ALMOST COMPLETE LYAPUNOV SPECTRUM IN STEP SKEW-PRODUCTS

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Abstract. We study the spectrum of Lyapunov exponents of a family of partially hyperbolic and topologically transitive local diffeomorphisms that are step skew-products over a horseshoe map, continuing previous investigations. These maps are genuinely non-hyperbolic and the central Lyapunov spectrum contains negative and positive values. We show that, besides one gap, this spectrum is complete. We also investigate how Lyapunov regular points with corresponding (central) exponents are distributed in phase space. The principal ingredients of our proofs are minimality of the underlying iterated function system and shadowing-like arguments.

Contents

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1. Introduction

Our work is motivated by recent studies [5, 6, 7] of a class of local diffeomorphisms having a topologically transitive set that possesses hyperbolic periodic points that have unstable manifolds with different dimensions. Such dynamical systems genuinely lack uniform hyperbolicity and present a number of interesting new phenomena such as, for example, a gap in the spectrum of Lyapunov exponents of ergodic measures even though that the system shows a strong form of topological transitivity (it is a homoclinic class, see below). This strongly contrasts with properties of uniformly hyperbolic systems for which the spectrum of Lyapunov exponents form a closed interval and Lyapunov regular points with a given fixed value are dense in the hyperbolic set. Such properties are immediate consequences of the existence of corresponding coexisting invariant Gibbs measures (each with corresponding exponents) that are supported on the entire uniformly hyperbolic set.

Motivated by such results, in the present paper we give a full analysis of orbital Lyapunov exponents. We show that the spectrum of exponents, besides one gap, is complete and, given any exponent in the spectrum, the corresponding Lyapunov regular points are densely distributed in space. This is a first step in understanding finer properties of the spectrum and perhaps its multifractal properties. Our results are based on an analysis of strong forms of minimality of the underlying iterated function system (IFS), an equidistribution of contracting and expanding periodic orbits, and shadowing arguments to explicitly construct orbits with given exponents. In particular, we invoke some general distortion arguments that allow to consider maps that are C^1 smooth only. We restrict our considerations to a specific class of maps, however the tools apply certainly in a much general setting. Observe that the principal ingredient is the existence of expanding (contracting) itineraries (see Sections 2.2, 2.3) that implies a strong form of minimality. The existence of such itineraries are related to blender-like structure that naturally appears in several partially hyperbolic dynamical systems [3].

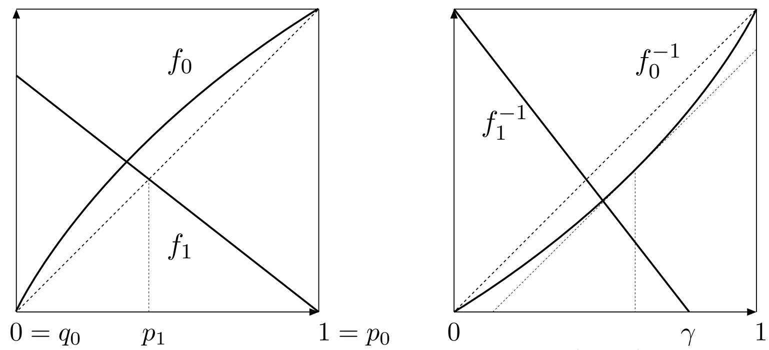

FIGURE 1. The maps f_0, f_1 and f_0^{-1}, f_1^{-1}.

The step skew-product structure enables us to restrict our attention to the underlying IFS generated by two maps. This IFS is not genuinely contracting, mixing contraction and expansion. Investigating random iterations of general non-contracting IFS, fractal properties and, in particular, relations between Lyapunov exponents, dimension, and entropy have been studied recently (see, for example, [8, 9, 11]). Such approaches focus on properties of measures that are stationary with respect to the IFS and are "essentially" contracting. Here we focus on the *orbital* behavior of the IFS, see Theorem 1.3.

1.1. **Step skew-product family.** We study the following class of maps. Given $\widehat{\mathbf{C}} = [0,1]^2$ and a diffeomorphism Φ of $\mathbb{R}^2$ having a horseshoe Γ in $\widehat{\mathbf{C}}$ that is conjugate to the full shift $\sigma\colon \Sigma_2 \to \Sigma_2$ on the symbolic space $\Sigma_2 \stackrel{\text{def}}{=} \{0,1\}^{\mathbb{Z}}$, let $\widehat{\mathbf{C}}_i$, $i = 0,1$, be naturally associated sub-cubes of $\widehat{\mathbf{C}}$ given by the "first level" rectangle in Γ, these rectangles are the connected components of $\Phi^{-1}([0,1]^2) \cap [0,1]^2$. Let $\mathbf{C} = [0,1]^3$ and $\mathbf{C}_i = \widehat{\mathbf{C}}_i \times [0,1]$ and consider the map $F\colon \mathbf{C} \to \mathbb{R}^3$ defined by

$$F(X) \stackrel{\text{def}}{=} (\Phi(\widehat{x}), f_i(x)) \quad \text{if} \quad X = (\widehat{x}, x) \in \mathbf{C}_i. \tag{1.1}$$

Here $f_0, f_1\colon [0,1] \to [0,1]$ be C^1 injective maps satisfying properties (F0), (F1), (F01), and ($\mathrm{F_B}$) specified below, compare Figure 1. To complete the definition of F in $\mathbf{C}$ we will consider some appropriate C^1-continuation of F such that $F(\mathrm{int}(\mathbf{C} \setminus \bigcup_i \mathbf{C}_i)) \cap \mathbf{C} = \emptyset$.

Let us briefly recall some dynamical properties of the local diffeomorphism F obtained in [5], compare Figure 2. We focus on the dynamics of F on the locally maximal invariant set Λ in $\mathbf{C}$

$$\Lambda \stackrel{\text{def}}{=} \Lambda^- \cap \Lambda^+, \qquad \text{where} \qquad \Lambda^{\pm} \stackrel{\text{def}}{=} \bigcap_{i\in\mathbb{N}} F^{\pm i}(\mathbf{C}).$$

By an appropriate choice of the horseshoe, $F|_\Lambda$ is partially hyperbolic with central direction $E^c = \{(0^s, 0^u)\} \times \mathbb{R}$ and in fact a special type of transitive

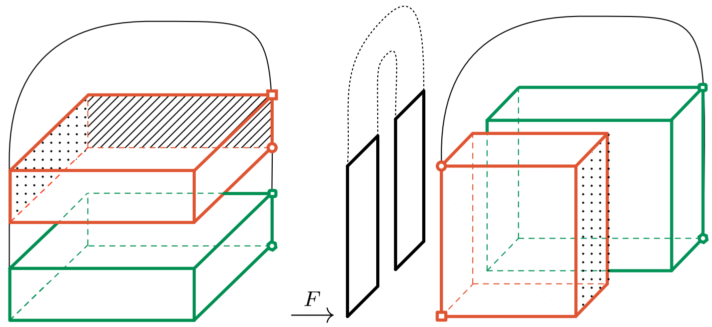

FIGURE 2. The dynamics in the cube **C**

set (called *homoclinic class*, see [3] for a discussion of the role of homoclinic classes in dynamics). Associated to the two fixed points θ_0, θ_1 of the horseshoe map Φ there are two fixed points $P_0 = (\theta_0, 1), P_1 = (\theta_1, p_1)$ (with contracting central direction) and one fixed point $Q_0 = (\theta_0, 0)$ (with expanding central direction) of the map F. The definition of F implies that the saddles P_0 and Q_0 are involved in a heterodimensional cycle, that is, the stable manifold of P_0 meets the unstable one of Q_0 and the unstable manifold of P_0 meets the stable one of Q_0 (see [5, Section 2] for all details). In Λ coexist intermingled hyperbolic sets with different dimension of their unstable manifolds that give rise to a plenitude of heterodimensional cycles associated to periodic points in Λ. This is the underlying mechanism to produce a rich dynamics mixing hyperbolicity of different types. This system also exhibits a very rich fibre structure. The topological properties of the non-hyperbolic homoclinic class Λ are studied in detail in [5].

Let us now specify our conditions.

(F0) The map f_0 is increasing and has exactly two hyperbolic fixed points, the point $q_0 = 0$ (repelling) and the point $p_0 = 1$ (attracting). Let $\beta = f_0'(0) > 1$ and $\lambda = f_0'(1) \in (0, 1)$. The derivative f_0' is decreasing in $[0, 1]$ and satisfies

$$\frac{\lambda^2}{\beta^2} \frac{1-\lambda}{1-\beta^{-1}} > 1. \tag{1.2}$$

(F1) The map f_1 is an affine orientation reversing contraction

$$f_1(x) \stackrel{\text{def}}{=} \gamma\,(1-x),$$

where $1 > \gamma \geq \lambda$. We denote by p_1 the attracting fixed point of f_1. Note that $f_1(1) = 0$.

(F01) There is $k_0 \geq 1$ such that

$$\gamma > \beta^{-k_0} > \beta^{-k_0-2} > \lambda^2. \tag{1.3}$$

($\mathbf{F_B}$) We have $f_0'(x) \in (0, 1)$ for all $x \in [\gamma, 1] = [f_1(0), 1]$.

Conditions (F0) and (F1) imply that the system of the fibre maps $\{f_0, f_1\}$ is of cycle type and mixes expanding and contracting behavior. Condition (F01) will guarantee the existence of appropriate contracting backward itineraries, and ($\mathrm{F}_{\mathbf{B}}$) the existence of appropriate expanding backward itineraries that use blender-like arguments (see Sections 2.3 and 2.2, respectively).

Remark 1.1. There are several ways to achieve the above four conditions simultaneously. Note that for given $\lambda \in (0,1)$ conditions (F0) and (F01) are clearly satisfied if $\beta > 1$ is sufficiently close to 1 and if γ is chosen accordingly to satisfy (1.3). To guarantee (F_{B}) it suffices to assume that the contraction constant γ in (F1) also satisfies $\gamma \in (f_0(y), 1)$ where $y \in (0,1)$ is the largest point with $f_0'(y) = 1$. Note that then $(f_0^{-1})'(x) > 1$ for all $x \in (f_0(y), 1]$.

1.2. **The underlying IFS.** The step skew-product structure of F allows us to reduce the study of its dynamics to the study of the IFS generated by the maps f_0, f_1. We use the following notation. Every sequence $\xi = (\ldots \xi_{-1}.\xi_0\xi_1 \ldots) \in \Sigma_2$ is given by $\xi = \xi^-.\xi^+$, where $\xi^+ \in \Sigma_2^+ \overset{\text{def}}{=} \{0,1\}^{\mathbb{N}_0}$ and $\xi^- \in \Sigma_2^- \overset{\text{def}}{=} \{0,1\}^{-\mathbb{N}}$. We denote by $(\xi_0 \ldots \xi_{m-1})^{\mathbb{Z}}$ the periodic sequence ξ of period m such that $\xi_i = \xi_{i+m}$ for all i and always refer to the smallest period of a sequence.

Given *finite* sequences $(\xi_0 \ldots \xi_n)$ and $(\xi_{-m} \ldots \xi_{-1})$, we let

$$f_{[\xi_0 \ldots \xi_n]} \overset{\text{def}}{=} f_{\xi_n} \circ \cdots \circ f_{\xi_1} \circ f_{\xi_0} \colon [0,1] \to [0,1]$$

and

$$f_{[\xi_{-m} \ldots \xi_{-1}.]} \overset{\text{def}}{=} (f_{\xi_{-1}} \circ \ldots \circ f_{\xi_{-m}})^{-1} = (f_{[\xi_{-m} \ldots \xi_{-1}]})^{-1}.$$

We also let

$$f_{[\xi_{-m} \ldots \xi_{-1}.\xi_0 \ldots \xi_n]} \overset{\text{def}}{=} f_{[\xi_0 \ldots \xi_n]} \circ f_{[\xi_{-m} \ldots \xi_{-1}.]}.$$

Note that the maps $f_{[\xi_{-m} \ldots \xi_{-1}.]}$ are in general only defined on a closed subinterval of $[0,1]$. A finite sequence $(\xi_{-m} \ldots \xi_{-1})$ is said to be *admissible* for a point x if the map $f_{[\xi_{-m} \ldots \xi_{-1}.]}$ is well-defined at x. A one-sided infinite sequence $(\ldots \xi_{-2}\xi_{-1}.) \in \Sigma_2^-$ is said to be *admissible* for x if $(\xi_{-m} \ldots \xi_{-1})$ is admissible for x for all $m \geq 1$. Note that the admissibility of a sequence ξ does not depend on the symbols $(\xi_0\xi_1 \ldots)$. A two-sided infinite sequence $\xi = (\ldots \xi_{-1}.\xi_0\xi_1 \ldots) \in \Sigma_2$ is said to be *admissible* for x if $(\ldots \xi_{-2}\xi_{-1}.)$ is admissible for x. By writing (x, ξ) we always suppose that ξ is admissible for x.

We study the spectrum of Lyapunov exponents of the IFS generated by the maps f_0, f_1. Given a point $p \in [0,1]$ and a one-sided sequence $\xi^+ = (\xi_0\xi_1 \ldots) \in \Sigma_2^+$, the *forward Lyapunov exponent* of p with respect to ξ^+ is defined by

$$\chi_+(p, \xi^+) \overset{\text{def}}{=} \lim_{n\to\infty} \frac{1}{n} \log \left|(f_{[\xi_0 \ldots \xi_{n-1}]})'(p)\right|$$

whenever this limit exists. Given a one-sided sequence $\xi^- = (\ldots \xi_{-1}.) \in \Sigma_2^-$ that is admissible for p, then the *backward Lyapunov exponent* of p with respect to ξ is defined by

$$\chi_-(p,\xi^-) \stackrel{\text{def}}{=} \lim_{n\to\infty} \frac{1}{-n} \log \big|(f_{[\xi_{-n}\ldots\xi_{-1}.]})'(p)\big|$$

whenever this limit exists. Otherwise we denote by $\underline{\chi}_\pm$ and $\overline{\chi}_\pm$ the *lower* and the *upper Lyapunov exponents* defined by taking the lower and the upper limit, respectively. Clearly, χ_+ (χ_-) does not depend on the backward (the forward) completion of the sequence. The same observation applies to $\underline{\chi}_\pm$ and $\overline{\chi}_\pm$. Given a two-sided sequence $\xi \in \Sigma_2$ we denote by $\chi_\pm(p,\xi)$ the corresponding values, whenever it makes sense (whenever ξ is admissible and the limit exists). Given a periodic sequence $(\xi_0\ldots\xi_{m-1})^{\mathbb{Z}}$ and a fixed point $p = p_{(\xi_0\ldots\xi_{m-1})^{\mathbb{Z}}} = f_{[\xi_0\ldots\xi_{m-1}]}(p_{(\xi_0\ldots\xi_{m-1})^{\mathbb{Z}}})$, we have

$$\chi_\pm(p,(\xi_0\ldots\xi_{m-1})^{\mathbb{Z}}) = \frac{1}{m}\log\big|(f_{[\xi_0\ldots\xi_{m-1}]})'(p)\big|. \tag{1.4}$$

The following result about the Lyapunov spectrum was shown in [5].

Proposition 1.2 (Spectral gap). *Under assumptions (F0), (F1), (F01), and ($F_{\mathbf{B}}$) the IFS satisfies*

$$\log\widetilde{\beta} \stackrel{\text{def}}{=} \sup\chi_+(p,\xi^+) < \log\beta = \log f_0'(0),$$

where the supremum is taken over all points $p \in [0,1]$ and sequences $\xi^+ \in \Sigma_2^+, \xi^+ \neq (\xi_0\ldots\xi_k0^{\mathbb{N}})$, $k \geq 0$.

By Proposition 1.2 the (forward) spectrum of Lyapunov exponents of the IFS contains a gap and has an isolated point at $\log\beta$. Note that, given any pair $(0,\xi)$ such that ξ^+ does not eventually consist of only $0's$, its forward orbit contains a point in the interval $(0,1)$ and by Proposition 1.2 the upper exponent of $(0,\xi)$ is less than $\log\beta$. Otherwise, if ξ^+ does eventually consist of only $0's$ then the Lyapunov exponent of $(0,\xi)$ is either $\log\beta$ and or $\log\lambda$. Similarly, given $(1,\xi)$ such that ξ^+ does not consist only of $0's$ then the first forward iterate is of the form $(0,\xi')$ and we are in the previous case obtaining the exponent $\log\beta$. Otherwise, the exponent of $(1,\xi)$ is $\log\lambda$.

Here we continue with these studies and provide full details about the (forward and backward) Lyapunov spectrum. In particular, we show that there are no further gaps. The following are our main results.

Theorem 1.3 (Lyapunov spectrum of the IFS). *Under assumptions (F0), (F1), (F01), and ($F_{\mathbf{B}}$) the IFS satisfies the following.*

Forward spectrum:

(i) *For every $\chi \in [0,\log\widetilde{\beta}\,]$ the set of points y for which there exists $\xi^+ \in \Sigma_2^+$ with $\chi_+(y,\xi^+) = \chi$ is dense in $[0,1]$.*

(ii) *For every $\chi \in [\log\lambda, 0]$ there exists $\xi^+ \in \Sigma_2^+$ with $\chi_+(y,\xi^+) = \chi$ for every $y \in [0,1]$.*

Backward spectrum:

(iii) *For every $\chi \in [0,\log\widetilde{\beta}\,]$ and for every $y \in (0,1)$ there exists $\xi^- \in \Sigma_2^-$ with $\chi_-(y,\xi^-) = \chi$.*

(iv) *For every* $\chi \in [\log\lambda, 0]$ *the set of points* y *for which there exists* $\xi^- \in \Sigma_2^-$ *with* $\chi_-(y,\xi^-) = \chi$ *is dense in* $[0,1]$.

The proof of Theorem 1.3 will be split into several steps and will be given in Section 5. The IFS that we study is genuinely non-contracting as it contains orbits with negative and orbits with positive Lyapunov exponent. Thus, *a priori* we do not have at hand shadowing properties classically used in hyperbolic theory. Nevertheless, because of the presence of certain expanding itineraries, we are able to establish a shadowing-like property to find (periodic) orbits with desired positive Lyapunov exponents. Likewise, we will proceed to find (periodic) orbits with desired negative exponents for which we will make use of certain contracting itineraries.

Remark 1.4. It might not be obvious why we need to consider backward Lyapunov exponents separately. After all, for any ergodic measure and any integrable potential, the forward and backward Birkhoff averages are the same, both equal to the integral of the potential with respect to the measure. So, one might assume that the forward and backward Lyapunov spectra must be equal. This intuition would be, however, wrong, as the example we give in the appendix shows.

1.3. **Back to the step skew-product map.** Due to the skew product structure and our hypotheses, the DF-invariant splitting $E^{ss}\oplus E^c\oplus E^{uu}$ given by

$$E^{ss} \stackrel{\text{def}}{=} \mathbb{R}\times\{(0^u,0)\},\quad E^c \stackrel{\text{def}}{=} \{(0^s,0^u)\}\times\mathbb{R},\quad E^{uu} \stackrel{\text{def}}{=} \{0^s\}\times\mathbb{R}\times\{0\}$$

is dominated (if expansion/contraction of Φ are strong enough) and for every Lyapunov regular point $R\in\Lambda$ coincides with the Oseledec splitting provided by the multiplicative ergodic theorem. In particular, the *Lyapunov exponent associated to the central direction* E^c at such a point R is well-defined and, in fact, is the Birkhoff average of the continuous function $R\mapsto \log\|dF|_{E^c_R}\|$

$$\chi_c(R) \stackrel{\text{def}}{=} \lim_{n\to\infty}\frac{1}{n}\log\|dF^n|_{E^c_R}\| = \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\log\|dF|_{E^c_{F^k(R)}}\|.$$

Given a Lyapunov regular point $R=(r^s,r^u,r)\in\Lambda$ and the sequence $\xi = (\ldots\xi_{-1}.\xi_0\xi_1\ldots)\in\Sigma_2$ associated to the point (r^s,r^u) in the two-dimensional horseshoe Γ, we have

$$\chi_c(R) = \chi(r,\xi) = \lim_{n\to\infty}\frac{1}{n}\log|(f_{[\xi_0\ldots\,\xi_{n-1}]})'(r)|$$

(analogous for forward/backward upper/lower Lyapunov exponent). Finally note that the remaining exponents are associated to the stable and the unstable directions E^{ss} and E^{uu}, respectively, and are uniformly bounded away from zero. The following is an immediate consequence of Theorem 1.3.

Corollary 1.5 (Complete spectrum)**.** *We have*

$$\{\chi_c(R)\colon R\in\Lambda\} = [\log\lambda, \log\widetilde\beta\,]\cup\{\log\beta\}.$$

Let us observe that as a consequence of our results and the methods for constructing non-hyperbolic ergodic measures with large support introduced in [10] and further developed in [1] we can prove the following result that strengthens Corollary 1.5.

Proposition 1.6. *For every* $\chi \in (\log \lambda, \log \widetilde{\beta}\,)$ *there is an ergodic measure* μ *with full support in* Λ *such that* $\chi_c(\mu) = \chi$.

As the proof closely follows the ideas in [10, 1] we only provide a sketch of it, see Section 5.5.

1.4. **Ingredients and organization.** To prove the above results, we study minimality of the IFS. Recall that, given a metric space X, a map $f\colon X \to X$ is minimal if each closed subset $Y \subset X$ such that $f(Y) \subset Y$ is either empty or coincides with X. Similarly, an iterated function system $\{f_i\}_i$ of maps $f_i\colon X \to X$ is said to be *forward minimal* if each closed set $Y \subset X$ such that $f_i(Y) \subset Y$ for all i is empty or coincides with X. Given a point $x \in X$, let

$$\mathcal{O}^+(x) \stackrel{\text{def}}{=} \{f_{[\eta_0 \ldots \eta_n]}(x)\colon \eta \in \Sigma_2^+, n \geq 0\}$$

be the set of all images of x. Note that an IFS $\{f_i\}_i$ of maps $f_i\colon X \to X$ is forward minimal if and only if for every $x \in X$ the set $\mathcal{O}^+(x)$ is dense in X.

In principle, we can also study backward minimality of an IFS studying the inverse maps f_i^{-1}. In our setting, when considering a characterization using dense backward orbits, we have to observe that not all backward concatenations are admissible. Let

$$\mathcal{O}^-(x) \stackrel{\text{def}}{=} \{f_{[\eta_{-n} \ldots \eta_{-1}.]}(x)\colon \eta \in \Sigma_2^- \text{ admissible for } x \text{ and } n \geq 1\}$$

be the set of all preimages of x under the IFS. Note that in our case we have $\mathcal{O}^-(1) = \{1\}$ and $\mathcal{O}^-(0) = \{0, 1\}$ and thus the IFS is not backward minimal as defined above. However, these two points are the only exceptional points as stated in the following proposition.

Proposition 1.7 (Forward and almost backward minimality)**.** *For every* $x \in [0, 1]$ *the set* $\mathcal{O}^+(x)$ *is dense in* $[0, 1]$. *For every* $x \in (0, 1)$ *the set* $\mathcal{O}^-(x)$ *is dense in* $[0, 1]$.

The investigation of minimal sets for an IFS is closely related to the investigation of minimal sets of Markov systems or pseudo Markov systems which occur in codimension one foliations. We further point out iterated function systems of maps over the circle and Duminy's theorem (see [14]).

The paper is organized as follows. In Section 2 we study dynamic properties of the underlying IFS such as expanding and contracting itineraries. In Section 3 we establish minimality of the IFS. In Section 4 we explicitly construct periodic orbits of the IFS with given approximate Lyapunov exponents. In Section 5 we provide the proof of Theorem 1.3 and explicitly construct forward/backward orbits of the IFS that have a given Lyapunov exponent. To do this we will make use of a number of shadowing arguments. We will also provide a sketch of Proposition 1.6.

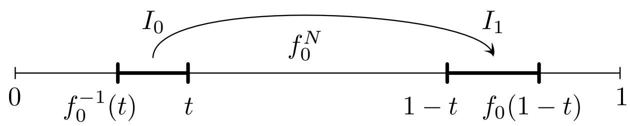

FIGURE 3. Choice of fundamental domains

2. DYNAMICS OF THE IFS

In this section we start to study dynamical properties of the IFS generated by the maps f_0 and f_1.

2.1. **Choice of fundamental domains.** For small $t > 0$, consider the fundamental domains of f_0

$$I_0(t) \stackrel{\text{def}}{=} [f_0^{-1}(t), t] \quad \text{and} \quad I_1(t) \stackrel{\text{def}}{=} [1-t, f_0(1-t)]$$

and define $N(t) \geq 1$ and $M(t) \geq 0$ the smallest integers such that

$$f_0^{N(t)}(I_0(t)) \subset [1-t, 1] \quad \text{and} \quad \big(f_0^{M(t)} \circ f_1 \circ f_0^{N(t)}\big)(f_0^{-1}(t)) \in I_0(t).$$

Note that $f_1 \circ f_0^{N(t)}(I_0(t)) \subset [0, \gamma\, t] \subset [0, t]$ and thus $M(t) \geq 0$.

Lemma 2.1. *There are arbitrarily small $t > 0$ with $f_0^{N(t)}(I_0(t)) = I_1(t)$.*

Proof. As $t \mapsto f_0^{N-1}(t)$ is continuous, $f_0^{N-1}(0) = 0$, and $f_0^{N-1}(1) = 1$, by the mean value theorem for any N there exists $t = t(N)$ such that $t + f_0^{N-1}(t) = 1$. It is not difficult to see that $t(N)$ accumulates at 0 as N increases. □

In the following we will assume that t is chosen as in Lemma 2.1 and is sufficiently close to 0 (see Proof of Lemma 2.3). Fixing such $t > 0$, in what follows we write

$$N = N(t), \quad M = M(t),$$
$$I_0 = [f_0^{-1}(t), t] = I_0(t), \quad I_1 = [1-t, f_0(1-t)] = f_0^N(I_0) = I_1(t).$$

2.2. **Forward expanding itineraries.** One of the key ingredients in our construction is the existence of so-called *expanding itineraries* (see Lemma 2.5 and [5, Section 3.2] for details). Roughly speaking, our IFS shows sufficient expansion such that any non-trivial interval under some appropriate iterate eventually covers an appropriate fundamental domain of f_0. Let us now show how this follows from our hypotheses.

Definition 2.2 (Expanded successor)**.** Given a closed subinterval J of $f_0^{-1}(I_0) \cup I_0$ let

$$N_+(J) \stackrel{\text{def}}{=} \begin{cases} N & \text{if } J \subset I_0, \\ N+1 & \text{otherwise.} \end{cases}$$

The *expanded successor* of J is defined by $f_{[0^{N_+(J)}1\,0^{M(J)}]}(J)$, where $M(J)$ is the smallest integer M with $f_{[0^{N_+(J)}1\,0^{M}]}(J) \cap I_0 \neq \emptyset$.

The next lemma justifies the terminology "expanded successor". Note that for its proof we require that t was chosen sufficiently small.

Lemma 2.3 (Uniform expansion for expanded successors)**.** *Let J be a closed subinterval of $f_0^{-1}(I_0)\cup I_0$. Then*

$$|(f_{[0^{N_+(J)}1\,0^{M(J)}]})'(x)| \ge \varkappa \overset{\text{def}}{=} \frac{\lambda^2}{\beta^2}\,\frac{1-\lambda}{1-\beta^{-1}} > 1 \quad \textit{for all } x\in J. \tag{2.1}$$

Recall that $\varkappa>1$ follows from condition (1.2) in (F0).

Proof. For simplicity, to avoid lengthy discussion involving approximations, let us assume that f_0 is linear in $[0,f_0(t)]$ and in $[1-t,f_0(1-t)]$ (indeed, as t is small this hypothesis is "almost true"). Recall that $\lambda=f_0'(1)\in(0,1)$ and $\beta=f_0'(0)>1$. We claim that

$$\big(f_0^N\big)'(x)\ge \frac{\lambda}{\beta}\,\frac{1-\lambda}{1-\beta^{-1}} \quad \text{for all } x\in I_0. \tag{2.2}$$

Observe now that by the mean value theorem there is $y\in f_0(I_0)$ such that

$$\big(f_0^N\big)'(y)=\frac{\big|f_0^N\big(f_0(I_0)\big)\big|}{\big|f_0(I_0)\big|}=\frac{\lambda\,t\,(1-\lambda)}{\beta\,t\,(1-\beta^{-1})}=\frac{\lambda\,(1-\lambda)}{\beta\,(1-\beta^{-1})}. \tag{2.3}$$

The monotonicity of f_0' and $y\in f_0(I_0)$ imply that $\big(f_0^N\big)'(x)\ge\big(f_0^N\big)'(y)$ for all $x\in I_0\cup f_0^{-1}(I_0)$ and hence (2.2).

Now observe that the definition of $N_+(J)$ implies that $f_0^{N_+(J)}(J)\subset[1-t,1-\lambda^2\,t]$. Hence we have

$$\big(f_0^{M(J)}\circ f_1\circ f_0^{N_+(J)}\big)(J)\subset f_0^{M(J)}\circ f_1\big([1-t,1-\lambda^2\,t]\big)\subset f_0^{M(J)}\big([\gamma\,\lambda^2\,t,\gamma\,t]\big).$$

Define M^- and M^+ as the smallest positive integers satisfying

$$\beta^{M^-}\gamma\,t\ge\beta^{-1}\,t \quad\text{and}\quad \beta^{M^+}\gamma\,\lambda^2\,t\ge t. \tag{2.4}$$

Note that $M^-\le M(J)\le M^+$ and thus

$$\beta^{M(J)}\ge(\beta\,\gamma)^{-1}.$$

Using (2.2) and recalling that $N_+(J)\le N+1$ and $\beta^{M(J)}\ge(\beta\,\gamma)^{-1}$, we get

$$\big|\big(f_0^{M(J)}\circ f_1\circ f_0^{N_+(J)}\big)'(x)\big|\ge\frac{1}{\beta\,\gamma}\cdot\gamma\cdot\lambda\,\frac{\lambda}{\beta}\,\frac{1-\lambda}{1-\beta^{-1}}=\frac{\lambda^2}{\beta^2}\,\frac{1-\lambda}{1-\beta^{-1}}=\varkappa,$$

for all $x\in J\subset f_0^{-1}(I_0)\cup I_0$. This ends the proof of the lemma. □

Using Lemma 2.3 and arguing as in [5, Lemma 3.5], we have the following.

Remark 2.4 (Forward expanding itineraries)**.** Given a closed interval $J\subset f_0^{-1}(I_0)\cup I_0$, we choose numbers $\iota(J),n(J)\in\mathbb{N}$, an itinerary $(\xi_0\ldots\xi_{n(J)})$ consisting of $\iota(J)$ concatenated loops of type $(0^{n_i}10^{m_i})$, $n_i\in\{N,N+1\}$ and $m_i\in\{M^-,\ldots,M^+\}$, and intervals $J_{\langle i\rangle}$, $i=0,\ldots,\iota(J)$, according to the following rules:

- Let $J_{\langle 0\rangle}=J$, $n_0=N_+(J_{\langle 0\rangle})$, and $m_0=M(J_{\langle 0\rangle})$.

- The interval $J_{\langle 1\rangle} = f_{[0^{m_0}\,1\,0^{n_0}]}(J_{\langle 0\rangle})$ is the expanded successor of $J_{\langle 0\rangle}$. Note that $J_{\langle 1\rangle} \cap I_0 \neq \emptyset$ and $J_{\langle 1\rangle} \subset (0,t]$.
- Arguing inductively, assume that for all $j \in \{1,\dots,i\}$ we have already defined the interval $J_{\langle j\rangle} \subset f_0^{-1}(I_0) \cup I_0$ with $J_{\langle j\rangle} \cap I_0 \neq \emptyset$ and numbers $n_j = N_+(J_{\langle j\rangle})$ and $m_j = M(J_{\langle j\rangle})$ such that $J_{\langle j+1\rangle} = f_{[0^{n_j}\,1\,0^{m_j}]}(J_{\langle j\rangle})$ is the expanded successor of $J_{\langle j\rangle}$. By Lemma 2.3 these intervals satisfy $|J_{\langle j+1\rangle}| > \varkappa\,|J_{\langle j\rangle}| > \varkappa^{j+1}\,|J|$.
- There is a smallest number $\iota = \iota(J)$ with $J_{\langle i\rangle} \subset f_0^{-1}(I_0) \cup I_0$ for all $i = 0,\dots,\iota(J)$ such that $J_{\langle \iota+1\rangle} = f_{[0^{n_\iota}\,1\,0^{m_\iota}]}(J_{\langle \iota\rangle}) = f_{[\xi_0\dots\,\xi_n]}(J)$ contains the fundamental domain $f_0^{-1}(I_0)$.
- By construction, the numbers $\iota(J)$ and $n(J)$ are bounded from above by constants depending only on the length of J.

The above construction result in the following slightly stronger version.

Lemma 2.5 (Forward expanding covering itineraries)**.** *There is $\varepsilon > 0$ such that for any $\tau > 0$ there is $n_+(\tau) \geq 1$ with the following property: given any interval $J \subset f_0^{-1}(I_0) \cup I_0$ of length $|J| \geq \tau$, there is an itinerary $(\rho_0 \dots \rho_n)$, $n \leq n_+(\tau)$, of concatenated loops of type $(0^{n_i}10^{m_i})$ such that*

- *$f_{[\rho_0\dots\,\rho_n]}(J)$ either contains $[f_0^{-2}(t)-\varepsilon, f_0^{-1}(t)]$ or $[f_0^{-2}(t), f_0^{-1}(t)+\varepsilon]$,*
- *$|(f_{[\rho_0\dots\,\rho_n]})'(x)| \geq \varkappa$ for all $x \in J$.*

Moreover, if the sequence $(\rho_0 \dots \rho_n)$ is a concatenation of ι loops $(0^{n_i}10^{m_i})$ then in the last estimate the expansion is bounded by $\varkappa^\iota$.

Proof. It suffices to observe that we can apply the steps in Remark 2.4 to the interval $\widetilde{J} = f_0^{-1}(I_0)$ obtaining an integer $\widetilde{\iota}$ and a sequence $(\eta_0 \dots \eta_{\widetilde{n}})$ such that the interval $\widetilde{J}_{\langle\widetilde{\iota}\rangle} = f_{[\eta_0\dots\,\eta_{\widetilde{n}}]}(\widetilde{J})$ covers $f_0^{-1}(I_0)$ and has length at least $\varkappa\,|f_0^{-1}(I_0)|$. Hence, in particular, there exists some universal number $\varepsilon > 0$ such that $\widetilde{J}_{\langle\widetilde{\iota}\rangle}$, and thus $f_{[\xi_0\dots\,\xi_n\eta_0\dots\,\eta_{\widetilde{n}}]}(J)$ with $n = n(\widetilde{J})$, either contains the interval $[f_0^{-2}(t)-\varepsilon, f_0^{-1}(t)]$ or the interval $[f_0^{-2}(t), f_0^{-1}(t)+\varepsilon]$. □

2.3. **Admissible backward expanding itineraries.** In this section we prove that to suitable intervals in $[0,1]$ we can associate an admissible composition of the maps f_0^{-1} and f_1^{-1} that is uniformly expanding. Using condition ($\mathrm{F}_{\mathbf{B}}$) we fix $\alpha > 1$ and $\gamma_* < \gamma = f_1(0)$ such that (compare Figure 1)

$$\begin{aligned} 1 < \alpha &< \min\left\{\gamma^{-1}, f_0'(f_0^{-1}(\gamma_*))\right\}, \\ (f_0^{-1})'(x) &> \alpha \quad \text{ for all } x \in [\gamma_*, 1]. \end{aligned} \tag{2.5}$$

The existence of expanding backward itineraries follows using an argument from [2] that is in the heart of the concept of a blender (see, for example, [3, Chapter 6.2]).

Lemma 2.6 (Backward expanding iterates)**.** *For every $\tau > 0$ there is an integer $n_-(\tau) > 0$ with the following property: given any closed interval*

$J \subset [0,1]$ of length $|J| \geq \tau$, there is an itinerary of the form

$$(\xi_{-n} \dots \xi_{-1}.) = (0^k\, 1^2\, \xi_{-n+k+2} \dots \xi_{-1}.), \quad n \leq n_-(\tau),$$

such that

- $f_{[0^k\,1^2.]}\big(f_{[\xi_{-n+k+2}\cdots\,\xi_{-1}.]}(J) \cap [\gamma_*, \gamma]\big) \supset [\gamma_*, 1]$,
- $|(f_{[\xi_{-j}\cdots\,\xi_{-1}.]})'(x)| \geq \alpha^j$ *for all* $x \in J \cap f_{[\xi_{-n}\cdots\,\xi_{-1}]}([\gamma_*, \gamma])$ *and all* $j = 1, \dots, n$.

Proof. Given $\tau > 0$, let $J \subset [0,1]$ be an interval of length $|J| \geq \tau$. Let us construct the itinerary $(\xi_{-n} \dots \xi_{-1}.)$. First, assume that $[\gamma_*, \gamma]$ is not contained in J. Then there are two cases: if $J \subset [\gamma_*, 1]$ we let $\xi_{-1} = 0$ and if $J \subset [0, \gamma]$ we let $\xi_{-1} = 1$. Now define recursively the first block in the itinerary, let $\ell = 1$ and apply either
i) if $f_{[\xi_{-\ell}\cdots\,\xi_{-1}.]}(J) \subset [\gamma_*, 1]$ then let $\xi_{-(\ell+1)} = 0$ and increase ℓ by 1, or
ii) if $f_{[\xi_{-\ell}\cdots\,\xi_{-1}.]}(J) \subset [0, \gamma]$ then let $\xi_{-(\ell+1)} = 1$ and increase ℓ by 1, or
iii) $f_{[\xi_{-\ell}\cdots\,\xi_{-1}.]}(J) \supset [\gamma_*, \gamma]$ then stop the recursion.
Recalling the definition of α in (2.5), in i) and ii) we have $|(f_{[\xi_{-k}\cdots\,\xi_{-1}.]})'(x)| \geq \alpha^k$ for every $x \in J$ and $k = 1, \dots, \ell$ and hence $|f_{[\xi_{-\ell}\cdots\,\xi_{-1}.]}(J)| \geq \alpha^\ell |J|$. Thus the recursion stops after finitely many times $\ell \leq (\tau \log \alpha)^{-1}$. Also observe

$$f_{[\xi_{-\ell}\cdots\,\xi_{-1}.]}(J) \cap [\gamma_*, \gamma] = [\gamma_*, \gamma].$$

In the remaining case $J \supset [\gamma_*, \gamma]$ there is no first block $(\xi_{-\ell} \dots \xi_{-1}.)$.

Note that $1 \in f_1^{-2}([\gamma_*, \gamma])$ and that the interval $[\gamma_*, \gamma]$ is α^2-expanded by f_1^{-2}. Let k be the first positive with $f_0^{-k}(f_1^{-2}(\gamma_*)) \leq \gamma_*$. Then by construction,

$$f_0^{-k}\big(f_1^{-2}(f_{[\xi_{-\ell}\cdots\,\xi_{-1}.]}(J) \cap [\gamma_*, \gamma])\big) \supset [\gamma_*, 1].$$

Consider now the sequence $(\xi_{-n} \dots \xi_{-1}.) = (0^k\, 1^2\, \xi_{-\ell} \dots \xi_{-1}.)$. Since by the choice of γ_* the map f_0^{-1} is α-expanding on $[\gamma_*, 1]$, one has that for all $x \in J$ with $f_{[\xi_{-n}\cdots\,\xi_{-1}.]}(x) \in [\gamma_*, \gamma]$ and for all $j = 1, \dots, n$ it holds

$$|(f_{[\xi_{-j}\cdots\,\xi_{-1}.]})'(x)| \geq \alpha^j.$$

This completes the proof of the lemma. $\square$

2.4. **Admissible backward contracting itineraries.** We now see that the IFS also shows sufficient backward contraction.

Definition 2.7 (Contracted predecessor)**.** Given a closed subinterval J of $I_1 \cup f_0(I_1)$, let

$$N_-(J) \stackrel{\text{def}}{=} \begin{cases} N & \text{if } J \subset I_1, \\ N+1 & \text{otherwise,} \end{cases} \tag{2.6}$$

where N is given in Section 2.1. The *contracted predecessor* of J is

$$f_1^{-1} \circ f_0^{-N_-(J)-k_0}(J) = f_{[1\,0^{N_-(J)+k_0}.]}(J),$$

where k_0 is defined in (1.3).

The next lemma justifies the terminology "contracted predecessor". Recall the constant $\varkappa$ defined in (2.1).

Lemma 2.8 (Uniform contraction for contracted predecessors)**.** *Let J be a closed subinterval of $I_1 \cup f_0(I_1)$. Then the contracted predecessor of J is contained in $I_1 \cup f_0(I_1)$ and we have*

$$|(f_{[1\,0^{N_-(J)+k_0}.]})'(x)| \le \varkappa^{-1} < 1 \quad \textit{for all } x \in J.$$

Proof. As in the proof of Lemma 2.3 let us assume linearity of f_0 close to 0 and 1. The definition of $N_-(J)$ implies that

$$f_0^{-N_-(J)-k_0}(J) \subset [\beta^{-k_0-2}\, t, \beta^{-k_0}\, t]$$

and thus

$$\begin{aligned}(f_1^{-1} \circ f_0^{-N_-(J)-k_0})(J) &\subset [1-\gamma^{-1}\,\beta^{-k_0}\, t, 1-\gamma^{-1}\,\beta^{-k_0-2}\, t] \\ &\subset [1-t, 1-\lambda^2\, t] = I_1 \cup f_0(I_1),\end{aligned}$$

where the last inclusion follows from (1.3). The estimate on the derivative is obtained exactly as in Lemma 2.3. □

3. Minimality

3.1. **Some preliminary results.** The following lemma will be used to establish backward minimality (Proposition 1.7) and to construct approximating repelling periodic orbits (Lemma 4.4) and shadowing orbits (Proposition 5.1 and Theorem 1.3 (i)).

Lemma 3.1. *Given $\Delta > 0$ and $\tau > 0$ sufficiently small, there are numbers $M = M_+(\Delta, \tau) \ge 1$ and $\delta = \delta_+(\Delta) > 0$ such that for every $p \in (\Delta, 1-\Delta)$ and every interval $J \subset (\Delta, 1-\Delta)$ of length $|J| \ge \tau$ there is a finite sequence $(\eta_0 \dots \eta_m)$, $m \le M$, such that*

$$f_{[\eta_0 \dots \eta_m]}(J) \supset [p-\delta/2, p+\delta/2].$$

Proof. The proof of the lemma is an almost immediate consequence of Lemma 2.5 if p is to the right of the fundamental domain I_0. However, it does require more care if p is to the left. To deal with the general case, we consider the coverings $\mathcal{S}^{\pm}$ of $[\Delta, 1-\Delta]$ defined below. First recall the choice of t in Section 2.1, let $\varepsilon > 0$ be given by Lemma 2.5, and define

$$I_\varepsilon^- \stackrel{\text{def}}{=} (f_0^{-2}(t) - \varepsilon, f_0^{-1}(t)), \quad I_\varepsilon^+ \stackrel{\text{def}}{=} (f_0^{-2}(t), f_0^{-1}(t) + \varepsilon).$$

We will use the following result whose proof we postpone.

Claim. *There are positive integers m_1, m_2, ℓ and two finite sets of finite sequences*

$$\mathcal{S}^{\pm} \stackrel{\text{def}}{=} \big\{(0^{m_1}\,1\,0^i),\, (0^{m_1+1}\,1\,0^i), \dots, (0^{m_2}\,1\,0^i), \quad i = 0, \dots \ell\big\}$$

such that

$$\mathcal{C}^{\pm} \stackrel{\text{def}}{=} \big\{f_{[\gamma_0 \dots \gamma_k]}(I_\varepsilon^{\pm}) \colon (\gamma_0 \dots \gamma_k) \in \mathcal{S}^{\pm}\big\}$$

form two coverings of $[\Delta, 1-\Delta]$ *by open intervals, respectively.*

Let $\delta \stackrel{\text{def}}{=} \delta_+(\Delta) > 0$ be a common Lebesgue number of the coverings $\mathcal{C}^\pm$.

Note that there are numbers $j_0, n_0, m_0 \ge 0$ such that $f_{[1^{j_0}\,0^{n_0}\,1\,0^{m_0}]}([0,1])$ is contained in $f_0^{-1}(I_0) \cup I_0$. Thus the same inclusion holds for any subinterval of $[0,1]$. We fix such numbers and some small $\tau > 0$ and consider a closed interval $J \subset (\Delta, 1-\Delta)$ with $|J| \ge \tau$ and a point $p \in (\Delta, 1-\Delta)$. By our choice of numbers $H = f_{[1^{j_0}\,0^{n_0}\,1\,0^{m_0}]}(J)$ is contained in $f_0^{-1}(I_0) \cup I_0$ and has length at least some number $\tau' > 0$ that depends only on τ.

By Lemma 2.5, there exists an itinerary $(\rho_0 \ldots \rho_n)$, whose length is uniformly bounded from above by some number depending only on τ', such that $f_{[\rho_0 \ldots \rho_n]}(H)$ covers either I_ε^- or I_ε^+. By the choice of δ, the interval $[p-\delta/2, p+\delta/2]$ is contained in some member $f_{[\gamma_0 \ldots \gamma_k]}(I_\varepsilon^\pm)$ of the coverings $\mathcal{C}^\pm$. Hence concatenating with this itinerary $(\gamma_0 \ldots \gamma_k)$ we have that

$$[p-\delta/2, p+\delta/2] \subset f_{[1^{j_0}\,0^{n_0}\,1\,0^{m_0}\,\rho_0\,\ldots\,\rho_n\,\gamma_0\,\ldots\,\gamma_k]}(J).$$

By construction, the length of $(1^{j_0}\,0^{n_0}\,1\,0^{m_0}\,\rho_0\,\ldots\,\rho_n\,\gamma_0\,\ldots\,\gamma_k)$ is bounded from above by some number $M_+(\Delta, \tau)$.

We now prove the claim. Choose $m_1 \ge 1$ large enough such that

$$f_{[0^{m_1}]}(I_\varepsilon^-) \subset (1-\Delta, 1) \quad \text{and} \quad f_{[0^{m_1}\,1]}(I_\varepsilon^-) \subset (0, \Delta).$$

Note that the set $f_{[0^{m_1}\,1]}(I_\varepsilon^-)$ may not cover a fundamental domain of f_0. However, since $f_1(1) = 0$ and I_ε^- contains a fundamental domain of f_0, we can choose a number $m_2 \ge m_1$ such that $f_1\big(f_{[0^{m_1}]}(I_\varepsilon^-) \cup \ldots \cup f_{[0^{m_2}]}(I_\varepsilon^-)\big)$ does cover a fundamental domain of f_0 in $(0, \Delta)$ (note that two consecutive iterates of I_ε^- by f_0 overlap). Indeed, the number of fundamental domains of f_0 covered tends to infinity as m_2 tends to infinity. Hence, we can consider $\ell \ge 1$ and the finite set of finite sequences

$$\mathcal{S}^- = \big\{(0^{m_1}\,1\,0^i),\,(0^{m_1+1}\,1\,0^i), \ldots, (0^{m_2}\,1\,0^i), \quad i = 0, \ldots \ell\big\}$$

such that

$$\mathcal{C}^- \stackrel{\text{def}}{=} \big\{f_{[\gamma_0 \ldots \gamma_k]}(I_\varepsilon^-) \colon (\gamma_0 \ldots \gamma_k) \in \mathcal{S}^-\big\}$$

forms a covering of $[\Delta, 1-\Delta]$ by open intervals. The construction of $\mathcal{C}^+$ is similar, possibly after increasing m_1, m_2, ℓ. This proves the claim and finishes the proof of the lemma. □

The next result is a version of Lemma 3.1 for backwards iterates. It will be used to establish forward minimality (Proposition 1.7) and to construct shadowing orbits (Theorem 1.3 (iv)).

Lemma 3.2. *Given* $\Delta > 0$ *and* $\tau > 0$ *sufficiently small, there is a number* $M = M_-(\Delta, \tau) \ge 1$ *such that for every closed interval* $J \subset (\Delta, 1-\Delta)$ *of length* $|J| \ge \tau$ *there is a finite sequence* $(\eta_{-m} \ldots \eta_{-1})$, $m \le M$, *such that for some* k *we have*

$$f_{[\eta_{-m} \ldots \eta_{-m+k+1}.]}\big(f_{[\eta_{-m+k+2} \ldots \eta_{-1}.]}(J) \cap [\gamma_*, 1]\big) \supset [\Delta/2, 1],$$

where γ_* is defined in (2.5).

Proof. Let $j = j(\Delta)$ be the smallest positive integer with $f_0^{-j}(\gamma_*) \in (0, \Delta/2)$. Given any interval $J \subset (\Delta, 1-\Delta)$ of length $|J| \geq \tau$, consider the number $n_-(\tau)$ and the finite sequence $(\xi_{-n} \dots \xi_{-n+k+2} \dots \xi_{-1})$ with $k \leq n \leq n_-(\tau)$ provided by Lemma 2.6 asserting that

$$f_{[\xi_{-n} \dots \xi_{-n+k+1}.]}(f_{[\xi_{-n+k+2} \dots \xi_{-1}.]}(J) \cap [\gamma_*, 1]) \supset [\gamma_*, 1].$$

Hence, by definition of j we have

$$\big(f_{[0^j.]} \circ f_{[\xi_{-n} \dots \xi_{-n+k+1}.]}\big)(f_{[\xi_{-n+k+2} \dots \xi_{-1}.]}(J) \cap [\gamma_*, 1]) \supset [\Delta/2, 1].$$

By construction, the length of the sequence $(0^j \xi_{-n} \dots \xi_{-1})$ is bounded from above by $M_-(\Delta, \tau) \stackrel{\text{def}}{=} j(\Delta) + n_-(\tau)$. To conclude the lemma it is enough to take $(\eta_{-m} \dots \eta_{-1}) = (0^j \xi_{-n} \dots \xi_{-1})$. □

The next lemma will help establishing a strong form of backward minimality. Recall the definition of γ in (F1) and of the fundamental domains I_0, I_1 in Section 2.1.

Lemma 3.3. *Given sufficiently small $\Delta > 0$, there are constants $\nu = \nu(\Delta) \in (0,1)$ and $K = K(\Delta) \geq 1$ satisfying the following property. Every closed interval $H \subset [\Delta, 1-\Delta]$ with $|H| \leq \nu$ is either contained in a fundamental domain of f_0 in $[1-t, 1-\Delta]$ or there is $n = n(H) \leq K$ such that $f_{[10^n.]}(H)$ is contained in a fundamental domain of f_0 in $[1-t, 1-\Delta]$.*

Proof. Let $\Delta > 0$ be sufficiently small such that $\Delta < f_0^{-2}(\gamma\, t)$. Fix $\overline{\nu} = \overline{\nu}(\Delta) > 0$ so that every interval in $[\Delta, 1-\Delta]$ with length less than $\overline{\nu}$ is contained in some fundamental domain of f_0. Let $k \geq 1$ be the smallest integer with $f_0^{-k}(I_1) \subset (0, \gamma\, t]$. Choose $\nu \in (0, \lambda^{k+1}\, \overline{\nu})$.

Let $H \subset [\Delta, 1-\Delta]$ be an interval with $|H| \leq \nu$. If $H \subset [1-t, 1-\Delta]$ then it is contained in a fundamental domain of f_0 and we are done. Let us assume that H intersects $[\Delta, 1-t)$ and let $n = n(H) \geq 1$ be the smallest number such that $f_0^{-n}(H) \subset [0, \gamma\, t]$. By our choice of Δ we have $f_0^{-n}(H) \subset [f_0^{-2}(\gamma\, t), \gamma\, t] \subset [\Delta, \gamma\, t]$ and thus

$$f_1^{-1} \circ f_0^{-n}(H) \subset [1-t, 1-\gamma^{-1}\,\Delta] \subset [1-t, 1-\Delta].$$

Note that $n \leq k$. Since λ^{-1} is the maximum expansion of the inverse maps of the IFS, we have

$$|f_1^{-1} \circ f_0^{-n}(H)| \leq \lambda^{-k-1}\, \nu < \overline{\nu}.$$

The choice of $\overline{\nu}$ guarantees that $f_1^{-1} \circ f_0^{-n}(H)$ is contained in a fundamental domain of f_0 in $[1-t, 1-\Delta]$, ending the proof of the lemma. □

3.2. **Minimality.** We now prove some results on minimality of the IFS. First we provide the proof of Proposition 1.7.

Proof of Proposition 1.7. To prove backward minimality (density of $\mathcal{O}^-(x)$) we argue by contradiction. Suppose that there is $x \in (0,1)$ such that $\mathcal{O}^-(x)$ is not dense in $[0,1]$. Then $[0,1] \setminus \mathcal{O}^-(x)$ contains a closed nontrivial interval $J \subset (0,1)$. Choose small $\Delta > 0$ such that $J \cup \{x\} \subset (\Delta, 1-\Delta)$ and fix small $\tau \in (0,|J|)$. By Lemma 3.1, there are $\delta > 0$ and a finite sequence $(\eta_0 \ldots \eta_m)$ so that $f_{[\eta_0 \ldots \eta_m]}(J)$ contains $(x-\delta/2, x+\delta/2)$. Thus J intersects $\mathcal{O}^-(x)$, contradicting the choice of J.

The forward minimality (density of $\mathcal{O}^+(x)$) follows from Lemma 3.2. We argue again by contradiction. Assume that there is $x \in (0,1)$ such that $[0,1] \setminus \mathcal{O}^+(x)$ contains a closed nontrivial interval $J \subset (0,1)$. We have that $x \in [\Delta/2, 1]$ for some $\Delta > 0$. Applying Lemma 3.2 to Δ and J we get a sequence $(\eta_{-m} \ldots \eta_{-1})$ such that $x \in [\Delta/2, 1] \subset f_{[\eta_{-m} \ldots \eta_{-1}.]}(J')$ for some subinterval J' of J on which the sequence is admissible. Thus $f_{[\eta_{-m} \ldots \eta_{-1}]}(x) \in J' \subset J$, contradicting the choice of J. □

We now establish some further results on uniform minimality for the IFS.

Lemma 3.4 (Uniform forward minimality)**.** *Given $\tau > 0$, there is $M = M(\tau)$ such that for every closed interval $J \subset [0,1]$ with $|J| \geq \tau$ there exists a finite sequence $(\eta_0 \ldots \eta_m)$, $m \leq M$, such that $f_{[\eta_0 \ldots \eta_m]}([0,1]) \subset J$.*

Proof. Recall that p_1 is the (attracting) fixed point of f_1. Consider a covering $I_1, \ldots, I_\ell$ of $[0,1]$ consisting of intervals of length $\tau/4$. By the forward minimality in Proposition 1.7, there are finite sequences $\eta^{(1)}, \ldots, \eta^{(\ell)}$ such that for every $i = 1, \ldots, \ell$ we have $f_{[\eta^{(i)}]}(p_1) \in I_i$ and the distance from $f_{[\eta^{(i)}]}(p_1)$ to the boundary of I_i is less than $\tau/10$. As f_1 is a uniform contraction, there is some sufficiently large number k (independent of i) such that $f_{[1^k \, \eta^{(i)}]}([0,1]) \subset I_i$. Note that the length of the finite sequences $(1^k \, \eta^{(i)})$, $i = 1, \ldots, \ell$, is bounded by some $M(\tau)$. To finish the proof it suffices to note that any interval J of length τ contains some interval I_i. □

Analogously, the following lemma that is an easy consequence of backward minimality in Proposition 1.7 and compactness. Its proof we omit.

Lemma 3.5 (Uniform backward minimality)**.** *Given $\tau > 0$ and $\Delta > 0$, there are numbers $M = M(\tau, \Delta)$ and $\varepsilon = \varepsilon(\tau, \Delta)$ such that for every pair of intervals $J, H \subset [\Delta, 1-\Delta]$ with $|J| \geq \tau$ and $|H| < \varepsilon$ there is a finite sequence $(\eta_{-m} \ldots \eta_{-1})$, $m \leq M$, that is admissible for all $x \in H$ and satisfies $f_{[\eta_{-m} \ldots \eta_{-1}.]}(H) \subset J$.*

We will also need the following slightly strengthened version of Lemma 3.5, controlling also the derivative.

Lemma 3.6 (Strong uniform backward minimality)**.** *Given $\tau > 0$ and $\Delta > 0$, there are numbers $M = M(\Delta, \tau) > 0$, $K = K(\tau, \Delta) \in (0,1)$, and $\nu =$*

$\nu(\Delta) \in (0,1)$ *such that for every pair of closed intervals*

$$J, H \subset [\Delta, 1-\Delta] \quad \textit{with} \quad |J| \geq \tau,\, |H| \leq \nu$$

there is a finite sequence $(\eta_{-m} \dots \eta_{-1})$*,* $m \leq M$*, that is admissible for all* $x \in H$ *and satisfies*

$$f_{[\eta_{-m}\dots\eta_{-1}.]}(H) \subset J \quad \textit{and} \quad |(f_{[\eta_{-m}\dots\eta_{-1}.]})'(x)| \leq K \textit{ for all } x \in H.$$

Proof. The idea to obtain uniform contraction is to follow for a long time a backward contracting itinerary (provided by Lemma 2.8) applied to H and to use uniform backward minimality from Lemma 3.5 to put H inside J. The first one involves an arbitrarily small derivative, while the second one gives some uniformly bounded correction term. We now provide the details.

Given τ, Δ, let $M = M(\tau, \Delta)$ and $\varepsilon = \varepsilon(\tau, \Delta)$ be the numbers provided by Lemma 3.5. Furthermore, let $\nu = \nu(\Delta)$ and $K = K(\Delta)$ be the numbers provided by Lemma 3.3.

Let $J, H \subset [\Delta, 1-\Delta]$ with $|J| \geq \tau$ and $|H| \leq \nu$. As in the proof of Lemma 3.3 define

$$H' = \begin{cases} H & \text{if } H \subset [1-t, 1-\Delta], \\ f_1^{-1} \circ f_0^{-n(H)}(H) & \text{otherwise,} \end{cases}$$

where $n(H)$ is the smallest number with $f_0^{-n(H)}(H) \subset [0, \gamma\, t]$. We let $n = n(H)$ and $\eta^{(1)}$ be the empty sequence in the first case and $\eta^{(1)} = (1\, 0^n)$ in the second case. Hence, by this choice, as in Lemma 3.3 the interval H' is contained in a fundamental domain of f_0 in $[1-t, 1-\Delta]$ and $n \leq K$.

Let $k = k(H) \geq 0$ be the smallest number such that $H'' = f_0^{-k}(H') \subset I_1 \cup f_0(I_1)$. Note that k is uniformly bounded from above by some number $L = L(\Delta)$. Recall the definition of $\varkappa$ in (2.1). Let $\ell \geq 1$ be the smallest number with

$$\varkappa^{-\ell}\, \lambda^{-(k+n+1+M)} \leq \varepsilon. \tag{3.1}$$

In particular, we have

$$\ell < 1 + \frac{(L+K+1+M)|\log\lambda| + |\log\varepsilon|}{\log\varkappa}.$$

Applying consecutively ℓ times Lemma 2.8 to the interval H'' and recalling (2.6), we obtain a finite sequence $\eta^{(2)}$ of maximal length

$$\ell\,(N_-(H'') + k_0 + 1) \leq \ell\,(N + k_0 + 2)$$

such that

$$H''' = f_{[\eta^{(2)}\, 0^k\, \eta^{(1)}.]}(H) \subset I_1 \cup f_0(I_1),$$

and

$$\left|(f_{[\eta^{(2)}\, 0^k\, \eta^{(1)}.]})'(x)\right| < \varkappa^{-\ell}\, \lambda^{-(k+n+1)} \quad \text{for all } x \in H.$$

This together with (3.1) and $\nu < 1$ imply

$$|H'''| < \varkappa^{-\ell}\, \lambda^{-(k+n+1)}\, \nu < \varepsilon.$$

We now apply Lemma 3.5 to the intervals J and H''' and get a finite sequence $\eta^{(3)}$ of maximal length M that is admissible for every $x \in H'''$ and such that

$$f_{[\eta^{(3)}.]}(H''') \subset J.$$

Hence, by (3.1) we obtain that for all $x \in H$ we have

$$|(f_{[\eta^{(3)}\,\eta^{(2)}\,0^k\,\eta^{(1)}.]})'(x)| \le \varkappa^{-\ell}\,\lambda^{-(k+n+1+M)} \le \varepsilon.$$

Now it is enough to take

$$(\eta_{-m}\dots\eta_{-1}) \stackrel{\text{def}}{=} (\eta^{(3)}\,\eta^{(2)}\,0^k\,\eta^{(1)})$$

and note that by the above choices m is uniformly bounded from above by some number that depends on τ and Δ only. This proves the lemma. $\square$

4. Approximating exponents with periodic orbits

We will show that for every point with an upper positive/negative Lyapunov exponent there exists a periodic point close-by with roughly the same exponent.

4.1. **Distortion estimates.**

We first provide some simple distortion results. Since we will apply them to both forward and backward iterates of the IFS, let us consider a general setting.

Given a set Z and a differentiable map g on Z, we denote by

$$\operatorname{Dist} g|_Z \stackrel{\text{def}}{=} \sup_{x,y\in Z} \frac{|g'(x)|}{|g'(y)|}$$

the *maximal distortion* of g on Z. Consider the IFS generated by two C^1 maps $g_0, g_1\colon J \to [0,1]$ for some closed sub-interval $J \subset [0,1]$. For $\vartheta > 0$ define

$$D(\vartheta) \stackrel{\text{def}}{=} \max_{x\in[0,1]}\,\max_i\,\big(\operatorname{Dist} g_i|_{[x-\vartheta/2,x+\vartheta/2]\cap J}\big). \tag{4.1}$$

Clearly, $D(\vartheta) \to 1$ as $\vartheta \to 0$.

Lemma 4.1 (Distortion at hyperbolic times)**.** *Let $(p,\xi^+) \in [0,1]\times\Sigma_2^+$, $\chi > 0$, $\varepsilon > 0$, $K \ge 1$, and $n \ge 1$ satisfy $\chi - 2\,\varepsilon > 0$,*

$$\begin{aligned} \Big|(g_{[\xi_0\dots\,\xi_n]})'(p)\Big| &\ge \frac{1}{K}\,e^{(n+1)(\chi-\varepsilon)}, \qquad \textit{and}\\ \Big|(g_{[\xi_{n-m+1}\dots\,\xi_n]})'(g_{[\xi_0\dots\,\xi_{n-m}]}(p))\Big| &\ge \frac{1}{K}\,e^{m(\chi-\varepsilon)} \qquad \textit{for all } m=1,\dots,n. \end{aligned} \tag{4.2}$$

Let $\vartheta > 0$ be small enough such that $D(\vartheta) < e^{\varepsilon}$ and $J_{n+1} \subset [0,1]$ be an interval of length less than $\vartheta\,K^{-1}$ containing $g_{[\xi_0\dots\,\xi_n]}(p)$ such that $(g_{[\xi_k\dots\,\xi_n]})^{-1}|_{J_{n+1}}$ is well-defined for every $k = 0, \dots, n$. Then for every $k = 0, \dots, n$ we have

$$|J_k| \le \vartheta\,e^{-(n+1-k)(\chi-2\varepsilon)}, \qquad \textit{where} \qquad J_k \stackrel{\text{def}}{=} (g_{[\xi_k\dots\,\xi_n]})^{-1}(J_{n+1}) \tag{4.3}$$

and

$$\operatorname{Dist} g_{\xi_k}|_{J_k} \le D(\vartheta) < e^{\varepsilon}. \tag{4.4}$$

Note that (4.3) is related to a hyperbolic time defined in (4.8).

Proof. Let J_{n+1} be an interval satisfying the hypothesis of the lemma and consider the sequence of intervals J_k. Arguing inductively, assume that there is some $k \in \{1, \ldots, n+1\}$ such that for every $\ell = k, \ldots, n+1$ we have

$$|J_\ell| \le \vartheta\, e^{-(n+1-\ell)(\chi-2\varepsilon)} \le \vartheta$$

and hence distortion is bounded in between. Using (4.2) and (4.3) we have

$$\begin{aligned} |J_{k-1}| &\le D(\vartheta)^{n-k+2}\, |(g_{[\xi_{k-1}\ldots\,\xi_n]})'(g_{[\xi_0\ldots\,\xi_{k-2}]}(p))|^{-1}|J_{n+1}| \\ &\le D(\vartheta)^{n-k+2}\, K e^{-(n-k+2)(\chi-\varepsilon)}|J_{n+1}| \le \vartheta\, e^{-(n+1-(k-1))(\chi-2\varepsilon)}. \end{aligned}$$

Note that the assumption holds for $k = n+1$. Hence, by induction, we get (4.3) for every $k = 1, \ldots, n+1$ and thus (4.4) also holds. This proves the lemma. □

Lemma 4.2 (Distortion along contracting orbit)**.** *Let* $(p, \xi^+) \in [0,1] \times \Sigma_2^+$, $\chi < 0$, $\varepsilon > 0$, *and* $K \ge 1$ *satisfy* $\chi + 2\,\varepsilon < 0$ *and*

$$|(g_{[\xi_0\ldots\,\xi_{n-1}]})'(p)| \le K\, e^{-n(|\chi|-\varepsilon)} \quad \text{for all } n \ge 1. \tag{4.5}$$

Let $\vartheta > 0$ *be small enough such that* $D(\vartheta) < e^\varepsilon$. *Let* $J \subset [0,1]$ *be an interval of length less than* $\vartheta\, K^{-1}$ *containing* p *and* $n \ge 1$ *be such that* $g_{[\xi_0\ldots\xi_{n-1}]}$ *is well-defined. Then*

$$|J_n| \le |J|\, K\, e^{-n(|\chi|-2\varepsilon)} < \vartheta, \quad \text{where} \quad J_n \overset{\text{def}}{=} g_{[\xi_0\ldots\,\xi_{n-1}]}(J) \tag{4.6}$$

and

$$\operatorname{Dist} g_{[\xi_0\ldots\,\xi_{n-1}]}|_J \le D(\vartheta)^n < e^{n\varepsilon}. \tag{4.7}$$

Proof. By hypothesis $|J| \le \vartheta\, K^{-1} \le \vartheta$. Given $n \ge 1$, assume that we have

$$|J_k| \le \vartheta\, e^{-k(|\chi|-2\varepsilon)} \le \vartheta \quad \text{and thus} \quad \operatorname{Dist} g_{\xi_k}|_{J_k} \le D(\vartheta) < e^\varepsilon$$

for every $k = 0, \ldots, n-1$. As $J_n = g_{\xi_{n-1}}(J_{n-1})$ we have

$$|J_n| \le |J|\, K e^{-n(|\chi|-\varepsilon)}\, D(\vartheta)^n \le |J|\, K\, e^{-n(|\chi|-2\varepsilon)} < \vartheta.$$

Hence, by induction, we yield (4.6) and thus (4.7) for every $n \ge 1$. □

4.2. **Positive spectrum.** We will use the previous distortion lemmas to establish results about shadowing by periodic orbits with positive exponent.

Given numbers $\chi > \varepsilon > 0$, a point $p \in [0,1]$, and an one-sided sequence $\xi^+ = (\xi_0\xi_1\ldots) \in \Sigma_2^+$, we call a number $n \ge 1$ a *Pliss hyperbolic time for* (p, ξ) *with exponent* $\chi - \varepsilon > 0$ if

$$\begin{aligned} |(f_{[\xi_0\ldots\,\xi_n]})'(p)| &\ge e^{(n+1)(\chi-\varepsilon)} \quad \text{and} \\ \big|(f_{[\xi_{n-m+1}\ldots\,\xi_n]})'(f_{[\xi_0\ldots\,\xi_{n-m}]}(p))\big| &\ge e^{m(\chi-\varepsilon)} \quad \text{for all } m = 1, \ldots, n. \end{aligned} \tag{4.8}$$

Remark 4.3 (Abundance of hyperbolic times)**.** It is easy to check that if $\overline{\chi}_+(p,\xi^+) > \chi - \varepsilon > 0$ then (p,ξ^+) has infinitely many hyperbolic times with exponent $\chi - \varepsilon$.

Lemma 4.4. *Given $\chi > 0$ and $\varepsilon \in (0,\chi)$, there is $\Delta > 0$ such that for every $p \in (0,1)$ and every sequence $\xi^+ \in \Sigma_2^+$ so that $\overline{\chi}_+(p,\xi^+) \in (\chi-\varepsilon,\chi+\varepsilon)$ we have the following:*

(i) *there are infinitely many Pliss hyperbolic times n for (p,ξ^+) with exponent $\chi-\varepsilon$ so that $f_{[\xi_0\ldots\xi_n]}(p) \in (\Delta, 1-\Delta)$,*

(ii) *there exist a point $q \in (0,1)$ arbitrarily close to p and a periodic sequence $\eta = (\eta_0\ldots\eta_{k-1})^{\mathbb{Z}} \in \Sigma_2$ such that $f_{[\eta_0\ldots\eta_{k-1}]}(q) = q$ and $\chi(q,\eta) \in (\chi-\varepsilon,\chi+\varepsilon)$.*

Proof. Choose small $\Delta, \Delta_0 \in (0,1)$ such that $\Delta < \Delta_0$ and

- $f_0'(x) > 1$ if $x \in [0,\Delta_0]$ and $f_0'(x) < 1$ if $x \in [1-\Delta_0, 1]$,
- $f_0(1-\Delta_0) < 1-\Delta$, and
- $f_1^2([0,1]) \subset (\Delta_0, 1-\Delta_0)$.

Let $\chi > 0$, $\varepsilon > 0$, $p \in (0,1)$, and $\xi^+ \in \Sigma_2^+$ such that $\overline{\chi}_+(p,\xi^+) > \chi-\varepsilon > 0$.

To prove (i), note that $\overline{\chi}_+(p,\xi^+) > 0$ and $p \in (0,1)$ together imply that ξ^+ contains infinitely many $1's$. Indeed, otherwise we would have $\overline{\chi}_+(p,\xi^+) = \log\lambda < 0$. Thus, in what follows we can consider hyperbolic times n such that the sequence $(\xi_0\ldots\xi_n)$ contains at least two symbols $1's$.

Considering one such time n, by the definition of hyperbolic time we have $|f_{\xi_n}'(f_{[\xi_0\ldots\xi_{n-1}]}(p))| > 1$ and thus $\xi_n = 0$. The choice of Δ_0 implies $f_{[\xi_0\ldots\xi_{n-1}]}(p) \notin [1-\Delta_0, 1]$. Thus $f_{[\xi_0\ldots\xi_n]}(p) = f_{[\xi_0\ldots\xi_{n-1}\,0]}(p) \notin [1-\Delta,1]$. Thus, it remains to check that either $f_{[\xi_0\ldots\xi_n]}(p)$ is not close to 0 or that the hyperbolic time n can be replaced by some possibly larger hyperbolic time n' with $f_{[\xi_0\ldots\xi_{n'}]}(p) \notin [0,\Delta]$. By our choice of n there is $\ell < n$ such that $\xi_\ell = 1$ and $\xi_i = 0$ for all $i \in \{\ell+1,\ldots,n\}$. We have to consider two cases:

(a) If $\xi_{\ell-1} = 1$ then $f_{[\xi_0\ldots\xi_{\ell-1}\xi_\ell]}(p) = f_{[\xi_0\ldots\xi_{\ell-2}\,1^2]}(p) \in f_1^2([0,1])$ and hence

$$f_{[\xi_0\ldots\xi_\ell\ldots\xi_n]}(p) \ge f_0^{n-\ell}(f_1^2(0)) \ge f_1^2(0) > \Delta_0 > \Delta$$

and we are done.

(b) If $\xi_{\ell-1} = 0$ then, since $(\xi_0\ldots\xi_n)$ contains at least two $1's$, there is $m \ge 1$ such that $\xi_i = 0$ for all $i \in \{m,\ldots,\ell-1\}$ and $\xi_{m-1} = 1$. Let

$$\overline{p}_{i+1} \stackrel{\text{def}}{=} f_{[\xi_0\ldots\xi_i]}(p) \quad \text{and} \quad \overline{q}_{i+1} \stackrel{\text{def}}{=} 1-\overline{p}_{i+1}.$$

If $\overline{p}_{n+1} \notin (0,\Delta]$ then $\overline{p}_{n+1} = f_{[\xi_0\ldots\xi_n]}(p) \in [\Delta, 1-\Delta]$ and we are done. Otherwise, if $\overline{p}_{n+1} \in (0,\Delta]$ we will get a contradiction. Indeed, assume that $\overline{p}_{n+1} \in (0,\Delta]$. Note that $\xi_{m-1} = 1$ implies $\overline{p}_m \in f_1([0,1]) = [0,\gamma]$. Since f_0 is "almost linear" close to 0 and close to 1, there are positive constants K_1 and K_2 independent of small Δ such that

$$(f_{[\xi_{\ell+1}\ldots\xi_n]})'(\overline{p}_{\ell+1}) = (f_0^{n-\ell})'(\overline{p}_{\ell+1}) \le K_1\,\frac{\overline{p}_{n+1}}{\overline{p}_{\ell+1}},$$

$$(f_{[\xi_m\ldots\xi_{\ell-1}]})'(\overline{p}_m) = (f_0^{\ell-m})'(\overline{p}_m) \le K_2\,\frac{1-\overline{p}_\ell}{1-\overline{p}_m} = K_2\,\frac{\overline{q}_\ell}{\overline{q}_m}.$$

Note also that $\overline{q}_\ell = \overline{p}_{\ell+1}/\gamma$ (by definition of f_1), $\overline{p}_m \le \gamma$ and thus $\overline{q}_m \ge (1-\gamma)$ (by choice of m), and $\overline{p}_{n+1} \le \Delta$. Putting together these inequalities we get

$$\begin{aligned}|(f_{[\xi_m \dots \xi_n]})'(\overline{p}_m)| &= |(f_0^{n-\ell} \circ f_1 \circ f_0^{\ell-m})'(\overline{p}_m)| \\ &\le \left(K_1 \frac{\Delta}{\overline{p}_{\ell+1}}\right) \gamma \left(K_2 \frac{\overline{p}_{\ell+1}}{\gamma(1-\gamma)}\right) \le K_1\, K_2 \frac{\Delta}{1-\gamma}.\end{aligned}$$

Noting that $n-m\to\infty$ as $\Delta\to 0$, the previous inequality implies that the sequence $(\xi_\ell \dots \xi_n)$ is not $(\chi-\varepsilon)$-expanding for $\overline{p}_m = f_{[\xi_0 \dots \xi_{m-1}]}(p)$ contradicting that n is a hyperbolic time with exponent $\chi-\varepsilon$. This contradiction concludes the proof of (i).

Let us now show (ii). By Remark 4.3, for sufficiently small $\varepsilon>0$ there exist infinitely many hyperbolic times n with exponent $\chi-\varepsilon$ and by (i) we can assume that $\overline{p}_{n+1} = f_{[\xi_0 \dots \xi_n]}(p) \in (\Delta, 1-\Delta)$. If n is a hyperbolic time, then (p,ξ^+), χ, ε, $K=1$, and n satisfy the hypothesis of Lemma 4.1. Moreover, as $\overline{\chi}_+(p,\xi^+) \in (\chi-\varepsilon, \chi+\varepsilon)$ if the hyperbolic time n is sufficiently large then

$$e^{(n+1)(\chi+\varepsilon)} \ge \left|(f_{[\xi_0 \dots \xi_n]})'(p)\right| \ge e^{(n+1)(\chi-\varepsilon)}. \tag{4.9}$$

Let $\vartheta \in (0, \Delta/2)$ be as in Lemma 4.1. Considering the interval of length ϑ,

$$J_{n+1} = \left[\overline{p}_{n+1} - \vartheta/2, \overline{p}_{n+1} + \vartheta/2\right] \subset (\Delta/2, 1-\Delta/2),$$

by this lemma we have that

$$|J_0| \le e^{-(n+1)(\chi-2\varepsilon)}\, \vartheta, \quad \text{where} \quad J_0 = (f_{[\xi_0 \dots \xi_n]})^{-1}(J_{n+1}). \tag{4.10}$$

We can assume that n is large enough so that $J_0 \subset [p-\delta/2, p+\delta/2]$, where $\delta = \delta(\Delta/2)$ is as in Lemma 3.1. Hence, applying this lemma to the interval $J_{n+1} \subset (\Delta/2, 1-\Delta/2)$ of length $|J_{n+1}| = \vartheta$ we get a finite sequence $\eta = (\eta_0 \dots \eta_m)$ so that

$$f_{[\eta_0 \dots \eta_m]}(J_{n+1}) \supset [p-\delta/2, p+\delta/2] \supset J_0$$

with m bounded by some number $M = M_+(\Delta/2, \vartheta)$ independent of n. Hence, by (4.10)

$$f_{[\xi_0 \dots \xi_n \eta_0 \dots \eta_m]}(J_0) \supset J_0$$

and thus there exists a point $q\in J_0$ that is fixed under the map $f_{[\xi_0 \dots \xi_n \eta_0 \dots \eta_m]}$. Using (4.9), the distortion bound (4.4), and $|f_i'| \ge \lambda$ in (F0), (F1), we obtain

$$\frac{\log\left|\left(f_{[\xi_0 \dots \xi_n \eta_0 \dots \eta_m]}\right)'(q)\right|}{n+m+2} \ge \frac{(n+1)\left(\chi - \varepsilon - \log D(\vartheta)\right)}{n+m+2} + \frac{(m+1)\log\lambda}{n+m+2}.$$

We get an analogous upper bound provided by (4.9) and $|f_i'| \le \beta$ in (F0). Since we can chose n sufficiently large and since m is independent of n, we verify that this exponent is sufficiently close to χ and that $|J_0|$ is small. Hence the periodic point q is close to p, ending the proof of Lemma 4.4. $\square$

We close this subsection by recalling the following result.

Proposition 4.5 ([5, Proposition 3.10]). *For every $\varepsilon > 0$ there exists a finite sequence $(\xi_0 \dots \xi_{n-1})$ such that the map $f_{[\xi_0 \dots \xi_{n-1}]}$ has an expanding fixed point whose Lyapunov exponent is in $(0, \varepsilon)$.*

4.3. **Negative spectrum.** Analogously to Lemma 4.4 we establish the following result about shadowing by periodic orbits with negative exponents. Note that here we require additionally that the forward exponent, that is to be approximated, is well-defined since we base our constructions on uniform forward contractions as in the estimates in (4.5).

Lemma 4.6. *Given $\chi < 0$ and $\varepsilon > 0$ small enough, for every $p \in (0,1)$ and every sequence $\xi^+ \in \Sigma_2^+$ so that $\chi_+(p, \xi^+) < \chi + \varepsilon < 0$, there exist a point $q \in (0,1)$ arbitrarily close to p and a periodic sequence $\eta = (\eta_0 \dots \eta_{k-1})^{\mathbb{Z}} \in \Sigma_2$ such that $f_{[\eta_0 \dots \eta_{k-1}]}(q) = q$ and $\chi(q, \eta) \in (\chi - \varepsilon, \chi + \varepsilon)$.*

Proof. As $\chi_+(p, \xi^+)$ is well-defined, for given $\varepsilon \in (0, |\chi|/2)$ there exists some constant $K > 1$ so that for every $n \geq 1$ we have

$$\frac{1}{K}\, e^{-n(|\chi|+\varepsilon)} \leq |(f_{[\xi_0 \dots\, \xi_{n-1}]})'(p)| \leq K\, e^{-n(|\chi|-\varepsilon)}. \tag{4.11}$$

Choose $\vartheta > 0$ so small that $D(\vartheta) < e^{\varepsilon}$, where $D(\vartheta)$ in defined in (4.1). Let $J \subset (0,1)$ be an open interval containing p with $|J| \leq \vartheta\, K^{-1}$. Recall that we denote by p_1 the attracting fixed point of f_1. By Proposition 1.7 there is a finite sequence $(\eta_0 \dots \eta_m)$ such that $f_{[\eta_0 \dots \eta_m]}(p_1)$ is in the interior of J. Hence for every ℓ big enough we have

$$f_{[1^\ell\, \eta_0 \dots \eta_m]}([0,1]) \subset J. \tag{4.12}$$

In particular, for every $n \geq 1$ we have

$$f_{[\xi_0 \dots\, \xi_{n-1}\, 1^\ell\, \eta_0 \dots \eta_m]}(J) \subset J$$

and thus there exists a point $q = q(n) \in J$ fixed by the map $f_{[\xi_0 \dots\, \xi_{n-1}\, 1^\ell\, \eta_0 \dots \eta_m]}$.

In view of (4.11), we can apply Lemma 4.2 to the interval J containing p and q and yield

$$\big|(f_{[\xi_0 \dots\, \xi_{n-1}]})'(q)\big| \leq K\, e^{-n\,(|\chi|-2\varepsilon)}.$$

Using $|f_i'| \leq \beta$, a crude estimate for the derivative of the remaining orbit is

$$\frac{1}{n+\ell+m+1} \log \big|(f_{[\xi_0 \dots\, \xi_{n-1}\, 1^\ell\, \eta_0 \dots \eta_m]})'(q)\big| \\ \leq \frac{\log K - n\,(|\chi| - 2\,\varepsilon)}{n+\ell+m+1} + \frac{(\ell+m+1)\, \log \beta}{n+\ell+m+1}.$$

We also get an analogous lower bound using (4.11) and $|f_i'| \geq \lambda$. Note that m and ℓ are fixed. Thus, by choosing n sufficiently large, this exponent is sufficiently close to χ. Finally, taking ϑ small, the point q can be taken arbitrarily close to p. This proves the lemma. $\square$

We close this subsection by recalling the following result.

Lemma 4.7 ([5, Proposition 3.9])**.** *For every $\varepsilon > 0$ there exists a finite sequence $(\xi_0 \dots \xi_{n-1})$ such that $f_{[\xi_0 \dots \xi_{n-1}]}$ is uniformly contracting in $[0,1]$ and has a fixed point in $(0,1)$ whose Lyapunov exponent is in $(-\varepsilon, 0)$.*

4.4. **Summary.** Observe that our methods derived in the previous sections immediately provide the following result.

Proposition 4.8. *Given $\chi \in (\log \lambda, \log \widetilde{\beta}\,)$ and $\varepsilon, \delta > 0$, there is a point $q \in (0,1)$ and a periodic sequence $\eta = (\eta_0 \dots \eta_{k-1})^{\mathbb{Z}} \in \Sigma_2$ such that $f_{[\eta_0 \dots \eta_{k-1}]}(q) = q$, $\{f_{[\eta_0 \dots \eta_\ell]}(q) \colon \ell = 0, \dots, k-1\}$ is δ-dense in $[0,1]$, and $\chi(q,\eta) \in (\chi - \varepsilon, \chi + \varepsilon)$.*

The rough idea of the proof of this proposition is the following. Note that our method to construct periodic orbits with some approximate exponent is to jump from a periodic orbit to another one using minimality. Observe also that given χ we can construct a δ-dense set of periodic orbits with exponent close to χ. Now, using minimality, we can jump from a periodic orbit to the next one to get a new periodic orbit with exponent close to χ. By construction, this periodic orbit is δ-dense.

5. Approximating Lyapunov exponents

In this section we will prove Theorem 1.3 showing that the spectrum of Lyapunov exponents is almost complete: it contains 0 and has no further gap besides the one in Proposition 1.2 (recall the definition of $\widetilde{\beta}$ given there).

5.1. **Positive forward spectrum.** Property (i) in Theorem 1.3 is a consequence of the minimality of the IFS and the next two propositions.

Proposition 5.1. *Consider sequences of points $(x_i)_i$, $x_i \in (0,1)$, and of sequences $(\xi^{(i)})_i$, $\xi^{(i)} \in \Sigma_2^+$, with positive upper exponents $\{\chi_i \stackrel{\text{def}}{=} \overline{\chi}_+(x_i, \xi^{(i)}) > 0\}$. Then for any accumulation point χ of $(\chi_i)_i$ there is $(y, \xi^+) \in (0,1) \times \Sigma_2^+$ with $\chi_+(y, \xi^+) = \chi$.*

Proposition 5.2. *For any $\chi \in (0, \log \widetilde{\beta}\,]$ the set of points z for which there exists a sequence $\xi^+ \in \Sigma_2^+$ such that $\overline{\chi}_+(z, \xi^+) = \chi$ is dense in $[0,1]$.*

We will first prove the above two propositions and then Theorem 1.3 (i).

Proof of Proposition 5.1. We can freely assume, possibly after passing to a subsequence, that $\lim_{i \to \infty} \chi_i = \chi$. By Lemma 4.4 (ii), we can also assume that all the pairs $(x_i, \xi^{(i)})$ are periodic, that is, $\xi^{(i)} = (\xi_0^{(i)} \dots \xi_{\ell(i)-1}^{(i)})^{\mathbb{Z}}$ and $f_{[\xi_0^{(i)} \dots \xi_{\ell(i)-1}^{(i)}]}(x_i) = x_i$ for some $\ell(i) \geq 1$. Indeed, otherwise we can replace each $(x_i, \xi^{(i)})$ by some periodic pair with Lyapunov exponent χ_i' sufficiently close to χ_i so that $\lim_{i \to \infty} \chi_i' = \chi$. Recall that Lyapunov exponents are constant along orbits. Hence, possibly after replacing x_i by some iterate, by Lemma 4.4 (i) we can assume that there exists $\Delta > 0$ so that $x_i \in (\Delta, 1 - \Delta)$ for every i.

Step 0: Choice of auxiliary sequences. Since the Lyapunov exponents χ_i converge to χ, there exists a sequence ε_i monotonically decreasing to 0 such that $|\chi_i-\chi|\le\varepsilon_i$ for every i. Recall also that each $(x_i,\xi^{(i)})$ is periodic and hence uniformly hyperbolic (backward and forward in time). Thus, given χ_i and ε_i, there exists $K_i\ge 1$ such that for every $k\ge 1$ we have

$$\frac{1}{K_i}\,e^{k\,(\chi_i-\varepsilon_i)}\le\left|(f_{[\xi^{(i)}_0\ldots\,\xi^{(i)}_{k-1}]})'(x_i)\right|\le K_i\,e^{k\,(\chi_i+\varepsilon_i)}.$$

In particular, for every multiple $m=L\,\ell_i$ of the period of $(x_i,\xi^{(i)})$ and every $k=1,\ldots,m$ we have

$$\frac{1}{K_i}\,e^{k\,(\chi_i-\varepsilon_i)}\le\left|(f_{[\xi^{(i)}_{m-k+1}\ldots\,\xi^{(i)}_{m}]})'(f_{[\xi^{(i)}_0\ldots\,\xi^{(i)}_{m-k}]}(x_i))\right|\le K_i\,e^{k\,(\chi_i+\varepsilon_i)}. \tag{5.1}$$

Further choose a sequence $\vartheta_i\to 0$ such that $D(\vartheta_i)<e^{\varepsilon_i}$ for every i. Finally, given Δ and $\vartheta_i\,K_i^{-1}$, let $M_i=M_+(\Delta,\vartheta_i\,K_i^{-1})$ and $\delta_i=\delta_+(\Delta)$ be the numbers provided by Lemma 3.1.

We will now choose sequences $(N_i)_{i\ge1}$ of integers and $(I_i)_{i\ge1}$ of intervals recursively. Each point in I_i will circle close to the periodic orbit $(x_i,\xi^{(i)})$ and then jump to the next interval I_{i+1}. The circling will be large compared to the jump. The final orbit will pass these intervals consecutively. See Figure 4.

Let I_1 be some small interval whose interior contains x_1. We now define I_i, $i\ge 2$, recursively. For $N_i=L_i\,\ell_i$ some sufficiently large multiple of the period of $(x_i,\xi^{(i)})$ we choose an interval $J(x_i,N_i)\subset I_i$ of length $\vartheta_i\,K_i^{-1}$ whose interior contains x_i such that

$$J^{(i)}_k\overset{\text{def}}{=}(f_{[\xi^{(i)}_k\ldots\,\xi^{(i)}_{N_i-1}]})^{-1}(J(x_i,N_i))\subset(0,1). \tag{5.2}$$

By (5.1) we can apply Lemma 4.1. In particular, (4.4) and the choice of ϑ_i above imply

$$\operatorname{Dist} f_{\xi^{(i)}_k}|_{J^{(i)}_k}\le D(\vartheta_i)\le e^{\varepsilon_i},\quad\text{for all } k=0,\ldots,N_i-1. \tag{5.3}$$

Moreover, by (4.3) we have

$$|J^{(i)}_0|\le\vartheta_i\,e^{-N_i(\chi_i-2\varepsilon_i)}. \tag{5.4}$$

We can demand that N_i was chosen large enough so that

$$J^{(i)}_0=(f_{[\xi^{(i)}_0\ldots\,\xi^{(i)}_{N_i-1}]})^{-1}(J(x_i,N_i))\subset I_i. \tag{5.5}$$

Note that this remains true if we increase N_i. Applying Lemma 3.1 to the interval $J(x_i,N_i)$ of length $\vartheta_i\,K_i^{-1}$ and the point x_{i+1}, we get a point $y_i\in J(x_i,N_i)$ and a sequence $(\eta^{(i)}_0\ldots\eta^{(i)}_{m_i-1})$ of length $m_i\le M_i$ such that

$$f_{[\eta^{(i)}_0\ldots\,\eta^{(i)}_{m_i-1}]}(y_i)=x_{i+1}.$$

Let

$$I_{i+1}\overset{\text{def}}{=}f_{[\eta^{(i)}_0\ldots\,\eta^{(i)}_{m_i-1}]}(J(x_i,N_i)).$$

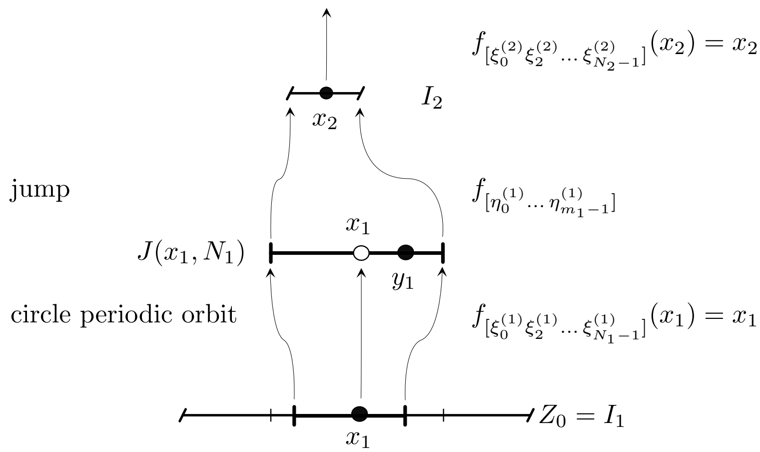

FIGURE 4. First step in the construction of (y,ξ^+) in the proof of Proposition 5.1.

Observe that if we choose L_i and hence N_i large (by circulating the corresponding periodic orbit several times) by (5.4) the interval $J_0^{(i)}$ can be taken arbitrarily small. By construction we have

$$(f_{[\xi_0^{(i)}\dots\xi_{N_i-1}^{(i)}\,\eta_0^{(i)}\dots\eta_{m_i-1}^{(i)}]})^{-1}(I_{i+1})\subset I_i.$$

This recursively defines a sequence of 9-tuples $(x_i,\xi^{(i)},\chi_i,\varepsilon_i,K_i,\vartheta_i,I_i,m_i,\eta^{(i)})$. Notice again that all the above stated properties remain true if we replace N_i by some larger multiple of the period of $(x_i,\xi^{(i)})$. We will adjust our choice of N_i in Step 2.

Step 1: Construction of the pair (y,ξ^+). Let

$$\begin{aligned}
Z_0 &\overset{\text{def}}{=} I_1,\\
Z_1 &= \left(f_{[\xi_0^{(1)}\dots\xi_{N_1-1}^{(1)}\,\eta_0^{(1)}\dots\eta_{m_1-1}^{(1)}]}\right)^{-1}(I_2),\\
Z_2 &\overset{\text{def}}{=} \left(f_{[\xi_0^{(1)}\dots\xi_{N_1-1}^{(1)}\,\eta_0^{(1)}\dots\eta_{m_1-1}^{(1)}]}\right)^{-1}\circ\left(f_{[\xi_0^{(2)}\dots\xi_{N_2-1}^{(2)}\,\eta_0^{(2)}\dots\eta_{m_2-1}^{(2)}]}\right)^{-1}(I_3)
\end{aligned}$$

and so on. By construction, the sequence $(Z_i)_{i\ge0}$ is a family of nested decreasing compact non-empty intervals. Thus, the intersection $\bigcap_{i\ge1}Z_i$ contains some point $y\in(0,1)$. Finally, we define the one-sided sequence ξ^+ by concatenating the segments $\xi_0^{(i)}\dots\xi_{N_i-1}^{(i)}\eta_0^{(i)}\dots\eta_{m_i-1}^{(i)}$ as follows

$$\xi^+\overset{\text{def}}{=}\xi_0^{(1)}\dots\xi_{N_1-1}^{(1)}\,\eta_0^{(1)}\dots\eta_{m_1-1}^{(1)}\,\xi_0^{(2)}\dots\xi_{N_2-1}^{(2)}\,\eta_0^{(2)}\dots\eta_{m_2-1}^{(2)}\dots.$$

To complete Step 1, define the auxiliary sequence $(n_i)_{i\ge0}$ by

$$n_0\overset{\text{def}}{=}0,\quad n_i\overset{\text{def}}{=}n_{i-1}+N_i+m_i.$$

Step 2: Lyapunov exponent of (y,ξ^+). By construction, the orbit of (y,ξ^+) "shadows" the orbit of $(x_{i+1},\xi^{(i+1)})$ for the time $n_i,\dots,n_i+N_{i+1}-1$

(with small distortion) and thereafter passes some "finite transition" for the time $n_i+N_{i+1},\ldots,n_i+N_{i+1}+m_{i+1}-1$ to arrive at a neighborhood of x_{i+2}. To estimate the "finite-time Lyapunov exponent", we distinguish two cases:

Case 1: Estimating $\log|(f_{[\xi_0\ldots\xi_{n_i+n}]})'(y)|$ **with** $n=0,\ldots,N_{i+1}-1$**.** By uniform expansion in (5.1), the distortion control in (5.3), and $\lambda\le|f_0'|,|f_1'|\le\beta$ for the derivative at the transitions we have

$$\prod_{k=1}^{i}\Big(e^{-N_k\varepsilon_k}\,K_k^{-1}e^{N_k(\chi_k-\varepsilon_k)}\lambda^{m_k}\Big)\,e^{-n\varepsilon_{i+1}}\,K_{i+1}^{-1}\,e^{n(\chi_{i+1}-\varepsilon_{i+1})}\le$$
$$\le\big|(f_{[\xi_0\ldots\xi_{n_i+n}]})'(y)\big|$$
$$\le\prod_{k=1}^{i}\Big(e^{N_k\varepsilon_k}\,K_k\,e^{N_k(\chi_k+\varepsilon_k)}\beta^{m_k}\Big)\,e^{n\varepsilon_{i+1}}\,K_{i+1}\,e^{n(\chi_{i+1}+\varepsilon_{i+1})}.$$

Hence

$$\frac{\log\big|(f_{[\xi_0\ldots\xi_{n_i+n}]})'(y)\big|}{n_i+n}\le\frac{\sum_{k=1}^{i}\big(\log K_k+m_k\log\beta\big)+\log K_{i+1}}{\sum_{k=1}^{i}(N_k+m_k)}$$
$$+\frac{\sum_{k=1}^{i}N_k(\chi_k+2\varepsilon_k)}{n_i+n}+\frac{n(\chi_{i+1}+2\varepsilon_{i+1})}{n_i+n}.$$

The first term can be made arbitrarily small, less than $\varepsilon_{i+1}/2$, if N_i was chosen sufficiently big. For the second term observe that

$$\frac{\sum_{k=1}^{i}N_k(\chi_k+2\varepsilon_k)}{n_i+n}\le\frac{\sum_{k=1}^{i-1}N_k(\chi_k+2\varepsilon_k)}{\sum_{k=1}^{i-1}N_k+N_i}+\frac{N_i(\chi_i+2\varepsilon_i)}{n_i+n}\le\frac{\varepsilon_{i+1}}{2}+\frac{N_i(\chi_i+2\varepsilon_i)}{n_i+n}$$

if N_i was chosen big enough. Thus, putting together the previous estimates, we obtain

$$\frac{1}{n_i+n}\log\big|(f_{[\xi_0\ldots\xi_{n_i+n}]})'(y)\big|\le$$
$$\le\varepsilon_{i+1}+\frac{(N_i+n)\big(\max\{\chi_i,\chi_{i+1}\}+2\max\{\varepsilon_i,\varepsilon_{i+1}\}\big)}{n_i+n}$$
$$<\max\{\chi_i,\chi_{i+1}\}+3\varepsilon_i.$$

The corresponding lower bound can be derived analogously.

Case 2: Estimating $\log|(f_{[\xi_0\ldots\xi_{n_i+N_{i+1}+n}]})'(y)|$ **with** $n=0,\ldots,m_{i+1}-1$**.** Similarily to Case 1, we can estimate

$$\prod_{k=1}^{i+1}\Big(e^{-N_k\varepsilon_k}\,K_k^{-1}e^{N_k(\chi_k-\varepsilon_k)}\Big)\cdot\prod_{k=1}^{i}\lambda^{m_k}\cdot\lambda^n\le$$
$$\le|(f_{[\xi_0\ldots\xi_{n_i+N_{i+1}+n}]})'(y)|\le\prod_{k=1}^{i+1}\Big(e^{N_k\varepsilon_k}\,K_k\,e^{N_k(\chi_k+\varepsilon_k)}\Big)\cdot\prod_{k=1}^{i}\beta^{m_k}\cdot\beta^n.$$

Hence,

$$\frac{\log|(f_{[\xi_0\ldots\,\xi_{n_i+N_{i+1}+n}]})'(y)|}{n_i+N_{i+1}+n} \le \frac{\sum_{k=1}^{i+1}\big(\log K_k+m_k\log\beta\big)}{N_{i+1}} + \frac{\sum_{k=1}^{i}N_k(\chi_k+2\varepsilon_k)}{N_{i+1}} + \chi_{i+1}+2\varepsilon_{i+1}.$$

As in the Case 1 the first two terms can be made arbitrarily small, less than $\varepsilon_{i+1}/2$, if N_{i+1} is chosen sufficiently big. Thus, we obtain

$$\frac{\log|(f_{[\xi_0\ldots\,\xi_{n_i+N_{i+1}+n}]})'(y)|}{n_i+N_{i+1}+n} \le \chi_{i+1}+3\,\varepsilon_{i+1}.$$

The corresponding lower bound can be derived analogously. This completes the estimates in Case 2.

Putting together Cases 1 and 2, we obtain that for every $i\ge 1$ and every $n=0,\ldots,N_{i+1}+m_{i+1}-1$ we have

$$\min\{\chi_i,\chi_{i+1}\}-3\,\varepsilon_i \le \frac{\log|(f_{[\xi_0\ldots\,\xi_{n_i+n}]})'(y)|}{n_i+n} \le \max\{\chi_i,\chi_{i+1}\}+3\,\varepsilon_i.$$

Since $\varepsilon_i\to 0$ and $\chi_i\to\chi$, this proves $\chi_+(y,\xi^+)=\chi$. This completes the proof of Proposition 5.1. $\square$

Proof of Proposition 5.2. Recall the definition of $\widetilde\beta$ in Proposition 1.2. By Proposition 5.1 there exists a pair $(y,\zeta^+)\in(0,1)\times\Sigma_2^+$ with forward Lyapunov exponent $\chi_+(y,\zeta^+)=\log\widetilde\beta$.

Let $\chi\in(0,\log\widetilde\beta\,]$ and fix some decreasing sequence $\varepsilon_i\to 0$. Choose any open interval $I_0\subset(0,1)$. We claim that this interval contains a point z such that $\overline\chi_+(z,\xi^+)=\chi$ for some sequence $\xi^+\in\Sigma_2^+$. Let $J_1=f_1(I_0)$. By Proposition 1.7 the set $\mathcal{O}^-(y)$ is dense in $[0,1]$ and hence there are a point $y_1\in J_1$ and a finite sequence $\eta^{(1)}$ such that

$$f_{[\eta^{(1)}]}(y_1)=y.$$

Consider the point x_1 and the one-sided sequence $\omega^{(1)}\in\Sigma_2^+$ defined by

$$x_1\stackrel{\text{def}}{=}f_1^{-1}(y_1)\in I_0 \quad\text{and}\quad \omega^{(1)}\stackrel{\text{def}}{=}1\,\eta^{(1)}\zeta^+.$$

Observe that $\chi_+(x_1,\omega^{(1)})=\widetilde\beta\ge\chi>0$. Since the first iterate of $(x_1,\omega^{(1)})$ is contracting, there exists a number $n_1\ge 2$ satisfying

$$\frac{1}{n_1}\log\big|(f_{[\omega_0^{(1)}\ldots\,\omega_{n_1-1}^{(1)}]})'(x_1)\big| > \chi-\frac{\varepsilon_1}{2}.$$

If n_1 is the smallest number with this property then

$$\frac{1}{k}\log\big|(f_{[\omega_0^{(1)}\ldots\,\omega_{k-1}^{(1)}]})'(x_1)\big| \le \chi-\frac{\varepsilon_1}{2} \qquad\text{for every } k=1,\ldots,n_1-1.$$

Recall that $|f_0'|,|f_1'|\le\beta$, thus

$$\frac{1}{n_1}\log\big|(f_{[\omega_0^{(1)}\ldots\,\omega_{n_1-1}^{(1)}]})'(x_1)\big| \le \chi-\frac{\varepsilon_1}{2}+\frac{\log\beta}{n_1}.$$

Hence, by continuity of the derivative, we can choose an interval $I_1 \subset I_0$ containing x_1 such that for all $x \in I_1$ we have

$$\chi - \varepsilon_1 < \frac{1}{n_1} \log \big|(f_{[\omega_0^{(1)} \dots \omega_{n_1-1}^{(1)}]})'(x)\big| < \chi + \frac{\log \beta}{n_1} + \varepsilon_1,$$
$$\frac{1}{k} \log \Big|(f_{[\omega_0^{(1)} \dots \omega_{k-1}^{(1)}]})'(x)\Big| < \chi + \varepsilon_1 \quad \text{for every } k = 1, \dots, n_1 - 1.$$

Let us now start a recursion. Given $i \geq 1$ assume that we have already constructed a one-sided finite sequence $\omega^{(i)}$ of length $n_i \geq 1$ and an interval $I_i \subset I_{i-1}$ such that for all $x \in I_i$ we have

$$\chi - \varepsilon_i \leq \frac{1}{n_i} \log \big|(f_{[\omega_0^{(i)} \dots \omega_{n_i-1}^{(i)}]})'(x)\big| < \chi + \frac{\log \beta}{n_i} + \varepsilon_i \tag{5.6}$$

and

$$\frac{1}{k} \log \Big|(f_{[\omega_0^{(i)} \dots \omega_{k-1}^{(i)}]})'(x)\Big| < \chi + \varepsilon_i \quad \text{for every } k = n_{i-1} + 1, \dots, n_i - 1. \tag{5.7}$$

To construct the segment of the one-sided sequence $\omega^{(i+1)}$ we proceed as above. For that recall again that f_1 is contracting and hence there exists a smallest number $m_i \geq 1$ for which

$$\log \big|(f_1^{m_i} \circ f_{[\omega_0^{(i)} \dots \omega_{n_i-1}^{(i)}]})'(x)\big| < 0, \quad \text{for all } x \in I_i.$$

Consider the closed interval

$$J_{i+1} \stackrel{\text{def}}{=} f_1^{m_i} \circ f_{[\omega_0^{(i)} \dots \omega_{n_i-1}^{(i)}]}(I_i).$$

By Proposition 1.7 the set $\mathcal{O}^-(y)$ is dense in $[0,1]$. Hence there exist a point $y_{i+1} \in J_{i+1}$ and a finite sequence $\eta^{(i+1)}$ such that $f_{[\eta^{(i+1)}]}(y_{i+1}) = y$. Let

$$x_{i+1} \stackrel{\text{def}}{=} \big(f_1^{m_i} \circ f_{[\omega_0^{(i)} \dots \omega_{n_i-1}^{(i)}]}\big)^{-1}(y_{i+1}) \in I_i$$

and define the one-sided infinite sequence

$$\omega^{(i+1)} \stackrel{\text{def}}{=} \omega_0^{(i)} \dots \omega_{n_i-1}^{(i)} 1^{m_i} \eta^{(i+1)} \zeta^+.$$

Since f_1 is contracting, for all $k = 1, \dots, m_i$ we have

$$\begin{aligned}\log \big|\big(f_{[\omega_0^{(i+1)} \dots \omega_{n_i-1+k}^{(i+1)}]}\big)'(x_{i+1})\big| &< \log \big|\big(f_{[\omega_0^{(i+1)} \dots \omega_{n_i-1}^{(i+1)}]}\big)'(x_{i+1})\big| \\ &= \log \big|\big(f_{[\omega_0^{(i)} \dots \omega_{n_i-1}^{(i)}]}\big)'(x_{i+1})\big|,\end{aligned}$$

where in the equality we use that $\omega_k^{(i+1)} = \omega_k^{(i)}$ for every $k = 1, \dots, n_i - 1$. Hence, since $x_{i+1} \in I_i$, for all $k = 1, \dots, m_i$ we hence have

$$\begin{aligned}&\frac{1}{n_i + k} \log \big|\big(f_{[\omega_0^{(i+1)} \dots \omega_{n_i+k-1}^{(i+1)}]}\big)'(x_{i+1})\big| \\ &\leq \frac{1}{n_i + k} \log \big|\big(f_{[\omega_0^{(i)} \dots \omega_{n_i-1}^{(i)}]}\big)'(x_{i+1})\big| < \frac{1}{n_i} \log \big|\big(f_{[\omega_0^{(i)} \dots \omega_{n_i-1}^{(i)}]}\big)'(x_{i+1})\big| \\ &< \chi + \varepsilon_i,\end{aligned} \tag{5.8}$$

where the last inequality follows from (5.7). Observe that the forward orbits of $(x_{i+1},\omega^{(i+1)})$ and (y,ζ^+) eventually coincide, hence they have the same exponent $\chi_+(x_{i+1},\omega^{(i+1)})=\log\widetilde\beta>\chi$. Thus, fixed small ε_{i+1}, there exists a first number $n_{i+1}>n_i+m_i$ for which

$$\frac{1}{n_{i+1}}\log\big|(f_{[\omega_0^{(i+1)}\dots\omega_{n_{i+1}-1}^{(i+1)}]})'(x_{i+1})\big|>\chi-\frac{\varepsilon_{i+1}}{2}.$$

Since n_{i+1} is the smallest number with this property we have

$$\frac{1}{k}\log\big|(f_{[\omega_0^{(i+1)}\dots\omega_{k-1}^{(i+1)}]})'(x_{i+1})\big|\le\chi-\frac{\varepsilon_{i+1}}{2}\qquad\text{for all } k=n_i+m_i+1,\dots,n_{i+1}-1. \tag{5.9}$$

Thus, as $|f_0'|,|f_1'|\le\beta$,

$$\frac{1}{n_{i+1}}\log\big|(f_{[\omega_0^{(i+1)}\dots\omega_{n_{i+1}-1}^{(i+1)}]})'(x_{i+1})\big|<\chi-\frac{\varepsilon_{i+1}}{2}+\frac{\log\beta}{n_{i+1}}. \tag{5.10}$$

By continuity, by (5.10) and (5.9) we can choose a closed interval $I_{i+1}\subset I_i$ containing x_{i+1} such that for all $x\in I_{i+1}$

$$\chi-\varepsilon_{i+1}<\frac{1}{n_{i+1}}\log\big|(f_{[\omega_0^{(i+1)}\dots\omega_{n_{i+1}-1}^{(i+1)}]})'(x)\big|<\chi+\frac{\log\beta}{n_{i+1}}+\varepsilon_{i+1} \tag{5.11}$$

and for all $x\in I_{i+1}$ and all $k=n_i+m_i+1,\dots,n_{i+1}-1$

$$\frac{1}{k}\log\Big|\big(f_{[\omega_0^{(i+1)}\dots\omega_{k-1}^{(i+1)}]}\big)'(x)\Big|<\chi+\varepsilon_{i+1}. \tag{5.12}$$

Moreover, by (5.8) we can assume that for all $x\in I_{i+1}$ and all $k=1,\dots,m_i$

$$\frac{1}{n_i+k}\log\big|\big(f_{[\omega_0^{(i+1)}\dots\omega_{n_i+k-1}^{(i+1)}]}\big)'(x)\big|<\chi+2\varepsilon_i. \tag{5.13}$$

In this way we construct a sequence $(I_i)_{i\ge1}$ of nested decreasing compact intervals such that $\bigcap_{i\ge1}I_i$ contains a point z. We also consider the one-sided "limit" sequence

$$\xi^+\overset{\text{def}}{=}\lim_{i\to\infty}\omega^{(i)}\in\Sigma_2^+.$$

By construction, (5.11), (5.12), and (5.13) guarantee that $\overline\chi_+(z,\xi^+)=\chi$. Finally, as the choice of the first interval I_0 was arbitrary, this proves that the set of points z with the claimed property is dense in $[0,1]$. □

We are now ready to prove Theorem 1.3 property (i).

Proof of Theorem 1.3 (i). Consider any $\chi\in(0,\log\widetilde\beta\,]$ and any $z\in(0,1)$. By Proposition 5.2 there is a pair (w,ϱ^+) such that $\overline\chi_+(w,\varrho^+)=\chi$. By Proposition 5.1 there is a pair $(y,\xi^+)\in(0,1)\times\Sigma_2^+$ such that $\chi_+(y,\xi^+)=\chi$. Take any $\delta>0$. By Proposition 1.7 the backward orbit $\mathcal O^-(y)$ of y is dense in $[0,1]$ and thus there are a point $\overline y\in(z-\delta,z+\delta)$ and a finite sequence η such that $f_{[\eta]}(\overline y)=y$. Consider the one-sided sequence $\eta\,\xi^+$ and note that the Lyapunov exponents of $(\overline y,\eta\,\xi^+)$ and (y,ξ^+) coincide. The fact that δ can be taken arbitrarily small proves the claim in the case $\chi\in(0,\log\widetilde\beta\,]$.

Likewise, for $\chi = 0$, by Proposition 4.5 and Proposition 5.1 there exists a pair $(y, \xi^+) \in (0,1) \times \Sigma_2^+$ such that $\chi_+(y, \xi^+) = 0$. Arguing as above, we conclude that the set of points z for which there exists $\xi^+ \in \Sigma_2^+$ with $\chi_+(z, \xi^+) = 0$ is dense in $[0,1]$. □

5.2. **Negative forward spectrum.** Property (ii) in Theorem 1.3 is a consequence of the minimality of the IFS and the following proposition.

Proposition 5.3. *For every $\chi \in [\log \lambda, 0)$ and every $\varepsilon \in (0, |\chi|)$ the set of points y for which there is a periodic sequence $\xi = (\xi_0 \ldots \xi_{k-1})^{\mathbb{Z}} \in \Sigma_2$ with*

- $f_{[\xi_0 \ldots \xi_{k-1}]}(y) = y$,
- $[0,1] \subset W^s_{\mathrm{loc}}(y, f_{[\xi_0 \ldots \xi_{k-1}]})$, *and*
- $\chi(y, \xi) \in (\chi - \varepsilon, \chi + \varepsilon)$

is dense in $[0,1]$.

Proof. Consider $\chi \in [\log \lambda, 0)$. Fix $\varepsilon > 0$ and a closed interval $J_1 \subset (0,1)$ of length $\vartheta_1 > 0$. We will construct a periodic pair $(y, (\xi_0 \ldots \xi_{k-1})^{\mathbb{Z}}) \in J_1 \times \Sigma_2$ with exponent in $(\chi - \varepsilon, \chi + \varepsilon)$ and $W^s_{\mathrm{loc}}(y, f_{[\xi_0 \ldots \xi_{k-1}]}) \supset [0,1]$.

Consider the fixed attracting pair $(1, 0^{\mathbb{Z}})$ with exponent $\chi(1, 0^{\mathbb{Z}}) = \log \lambda$. By Lemma 4.7 there is $p \in (0,1)$ and a finite sequence $(\zeta_0 \ldots \zeta_n)$ such that $(p, (\zeta_0 \ldots \zeta_n)^{\mathbb{Z}})$ is periodic attracting with negative exponent $\chi(p, (\zeta_0 \ldots \zeta_n)^{\mathbb{Z}})$ close to 0.

Fix $\vartheta_2 > 0$ such that

$$J_2 = [p - \vartheta_2/2, p + \vartheta_2/2] \subset W^s_{\mathrm{loc}}(p, f_{[\zeta_0 \ldots \zeta_n]}) \subset (0,1).$$

Let $M \overset{\text{def}}{=} \max\{M(\vartheta_1), M(\vartheta_2)\}$, where $M(\vartheta_1), M(\vartheta_2) \geq 1$ are the numbers provided by Lemma 3.4. Applying Lemma 3.4 to the interval J_2 we get a finite sequence $\eta^{(2)}$ of length $|\eta^{(2)}| \leq M(\vartheta_2) \leq M$ such that

$$f_{[\eta^{(2)}]}([0,1]) \subset J_2 \subset W^s_{\mathrm{loc}}(p, f_{[\zeta_0 \ldots \zeta_n]})$$

and thus for any $\ell \geq 1$ we have

$$f_{[\eta^{(2)}\,(\zeta_0 \ldots \zeta_n)^\ell]}([0,1]) = \big(f_{[\zeta_0 \ldots \zeta_n]}\big)^\ell \big(f_{[\eta^{(2)}]}([0,1])\big) \subset \big(f_{[\zeta_0 \ldots \zeta_n]}\big)^\ell (J_2) \subset J_2.$$

Applying Lemma 3.4 to the interval J_1 we get a finite sequence $\eta^{(1)}$ of length $|\eta^{(1)}| \leq M(\vartheta_1) \leq M$ such that for all $r \geq 1$

$$f_{[\eta^{(2)}(\zeta_0 \ldots \zeta_n)^\ell\, 0^r\, \eta^{(1)}]}([0,1]) = f_{[\eta^{(1)}]}\big(f_{[\eta^{(2)}(\zeta_0 \ldots \zeta_n)^\ell\, 0^r]}([0,1])\big) \subset J_1$$

and thus

$$f_{[\eta^{(2)}(\zeta_0 \ldots \zeta_n)^\ell\, 0^r\, \eta^{(1)}]}(J_1) \subset J_1.$$

Considering the periodic sequence $\xi = \xi(r)$ defined by

$$\xi = (\xi_0 \ldots \xi_{k-1})^{\mathbb{Z}} \overset{\text{def}}{=} (\eta^{(2)}(\zeta_0 \ldots \zeta_n)^\ell\, 0^r\, \eta^{(1)})^{\mathbb{Z}}$$

we get a periodic point $y = y(r) \in J_1$ such $y = f_{[\eta^{(2)}(\zeta_0 \ldots \zeta_n)^\ell\, 0^r\, \eta^{(1)}]}(y)$.

We now show that r and ℓ can be chosen such that (y, ξ) is an attracting pair with exponent $\chi(y, \xi)$ close to χ and $[0,1] \subset W^s(y, f_{[\xi_0 \ldots \xi_{k-1}]})$. Let us

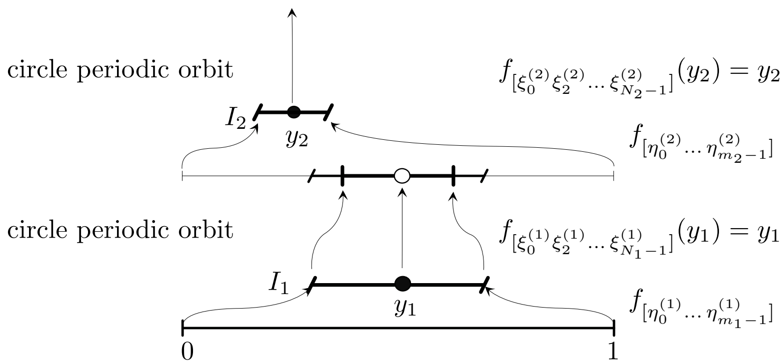

FIGURE 5. First steps in the construction of ξ^+ in the proof of Theorem 1.3 (ii).

refrain from all details and only point out the essential steps. Recall that $J_2 \subset (0,1)$. Hence, if $r \geq 1$ is large then $f_{[\eta^{(2)}\,(\zeta_0\dots\zeta_n)^\ell\,0^r]}([0,1])$ is close to 1.

The number r marks some fixed finite transition from J_2 to a small neighborhood of the fixed point 1. The finite sequences $\eta^{(2)}$ and $\eta^{(1)}$ mark the transitions from J_1 to J_2 (where the exponent is close to 0) and from a neighborhood close to 1 (where the exponent is $\log\lambda$) to the interval J_1, respectively. Recall that the lengths of these sequences are bounded by M. Finally, the numbers r and ℓ mark the repetition of loops at 1 (exponent close to $\log\lambda$) and at the periodic point p (exponent 0). Note that $\chi/\log\lambda$ can be approximated arbitrarily closely by rational numbers. Thus we can choose the numbers r, $\ell \geq 1$ large and such that

$$\frac{r\cdot\log\lambda+\ell\cdot 0}{r+\ell}\sim\chi. \tag{5.14}$$

Hence, when the numbers r and ℓ are chosen large enough, though respecting the approximation in (5.14), by simple distortion estimates we can guarantee that this composed map $f_{[\xi_0\dots\xi_{k-1}]}$ is a contraction in J_1 and hence has a unique fixed point y with $\chi(y,\xi)\sim\chi$. This also guarantees that $[0,1]\subset W^s(y,f_{[\xi_0\dots\xi_{k-1}]})$.

Finally, as the choice of the initial interval $J_1\subset(0,1)$ was arbitrary, the set of points y for which there is $\xi\in\Sigma_2$ such that (y,ξ) is periodic, $\chi(y,\xi)\in(\chi-\varepsilon,\chi+\varepsilon)$, and $[0,1]\subset W^s_{\mathrm{loc}}(y,f_{[\xi_0\dots\xi_{k-1}]})$ is dense in $[0,1]$, proving the proposition. □

Proof of Theorem 1.3 (ii). Fix $\chi\in[\log\lambda,0]$. We will construct a sequence $\xi^+\in\Sigma_2^+$ with $\chi_+(y,\xi^+)=\chi$ for every $y\in[0,1]$. See Figure 5 for an illustration.

Step 0: Choice of auxiliary sequences. Let $J_0=[0,1]$. Fixing some monotonically decreasing sequence $\varepsilon_i\to 0$, we choose sequences of points y_i, numbers K_i, ϑ_i, M_i, and intervals J_i as follows.

By Proposition 5.3, for every $i\geq 1$ there is a periodic pair $(y_i,\xi^{(i)})\in(0,1)\times\Sigma_2$ of period ℓ_i with negative exponent $\chi_i\in(\chi-\varepsilon_i,\chi+\varepsilon_i)$. By shrinking ε_i, we can assume that $\chi_i + 3\varepsilon_i < 0$. Further, by uniform hyperbolicity (backward and forward in time) of the closed orbit of $(y_i, \xi^{(i)})$, there exists $K_i > 1$ such that for every $n \geq 1$ we have

$$\frac{1}{K_i} e^{-n(|\chi_i|+\varepsilon_i)} \leq |(f_{[\xi_0^{(i)} \dots \xi_{n-1}^{(i)}]})'(y_i)| \leq K_i e^{-n(|\chi_i|-\varepsilon_i)}. \tag{5.15}$$

Choose a sequence $\vartheta_i \to 0$ such that $D(\vartheta_i) < e^{\varepsilon_i}$ for every i. Let J_i be the interval centered at y_i of length $\vartheta_i K_i^{-1}$. Without loss of generality, we can assume that ϑ_i is so small that $J_i \subset (0,1)$. Let N_i be some multiple of the period of $(y_i, \xi^{(i)})$ that will be specified below. By Lemma 3.4, for each J_i, there are a number $M_i \stackrel{\text{def}}{=} M(\vartheta_i K_i^{-1}) \geq 1$ and a finite sequence $\eta^{(i)}$ of length $m_i \stackrel{\text{def}}{=} |\eta^{(i)}| \leq M_i$ such that $f_{[\eta^{(i)}]}([0,1]) \subset J_i$.

For every $i \geq 1$ we apply Lemma 4.2 to $(y_i, \xi^{(i)})$ and the interval J_i and, by our choice of ϑ_i and (5.15), we obtain that for every $x \in J_i$ and $n = 1, \dots, N_i$

$$\operatorname{Dist} f_{[\xi_0^{(i)} \dots \xi_{n-1}^{(i)}]}|_{J_i} \leq D(\vartheta_i)^n < e^{n\,\varepsilon_i}. \tag{5.16}$$

By (5.15) and (5.16) for every $x \in J_i$ and $n = 1, \dots, N_i$ we have

$$\frac{1}{K_i} e^{-n(|\chi_i|+2\varepsilon_i)} \leq \big|(f_{[\xi_0^{(i)} \dots \xi_{n-1}^{(i)}]})'(x)\big| \leq K_i e^{-n(|\chi_i|-2\varepsilon_i)}. \tag{5.17}$$

In particular, as y_i is periodic and N_i is a multiple of its period, if

$$N_i \geq \frac{2\log K_i - \log \vartheta_i}{|\chi_i| - 2\varepsilon_i}$$

(we will further specify N_i below) then

$$f_{[\xi_0^{(i)} \dots \xi_{N_i-1}^{(i)}]}(J_i) \subset J_i.$$

Moreover, by our choice of $\eta^{(i)}$, in particular $f_{[\eta^{(i)}\xi_0^{(i)} \dots \xi_{N_i-1}^{(i)}]}(J_{i-1}) \subset J_i$. Recursively, this fixes a sequence of 8-tuples $(y_i, \xi^{(i)}, \chi_i, \varepsilon_i, K_i, \vartheta_i, m_i, \eta^{(i)})$. Notice that all the above stated properties remain true if we replace N_i be some larger multiple of the period of $(y_i, \xi^{(i)})$. We will adjust our choice of N_i in Step 2.

Step 1: Construction of ξ^+. The one-side sequence $\xi^+ \in \Sigma_2^+$ is obtained by concatenating finite repetitions of periodic parts of $\xi^{(i)}$ and some transition sequences $\eta^{(i)}$. More precisely, let

$$\xi^+ \stackrel{\text{def}}{=} \eta_0^{(1)} \dots \eta_{m_1-1}^{(1)} \xi_0^{(1)} \dots \xi_{N_1-1}^{(1)} \eta_0^{(2)} \dots \eta_{m_2-1}^{(2)} \xi_0^{(2)} \dots \xi_{N_2-1}^{(2)} \dots . \tag{5.18}$$

To complete Step 1 define the auxiliary sequence $(n_i)_{i\geq 0}$ by

$$n_0 \stackrel{\text{def}}{=} 0, \quad n_i \stackrel{\text{def}}{=} n_{i-1} + m_i + N_i.$$

Step 2: Lyapunov exponent of (y, ξ^+) for arbitrary $y \in [0,1]$. Observe that, after some finite transition of length $m_1 \leq M_1$, the trajectory of (y, ξ^+) jumps to the neighborhood J_1 of y_1. Further the orbit of (y, ξ^+) "shadows" the orbit of $(y_{i+1}, \xi^{(i+1)})$ for the time $n_i, \dots, n_i + N_{i+1} - 1$ (with small distortion) and thereafter passes some "finite transition" for the time $n_i+N_{i+1},\dots,n_i+N_{i+1}+m_{i+1}-1$ to arrive at a neighborhood of y_{i+2}. Recall that we have chosen a sequence of 8-tuples $(y_i,\xi^{(i)},\chi_i,\varepsilon_i,K_i,\vartheta_i,m_i,\eta^{(i)})$. We will now estimate the exponent and, in particular, specify the choice of N_i (this choice is done *after* selecting the sequence of 8-tuples). The calculations are essentially the same as in the proof of Proposition 5.1. We distinguish two cases.

Case 1: Estimating $\log|(f_{[\xi_0\dots\,\xi_{n_i+n}]})'(y)|$ with $n=0,\dots,m_{i+1}-1$. By uniform contraction (5.17) and $\lambda\le|f_0'|,|f_1'|\le\beta$ we have

$$|(f_{[\xi_0\dots\,\xi_{n_i+n}]})'(y)|\ge\prod_{k=1}^{i}\left(K_k^{-1}\,e^{-N_k(|\chi_k|+2\varepsilon_k)}\right)\cdot\prod_{k=1}^{i}\lambda^{m_k}\cdot\lambda^n.$$

Hence, as $n\le m_{k+1}$,

$$\begin{aligned}\frac{\log\left|(f_{[\xi_0\dots\,\xi_{n_i+n}]})'(y)\right|}{n_i+n}\ge&\;\frac{-\sum_{k=1}^{i}\log K_k+\sum_{k=1}^{i+1}m_k\log\lambda}{\sum_{k=1}^{i}(m_k+N_k)}\\&-\frac{\sum_{k=1}^{i}N_k(|\chi_k|+2\varepsilon_k)}{n_i+n}.\end{aligned}$$

As the first term of the right-hand side depends only on those values of the initially fixed 8-tuple which have index less or equal than $i+1$, it can be made arbitrarily small in absolute value, less than $\varepsilon_{i+1}/2$, if N_i is chosen sufficiently big. For the second term observe that

$$\begin{aligned}-\frac{\sum_{k=1}^{i}N_k(|\chi_k|+2\varepsilon_k)}{n_i+n}&=-\frac{\sum_{k=1}^{i-1}N_k(|\chi_k|+2\varepsilon_k)}{\sum_{k=1}^{i}(m_k+N_k)+n}-\frac{N_i(|\chi_i|+2\varepsilon_i)}{\sum_{k=1}^{i}(m_k+N_k)+n}\\&\ge-\frac{\varepsilon_{i+1}}{2}-\frac{N_i(|\chi_i|+2\varepsilon_i)}{\sum_{k=1}^{i}(m_k+N_k)+n}\end{aligned}$$

if N_i was chosen big enough. Thus,

$$\frac{\log|(f_{[\xi_0\dots\,\xi_{n_i+n}]})'(y)|}{n_i+n}\ge-|\chi_i|-3\max\{\varepsilon_i,\varepsilon_{i+1}\}.$$

The corresponding upper bound can be derived analogously.

Case 2: Estimating $\log|(f_{[\xi_0\dots\,\xi_{n_i+m_{i+1}+n}]})'(y)|$ with $n=0,\dots,N_{i+1}-1$. By uniform contraction (5.17) and $\lambda\le|f_0'|,|f_1'|\le\beta$ we have

$$\left|(f_{[\xi_0\dots\,\xi_{n_i+m_{i+1}+n}]})'(y)\right|\ge\prod_{k=1}^{i}\left(K_k^{-1}\,e^{-N_k(|\chi_k|+2\varepsilon_k)}\lambda^{m_k}\right)K_{i+1}^{-1}\,e^{-n(|\chi_{i+1}|+2\varepsilon_{i+1})}.$$

Hence

$$\frac{\log\big|(f_{[\xi_0\ldots\,\xi_{n_i+m_{i+1}+n}]})'(y)\big|}{n_i+m_{i+1}+n} \ge \frac{\sum_{k=1}^{i}\big(-\log K_k+m_k\log\lambda\big)-\log K_{i+1}}{\sum_{k=1}^{i}(N_k+m_k)}$$
$$-\frac{\sum_{k=1}^{i}N_k(|\chi_k|+2\varepsilon_k)}{n_i+m_{i+1}+n}-\frac{n\big(|\chi_{i+1}|+2\varepsilon_{i+1}\big)}{n_i+m_{i+1}+n}.$$

The first term on the right-hand side can be made small in absolute value, less than $\varepsilon_{i+1}/2$, if N_i was chosen big. As before, for the second term observe

$$-\frac{\sum_{k=1}^{i}N_k(|\chi_k|+2\varepsilon_k)}{n_i+m_{i+1}+n}\ge-\frac{\varepsilon_{i+1}}{2}-\frac{N_i(|\chi_i|+2\varepsilon_i)}{n_i+m_{i+1}+n}$$

if N_i was chosen big enough. Thus, if N_i is large enough, we obtain

$$\frac{\log\big|(f_{[\xi_0\ldots\,\xi_{n_i+m_{i+1}+n}]})'(y)\big|}{n_i+m_{i+1}+n}\ge-\max\{|\chi_i|,|\chi_{i+1}|\}-3\max\{\varepsilon_i,\varepsilon_{i+1}\}.$$

The corresponding upper bound can be derived analogously.

Putting together Cases 1 and 2 we obtain that for every $i\ge1$ and every $n=0,\ldots,N_{i+1}+m_{i+1}-1$ we have

$$-\max\{|\chi_i|,|\chi_{i+1}|\}-3\varepsilon_i\le\frac{\log|(f_{[\xi_0\ldots\,\xi_{n_i+n}]})'(y)|}{n_i+n}\le-\min\{|\chi_i|,|\chi_{i+1}|\}+3\varepsilon_i.$$

Since $\varepsilon_i\to0$ and $\chi_i\to\chi$, this proves that $\chi_+(y,\xi^+)=\chi$. This concludes the proof of (ii) of Theorem 1.3. □

5.3. **Positive backward spectrum.**

Proof of Theorem 1.3 (iii). The proof is very similar to the one of Theorem 1.3 (i). We will only sketch the essential changes. First, given an exponent, we will find a sequence of periodic orbits with approximately that exponent by taking into account results above. Then we will follow these periodic orbits in the reverse direction and construct an admissible backward trajectory consecutively following these orbits. In this step we will make use of the Lemma 3.6 (strong uniform backward minimality) that provides *admissible* trajectories following the analogous steps of Proposition 5.1 (note that admissibility was not a problem for forward trajectories).

Fix $\chi\in[0,\log\widetilde\beta\,]$, $y\in(0,1)$, and $\Delta>0$ with $y\in[\Delta,1-\Delta]$. Consider the IFS generated by the maps $g_i=f_i^{-1}$ and consider the associated distortion constant $D(\cdot)$ as defined in (4.1). Fix sequences $\varepsilon_i\to0$, $\vartheta_i\to0$ with $D(\vartheta_i)<e^{\varepsilon_i}$. Let $\nu=\nu(\Delta)>0$ be given as in Lemma 3.6. Let $J_0\subset[\Delta,1-\Delta]$ be centered at y and of length $|J_0|\le\nu$.

Observe that $|\log\lambda|\le\log\widetilde\beta$ and thus $-\chi\in[\log\lambda,0]$. This can, for example, be seen from Proposition 5.2 and Lemma 4.4 and the fact that Lyapunov exponents of a periodic orbit reverse sign when considering the inverse orbit.

By Proposition 5.3, for every $i\ge1$ there is a periodic pair $(y_i,\xi^{(i)})$ of period ℓ_i with negative exponent $\chi_i\in(\chi-\varepsilon_i,\chi+\varepsilon_i)$. Further, by uniform hyperbolicity of the orbit of $(y_i, \xi^{(i)})$, there exists $K_i > 1$ such that for every $n \geq 1$ we have (5.15). Choose an interval J_i centered at y_i and of length $\vartheta_i K_i^{-1}$. By the choice of y_i and ϑ_i we can assume that $J_i \subset [\Delta, 1-\Delta]$ and $|J_i| \leq \nu$ for every $i \geq 1$. We apply Lemma 3.6 to the intervals $J = J_i$ and $H = J_{i-1}$. By this lemma, for each J_i there are $M_i \stackrel{\text{def}}{=} M(\Delta, \vartheta_i K_i^{-1}) \geq 1$ and a finite sequence $\eta^{(i)}$ of length $m_i \stackrel{\text{def}}{=} |\eta^{(i)}| \leq M_i$ admissible for all $x \in J_{i-1}$ so that $f_{[\eta^{(i)}.]}(J_{i-1}) \subset J_i$.

This fixes a sequence of 9-tuples $(y_i, \xi^{(i)}, \chi_i, \varepsilon_i, K_i, \vartheta_i, I_i, m_i, \eta^{(i)})$ to which we can apply Steps 1 and 2 as in proof of Proposition 5.1. $\square$

5.4. **Negative backward spectrum.**

Proof of Theorem 1.3 (iv). Consider any $\chi \in [\log \lambda, 0]$ and any $z \in (0,1)$. Let us first assume that

$$\text{there is a pair } (y, \xi^-) \in (0,1) \times \Sigma_2^- \text{ with } \chi_-(y, \xi^-) = \chi. \tag{5.19}$$

By Proposition 1.7 the forward orbit $\mathcal{O}^+(y)$ of y is dense in $(0,1)$. Thus, given $\delta > 0$, there are a point $\overline{y} \in (z-\delta, z+\delta)$ and a finite sequence η such that $f_{[\eta.]}(\overline{y}) = y$. Consider the one-sided sequence $\xi^-\eta \in \Sigma_2^-$ and note that the backward Lyapunov exponents of $(\overline{y}, \xi^-\eta)$ and (y, ξ^-) coincide. As δ can be arbitrarily small, this will prove the theorem.

What remains to show is (5.19). Fix $\Delta > 0$ small. By Theorem 1.3 (ii) and Lemma 4.6, there exists a sequence of periodic pairs $(x_i, \xi^{(i)})$ such that $x_i \in (2\Delta, 1-2\Delta)$ and $\lim_i \chi_i = \chi$. As in the proof of item (iii) of the Theorem, consider the IFS generated by the maps $g_k = f_k^{-1}$ and the associated distortion constant $D(\cdot)$ defined in (4.1). Fix $\varepsilon_i \to 0$ and $\vartheta_i \to 0$ satisfying $D(\vartheta_i) < e^{\varepsilon_i}$ and $|\chi_i - \chi| \leq \varepsilon_i$. By uniform hyperbolicity of the closed orbit of $(x_i, \xi^{(i)})$, there exists $K_i > 1$ such that for every $n \geq 1$ we have (5.1).

As in Step 0 in the proof of Proposition 5.1, for every $i \geq 1$ let I_i be some interval containing x_i in its interior and choose $N_i \geq 1$ and an interval $J_i = J(x_i, N_i) \subset (\Delta, 1-\Delta)$ of length $\vartheta_i K_i^{-1}$ containing x_i such that

$$|f_{[\xi^{(i)}_{-N_i} \cdots \, \xi^{(i)}_{-1}]}(J_i)| = |(f_{[\xi^{(i)}_{-N_i} \cdots \, \xi^{(i)}_{-1}.]})^{-1}(J_i)| \leq \vartheta_i \, e^{-N_i(|\chi_i| - 2\varepsilon)}$$

as in (5.4) and (5.5). Given Δ and $\vartheta_i K_i^{-1}$, let $M_i = M_-(\Delta, \vartheta_i K_i^{-1})$ be as in Lemma 3.2. Applying this lemma to J_i and the point x_{i+1}, we find $y_i \in J_i$ and a finite sequence $\eta^{(i)}$ with $|\eta^{(i)}| \leq M_i$ such that $(\eta^{(i)}.)$ is admissible for y_i and $x_{i+1} = f_{[\eta^{(i)}.]}(y_i)$. By shrinking J_i, we can assume that J_i contains y_i and that $(\eta^{(i)}.)$ is admissible everywhere on J_i. Finally, assuming that N_i was chosen large enough, we have

$$f_{[\xi^{(i)}]}(J_i) \subset I_i$$

and thus, by construction,

$$f_{[\eta^{(i)} \, \xi^{(i)}]}(I_{i+1}) \subset I_i, \qquad \text{where} \qquad I_{i+1} \stackrel{\text{def}}{=} f_{[\eta^{(i)}.]}(J_i).$$

This fixes a sequence of 9-tuples $(x_i,\xi^{(i)},\chi_i,\varepsilon_i,K_i,\vartheta_i,I_i,m_i,\eta^{(i)})$ to which we can apply Steps 1 and 2 as in proof of Proposition 5.1 to construct a pair (y,ξ^-) satisfying $\chi_-(y,\xi^-)=\chi$. This finishes the proof of the theorem. □

5.5. **Measures with full support.** The proof of Proposition 1.6 follows combining our construction of shadowing periodic orbits, using minimality, and the approximation methods in [1, 10]. We refrain ourselves from providing all details and only sketch the key ingredients.

The case of non-hyperbolic measures $\chi=0$ is a bit more subtle but follows as in [1].

Let us consider the case $\chi\in(\log\lambda,0)$.

Step 1. Given two periodic points P_1,P_2 with negative central exponents, given numbers $\theta_1,\theta_2\in(0,1)$ with $\theta_1+\theta_2<1$ and $\delta>0$, following our constructions in Section 4, we obtain a new periodic point P_3 whose orbit is δ-dense in Λ and mimics the orbit of P_1 during a fraction of time $\sim\theta_1$ and the orbit of P_2 during a fraction of time $\sim\theta_2$. As this point can be chosen with arbitrarily large period, if $\theta_1+\theta_2\sim 1$ then the central Lyapunov exponent of P_3 is approximately $\theta_1\chi_c(P_1)+\theta_2\chi_c(P_2)$.

We now sketch how to construct this point. Let $P_i=(p_i,(\xi_0^i\dots\xi_{n_i}^i)^{\mathbb Z})$, $i=1,2$. Our construction (see Lemma 4.6) provides a point $p_3\sim p_1$ that is periodic for a sequence $\eta=(\eta_0\dots\eta_k)^{\mathbb Z}$ of the form $\eta_0\dots\eta_r=(\xi_0^1\dots\xi_{n_1}^1)^\ell$ and $\eta_i\dots\eta_j=(\xi_0^2\dots\xi_{n_2}^2)^m$ for some large numbers ℓ,m and $r<i<j<k$. This implies that the orbit of the point $P_3=(p_3,\eta)$ is close to the one of P_1 for the initial time and passes close to the one of P_2 for some intermediate time.

Step 2. Consider now a sequence of numbers $\varkappa_n\in(0,1)$ with $\prod_{n\ge1}\varkappa_n>0$. Assume that there is a sequence of periodic points R_n such that $\chi_c(R_0)\in(\chi,0)$ and that for each n the orbit of R_n is $1/n$-dense in Λ, satisfies $\chi_c(R_n)\sim(\chi+\chi_c(R_{n-1}))/2$ and that the orbit of R_n shadows the orbit of R_{n-1} during a proportion of time $\sim 1-\varkappa_n$. Now for each n consider the periodic measure μ_n uniformly distributed on the orbit of R_n. Then the sequence μ_n converges to an ergodic measure μ with full support in Λ and whose central Lyapunov exponent is χ. For details see [1, Lemmas 2.5 and 2.1].

We now explain how the orbits R_n are constructed. Assume that R_n is already constructed. Since the spectrum is complete in $(\log\lambda,0)$, there is a central contracting point Q_{n+1} whose central Lyapunov exponent is

$$\chi_c(Q_{n+1})\sim\frac{\chi-\chi_c(R_n)+2\varkappa_n\,\chi_c(R_n)}{2\varkappa_n}.$$

Note that this number is negative and smaller than $\chi_c(R_n)$ and therefore there is a periodic point with approximately such an exponent. Using Step 1, we construct the periodic orbit R_{n+1} that mimics the orbit R_n during a fraction $\sim 1-\varkappa_n$, the orbit of Q_{n+1} during a fraction $\sim\varkappa_n$, and is $1/n$-dense in Λ. By construction, the exponent of R_{n+1} is approximately

$$(1-\varkappa_n)\chi_c(R_n)+\varkappa_n\chi_c(Q_{n+1})\sim(\chi+\chi_c(R_n))/2$$

as desired.

For the case $\chi \in (0, \log \widetilde{\beta})$ we proceed analogously, considering the inverse map.

This proves the proposition.

Appendix

Here we give an example of C^∞ invertible map on a compact set, for which the set of possible forward Lyapunov exponent differs from the set of backward Lyapunov exponents. This will be a map on $\mathbb{R}^4$, but we will present it as a map on $\Sigma_2 \times [0,1]^2$ in the same way as in the main body of the paper we presented a map acting on $\mathbb{R}^3$ as a map on $\Sigma_2 \times [0,1]$.

Define $T\colon \Sigma_2 \times [0,1]^2 \to \Sigma_2 \times \mathbb{R}[0,1]$ by

$$T(\xi, x, t) \stackrel{\text{def}}{=} (\sigma\xi, f_{\xi_0,t}(x), g(t)).$$

We will assume $g(t)$ to have a (topologically) attracting fixed point at 0, a repelling fixed point at 1, and no other fixed points. The attracting point at 0 will be a very flat (infinite degree) parabolic point. The homeomorphisms $f_{i,t}$ are defined by

- $f_{0,t} = f_0, f_{1,t} = f_1 - t$ for $t \le \varepsilon$,
- $f_{0,1} = \{x \to x\}, f_{1,1} = f_1$,
- $f_{0,t}([0,1]) = [0,1]$ for all t,

where f_0, f_1 are like in the main part of the paper (hence T restricted to $\{t = 0\}$ is exactly the map we studied).

Let us denote by A the set of points whose trajectories never leave $\Sigma_2 \times [0,1]^2$, this set is open and nonempty. We can divide A into $A_0 = A \cap \{t = 0\}$, $A_1 = A \cap \{t = 1\}$ and $A_r = A \setminus (A_0 \cup A_1)$.

We are now ready to calculate the (forward and backward) Lyapunov exponents that are simply the forward and backward Birkhoff averages of the potential $\log |\frac{d}{dx} f_{\omega_0,t}|$, like in the main part of the paper. For points in A_0 the forward and backward spectra are both equal to $[\log \lambda, \widetilde{\beta}\,] \cup \{\log \beta\}$. For points in A_1 the forward and backward spectra are both equal to $[\log \lambda, 0]$. For points in A_r the backward spectrum is also $[\log \lambda, 0]$ because those points converge to A_1 under backward iterations of T.

The difference is the forward spectrum on A_r. With the right choice of g it will be equal to $[\log \lambda, \log \beta]$, that is, it will not have a gap.

The reason for the gap in A_0 is that we cannot spend a long time around $x = 0$ (using the map f_0) and then come back in a short time – whenever we are close to 0, we must have spent a lot of time around $x = 1$ (using the map f_0 to stay there) before and the Birkhoff sums of $\log |f'|$ gathered in those two time periods cancel each other almost completely, leading to the drop of the Lyapunov exponent (we have to be exactly at $x = 0$ or the Lyapunov exponent is at most $\log \widetilde{\beta}$). This is no longer true for points in A_r. Indeed, when we have small $t > 0$, the preimage $y_t = f_{1,t}^{-1}(0)$ is strictly inside $[0,1]$, and hence we can get there using only bounded number of iterations of

$f_{0,t} = f_0$. If $g^n(t)$ converge to 0 sufficiently slowly, this lets us to construct a point the forward Lyapunov spectrum at which takes any prescribed value in $(\log \widetilde{\beta}, \log \beta)$.

Remark 5.4. A simpler example was shown to us by Michał Misiurewicz, [13].

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