stable and unstable Green subbundles are linearly independent if and only if:*
   - *The norm $\| J(t) \|$ of every stable Jacobi field tends to a finite limit as $t \to +\infty$, and tends to ∞ if $t \to -\infty$.*
   - *The norm $\| J(t) \|$ of every unstable Jacobi field tends to a finite limit as $t \to -\infty$, and tends to ∞ if $t \to \infty$.*

A special feature of surfaces is that given a Jacobi field $J(t)$ defined along a geodesic γ and everywhere perpendicular to $\gamma'(t)$ can be written as

$$J(t) = f(t)e(t)$$

where $e(t)$ is a unit, parallel vector field in γ perpendicular to $\gamma'(t)$ and $f(t)$ is a scalar solution of the Jacobi equation

$$f''(t) + K(t)f(t) = 0.$$

So without loss of generality we shall identify Jacobi fields with scalar solutions of the Jacobi equation. The well known Riccati equation associated to the Jacobi equation

$$u'(t) + u^2(t) + K(t) = 0,$$

whose solutions are of the form $u(t) = \frac{f'(t)}{f(t)}$, $f(t)$ a scalar solution of the Jacobi equation, has the following properties:

Lemma 4.7. *Let (M, g) be a compact surface without focal points. Then*

(1) *Given a geodesic γ_θ, the stable Jacobi fields define a unique solution $u^s_\theta(t)$ of the Riccati equation that is defined for every $t \in \mathbb{R}$. This solution is non-positive. Analogously, the unstable Jacobi fields give rise to a unique solution $u^u_\theta(t)$ of the Riccati equation that is defined for every $t \in \mathbb{R}$ and is everywhere non-negative.*
(2) *There exists a constant $K_0 > 0$ such that $\mid u^s_\theta(t) \mid$ and $\mid u^u_\theta(t) \mid$ are bounded above by K_0 for every $t \in \mathbb{R}$.*
(3) *Every other solution of the Riccati equation tends to $u^u_\theta(t)$ as $t \to +\infty$, and tends to $u^s_\theta(t)$ as $t \to +\infty$.*
(4) *The solutions $u^s_\theta(t)$, $u^u_\theta(t)$ depend continuously on θ.*

4.2. **Non-degenerate bubbles.**

Definition 4.8. Let (M, g) be a complete, C^∞ surface. A non-degenerate bubble $B \subset M$ is the closure of a non-empty open, simply connected set A of positive curvature satisfying the following properties:

(1) The boundary of B is a C^1 strictly convex, simple closed curve where the Gaussian curvature of the surface vanishes everywhere.
(2) There exists an open neighborhood U of B such that the complement $U-B$ of B in U is an open annulus of negative curvature.

The definition of a non-degenerate bubble B implies that every closed loop in B is contractible in the surface. Moreover, the points in a neighborhood U of B with vanishing curvature are isolated from other points with vanishing curvature in the surface.

For simplicity, we shall say that a complete surface is **non-degenerate** if the regions of positive curvature of the surface are contained in the union of a finite number of non-degenerate bubbles. From now on, we shall consider surfaces without focal points which are non-degenerate.

We shall apply the basic lemmas of the previous subsection to get some special properties of Jacobi fields under this assumption. Recall that the notation $\gamma_{(p,v)}(t)$ designates the geodesic with initial conditions $\gamma_{(p,v)}(0)=p$, $\gamma'_{(p,v)}(0)=v$. The next two lemmas are straightforward consequences of Lemma 4.6.

Lemma 4.9. *Let (M,g) be a non-degenerate compact surface without focal points. Then the Green bundles along every geodesic γ_θ which intersects a bubble are linearly independent.*

Proof. This is simply because a geodesic where stable and unstable Jacobi fields agree is a flat geodesic according to Lemma 4.6 (4), and there is no flat geodesic totally contained in a bubble since bubbles are simply connected. Thus, a geodesic meeting a bubble must meet its boundary and therefore has nonflat points. □

Lemma 4.10. *Let (M,g) be a non-degenerate compact surface without focal points. Let γ be a geodesic which crosses a bubble. Then every perpendicular Jacobi field of γ can be written as the sum of a stable Jacobi field and an unstable Jacobi field. That is, every perpendicular Jacobi field $J(t)$ satisfies one of the following:*

(1) *$J(t)$ is stable, and hence its norm is non-increasing,*
(2) *$J(t)$ is unstable and its norm is non-decreasing,*
(3) *$J(t)=AJ^s(t)+BJ^u(t)$, where $J^s(t)$, $J^u(t)$ are respectively, a stable and an unstable Jacobi field with $\| J^s(0) \| = \| J^u(0) \| = 1$, and $AB \neq 0$.*

In the case (3) where $J(t)$ has a zero we call it a *radial* Jacobi field.

Lemma 4.11. *Let (M,g) be a non-degenerate compact surface without focal points. Then there exists $A>0$ satisfying the following property:*

For every $p \in M$ in the boundary of a bubble, and every $v \in T_1M$ that is either tangent to the boundary of the bubble or points into the interior of the bubble, we have $u^u_{(p,v)}(0) \geq A$.

Proof. Let us first show that $u^u_{(p,v)}(0) \neq 0$ for every (p,v) satisfying the hypothesis in the statement.

We know that $u^u_{(p,v)}(t) = \frac{f'(t)}{f(t)}$, where $f(t)$ is a scalar solution with no zeroes of the Jacobi equation in the geodesic $\gamma_{(p,v)}(t)$, and $f'(t) \geq 0$ for every $t \in \mathbb{R}$. The critical points of $f(t)$ coincide with the zeroes of $u^u_{(p,v)}(t)$. According to the assumption on (p,v) we have two cases:

Case 1: $\gamma_{(p,v)}(t)$ is tangent to a boundary component $\partial(B)$ of a bubble B at $t = 0$.

In this case, since ∂B is a convex curve, there exists $\epsilon > 0$, $\epsilon = \epsilon(B)$, such that $\gamma_{(p,v)}(t) \notin B$ for every $t \in (-\epsilon, 0) \cup (0, \epsilon)$. The number ϵ can be chosen uniformly on the set of unit vectors (p, v), $p \in \partial B$, v tangent to ∂B. Because by the definition of the bubble, there exists an annulus of negative curvature surrounding B. By the Jacobi equation, $f''(t) = -K(t)f(t)$, where $K(t) = K(\gamma_{(p,v)}(t))$, which means that $f(t)$ is convex in $(-\epsilon, \epsilon)$. Thus, if $f(t)$ has a critical point in $(-\epsilon, \epsilon)$, this critical point would be a strict local minimum contradicting the fact that $f(t)$ is always non-decreasing.

Case 2: $\gamma_{(p,v)}(t)$ crosses ∂B, namely, there exists $\delta > 0$ such that $\gamma_{(p,v)}(t) \notin B$ for every $t \in (-\delta, 0)$ and $\gamma_{(p,v)}(t) \in B$ for every $t \in [0, \delta)$.

We can assume without loss of generality that the Gaussian curvature of $\gamma_{(p,v)}(t)$ is negative for $t \in (-\delta, 0)$, and non-negative for $t \in (0, \delta)$. Under such conditions, the function $f(t)$ is not only increasing but also convex for every $t \in (-\delta, 0)$. This clearly implies that $f'(0) \neq 0$, because convexity yields that the derivative of $f(t)$ is increasing in $(-\delta, 0)$.

The above arguments show that $u^u_{(p,v)}(0) \neq 0$ for every unit vector (p, v) that is either tangent to a boundary component of a bubble or points into the interior of a bubble. Since the surface has no focal points we actually have that $u^u_{(p,v)}(0) \neq 0$ for such vectors (p, v). Since the surface has a finite number of bubbles, with convex boundary components, the set of these unit vectors (p, v) is compact. Since the unstable Riccati solution $u^u_\theta(0)$ depends continuously on $\theta \in T_1M$ (Lemma 4.7), we get a positive lower bound $A > 0$ for $u^u_{(p,v)}(0)$, for every (p, v) satisfying the assumptions of the statement. □

We get an analogous result for stable solutions of the Riccati equation.

Lemma 4.12. *Let (M, g) be a non-degenerate compact surface without focal points. Then there exists $A' > 0$ satisfying the following property:*

For every $p \in M$ in the boundary of a bubble, and every $v \in T_1M$ that is either tangent to the boundary of the bubble or points outside the interior of the bubble, we have $u^s_{(p,v)}(0) \leq -A'$.

Notice that in the latter lemma, the geodesics $\gamma_{(p,v)}(t)$ are either tangent to the boundary of a bubble or leaving a bubble, while in the former lemma the geodesics $\gamma_{(p,v)}(t)$ are either tangent to the boundary of a bubble or entering a bubble.

Lemma 4.11 allows us to describe the zeroes of the stable and unstable solutions of the Riccati equation along a geodesic meeting a bubble: the set of such zeroes is very simple.

Lemma 4.13. *Let (M, g) be a non-degenerate compact surface without focal points. Then, there exists $\lambda > 0$ such that given any bubble B, and $\theta \in T_1M$ such that $\gamma_\theta(t)$, $t \in [a, b]$ is an embedded connected component of $\gamma_\theta \cap B$, we have*

1. *The unstable Riccati solution $u^u_\theta(t)$ has a zero at some $t \in [a, b]$ if and only if $t = b$. Moreover, if $u^u_\theta(b) = 0$ then $t = b$ is the only zero in the interval $[a - \lambda, b + \lambda]$.*

(2) *The stable Riccati solution* $u^s_\theta(t)$ *has a zero at some* $t \in [a,b]$ *if and only if* $t = a$*. If* $u^s_\theta(a) = 0$ *then* $t = a$ *is the only zero in the interval* $[a-\lambda, b+\lambda]$*.*

Proof. Indeed, let us suppose that $u^u_\theta(t_0) = 0$ for some $t_0 \in (a,b]$ ($t \neq a$ by Lemma 4.11). Then t_0 is a minimum of $u^u_\theta(t)$ and therefore $\frac{d}{dt}u^u_\theta(t_0) = 0$. Thus, the Riccati equation yields that $K(\gamma_\theta(t_0)) = 0$ and this can only occur at the boundary of the bubble. Since the geodesic $\gamma_\theta(t)$, $t \in [a,b]$, has just two points corresponding to $t = a, t = b$ in the boundary of the bubble and $u^u_\theta(a) > 0$ by Lemma 4.11, we must have that $u^u_\theta(b) = 0$ as claimed. The proof of item (ii) is analogous.

The zeroes of both solutions of the Riccati equation are isolated in the boundary of a bubble, since we have just shown that the Gaussian curvature vanishes at these zeroes and this happens if and only if the zeroes are in the boundary curve of the bubble. But geodesics meet at isolated points the boundary curves of bubbles: a geodesic meets such a curve either transversally or tangentially. In the former case the intersection is clearly isolated in the geodesic, in the latter case the intersection is isolated because the boundary curves of bubbles are convex curves of the surface. There is a number $\lambda(B) = \lambda$, namely, the minimum value among the radius of the annulus of negative curvature surrounding the bubbles B, where $t = b$ is the only zero of $u^u_\theta(t)$ in the interval $[a - \lambda(B), b + \lambda(B)]$. Since the number of bubbles is finite, the number λ in the statement can be chosen to be the minimum among the bubbles of the numbers $\lambda(B)$. This finishes the proof of the Lemma. □

Corollary 4.14. *Let* $0 < \lambda$ *be as in Lemma 4.13. Then there exists* $L = L(\lambda) > 0$ *such that if* $\gamma_\theta[a,b]$ *is an embedded connected component of the intersection of the geodesic* γ_θ *with a bubble* B*, we have that the unstable Jacobi field* $J(t)$ *in* γ_θ *with* $\parallel J(a) \parallel = 1$ *satisfies*

$$\parallel J(b+\lambda) \parallel > \parallel J(t) \parallel + L \geq 1 + L,$$

for every $t \in [a,b]$*, and*

$$\parallel J(b+\lambda) \parallel > \parallel J(t) \parallel + L,$$

for every $t \in [0,b]$*.*

Proof. Let $A > 0$ be the constant given in Lemma 4.11. There exist $\tau > 0$ such that for each $\theta \in T_1M$ we have

$$\parallel J(a+\tau) \parallel > 1 + \frac{A}{2}\tau,$$

where $J(t)$ is the unstable Jacobi field of γ_θ with $\parallel J(a) \parallel = 1$. This is because

$$\parallel J(a) \parallel' \geq A,$$

so the continuity of $u^s(t)$ with respect to t and θ implies that there exists $\tau > 0$ where $\parallel J(a+t) \parallel' > \frac{A}{2}$ for every $t \in (0,\tau]$, the number τ being at least the width of the annulus of negative curvature surrounding the bubble B. Integrating this equation we get the estimate for the norm of $J(a+\tau)$.

So we have two cases to consider: either $a = b$, and in this case we just take $L = \frac{A}{2}\tau$, or $a \neq b$. In this latter case, by Lemma 4.13 the norm of $J(t)$ in the interval $t \in (a-\lambda, b+\lambda)$ can only have a critical point at $t = b$. Moreover, $\parallel J(t) \parallel$ strictly increases in the interval $t \in (b, b+\lambda)$. By the continuous dependence of unstable solutions with respect to θ and compactness, we get a number $\sigma > 0$ such that

$$\parallel J(b+\lambda) \parallel > \parallel J(b) \parallel + \sigma,$$

for every $\theta \in T_1M$ under the assumptions of the lemma. This yields

$$\| J(b+\lambda) \| > \| J(t) \| + 1 \geq 1 + \frac{A}{2}\tau + \sigma,$$

for every $t \in [a,b]$, where the last inequality follows from the fact that unstable Jacobi fields are nondecreasing. So $L = \frac{A}{2}\tau$ satisfies the inequality in the statement for $t \in [a,b]$. The second inequality in the statement follows from the first one and the nondecreasing property of unstable Jacobi fields. □

With the help of Lemma 4.13 we can estimate the size of intervals where the slope of non-stable Jacobi fields changes sign. Recall that non-stable Jacobi fields are of 3 types: radial, unstable and sums of stable and unstable Jacobi fields. The norms of the first two are non-decreasing, but the norms of the third type may have many minimum points.

Lemma 4.15. *Let (M,g) be a non-degenerate compact surface without focal points. Let $A > 0$ be the constant given in Lemma 4.11, and let $0 < \lambda$, $L = L(\lambda) > 0$ be the constants given by Corollary 4.14. Then there exist $\eta = \eta(\lambda) > 0$ such that for every given $0 < \sigma < \eta$ there exist $c > 0$, $T_\sigma > 0$, $1 > \epsilon = \epsilon(\sigma) > 0$ with the following property:*

Let $\gamma_\theta(t)$ be a geodesic such that $\gamma_\theta([a,b])$ is an embedded connected component of the intersection of γ_θ with a bubble. Let $J(t)$ be a perpendicular Jacobi field of γ_θ such that $\| J(a) \| = 1$ and $\sup_{t\in[a-T_\sigma,a]} \| J(t) \| \leq 2$. Then

1. *$\| J(t) \|$ is increasing for every $t \in [a, b-\sigma) \cup (b+\sigma, b+\lambda)$. Moreover, $\| J(a) \|' > \frac{A}{2}$,*
2. *$\| J(b+\lambda) \| > 1 + \frac{L}{2}$ for every $t \in [a-\lambda, b)$, and $\| J(b+\lambda) \|$ is the maximum value of $\| J(t) \|$ in the interval $t \in [a-\lambda, b+\lambda]$.*
3. *$\| J(t) \| \geq 1 - \epsilon > 0$ for every $t \in (a,b)$.*
4. *$\| J(b+\lambda) \|' \geq c > 0$.*

Proof. Let us consider a bubble B, and let us consider the set C of points $\theta = (p,v) \in T_1M$ such that p is in the boundary of B and (p,v) is either tangent to the boundary of B or points into B. Let us consider a geodesic γ_θ such that $\gamma_\theta([a,b])$ is an embedded connected component of $\gamma_\theta \cap B$. If $[a,b]$ is nontrivial, then γ_θ crosses $\partial(B)$ transversally. By Lemmas 4.11, 4.12, the solutions $u^u_\theta(t)$, $u^s_\theta(t)$ differ at $t = a$ by a number $A - A' > 0$, since $u^u_\theta(a) \geq A$ and $u^s_\theta(a) \leq A' < 0$.

Let $\lambda > 0$ be as in Lemma 4.13. By Lemma 4.13, there might be just one zero of $u^u_\theta(t)$ in $t \in [a,b]$, namely, at $t = b$, and this would be the only zero in the interval $[a-\lambda, b+\lambda]$. Let $J^u_\theta(t)$ be the unstable Jacobi field of γ_θ such that $\| J^u_\theta(a) \| = 1$ (it is unique up to multiplication by -1). Since $u^u_\theta(t)$ has the same sign of the derivative of $\| J^u_\theta(t) \|$, we have that $\| J^u_\theta(t) \|$ is increasing in $t \in [a-\lambda, b) \cup (b, b+\lambda]$. So by continuity of $u^u(t)$ with respect to θ and compactness there exists $\Delta > 0$ such that

$$\| J(b+\lambda) \| > \Delta$$

for every geodesic γ_θ, $\theta \in C$.

Moreover, $\| J^u_\theta(b+\lambda) \| > \| J^u_\theta(t) \|$ for every $t \in [0, b+\lambda)$ since the norm of unstable Jacobi fields is nondecreasing, and $\| J^u_\theta(b+\lambda) \| > \| J^u_\theta(b) \| + L = 1 + L$.

By continuity of Jacobi fields with respect to initial conditions, there exists $\sigma_0 > 0$ such that for any given $0 < \sigma \leq \sigma_0$, there exists $\epsilon = \epsilon(\sigma, B) > 0$, $\delta = \delta(\sigma) > 0$, with the following property:

For every $\theta \in C$, for every perpendicular Jacobi field $J(t)$ of γ_θ such that $\| J(0) \|= 1$, and $\| J_\theta^{u'}(0) - J'(0) \|< \delta$, we have that

(1) $\| J(t) \|' > 0$ for every $t \in [a-\lambda, b-\sigma) \cup (b+\sigma, b+\lambda]$,
(2) $\| J(b+\lambda) \| > \| J(t) \| + \frac{L}{2}$ for every $t \in [a-\lambda, b)$.
(3) $\| J(t) \| > 1-\epsilon$ for every $t \in [a,b]$,
(4) $\| J(b+\lambda) \|' > \frac{\Delta}{2}$.

In other words, Jacobi fields $J(t)$ close enough to the unstable Jacobi field $J_\theta^u(t)$ essentially increase in the interval $[a-\lambda, b+\lambda]$, and the norm $\| J(b+\lambda) \|$ is the maximum norm for $t \in [a-\lambda, b+\lambda]$. Moreover, the derivative of $\| J(t) \|$ at the maximum $t = b+\lambda$ is strictly positive greater than $c = \frac{\Delta}{2}$.

Notice that we can chose δ independent of B by compactness and continuity of unstable Jacobi fields. To finish the proof of the lemma, let $\eta = \sigma_0$ and apply Lemma 4.5 taking $T_\sigma = T(0,\delta)$.

□

Remark: Observe that items (1), (2), (3), (4) are properties of Jacobi fields which persist after small perturbations of the Jacobi field. This will be crucial to study Jacobi fields of geodesics of conformal metrics which are close enough to g.

5. The action of Subharmonic RYM flows on non-degenerate surfaces without focal points

The goal of this section is to describe the action of the Ricci-Yang-Mills flow on the Jacobi fields of compact surfaces without focal points under the assumption $\Delta f < 0$ in regions of non-negative curvature for the initial magnetic potential f. The main result of the section is the following:

Proposition 5.1. *Let (M,g) be a compact, C^∞ non-degenerate surface without focal points. Let us consider the RYM flow g_ρ for $\rho \in [0,\tau]$, $g_0 = g$, associated to a subharmonic function as in Proposition 3.1. Let $0 < \lambda$, $\eta = \eta(\lambda) > 0$ be as in Lemma 4.15. Then for every $0 < \sigma < \eta$ there exists $0 < \rho(\sigma) \leq 0$, with $\rho(\sigma) \to 0$ if $\sigma \to 0$, such that,*

(1) *g_ρ has no conjugate points for every $0 < \rho \leq \rho(\sigma)$.*
(2) *The set of points where the Gaussian curvature K_ρ of (M, g_ρ) vanishes is a finite collection of closed, convex curves contained in the bubbles of (M,g), for every $0 < \rho \leq \rho(\sigma)$.*
(3) *Let $\gamma_\theta^\rho(t)$ be a geodesic of (M, g_ρ), and let $[a(\theta)_{\rho,n}, b(\theta)_{\rho,n}]$, $n \in S(\theta,\rho) \subset \mathbb{Z}$ be the collection of intervals parametrizing the successive intersections of $\gamma_\theta^\rho(t)$ with the regions of positive K_ρ-curvature. Then a non-trivial Jacobi field $J_\rho(t)$ that is both stable and unstable is parallel in the set*
$$\Sigma(\rho,\theta) = \mathbb{R} - \bigcup_{n \in S(\theta,\rho)} (a(\theta)_{\rho,n} - \sigma, b(\theta)_{\rho,n} + \sigma),$$
for every $0 < \rho \leq \rho(\sigma)$.
(4) *Let $\gamma_\theta^\rho(t)$ be a geodesic of (M, g_ρ) with a non-trivial Jacobi field that is both stable and unstable. Then $\gamma_\theta^\rho(t)$ is flat in the set $P(\rho,\theta)$ defined in the previous item for every $0 < \rho \leq \rho(\sigma)$.*

The idea of the proof of Proposition 5.1 is the following: the action of RYM flow associated to the appropriate subharmonic function shrinks the regions of positive curvature of the surface and decreases the positive curvature in such regions as well (Proposition 3.1). In fact, the proof of item (2) follows from this assertion and the assumptions on the bubbles of (M,g). The norm of Jacobi fields in (M,g) with a zero is strictly increasing, even in the regions of positive curvature of (M,g) where the norm of Jacobi fields is not a convex function. Since the curvature decreases in these "bad regions" we could expect that the norm of Jacobi fields in g_t would become closer to convex than in g and would get steeper. The delicate issue of this idea is that as the metric moves the geodesics move too, and hence it becomes difficult to control the evolution of the curvatures along perturbed geodesics. We shall subdivide the proof of the Proposition into many steps.

5.1. **Local Sturm-Liouville comparison between Jacobi fields in (M,g) and (M,g_ρ).** One of the pieces of the proof of Proposition 5.1 is Sturm-Liouville comparison theory. In fact, to show that Jacobi fields of (M,g_ρ) with a zero diverge, we look at the regions of M where the curvature K_ρ has fixed sign. When K_ρ is non-positive, the norm of Jacobi fields is a convex function and therefore, once the norm of a Jacobi field has positive derivative at a point of non-positive curvature of a geodesic, then this norm keeps increasing as long as the geodesic stays in a region of non-positive curvature. However, the growth of Jacobi fields is not at all obvious in regions of positive curvature. Since the initial surface (M,g) has no focal points, we know that "most " of its Jacobi fields increase despite of the sign of the curvature (Lemmas 4.10 and 4.15).

Since the curvature K_ρ decreases in the regions of positive K-curvature, we might expect that Sturm-Liouville comparison theory would imply that g_ρ-Jacobi fields increase more than g-Jacobi fields in regions of positive K_ρ-curvature. The main problem to apply this idea is that comparison works if we can compare the curvatures along g_ρ-geodesics and g-geodesics. The goal of this subsection is to show a technical result to ensure that at least locally, we can compare the curvatures of g_ρ and g along geodesics. We start with an elementary lemma whose proof follows from standard theory of ordinary differential equations.

Lemma 5.2. *Let $a > 0$, $\epsilon > 0$. There exists $\delta = \delta(a,\epsilon) > 0$ such that if (M,g), $(M,\bar{g})$ are two metrics in M such that $\| g - \bar{g} \|_{C^2} \leq \delta$ then*

$$d(\gamma(t), \beta(t)) \leq \epsilon$$

for every $t \in [0,a]$, where γ is a unit speed g-geodesic, β is a unit speed $\bar{g}$-geodesic, $\gamma(0) = \beta(0)$, $\gamma'(0) = \frac{\beta'(0)}{\|\beta'(0)\|_g}$.

Next, let $\theta = (p,v) \in T_1(M,g_\rho)$, $\gamma_\theta^\rho(t)$ be a g_ρ-geodesic, and let $(0,y_\theta)$ be an interval such that $\gamma_\theta^\rho([0,y_\theta])$ is a connected component of the intersection of $\gamma_\theta^\rho(t)$ with $K_\rho > 0$. Of course, the number y_θ depends on ρ but we shall omit this dependence through the subsection to simplify notation.

By Proposition 3.1, we know that $\gamma_\theta^\rho([0,y_\theta])$ is contained in the interior B_0 of a bubble B of (M,g): there exist $x_\theta > 0$ and $\bar{y}_\theta > y_\theta$ such that .

$$\gamma_\theta^\rho([-x_\theta, \bar{y}_\theta])$$

is the connected component of $\gamma_\theta^\rho \cap B_0$ containing $\gamma_\theta^\rho([0,y_\theta])$.

Moreover, there exists $\nu_\rho > 0$ such that

$$x_\theta > \nu_\rho, \ \bar{y}_\theta - y_\theta > \nu_\rho$$

for every $\theta = (p, v)$ such that p is in the boundary of the connected component of $K_\rho > 0$ contained in B_0.

For any given $s \in [0, y_\theta]$, let $\gamma_{\theta,s}$ be a g-geodesic such that $\gamma_{\theta,s}(s) = \gamma^\rho(s)$, $\gamma'_{\theta,s}(s) = \frac{1}{\|\gamma^{\rho'}(s)\|_g}\gamma^{\rho'}(s)$.

Let $[0, \tau]$ be the interval of definition of the flow g_ρ given in Proposition 3.1, and let P_ρ be the set of non-negative curvature of (M, g_ρ). According to Proposition 3.1, there exists $C > 0$ such that $K_\rho(p) < K(p) - C\rho$ for every $p \in P_\rho$ and $\rho \in [0, \tau]$. Since $\gamma_{\theta,s}(s) = \gamma^\rho(s)$ for every $s \in [0, y_\theta]$, we have that $K(\gamma_{\theta,s}(s)) > 0$ and $K_\rho(\gamma^\rho_\theta(s)) < K(\gamma^\rho_\theta(s))$ for every $s \in [0, y_\theta]$.

By continuity of Gaussian curvatures and geodesics, and the compactness of M we get,

Lemma 5.3. *For each $\rho \in (0, \tau]$, there exists $t_\rho > 0$, such that*

$$K_\rho(\gamma^\rho_\theta(t)) < K(\gamma_{\theta,s}(t)),$$

for every $t \in [s, s + t(\rho)]$.

Proof. Indeed, let $\gamma_{\theta,s}(t_s, \sigma_s)$ be an open connected component of $\gamma_{\theta,s}$ in B_0, where $s \in (t_s, \sigma_s)$. Obviously, the interval (t_s, σ_s) depends on θ too, but we shall omit this dependence to shorten notation. Let $\nu_\rho > 0$ be the number defined above, namely

$$\gamma^\rho_\theta([-\nu_\rho, y_\theta + \nu_\rho]) \subset B_0.$$

Let $\delta_\rho > 0$ be such that

$$K_\rho(x) < K(x) - \frac{C}{2}\rho$$

for every $x \in V_\rho$, where V_ρ is a δ_ρ-tubular neighborhood of $\gamma_{\theta,s}(-t_s, \sigma_s)$ given by

$$V_\rho = \{x \in M, \ , s.t. \inf_{t\in(-t_s,\sigma_s)} d(x, \gamma_{\theta,s}(t)) < \delta_\rho\}.$$

The number δ_ρ can be chosen uniformly in B by compactness. Since the geodesics $\gamma^\rho_\theta(t)$, $\gamma_{\theta,s}(t)$ are tangent at $t = s$ and have bounded C^∞ geometry, there exists $\tau = \tau_\rho > 0$ such that $\gamma^\rho_\theta(t) \in V_\rho$ for every $t \in [s, s + \tau]$, and every $s \in [0, y_\theta]$. Moreover, there exists a constant $F > 0$ depending on an upper bound for the curvatures K and K_ρ, and $a_\rho > 0$ such that

$$\mid K(\gamma^\rho_\theta(t)) - K(\gamma_{\theta,s}(t)) \mid \leq F(t - s)^2,$$

for every $t \in [s - a_\rho, s + a_\rho]$. The number a_ρ does not depend on θ by compactness. So we get,

$$\begin{aligned} K_\rho(\gamma^\rho_\theta(t)) &< K(\gamma^\rho_\theta(t)) - \frac{C}{2}\rho \\ &\leq K(\gamma_{\theta,s}(t)) + (K(\gamma^\rho_\theta(t)) - K(\gamma_{\theta,s}(t)) \mid - \frac{C}{2}\rho \\ &\leq K(\gamma_{\theta,s}(t)) + F(t - s)^2 - \frac{C}{2}\rho, \end{aligned}$$

and hence for every $s \in [0, y_\theta]$, for every $\mid t - s \mid < t_\rho = \min\{\sqrt{\frac{C\rho}{3F}}, \tau_\rho, a_\rho, \nu_\rho\}$ we have

$$K_\rho(\gamma^\rho_\theta(t)) < K(\gamma_{\theta,s}(t)),$$

as claimed. □

Lemma 5.3 allows us to apply Sturm-Liouville comparison theory to g-Jacobi fields and g_ρ-Jacobi fields in small pieces of geodesics.

Lemma 5.4. *Let $\rho \in (0, \tau]$, $t_\rho > 0$ be as in Lemma 5.3. Let $J_\rho(t)$ be a perpendicular Jacobi field of γ^ρ_θ with $\| J_\rho(s) \|_{g_\rho} = 1$, let $J(t)$ be a perpendicular Jacobi field of $\gamma_{\theta,s}$, such that $J_\rho(s) = J(s)$, $0 < \| J'_\rho(s) \|_{g_\rho} = \| J'(s) \|_g$. Then,*

1. *$J(t)$ is not a stable Jacobi field of (M, g).*
2. *If $J(t)$ is either unstable or radial, then $\| J'_\rho(t) \|_{g_\rho} > 0$ for every $t \in (s, s+t_\rho]$ and at each $s \in [0, y_\theta]$.*
3. *There exists $\eta > 0$ small with the following property:*
   Given $0 < \sigma \leq \eta$ there exists $\delta = \delta(\sigma) > 0$ such that if $\| J'(s) - J^{u'}(s) \| < \delta$, where $J^u(t)$ is an unstable Jacobi field in $\gamma_{\theta,s}$ satisfying $J(s) = J^u(s)$, and $\gamma^\rho_\theta([s, s + t_\rho])$ is in the interior B_0 of a bubble B with
   $$\bar{y}_\theta - (s + t_\rho) \geq \sigma$$
   then
   $$\| J'_\rho(t) \|_{g_\rho} > 0$$
   for every $t \in [s, s + t_\rho]$.

Proof. Item (1) follows from Lemma 4.6 item(1); the norms of stable Jacobi fields of (M, g) are non-increasing. Item (2) follows from Lemma 4.6 and Sturm-Liouville comparison theorems. Item (3) follows from Lemma 4.15, which gives us precise control over how the norm of $J(t)$ increases inside a bubble.

□

5.2. **RYM flow and growth of radial Jacobi fields in (M, g_ρ).** In this subsection we show item (1) of Proposition 5.1. We shall show that Jacobi fields have just one zero, which is equivalent to the absence of conjugate points.

Lemma 5.5. *Let (M, g) be a compact, C^∞ non-degenerate surface without focal points. Let us consider the RYM flow g_ρ for $\rho \in [0, \tau]$, $g_0 = g$, associated to a subharmonic function as in Proposition 3.1. Let $0 < \lambda \leq \lambda_0$, $\eta = \eta(\lambda) > 0$ be as in Lemma 4.15. Then, given $\sigma < \eta$ there exists $\rho(\sigma) \leq \tau$, with $\rho(\sigma) \to 0$ if $\sigma \to 0$, such that for every $0 < \rho \leq \rho(\sigma)$ we have the following properties:*

Let $\gamma^\rho_\theta(t)$ be a geodesic of (M, g_ρ), and let $J_\rho(t)$ be a non-trivial, perpendicular Jacobi field such that $J_\rho(0) = 0$, $\| J'_\rho(0) \| = 1$. Then

1. *Let $[a(\theta)_{\rho,n}, b(\theta)_{\rho,n}]$, $n \in S(\theta, \rho) \subset \mathbb{Z}$ be the collection of intervals parametrizing the successive intersections of $\gamma^\rho_\theta(t)$ with the regions of positive K_ρ-curvature. Then $\| J_\rho(t) \|$ is increasing in the set*
   $$\Sigma^+(\rho, \theta) = \mathbb{R}^+ - \bigcup_{n \in S(\theta,\rho)} (b(\theta)_{\rho,n} - \sigma, b(\theta)_{\rho,n} + \sigma),$$
   for every $\rho \in [0, \rho(\sigma)]$.
2. *The points $\gamma^\rho_\theta(b(\theta)_{\rho,n} + \lambda)$ belong to a region of negative K_ρ-curvature and $\| J'_\rho(b(\theta)_{\rho,n} + \lambda) \| > 0$,*
3. *The norm of $J_\rho(t)$ at the sequence $t = a(\theta)_{\rho,n}$ increases:*
   $$\| J_\rho(a(\theta)_{\rho,n}) \| < \| J_\rho(b(\theta)_{\rho,n} + \lambda) \| < \| J_\rho(a(\theta)_{\rho,n+1}) \|$$

for every $n \in S(\theta, \rho)$*, and*

$$\sup_{t\in[0,a(\theta)_{\rho,n}]} \| J_\rho(t) \| = \| J_\rho(a(\theta)_{\rho,n}) \| .$$

(4) $J_\rho(t) > 0$ *for every* $t > 0$*, and every* $\rho \in [0, \rho(\sigma)]$.

The strategy of the proof is the following: all the difficulty is concentrated in the regions of positive K_ρ-curvature, since Jacobi fields lose convexity in such regions.

First of all, let us recall some of the constants defined in previous lemmas. Let $A > 0$ be as in Lemma 4.11, let $\lambda > 0$, $L = L(\lambda) > 0$, $\eta = \eta(\lambda) > 0$, $T_\sigma > 0$ for $0 < \sigma < \eta$ be as in Lemma 4.15. Let $\theta = (p, v) \in T_1(M, g_\rho)$, let $r \in \mathbb{R}$. Let $\theta_r \in T_1(M, g)$ be defined by the following conditions:

(1) $\gamma_{\theta_r}(r) = \gamma^\rho_\theta(r)$,
(2) $\gamma'_{\theta_r}(r) = \gamma^{\rho'}_\theta(r)/ \| \gamma^{\rho'}_\theta(r) \|_g$.

Clearly, as ρ goes to zero γ^ρ_θ converges to γ_{θ_r} uniformly on compact sets for each fixed r.

Let J_ρ be a radial Jacobi field in γ^ρ_θ with $J_\rho(0) = 0$, and let us consider the normalized Jacobi fields $J_{\rho,r}(t) = J_\rho(t)/ \| J_\rho(r) \|_\rho$, whenever $\| J_\rho(r) \|_\rho \neq 0$. Of course, as a consequence of the proof we shall show that these norms never vanish.

Let $J^g_r(t)$ be the (M, g)-perpendicular Jacobi field in the (M, g)-geodesic $\gamma_{\theta_r}(t)$ defined by the initial conditions

$$\| J^g_r(r) \| = 1, \quad \| J^g_r(r) \|' = \| J_{\rho,r}(r) \|' .$$

Before proceeding with the proof of Lemma 5.5 we expand our set of uniform bounds depending only on M, g and σ.

Lemma 5.6. *Given* $\sigma > 0$*, there exists a* positive $\rho(\sigma)$ *sufficiently small such that for every* $0 < \rho \le \rho(\sigma)$ *the following conditions are satisfied for each* $\theta \in T_1(M, g_\rho)$*:*

(1) *The maximum of the* g *and* g_ρ *distances from the boundaries of the bubbles of* (M, g) *to the boundaries of the bubbles in* (M, g_ρ) *is less than* σ*.*
(2) *Let* $D > 0$ *be the diameter of* (M, g)*. If* $1 < r \le T_\sigma$*, then* $J_\rho(t) \neq 0$ *for every* $t \in (0, r]$ *and* $\| J'_{\rho,r}(t) \| > 0$ *for every* $t \in [0, r + D]$.
(3) *Given* $T > 0$*, if* $\| J_{\rho,T}(t) \| \le \frac{3}{2}$ *for every* $t \in [T - T_\sigma, T]$*, then* J^g_T *satisfies*

$$\| J^g_T(t) \| \le 2$$

for every $t \in [T - T_\sigma, T]$*, and* $J_{\rho,T}(T + s) > 0$ *for every* $s \in [0, \bar{D}]$*, where* $\bar{D}$ *is the maximum diameter of a bubble of* (M, g)*.*
(4) *Let* D *be the diameter of* (M, g)*. Given* $T > 0$*, if there exists* $t_0 \in [T, T + D]$ *such that* $\| J^{g\prime}_T(t_0) \| > c > 0$ *then* $\| J'_{\rho,T}(t_0) \| > \frac{c}{2}$*.*
(5) *Let* $b - D \le T < b$*. If*
   - $\| J^g_T(b + \lambda) \| > \| J^g_T(t) \| + \frac{L}{2}$ *for every* $t \in [T, b]$*,*
   - $\| J^g_T(b+\lambda) \|$ *is a strict maximum for the values* $\| J^g_T(t) \|$*,* $t \in [T, b+\lambda]$ *with* $\| J^{g'}_T(b + \lambda) \| > c > 0$*.*

   Then $\| J_{\rho,T}(b + \lambda) \| > \| J_{\rho,T}(T) \| + \frac{L}{2} = 1 + \frac{L}{2}$*, and* $\| J_{\rho,T}(b + \lambda) \|$ *is a strict maximum value for* $\| J_{\rho,T}(t) \|$*,* $t \in [T, b + \lambda]$*.*

Proof. Item (1) follows from the continuity of curvature under the RYM flow.

Item (2) is a consequence of the continuous dependence of geodesics and Jacobi fields with respect to ρ and the absence of focal points for (M, g): Jacobi fields J

in (M,g) with $J(0)=0$, $\| J'(0) \| = \| J'_\rho(0) \| \neq 0$ form a cocompact set of increasing Jacobi fields.

Item (3) is again by continuous dependence of geodesics and Jacobi fields with respect to the parameter ρ, and the bound T_σ on the length of the intervals $[T - T_\sigma, T]$ (Lemma **??**). For items 3,4,5 it is important to notice that the argument refers to the image of the geodesic in the compact manifold T_1M, rather than relying on a compact subset of the geodesic itself.

Item (5) follows from the persistence under C^2 perturbations of the family $\mathcal{F}_{N,d_1,d_2}$, $N, d_i > 0$, of continuous, positive C^2 functions $f : [0,x] \longrightarrow \mathbb{R}$, $x \leq D$, with $\| f \|_{C^2} \leq N$ such that

1. $f(x)$ is a strict maximum ($f(x) > f(s)$ for every $s \in [0,x]$), and $f'(x) > \frac{c}{2}$.
2. $f'(s) > 0$ in the interval $[x-d_1, x]$,
3. $f(x) > f(s) + d_2$ for every $s \in [0, x-d_1]$.

The proof of this fact is an easy exercise of elementary calculus. Observe that the C^2-norms of normalized Jacobi fields at $t=0$ are uniformly bounded in $[0,x]$ for $x \leq D$. □

We would like to stress that conditions (3) and (4) are just continuous extensions of the geometric properties of (M,g)-Jacobi fields stated in Lemma 4.15, and that the choice of ρ in (3), (4) does not depend on σ.

We now begin the proof of Lemma 5.5. We shall proceed by induction on the nonnegative, increasing sequence $a(\theta)_{\rho,n}$ where the geodesic γ_θ^ρ enters a region of K_ρ-positive curvature. Namely, we shall show that items (1), (2), (3) in Lemma 5.5 hold for every $t \leq a(\theta)_{\rho,n+1}$ if they hold for every $t \leq a(\theta)_{\rho,n}$, for every $n \geq 1$.

It is clear that if γ_θ^ρ never meets a region of K_ρ positive curvature then radial Jacobi fields are convex and therefore increasing, so Lemma 5.5 holds. So let us suppose that γ_θ^ρ meets the region of non-negative K_ρ curvature at the point $\gamma_\theta^\rho(a(\theta)_{\rho,1})$.

We have two cases:

1) Either $a(\theta)_{\rho,1} \leq T_\sigma$, so we apply condition (2) in Lemma 5.6 to deduce items (1), (2), (3) in Lemma 5.5 for every $t \in [0, a(\theta)_{\rho,1}]$ and the first induction step is proved;

2) Or $a(\theta)_{\rho,1} > T_\sigma$.

In this latter case the Jacobi field J_ρ is increasing in $[0, a(\theta)_{\rho,1}]$, an interval with length greater than T_σ. By condition (3) in Lemma 5.6 the (M,g)-Jacobi field $J^g_{a(\theta)_{\rho,1}}(t)$ satisfies

$$\| J^g_{a(\theta)_{\rho,1}}(t) \| \leq 2$$

for every $t \in [a(\theta)_{\rho,1} - T_\sigma, a(\theta)_{\rho,1}]$. By Lemma 4.15 and Lemma 5.6 (2), (3), (4) (the definition of $\rho(\sigma)$) we get the following statement:

Lemma 5.7. *Let $\rho \leq \rho(\sigma)$, and suppose that for some $a(\theta)_{\rho,n} > T_\sigma$ we have $J_\rho(a(\theta)_{\rho,n}) \neq 0$ and*

$$\sup_{t \in [0, a(\theta)_{\rho,n}]} \| J_{\rho, a(\theta)_{\rho,n}}(t) \| \leq 1.$$

Then $J_{\rho,a(\theta)_{\rho,n}}(t)$ satisfies

1. $\| J'_{\rho,a(\theta)_{\rho,n}}(a(\theta)_{\rho,n}) \|_{g_\rho} > \frac{A}{2}$ *where $A > 0$ is the constant defined in Lemma 4.11.*
2. $\| J_{\rho,a(\theta)_{\rho,n}}(b(\theta)_{\rho,n} + \lambda) \| > \| J_{\rho,a(\theta)_{\rho,n}}(t) \| > 0$ *for every $t \in [0, b(\theta)_{\rho,n} + \lambda)$.*

(3) $\| J'_{\rho,a(\theta)_{\rho,n}}(b(\theta)_{\rho,n}+\lambda) \| > \frac{c}{2}$.

Proof. Recall that $\| J_{\rho,a(\theta)_{\rho,n}}(a(\theta)_{\rho,n}) \|_{g_\rho} = 1$. By the induction assumption (item (3) in Lemma 5.5) the Jacobi field $J_{\rho,a(\theta)_{\rho,n}}$ satisfies

$$\| J_{\rho,a(\theta)_{\rho,n}}(t) \|_{g_\rho} \leq 1$$

for every $t \in [0, a(\theta)_{\rho,n}]$.

Let $J(t)$ be the perpendicular g-Jacobi field defined in the g geodesic $\gamma_{\theta,a(\theta)_{\rho,n}}(t)$ whose initial conditions are

$$\| J(a(\theta)_{\rho,n}) \|_g = 1, \| J'(a(\theta)_{\rho,n} \|_g = \| J'_\rho(a(\theta)_{\rho,n}) \|_{g_\rho} .$$

As these conditions only determine $J(t)$ up to orientation, let us take $J(t)$ with the same orientation as $J_\rho(t)$.

Thus, by Lemma 5.6 (2), the choice of $\rho(\sigma)$ implies

$$\| J(t) \| \leq 2$$

for every $t \in [a(\theta)_{\rho,n} - T_\sigma, a(\theta)_{\rho,n}]$, which implies by Lemma 4.15 that

$$\| J'(a(\theta)_{\rho,n}) \| > \frac{A}{2},$$

where $A > 0$ is the constant defined in Lemma 4.11. Moreover, from Lemma 4.14, items (1), (2), (3), (4) we have

$$\| J((b(\theta)_{\rho,n} + \lambda) \| > \| J(a(\theta)_{\rho,n}) \| + \frac{L}{2},$$

$$\sup_{t\in[a(\theta)_{\rho,n}, b(\theta)_{\rho,n}+\lambda]} \| J(t) \| = \| J(b(\theta)_{\rho,n} + \lambda) \|,$$

$$\| J(t) \| > 0$$

for every $t \in (0, b(\theta)_{\rho,n})$ and

$$\| J'((b(\theta)_{\rho,n} + \lambda) \| > c.$$

By the induction assumption (item (3)), and conditions (1),(3),(4) and (5) in Lemma 5.6 we conclude the proof of the lemma. □

So we have just shown that the derivative of the norm of $J_\rho(t)$ at $t = a(\theta)_{\rho,n}$ is positive. Further, this norm leaves the region of positive $K_\rho > 0$ in good shape: it has positive derivative just after leaving this region, at the point $\gamma_{\theta,\rho}(b(\theta)_{\rho,n} + \lambda)$ where the K_ρ-curvature is negative. This implies that

Corollary 5.8. *Under the assumptions of Lemma 5.7, we have that*

(1) $\| J'_\rho(t) \|_{g_\rho} > 0$ *for every* $t \in [b(\theta)_{\rho,n} + \lambda, a(\theta)_{\rho,n+1})$,
(2) $\| J_\rho(s) \| \geq \| J_\rho(t) \|$ *for every* $0 \leq t \leq s$, *and* $s \in [b(\theta)_{\rho,n} + \lambda, a(\theta)_{\rho,n+1})$.

Proof. Item (1) follows from the convexity of the norm of Jacobi fields in regions of negative Gaussian curvature, so if in an interval $[x, y]$ the curvature of a geodesic β is negative, once the norm of a Jacobi field $J(t)$ of β has positive derivative at x, it has positive derivative for every $t \in [x, y]$.

Item (2) follows from item (1) and Lemma 5.7 (item (2)). □

Corollary 5.9. *(M, g_ρ) has no conjugate points, for $0 \leq \rho \leq \rho(\sigma)$, for every $0 \leq \sigma \leq \eta(\lambda)$, where η is the constant given in Lemma 4.15 and $\rho(\sigma)$ is given in Lemma 5.6.*

Proof. Corollary 5.8 gives us a good control of the growth of $\| J_\rho(t) \|_{g_\rho}$ after leaving a bubble: it might not be always increasing immediately after exiting, but after traveling a distance λ from the bubble the norm is again larger than it has ever been before.

Further, Lemma 5.7 item (2) tells us that $J_\rho(t)$ does not vanish inside a bubble. Combining the above two remarks we get that $J_\rho(t)$ vanishes only at $t = 0$ in the interval $[0, +\infty)$, thus implying that (M, g_ρ) has no conjugate points. □

We must look at the growth properties of $\| J_\rho(t) \|_{g_\rho}$ **inside** the bubble B_0 to strengthen Corollary 5.9 to the Anosov condition.

Lemma 5.10. *Let $\lambda > 0$, $0 < \sigma < \eta$ be as in Lemma 4.15. Then the norm of $J_\rho(t)$ is increasing for every $t \in [a(\theta)_{\rho,n}, b(\theta)_{\rho,n} - \sigma)$, and every $\rho \leq \rho(\sigma)$.*

Proof. Let us take a partition of $[a(\theta)_{\rho,n}, b(\theta)_{\rho,n}]$ by intervals of the form $[x_i, x_{i+1}]$, $i = 0, 1, ..., m_n$, where

(1) $x_0 = a(\theta)_{\rho,n}$,
(2) $x_{m_n} = b(\theta)_{\rho,n}$,
(3) $| x_{i+1} - x_i | = t_\rho$, for every $i = 0, 1, .., n - 2$, where $t_\rho > 0$ is defined in Lemma 5.3,
(4) $| x_n - x_{n-1} | \leq t_\rho$.

Following the notation of Lemma 5.4, let us consider the geodesics $\gamma_{\theta,x_i}(t)$, the perpendicular g-Jacobi fields $J_i(t)$ given by

$$\| J_i(x_i) \| = \| J_\rho(x_i) \|,$$

$$\| J'_i(x_i) \| = \| J'_\rho(x_i) \|,$$

for every $i = 0, 1, .., m_n$, chosen in a way that all the J_i's have the same orientation as J_ρ. Applying Lemmas 5.7 and 4.15 we have that

(1) The norm of $J_0(t)$ is increasing for every $t \in [a(\theta)_{\rho,n}, b(\theta)_{\rho,n} - \sigma)$,
(2) $\| J_\rho(t) \|'_{g_\rho} > \| J_0(t) \|$ for every $t \in [a(\theta)_{\rho,n}, x_1]$.

By induction on $i \leq n - 1$, the same assertion holds for every $J_i(t)$, thus proving the lemma. □

The proof of Lemma 5.5 follows from Lemma 5.7, Corollary 5.8, and Lemma 5.10.

5.3. **Proof of Proposition 5.1.** Item (1) of Proposition 5.1 is Corollary 5.9

Item (2) follows from Proposition 3.1 and the definition of the bubbles of (M, g).

The proof of item (3) follows from two remarks. First of all, we know that (M, g_ρ) has no conjugate points, so each geodesic has stable and unstable Jacobi fields as defined in Section 4. Moreover, a nontrivial, perpendicular Jacobi field $J_\rho(t)$ that is both stable and unstable has to be the limit of radial Jacobi fields

$$J_\rho(t) = \lim_{T \to +\infty} J_T(t) = \lim_{T \to -\infty} J_T(t)$$

of (M, g_ρ) with $J_\rho(0) = J_T(0)$, $J_T(T) = 0$.

Secondly, Lemma 5.5 applied to $J_T(t)$, $T < 0$, gives us that $\| J_T(t) \|'$ is positive in the set

$$\Sigma^+(\rho,\theta) = \mathbb{R} - \bigcup_{n\in S(\theta,\rho)} (b(\theta)_{\rho,n} - \sigma(\rho), b(\theta)_{\rho,n} + \sigma(\rho)).$$

Since stable Jacobi fields are unstable Jacobi fields on reversing the orientation of geodesics, Lemma 5.5 implies that $\| J_T(t) \|'$ is negative in the set

$$\Sigma^-(\rho,\theta) = \mathbb{R} - \bigcup_{n\in S(\theta,\rho)} (a(\theta)_{\rho,n} - \sigma(\rho), a(\theta)_{\rho,n} + \sigma(\rho)).$$

Hence, the norm of the limiting Jacobi field $J_\rho(t)$ is both increasing and decreasing in $\Sigma^+(\rho,\theta)\cap\Sigma^-(\rho,\theta) \supset \Sigma(\rho,\theta)$, which means that the norm of $J_\rho(t)$ is constant in each connected component of $\Sigma(\rho,\theta)$. This shows (more than) item (3).

Finally, item (4) follows from item (3) and the Jacobi equation, since the existence of a nontrivial, perpendicular Jacobi field with constant norm in an interval $[x, y]$ implies that the Gaussian curvature of the subjacent geodesic in this interval is zero.

6. Metrics without focal points and non-degenerate bubbles are in the closure of Anosov metrics

The proof of Theorem 1.2 follows from the following statement.

Theorem 6.1. *Take (M, g_ρ) as in Proposition 5.1. Then the geodesic flow of (M, g_ρ) is Anosov for every $\rho \in (0, \rho(\sigma))$.*

Let us recall a well known characterization of Anosov geodesic flows of manifolds without conjugate points due to P. Eberlein [6].

Theorem 6.2. *Let (M, g) be a compact Riemannian manifold without conjugate points. The geodesic flow is Anosov if and only if the Green bundles are linearly independent at every point of the unit tangent bundle.*

So to show Theorem 6.1 it is enough to show that there are no nontrivial Jacobi fields which are at the same time stable and unstable Jacobi fields. According to Proposition 5.1, such a Jacobi field would give rise to a geodesic γ_θ of (M, g_ρ) that is flat in $P(\rho,\theta)$. By Proposition 3.1 and the definition of the bubbles of (M, g), the regions of vanishing curvature of (M, g_ρ) are contained in the interior of the bubbles of (M, g). The interior of each bubble is simply connected, and it must contain connected flat g_ρ-geodesics.

Since the complement of $\gamma_\theta(P(\rho,\theta))$ is formed by a union of small intervals whose lengths are bounded above by $L\sigma(\rho)$, we can choose $\sigma(\rho)$ suitably small in Proposition 5.1 such that $L\sigma(\rho)$ is strictly less than the minimum distance between the bubbles of (M, g). This implies that γ_θ has to be in a $L\sigma(\rho)$-tubular neighborhood of a **single** bubble of (M, g). Therefore, the existence of a nontrivial Jacobi field that is both stable and unstable leads us to the existence of a geodesic of (M, g_ρ) that is contained in a simply connected region. Such a fact contradicts the absence of conjugate points in (M, g_ρ), since geodesics in complete manifolds without conjugate points have unbounded lifts in the universal covering.

This proves Theorem 6.1 and thus Theorem 1.1.

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