Regularity of the extremal solutions for the Liouville system

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In this short note, we study the smoothness of the extremal solutions to the following system of equations:

$$\begin{cases} -\Delta u = \mu e^{v} & \text{in } \Omega, \\ -\Delta v = \lambda e^{u} & \text{in } \Omega, \\ \quad u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\lambda, \mu > 0$ are parameters and Ω is a smoothly bounded domain of $\mathbb{R}^N$, $N \geq 1$. As shown by M. Montenegro (see [6]), there exists a limiting curve Υ in the first quadrant of the (λ, μ)-plane serving as borderline for existence of classical solutions of (1). He also proved the existence of a weak solution u^* for every (λ^*, μ^*) on the curve Υ and left open the question of its regularity. Following standard terminology (see e.g. the books [3], [5] for an introduction to this vast subject), u^* is called an extremal solution. Our result is the following.

Theorem 1 *Let $1 \leq N \leq 9$. Then, extremal solutions to (1) are smooth.*

Remark 2 *C. Cowan ([1]) recently obtained the same result under the further assumption that $(N-2)/8 < \lambda/\mu < 8/(N-2)$.*

Any extremal solution u^* is obtained as the increasing pointwise limit of a sequence of regular solutions (u_n) associated to parameters $(\lambda_n, \mu_n) = (1 - 1/n)(\lambda^*, \mu^*)$. In addition, see [6], u_n is stable in the sense that the principal eigenvalue of the linearized operator associated to (1) is nonnegative. In other words, there exist $\lambda_1 \geq 0$ and two positive functions $\varphi_1, \psi_1 \in C^2(\overline{\Omega})$ such that

$$\begin{cases} -\Delta\varphi_1 - g'(v)\psi_1 = \lambda_1\varphi_1 & \text{in } \Omega, \\ -\Delta\psi_1 - f'(u)\varphi_1 = \lambda_1\psi_1 & \text{in } \Omega. \\ \qquad \varphi_1 = \psi_1 = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where, in the context of (1), $g(v) = e^v$ and $f(u) = e^u$. This motivates the following useful inequality.

Let f, g denote two nondecreasing C^1 functions and consider the more general system

$$\begin{cases} -\Delta u = g(v) & \text{in } \Omega, \\ -\Delta v = f(u) & \text{in } \Omega, \\ \quad u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

Lemma 3 *Let $N \geq 1$ and let $(u, v) \in C^2(\overline{\Omega})^2$ denote a stable solution of* (3)*. Then, for all $\varphi \in C_c^1(\Omega)$, there holds*

$$\int_\Omega \sqrt{f'(u)g'(v)}\varphi^2 \, dx \leq \int_\Omega |\nabla\varphi|^2 \, dx \tag{4}$$

Remark 4 *As we just learnt, the same inequality has been obtained independently by C. Cowan and N. Ghoussoub. See* [2]*.*

Proof. Since (u, v) is stable, there exist $\lambda_1 \geq 0$ and two positive functions $\varphi_1, \psi_1 \in C^2(\overline{\Omega})$ solving (2). Given $\varphi \in C_c^1(\Omega)$, multiply the first equation in (2) by φ^2/φ_1 and integrate. Then,

$$\begin{aligned} \int_\Omega g'(v)\frac{\psi_1}{\varphi_1}\varphi^2 \, dx &\leq \int_\Omega \frac{\varphi^2}{\varphi_1}(-\Delta\varphi_1) \\ &= -\int_\Omega |\nabla\varphi_1|^2\Big(\frac{\varphi}{\varphi_1}\Big)^2 + 2\int_\Omega \frac{\varphi}{\psi_1}\nabla\varphi\nabla\varphi_1 \\ &= -\int_\Omega \Big|\frac{\varphi}{\varphi_1}\nabla\varphi_1 - \nabla\varphi\Big|^2 + \int_\Omega |\nabla\varphi|^2 \leq \int_\Omega |\nabla\varphi|^2. \end{aligned} \tag{5}$$

Working similarly with the second equation, we also have

$$\int_\Omega f'(u)\frac{\varphi_1}{\psi_1}\varphi^2 \, dx \leq \int_\Omega |\nabla\varphi|^2 \, dx \tag{6}$$

(4) then follows by combining the Cauchy-Schwarz inequality and (5)- (6). □

Thanks to the inequality (4), we obtain the following estimate.

Lemma 5 *Let $N \geq 1$. There exists a universal constant $C > 0$ such that any stable solution of* (1) *satisfies*

$$\int e^{u+v} \, dx \leq C\,|\Omega| \left(\frac{\lambda}{\mu} + \frac{\mu}{\lambda}\right). \tag{7}$$

Proof. Multiply the second equation in (1) by $e^v - 1$ and integrate.

$$\begin{aligned} \lambda\int_\Omega e^{u+v}dx \geq \lambda\int_\Omega e^u(e^v - 1)dx &= \int_\Omega \nabla v\nabla(e^v - 1) \, dx \\ &= 4\int_\Omega \left|\nabla(e^{v/2} - 1)\right|^2 \, dx. \end{aligned} \tag{8}$$

Using (4) with test function $\varphi = e^{v/2} - 1$, it follows that

$$\lambda \int_\Omega e^{u+v}\, dx \geq 4\sqrt{\lambda\mu} \int_\Omega e^{\frac{u+v}{2}} (e^{v/2} - 1)^2\, dx \geq 4\sqrt{\lambda\mu} \int_\Omega e^{\frac{u+v}{2}} e^v\, dx - 8\sqrt{\lambda\mu} \int_\Omega e^{\frac{u+v}{2}} e^{v/2}\, dx. \tag{9}$$

By Young's inequality, $e^{v/2} = \frac{1}{\sqrt{2}} e^{v/2} \cdot \sqrt{2} \leq \frac{1}{4} e^v + 1$. So,

$$\int_\Omega e^{\frac{u+v}{2}} e^{v/2}\, dx \leq \frac{1}{4} \int_\Omega e^{\frac{u+v}{2}} e^v\, dx + \int_\Omega e^{\frac{u+v}{2}}\, dx.$$

Plugging this in (9), we obtain

$$\lambda \int_\Omega e^{u+v}\, dx + 8\sqrt{\lambda\mu} \int_\Omega e^{\frac{u+v}{2}}\, dx \geq 2\sqrt{\lambda\mu} \int_\Omega e^{\frac{u+v}{2}} e^v\, dx. \tag{10}$$

Similarly,

$$\mu \int_\Omega e^{u+v}\, dx + 8\sqrt{\lambda\mu} \int_\Omega e^{\frac{u+v}{2}}\, dx \geq 2\sqrt{\lambda\mu} \int_\Omega e^{\frac{u+v}{2}} e^u\, dx. \tag{11}$$

Multiply (10) and (11) to get

$$\lambda\mu \left(\int_\Omega e^{u+v}\, dx \right)^2 + 64\lambda\mu \left(\int_\Omega e^{\frac{u+v}{2}}\, dx \right)^2 + 8\sqrt{\lambda\mu}(\lambda+\mu) \int_\Omega e^{u+v}\, dx \int_\Omega e^{\frac{u+v}{2}}\, dx \geq 4\lambda\mu \int_\Omega e^{\frac{u+v}{2}} e^u\, dx \int_\Omega e^{\frac{u+v}{2}} e^v\, dx. \tag{12}$$

Using Young's inequality, the left-hand side in the above inequality is bounded above by

$$2\lambda\mu \left(\int_\Omega e^{u+v}\, dx \right)^2 + C(\lambda+\mu)^2 \left(\int_\Omega e^{\frac{u+v}{2}}\, dx \right)^2, \tag{13}$$

where C is a universal constant. In addition, by the Cauchy-Schwarz inequality,

$$\int_\Omega e^{\frac{u+v}{2}} e^u\, dx \int_\Omega e^{\frac{u+v}{2}} e^v\, dx \geq \left(\int_\Omega e^{u+v}\, dx \right)^2. \tag{14}$$

Plugging (14) in (13) and remembering that (13) is an upper bound of the left-hand side in (12), we obtain

$$C(\lambda+\mu)^2 \left(\int_\Omega e^{\frac{u+v}{2}}\, dx \right)^2 \geq 2\lambda\mu \int_\Omega e^{\frac{u+v}{2}} e^u\, dx \int_\Omega e^{\frac{u+v}{2}} e^v\, dx. \tag{15}$$

By the Cauchy-Schwarz inequality and (14), we have

$$\begin{aligned}(16)\qquad \left(\int_\Omega e^{\frac{u+v}{2}}\,dx\right)^2 &\le |\Omega|\int_\Omega e^{u+v}\,dx\\ &\le |\Omega|\left(\int_\Omega e^{\frac{u+v}{2}}e^u\,dx\int_\Omega e^{\frac{u+v}{2}}e^v\,dx\right)^{1/2}.\end{aligned}$$

Using (16) in (15), we obtain

$$(17)\qquad C\frac{(\lambda+\mu)^2}{\lambda\mu}\,|\Omega| \ge \left(\int_\Omega e^{\frac{u+v}{2}}e^u\,dx\int_\Omega e^{\frac{u+v}{2}}e^v\,dx\right)^{1/2}.$$

Applying once more (14), we obtain the desired estimate. □

We can now prove Theorem 1.

Step 1. Case $1 \le N \le 3$. It is enough to treat the case $N = 3$, the cases $N = 1, 2$ being easier. By (8) and (7), $e^{v/2} - 1$ is bounded in $H^1_0(\Omega)$ (with a uniform bound with respect to λ and μ). By the Sobolev embedding, it follows that e^v is bounded in $L^{\frac{N}{N-2}}(\Omega)$. By (8) and elliptic regularity, u is bounded in $W^{2,\frac{N}{N-2}}$. For $N = 3$, $\frac{N}{N-2} > \frac{N}{2}$. By Sobolev's embedding, we deduce that u is bounded, and so must be v. This implies the desired conclusion for the corresponding extremal solution.

Step 2. General case. We adapt a method introduced in [4]. Fix $\alpha > 1/2$ and multiply the first equation in (1) by $e^{\alpha u} - 1$. Integrating over Ω, we obtain

$$\mu\int_\Omega (e^{\alpha u}-1)\,e^v\,dx = \alpha\int_\Omega e^{\alpha u}|\nabla u|^2\,dx = \frac{4}{\alpha}\int_\Omega \left|\nabla\left(e^{\frac{\alpha u}{2}}-1\right)\right|^2\,dx$$

By (4),

$$\sqrt{\lambda\mu}\int_\Omega e^{\frac{u+v}{2}}\left(e^{\frac{\alpha u}{2}}-1\right)^2\,dx \le \int_\Omega \left|\nabla\left(e^{\frac{\alpha u}{2}}-1\right)\right|^2\,dx.$$

Combining these two inequalities, we deduce that

$$(18)\qquad \sqrt{\lambda\mu}\int_\Omega e^{\frac{u+v}{2}}\left(e^{\frac{\alpha u}{2}}-1\right)^2\,dx \le \frac{\alpha}{4}\mu\int_\Omega (e^{\alpha u}-1)\,e^v\,dx$$

Hence,

$$(19)\qquad \sqrt{\lambda\mu}\int_\Omega e^{\frac{2\alpha+1}{2}u}e^{\frac{v}{2}}\,dx \le \frac{\alpha}{4}\mu\int_\Omega e^{\alpha u}e^v\,dx + 2\sqrt{\lambda\mu}\int_\Omega e^{\frac{\alpha+1}{2}u}e^{\frac{v}{2}}\,dx$$

Let us estimate the terms on the right-hand side. By Hölder's inequality,

$$(20)\qquad \int_\Omega e^{\alpha u}e^v\,dx \le \left(\int_\Omega e^{\frac{2\alpha+1}{2}u}e^{\frac{v}{2}}\,dx\right)^{\frac{2\alpha-1}{2\alpha}}\left(\int_\Omega e^{\frac{u}{2}}e^{\frac{2\alpha+1}{2}v}\,dx\right)^{\frac{1}{2\alpha}}$$

Given $\varepsilon > 0$, it also follows from Young's inequality that

$$\int_\Omega e^{\frac{\alpha+1}{2}u} e^{\frac{v}{2}}\,dx \le \frac{\varepsilon}{2}\sqrt{\frac{\mu}{\lambda}}\int_\Omega e^{\alpha u}e^{v}\,dx + \frac{1}{2\varepsilon}\sqrt{\frac{\lambda}{\mu}}\int_\Omega e^{u}\,dx.$$

Using (7), we deduce that

$$\int_\Omega e^{\frac{\alpha+1}{2}u} e^{\frac{v}{2}}\,dx \le \frac{\varepsilon}{2}\sqrt{\frac{\mu}{\lambda}}\int_\Omega e^{\alpha u}e^{v}\,dx + \frac{1}{2\varepsilon}\sqrt{\frac{\lambda}{\mu}}C\,|\Omega|\left(\frac{\lambda}{\mu}+\frac{\mu}{\lambda}\right). \tag{21}$$

where C is the universal constant of Lemma 5.

So, gathering (19), (20), (21), and letting

$$X = \int_\Omega e^{\frac{2\alpha+1}{2}u}e^{\frac{v}{2}}\,dx \quad\text{and}\quad Y = \int_\Omega e^{\frac{2\alpha+1}{2}v}e^{\frac{u}{2}}\,dx,$$

we obtain

$$\sqrt{\lambda\mu}\,X \le \left(\frac{\alpha}{4}+\varepsilon\right)\mu\,X^{\frac{2\alpha-1}{2\alpha}}Y^{\frac{1}{2\alpha}} + C\frac{\lambda}{\varepsilon}\,|\Omega|\left(\frac{\lambda}{\mu}+\frac{\mu}{\lambda}\right).$$

By symmetry, we also have

$$\sqrt{\lambda\mu}\,Y \le \left(\frac{\alpha}{4}+\varepsilon\right)\lambda\,Y^{\frac{2\alpha-1}{2\alpha}}X^{\frac{1}{2\alpha}} + C\frac{\mu}{\varepsilon}\,|\Omega|\left(\frac{\lambda}{\mu}+\frac{\mu}{\lambda}\right).$$

Multiplying these inequalities, we deduce that

$$\left(1-\left(\frac{\alpha}{4}+\varepsilon\right)^2\right)X\,Y \le C_1\left(\frac{\lambda}{\mu}+\frac{\mu}{\lambda}\right)^2\left(1+X^{\frac{2\alpha-1}{2\alpha}}Y^{\frac{1}{2\alpha}}+Y^{\frac{2\alpha-1}{2\alpha}}X^{\frac{1}{2\alpha}}\right).$$

where $C_1 = C\frac{|\Omega|}{\varepsilon}\left(\frac{\alpha}{4}+\varepsilon\right) > 0$. Hence, for every $\alpha < 4$, either X or Y must be bounded (with a uniform bound with respect to λ and μ).
Without loss of generality, $\lambda \ge \mu$ and by the maximum principle, $v \ge u$. It follows that e^u is bounded in $L^p(\Omega)$ for every $p = \alpha + 1 < 5$. Using standard elliptic regularity, the result follows. □

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