Parallel Opposite Sides Polygons

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Abstract. In this paper we consider planar polygons with parallel opposite sides. This type of polygons can be regarded as discretizations of closed convex planar curves by taking tangent lines at samples with pairwise parallel tangents. For this class of polygons, we define discrete versions of the area evolute, central symmetry set, equidistants and area parallels and show that they behave quite similarly to their smooth counterparts.

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1. Introduction

In the emerging field of discrete differential geometry, one tries to define discrete counterparts of concepts from differential geometry that preserve most of the properties of the smooth case. The idea is to discretize not just the equations, but the whole theory ([1]). Besides being important in computer applications, good discrete models may throw light in some aspects of the theory that remain hidden in the smooth context. It is also a common belief that a good discrete counterpart of a concept from differential geometry leads to efficient numerical algorithms for computing it.

There are several affine symmetry sets and evolutes associated with closed planar curves. In [2], we have considered convex equal area polygons, which can be regarded as discretizations of closed convex planar curves by uniform sampling with respect to affine arc-length. For this type of polygon, discrete notions of affine normal and curvature, affine evolute, parallels and affine distance symmetry set are quite natural.

In this paper we consider convex planar polygons with parallel opposite sides (CPOS). This type of polygons can be regarded as discretizations of closed convex planar curves by taking tangent lines at samples with pairwise

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parallel tangents. For this class of polygons, we shall define discrete notions of the area evolute, central symmetry set, equidistants and area parallels and show that they behave quite similarly to the smooth case.

Given a closed convex planar smooth curve, the area evolute (AE) is defined as the locus of midpoints of chords with parallel tangents. The central symmetry set (CSS) is the envelope of chords with parallel tangents ([4],[7]). The AE and CSS have the following properties:

- For a point x inside the curve, consider chords with x as midpoint and denote by $N(x)$ the number of such chords. Then $N(x)$ changes by 2 when you traverse the AE ([4]).
- Both AE and CSS have an odd number of cusps, at least three ([5]).
- Both AE and CSS reduce to a point if and only if the curve is symmetric with respect to a point ([8]).

For a CPOS polygon $\mathcal{P}$, we call great diagonals the lines connecting opposite vertices. For such polygons, we define the *area evolute* (AE) and *central symmetry set* (CSS) as follows: The CSS is the polygonal line whose vertices are the intersection of great diagonals, while the AE is the polygonal line connecting midpoints of the same great diagonals. We also define the notion of cusps for these polygonal lines and then prove that all the above mentioned properties remain true.

An equidistant of a closed convex planar smooth curve is the locus of points that belong to a parallel tangents chord with a fixed ratio with respect to its extremities. We have the following property:

- CSS is the locus of cusps of equidistants ([4]).

We define the equidistants of a CPOS polygon $\mathcal{P}$ as the polygon whose vertices are along the great diagonals with a fixed ratio λ with respect to the original vertices. Observe that each polygon $\mathcal{P}_\lambda$ of the equidistants family has the same CSS and AE as $\mathcal{P}$. We shall verify that, as in the smooth case, cusps of the equidistants coincides with the CSS. The locus of self-intersections of the equidistants is also an interesting set to consider. We call it the *equidistant symmetry set* (ESS) and prove that the branches of the ESS have cusps at the CSS and endpoints at cusps of the CSS or cusps of the AE.

For a point x inside the curve and c a chord with x as midpoint, let $A(x,c)$ denote the smallest area between the two regions inside the curve bounded by c. Although there may exist more that one c for a given x, we define the area parallels as level sets of A ([3],[9]). The following property is well known:

- The locus of cusps of the area parallels is exactly the AE ([6], sec.2.7).

In the CPOS polygons context, the definition of area parallels can be carried out without changes. We show that, as in the smooth case, cusps of the area parallels belong to the area evolute.

As an important tool in understanding area parallels, we show the existence a one-parameter family of polygons $\mathcal{Q}_\mu$ whose CSS is exactly the AE of $\mathcal{P}_\lambda$ and whose sides are parallel to the great diagonals of $\mathcal{P}_\lambda$. This family is uniquely defined and we call it the *dual family*. The AE of any polygon of the family $\mathcal{Q}_\mu$ does not depend on μ, and we shall denote it by $\mathcal{N}$. As far as we know, a smooth counterpart of the dual family has not been studied until now.

In the smooth case, the locus of self-intersections of the area parallels is called *affine area symmetry set* (AASS) ([6], sec.2.7) and we can define the AASS for a CPOS in a similar way. This set is difficult to understand in general, but we can describe it when the original polygon satisfies an *almost symmetry* hypothesis. For a CPOS polygon, we call a 1-diagonal any line connecting vertices such that one is adjacent to the opposite of the other. The almost symmetry hypothesis states that for some μ_0, $\mathcal{Q}_{\mu_0}$ contains the AE of $\mathcal{P}$ in its interior and the midpoints of the 1-diagonals are outside it. Under this hypothesis, the equidistants of $\mathcal{Q}_\mu$ for $\mu \geq \mu_0$ coincide with the area parallels of $\mathcal{P}$ and thus we conclude that the branches of the AASS have cusps at the AE and endpoints at the cusps of the AE or cusps of $\mathcal{N}$. Observe also that, in this context, $\mathcal{N}$ is exactly the area parallel of level half of the total area of $\mathcal{P}$.

The paper is organized as follows: In section 2 we give the main definitions for parallel opposite sides polygons and prove the basic properties. In section 3 we study the equidistants and the corresponding symmetry set. In section 4 we define and prove the existence of the dual family. In section 5 we discuss the area parallels and the affine area symmetry set.

We have used the free software GeoGebra ([10]) for all figures and many experiments during the preparation of the paper. We would like to thank the GeoGebra team for this excellent mathematical tool.

2. Basic notions

A closed planar polygon $\mathcal{P}$ is called convex if it bounds a convex region and has not parallel adjacent sides. Let $\{P_1, ..., P_n, P_{n+1}, ..., P_{2n}\}$ denote the vertices of a convex planar $2n$-gon $\mathcal{P}$. The polygon has parallel opposite sides if

$$(P_{i+n+1} - P_{i+n}) \parallel (P_{i+1} - P_i),$$

for any $1 \leq i \leq n$. Throughout the paper, indices will be taken modulo $2n$. We shall use the acronym CPOS to indicate a convex parallel opposite sides polygon. We shall consider that CPOS polygons $\mathcal{P}$ are positively oriented, i.e.,

$$[P_{i+1} - P_i, P_{j+1} - P_j] > 0,$$

for any $1 \leq i < j \leq n$.

2.1. Area Evolute and Central Symmetry Set

Denote by d_i the great diagonal P_iP_{i+n} and by $D(i+\frac{1}{2})$ the point of intersection of d_i and d_{i+1}. The *central symmetry set* (CSS) of the polygon $\mathcal{P}$ is the polygon whose vertices are $D(i+\frac{1}{2})$, $1 \leq i \leq n$.

Denote by $M_i = \frac{1}{2}(P_i + P_{i+n})$ the midpoints of the segments P_iP_{i+n} and by $m(i+\frac{1}{2})$ the lines through M_i parallel to P_iP_{i+1}, which we shall call mid-parallel lines. The polygon whose vertices are M_i, $1 \leq i \leq n$, is called the *area evolute* (AE) of the polygon (see figure 1).

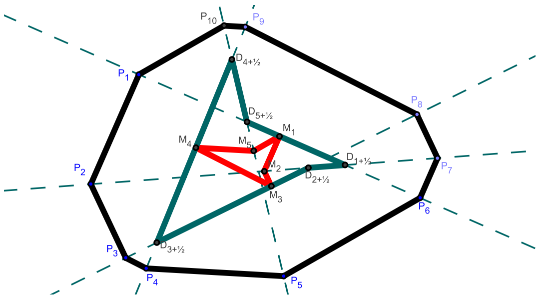


FIGURE 1. Great diagonals traced (green). The area evolute has vertices M_i (red) while the central symmetry set has vertices $D_{i+1/2}$.(green).

A CPOS is called *symmetric* with respect to a point O if $P_{i+n} - O = O - P_i$, for any $1 \leq i \leq n$.

Proposition 2.1. *The central symmetry set of a CPOS polygon reduces to a point if and only if the polygon is symmetric. Similarly, the area evolute of a CPOS polygon reduces to a point if and only if the polygon is symmetric.*

Proof. It is clear that if the CPOS polygon is symmetric with respect to a point O, then the AE and the CSS reduce to this point. It is also clear that if the AE of a polygon reduces to a point O, then the polygon is symmetric with respect to O.

Assume now that the CSS $\mathcal{P}$ reduces to a point O. Then necessarily

$$P_{i+n+1} - P_{i+n} = -\alpha\left(P_{i+1} - P_i\right),$$

for some $\alpha > 0$. But such a polygon can close only if $\alpha = 1$, and thus $\mathcal{P}$ is symmetric. □

2.2. The mid-point property

We shall denote by $e(i+\frac{1}{2})$ the edge whose endpoints are P_i and P_{i+1}. A pair of edges $e(i+\frac{1}{2})$ and $e(j+\frac{1}{2})$ determines a parallelogram $M(i+\frac{1}{2}, j+\frac{1}{2})$ consisting of the mid-points of pairs (y_1, y_2), $y_1 \in e(i+\frac{1}{2})$ and $y_2 \in e(j+\frac{1}{2})$.

Denote by $N(x)$ the number of chords with midpoint x, identifying those chords whose endpoints belong to the same sides of the polygon.

Proposition 2.2. *$N(x)$ is locally constant except at the area evolute. When you traverse a segment of the area evolute, $N(x)$ increases or decreases by 2, depending on the orientation.*

Proof. Write $x = \frac{1}{2}(x_1 + x_2)$, where $x_1 \in e(i + \frac{1}{2})$ and $x_2 \in e(j + \frac{1}{2})$. We shall assume that x_1 and x_2 are not both endpoints of $e(i + \frac{1}{2})$ and $e(j + \frac{1}{2})$, or equivalently, x is not a vertex of $M(i + \frac{1}{2}, j + \frac{1}{2})$.

We must consider the following cases:

1. If x_1 and x_2 are both in the interior of $e(i + \frac{1}{2})$ and $e(j + \frac{1}{2})$, $j \neq i + n$, or equivalently, x is in interior of $M(i + \frac{1}{2}, j + \frac{1}{2})$, then for y in a neighborhood of x, we can find unique $y_1 \in e(i + \frac{1}{2})$ and $y_2 \in e(j + \frac{1}{2})$ such that $y = \frac{1}{2}(y_1 + y_2)$.
2. If $x_1 = P_i$ and $j \neq i + n$, $j \neq i + n - 1$, then again there exists a neighborhood U of x such that for $y \in U$ we can find unique $y_1 \in e(i - \frac{1}{2}) \cup e(i + \frac{1}{2})$ and $y_2 \in e(j + \frac{1}{2})$ such that $y = \frac{1}{2}(y_1 + y_2)$. In fact, for $y \in U \cap M(i - \frac{1}{2}, j + \frac{1}{2})$ we choose $y_1 \in e(i - \frac{1}{2})$ while for for $y \in U \cap M(i + \frac{1}{2}, j + \frac{1}{2})$ we choose $y_1 \in e(i + \frac{1}{2})$.
3. Suppose that $x_1 = P_i$ and $x_2 \in e(i+n-\frac{1}{2})$ with $|e(i-\frac{1}{2})| > |e(i+n-\frac{1}{2})|$ (see figure 2). In this case x is also the midpoint of $x_2' = P_{i+n-1}$ and some $x_1' \in e(i - \frac{1}{2})$. Thus, for y in a neighborhood of x, we distinguish between $y \in M(i + \frac{1}{2}, i + n - \frac{1}{2})$ and $y \in M(i - \frac{1}{2}, i + n - \frac{3}{2})$. In the first case, y is the midpoint of $y_1 \in e(i + \frac{1}{2})$ and $y_2 \in e(i + n - \frac{1}{2})$. In the second case, y is the midpoint of $y_1 \in e(i - \frac{1}{2})$ and $y_2 \in e(i + n - \frac{3}{2})$.

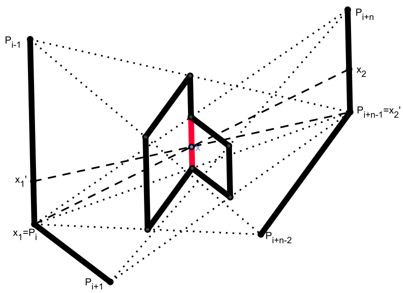


FIGURE 2. Case 3. x belongs to the red segment.

4. It remains to analyze the case in which x is on the area evolute. Suppose that $x_2 = P_{i+n}$ and $x_1 \in e(i - \frac{1}{2})$ with $|e(i - \frac{1}{2})| > |e(i + n - \frac{1}{2})|$ (see figure 3). In this case x is also the midpoint of $x_2' = P_{i+n-1}$ and some $x_1' \in e(i - \frac{1}{2})$ and the parallelograms with x in the boundary are $M(i - \frac{1}{2}, i + n - \frac{3}{2})$ and $M(i - \frac{1}{2}, i + n + \frac{1}{2})$. Consider now y in a neighborhood of x. If y is closer to $e(i - \frac{1}{2})$ then y is the midpoint of some pair (y_1, y_2), $y_1 \in e(i - \frac{1}{2})$ and $y_2 \in e(i + n + \frac{1}{2})$ and is also the midpoint of another pair (y_1', y_2'), $y_1' \in e(i - \frac{1}{2})$ and $y_2' \in e(i + n - \frac{3}{2})$. On the other hand, if y is closer to $e(i + n - \frac{1}{2})$, there are no pairs (y_1, y_2) in the neighborhood of (x_1, x_2) or (x_1', x_2') with y as midpoint. We conclude that when we traverse a segment of the area evolute from the smaller to the bigger side, $N(x)$ is increased by 2.

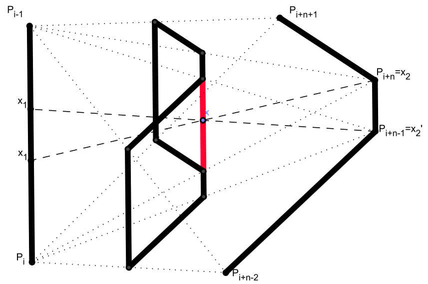


FIGURE 3. Case 4. A fold at the area evolute.

□

2.3. Cusps of the central symmetry set

In this section, we make the simplifying assumption that $D(i-\frac{1}{2}) \neq D(i+\frac{1}{2})$, for every $1 \leq i \leq n$. Define $\lambda(i+\frac{1}{2})$, $1 \leq i \leq 2n$, by

$$D(i+\tfrac{1}{2}) = P_i + \lambda(i+\tfrac{1}{2})\,(P_{i+n} - P_i)\,,$$

where $D(i+n+\frac{1}{2}) = D(i+\frac{1}{2})$. Observe that

$$e(i+n+\tfrac{1}{2}) = \frac{1-\lambda(i+\frac{1}{2})}{\lambda(i+\frac{1}{2})}\; e(i+\tfrac{1}{2}),$$

for any $1 \leq i \leq 2n$ and that $\lambda(i+n+\frac{1}{2}) = 1 - \lambda(i+\frac{1}{2})$.

We say that $D(i_0+\frac{1}{2})$ is a *cusp* of the CSS whenever $\lambda(i_0+\frac{1}{2})$ is a local extremum of the cyclic sequence $\lambda(i+\frac{1}{2})$, $1 \leq i \leq 2n$.

Proposition 2.3. *The number of cusps of the CSS is odd and* ≥ 3.

Proof. Since $\lambda(i+n+\frac{1}{2}) = 1 - \lambda(i+\frac{1}{2})$, there are an odd number of local extrema in the sequence $\lambda(i+\frac{1}{2})$, $1 \leq i \leq n$. Thus the number of cusps of the CSS is odd.

Assume now by contradiction that the sequence $\lambda(i+\frac{1}{2})$, $1 \leq i \leq 2n$, has only one local extremum. Then it would necessarily have n consecutive values bigger than $1/2$ followed by n consecutive values smaller than $1/2$. Thus the sides of the polygon would be of the form $w_1, ..., w_n, -\alpha_1 w_1, ..., -\alpha_n w_n$, with $\alpha_i > 1$, $1 \leq i \leq n$, and

$$\sum_{i=1}^{n} w_i = \sum_{i=1}^{n} \alpha_i w_i.$$

Taking the determinant product with w_1 we get

$$\sum_{i=2}^{n} [w_1, w_i] = \sum_{i=2}^{n} \alpha_i [w_1, w_i],$$

which is a contradiction since $[w_1, w_i] > 0$, for $2 \leq i \leq n$. □

In section 4 we shall show that the area evolute of $\mathcal{P}$ is the CSS of another CPOS polygon $\mathcal{Q}$ (see corollary 4.4). Thus the number of cusps of the area evolute is also odd and ≥ 3.

2.4. Symmetric and non-symmetric CPOS equal-area polygons

A polygon is called equal-area if

$$[P_{i+1} - P_i, P_i - P_{i-1}] = [P_{j+1} - P_j, P_j - P_{j-1}]$$

for any i, j ([2]). It is easy to verify that any symmetric CPOS is equal-area, but, as we shall see below, the reciprocal is true only for even n.

For n odd, consider vectors $w_1, w_2, ..., w_n$ satisfying $[w_i, w_{i+1}] < 0$ and $\sum_{i=1}^{n} w_i = 0$. For $\alpha > 0$, consider a $2n$-gon $\mathcal{P}$ whose sides are $e_i = w_i$, $e_{n+i} = -\alpha w_i$, for $1 \leq i \leq n$ odd, and $e_i = -\alpha w_i$, $e_{n+i} = w_i$, for $1 \leq i \leq n$ even. Then the polygon $\mathcal{P}$ is CPOS and equal-area.

Proposition 2.4. *Any CPOS equal-area is symmetric or obtained by the above construction.*

Proof. Let $e_1, e_2 ..., e_n, -\alpha_1 e_1, ..., -\alpha_n e_n$ denote the sides of an equal area polygon $\mathcal{P}$. Then

$$[e_1, e_2] = [e_2, e_3] = ... = -\alpha_1[e_n, e_1] = \alpha_1\alpha_2[e_1, e_2] = ... = -\alpha_n[e_n, e_1].$$

Thus $\alpha_1 = \alpha_n$ and $\alpha_i\alpha_{i+1} = 1$. If $\alpha_i = 1$ for some i, then $\alpha_i = 1$ for any i and the polygon is symmetric. If $\alpha_i \neq 1$, the necessarily n is odd and $\alpha_{i+1} = \alpha_i^{-1}$. Thus the polygon is obtained by the above construction. □

We shall refer to the above constructed polygons as non-symmetric equal-area CPOS polygons (see figure 4).

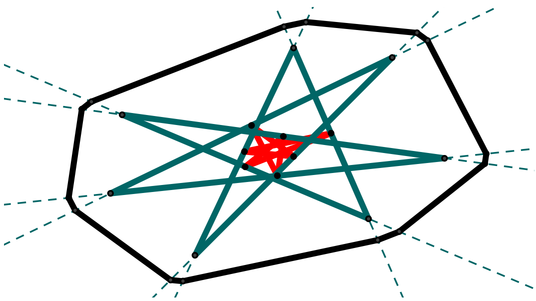


FIGURE 4. A CPOS 14-gon non-symmetric equal-area.

Proposition 2.5. *Consider a non-symmetric equal-area CPOS polygon $\mathcal{P}$. Then M_i is the midpoint of $D(i - \frac{1}{2})D(i + \frac{1}{2})$. As a consequence, the CSS have n cusps, i.e., all vertices are cusps.*

Proof. Write $P_{i+1} - P_i = -\alpha\,(P_{i+n+1} - P_{i+n})$ for some $\alpha > 0$. Then $P_{i+n} - P_{i+n-1} = -\alpha\,(P_i - P_{i-1})$. Taking $\lambda = \frac{\alpha-1}{2(1+\alpha)}$, straightforward calculations show that

$$D(i \pm \tfrac{1}{2}) = M_i \pm \lambda(P_{i+n} - P_i),$$

which proves that M_i is the mid-point of $D(i-\frac{1}{2})D(i+\frac{1}{2})$. From this, one easily concludes that each point of the CSS is a cusp. □

As we shall see in section 3, the fact that all M_i belong to the CSS implies that they are all cusps of the AE.

3. Equidistants

Even if a parallel opposite sides (POS) polygon is not convex, we can define its center symmetry set and area evolute in a similar way. We say that two POS polygons are *equivalent* if they have the same area evolute and central symmetry set. Consider the one-parameter family $\mathcal{P}(\lambda)$ whose vertices are

$$P_i(\lambda) = P_i + \lambda(P_{n+i} - P_i), \quad \lambda \in \mathbb{R}.$$

It is not difficult to see that the POS polygons of this family are equivalent, although they are not all convex. Nevertheless, we shall maintain the acronym CPOS to a one parameter family $\mathcal{P}_\lambda$ of equivalent POS polygons such that at least one $\mathcal{P}_{\lambda_0}$ is convex.

A polygon of the CPOS family $\mathcal{P}_\lambda$ associated with $\mathcal{P}$ is called an *equidistant*. For $\lambda = \frac{1}{2}$, the equidistant is exactly the area evolute (see figure 5).

3.1. Cusps of the equidistants and the CSS

Denote $f_i(\lambda) = [e(i-\frac{1}{2})(\lambda), e(i+\frac{1}{2})(\lambda)]$. Observe that $f_i(0) > 0$ and f_i changes sign only when $e(i-\frac{1}{2})(\lambda)$ or $e(i+\frac{1}{2})(\lambda)$ vanishes. Thus $f_i(\lambda) < 0$ if an only if $e(i-\frac{1}{2})(\lambda)$ and $e(i+\frac{1}{2})(\lambda)$ are in the same half-plane determined by d_i. We say that a vertex $P_i(\lambda)$ is a *cusp* of the equidistant $\mathcal{P}_\lambda$ if $f_i(\lambda) < 0$, or equivalently, if $e(i-\frac{1}{2})(\lambda)$ and $e(i+\frac{1}{2})(\lambda)$ are in different half-planes determined by d_i.

Proposition 3.1. *The set formed by the cusps of the λ-equidistants coincides with the CSS.*

Proof. We write

$$e(i \pm \tfrac{1}{2})(\lambda) = (1-\lambda)e(i \pm \tfrac{1}{2}) + \lambda e(i+n \pm \tfrac{1}{2}).$$

Thus $f_i(\lambda)$ is a quadratic function that vanishes at $\lambda(i-\frac{1}{2})$ and $\lambda(i+\frac{1}{2})$ corresponding to the intersections of d_i with d_{i-1} and d_{i+1}, respectively. Thus $f_i(\lambda) < 0$ only for λ between $\lambda(i-\frac{1}{2})$ and $\lambda(i+\frac{1}{2})$, which correspond to points of the CSS. □

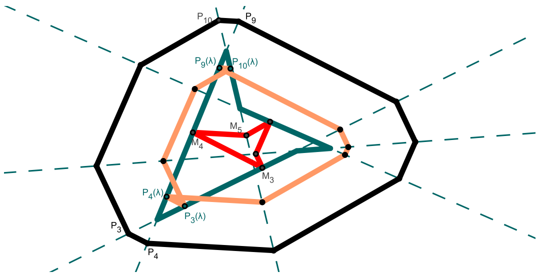


FIGURE 5. An equidistant with vertices $P_i(\lambda)$ with $\lambda = 0.2$. The labeled vertices $P_i(\lambda)$ are cusps for this equidistant. When $\lambda = 1/2$, note that M_1, M_3 and M_4, which are on the CSS, are cusps, but M_2 and M_5 are not.

3.2. Equidistant symmetry set

The self intersection of the equidistants form a set that we call *equidistant symmetry set* (ESS).

Given non-parallel sides $e(i+\frac{1}{2})$, $e(j+\frac{1}{2})$, denote by $P(i+\frac{1}{2}, j+\frac{1}{2})$ the intersection of their support lines. Denote also by $l(i+\frac{1}{2}, j+\frac{1}{2})$ the line passing through $P(i+\frac{1}{2}, j+\frac{1}{2})$ and $P(i+n+\frac{1}{2}, j+n+\frac{1}{2})$. Observe that the support line of an edge of the ESS associated with a pair of sides $(i+\frac{1}{2}, j+\frac{1}{2})$ is exactly $l(i+\frac{1}{2}, j+\frac{1}{2})$.

Consider two edges $(i-\frac{1}{2}, j+\frac{1}{2})$ and $(i+\frac{1}{2}, j+\frac{1}{2})$ of the ESS with a common vertex $(i, j+\frac{1}{2})$. We say that the vertex is a *cusp* if both edges are at the same side of d_i.

Lemma 3.2. *A vertex of the ESS which is not an endpoint is a cusp if and only if it belongs to the CSS.*

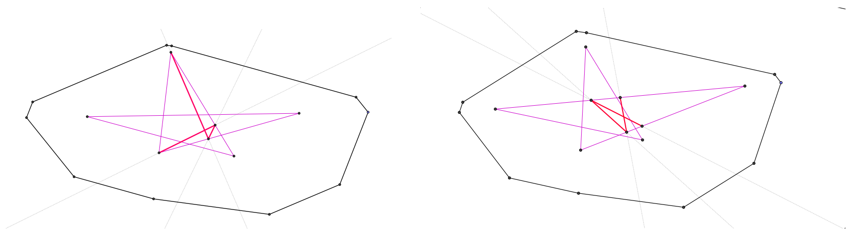
Proof. The support lines of $e(i-\frac{1}{2})(\lambda)$ and $e(i+\frac{1}{2})(\lambda)$ determine a planar region which do not contain any other support line of $e(j+\frac{1}{2})(\lambda)$, $j \neq i-1, i$. We shall denote this region by $R_i(\lambda)$.

Consider a vertex $(i, j+\frac{1}{2})$ at level λ_0. Assume the $P_i(\lambda_0)$ is at the CSS. Then the λ_0-equidistant have a cusp at this point. Observe that the support line of $e(j+\frac{1}{2})(\lambda)$ is not contained in $R_i(\lambda)$. Thus, for λ close to λ_0, the edge $e(j+\frac{1}{2})(\lambda)$ intersects $e(i+\frac{1}{2})(\lambda)$ and $e(i-\frac{1}{2})(\lambda)$ only for $\lambda > \lambda_0$ or $\lambda < \lambda_0$. For definiteness, we shall assume that these intersections occur for $\lambda > \lambda_0$. Thus, for $\lambda > \lambda_0$ we have two different intersections of the equidistants and thus the ESS will be in one side of d_i, which means a cusp.

Reciprocally, if the vertex $(i, j+\frac{1}{2})$ is not in the CSS, then the λ_0-equidistant have a regular point. For λ close to λ_0, the edge $e(j+\frac{1}{2})(\lambda)$ intersects $e(i+\frac{1}{2})(\lambda)$ and $e(i-\frac{1}{2})(\lambda)$ once. Thus the corresponding edges of the CSS crosses the line d_i, and thus is a regular point of the ESS. □

Proposition 3.3. *Every branch of the ESS can be continued until it reaches a cusp of the CSS or a cusp of the AE (see figure 6).*

Proof. Consider an edge $e(i+\frac{1}{2}, j+\frac{1}{2})$ of the ESS. This edge will end at the great diagonal d_i, d_{i+1}, d_j or d_{j+1}. For definiteness, assume it is d_{j+1}. Then we can continue the ESS by the edge $e(i+\frac{1}{2}, j+1+\frac{1}{2})$. This is always possible except in two cases: (1) $|j+1-i| = 1$ and we are at a cusp of the CSS or (2) $j+1 = i+n$ and we are at a cusp of the AE. □



(a) A branch of the ESS with both endpoints at the cusps of the CSS.

(b) A branch of the ESS with both endpoints at the cusps of the AE.

FIGURE 6. Branches of the ESS.

4. The dual family of polygons

To a family of equivalent CPOS polygons $\mathcal{P}_\lambda$, we can associate another family of equivalent CPOS polygons $\mathcal{Q}_\mu$ whose vertices are at the mid-parallel lines and whose sides are parallel to the great diagonals of $\mathcal{P}(\lambda)$ (see figure 7). In this section we prove the existence of such a family $\mathcal{Q}_\mu$, called the *dual family*.

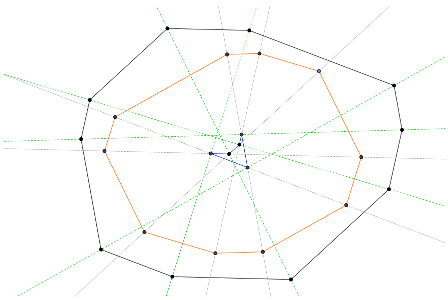


FIGURE 7. The dual family.

Denote $v(i+\frac{1}{2}) = M_{i+1} - M_i$ and $u_i = P_i - M_i$, $1 \leq i \leq n$. Define also $v(i+n+\frac{1}{2}) = v(i+\frac{1}{2})$ and $u_{i+n} = -u_i$. Observe that

$$\left[v(i+\tfrac{1}{2}), u_i - u_{i+1}\right] = 0 \tag{4.1}$$

and

$$\sum_{k=1}^{n} v(k+\tfrac{1}{2}) = 0. \tag{4.2}$$

Lemma 4.1. *Denote by A^1 and A^2 the areas of the regions of the polygon $\mathcal{P}$ bounded by the great diagonal d_1. Then*

$$A^2 - A^1 = 4\sum_{j=1}^{n}[v(j+\tfrac{1}{2}), u_j]$$

Proof. Observe that

$$A(M_1P_jP_{j+1}) = \left[\sum_{k=1}^{j-1} v(k+\tfrac{1}{2}) + u_j, \sum_{k=1}^{j} v(k+\tfrac{1}{2}) + u_{j+1}\right]$$

$$A(M_1P_{j+n}P_{j+n+1}) = \left[\sum_{k=1}^{j-1} v(k+\tfrac{1}{2}) - u_j, \sum_{k=1}^{j} v(k+\tfrac{1}{2}) - u_{j+1}\right]$$

and thus the difference between these areas is

$$\frac{1}{2}\Delta A(j) = \left[\sum_{k=1}^{j} v(k+\tfrac{1}{2}), u_j\right] - \left[\sum_{k=1}^{j-1} v(k+\tfrac{1}{2}), u_{j+1}\right].$$

So $A^2 - A^1$ is given by

$$\sum_{j=1}^{n}\Delta A(j) = 2\sum_{k=1}^{n-1}\left[v(k+\tfrac{1}{2}), \sum_{j=k+1}^{n} u_j - u_{j+1}\right] + 2\sum_{j=1}^{n}\left[v(j+\tfrac{1}{2}), u_j\right]$$

$$= 2\sum_{k=1}^{n-1}\left[v(k+\tfrac{1}{2}), u_{k+1} + u_1\right] + 2\sum_{j=1}^{n}\left[v(j+\tfrac{1}{2}), u_j\right] = 4\sum_{j=1}^{n}\left[v(j+\tfrac{1}{2}), u_j\right],$$

where we have used (4.1) and (4.2). □

Denote by $N(i+\frac{1}{2})$ the point of the mid-parallel $m(i+\frac{1}{2})$ satisfying

$$[N(i+\tfrac{1}{2}) - M_i, P_{i+n} - P_i] = \frac{1}{4}\left(A_i^2 - A_i^1\right),$$

where A_i^1 and A_i^2 denote the areas of the regions of the polygon $\mathcal{P}$ bounded by the great diagonal d_i. Denote by $\mathcal{N}$ the polygon with vertices $N_{i+\frac{1}{2}}$.

Proposition 4.2. *Given a CPOS polygon $\mathcal{P}$, we can associate a one parameter CPOS family $\mathcal{Q}_\mu$ with vertices at the mid-parallel lines and sides parallel to the great diagonals of $\mathcal{P}$. The AE of this family is exactly $\mathcal{N}$.*

Proof. Start with a point

$$Q(1-\tfrac{1}{2}) = M_n + \mu v(1-\tfrac{1}{2}) \in m(1-\tfrac{1}{2})$$

and then follow the parallel to the great diagonal d_1 until it reaches the line $m(1+\frac{1}{2})$ at $Q(1+\frac{1}{2})$. Then take the parallel to d_2 until intersect the line $m(2+\frac{1}{2})$ at $Q(2+\frac{1}{2})$. Follow this algorithm and stop after $2n$ steps, when the polygon reaches the line $m(1-\frac{1}{2})$ at $Q(2n+\frac{1}{2})$. We must prove that $Q(2n+\frac{1}{2})=Q(1-\frac{1}{2})$.

Define $\mu(i+\frac{1}{2})$, $1\leq i\leq 2n$, by the relation

$$Q(i+\tfrac{1}{2})=M_i+\mu(i+\tfrac{1}{2})v(i+\tfrac{1}{2}).$$

We must then prove that $\mu(2n+\frac{1}{2})=\mu(1-\frac{1}{2})$, where $\mu(1-\frac{1}{2})=\mu$. Denoting $q_i=Q(i+\frac{1}{2})-Q(i-\frac{1}{2})$, $1\leq i\leq 2n$, we have

$$q_i=\left(1-\mu(i-\tfrac{1}{2})\right)v(i-\tfrac{1}{2})+\mu(i+\tfrac{1}{2})v(i+\tfrac{1}{2}).$$

Since q_i must be parallel to u_i we obtain

$$\mu(i+\tfrac{1}{2})[v(i+\tfrac{1}{2}),u_i]+\left(1-\mu(i-\tfrac{1}{2})\right)[v(i-\tfrac{1}{2}),u_i]=0. \tag{4.3}$$

Denoting $\beta(i+\frac{1}{2})=\mu(i+\frac{1}{2})-\mu(i+n+\frac{1}{2})$, we get

$$\beta(i+\tfrac{1}{2})[v(i+\tfrac{1}{2}),u_i]=\beta(i-\tfrac{1}{2})[v(i-\tfrac{1}{2}),u_{i-1}],$$

where we have used (4.1). We conclude that $\beta(1-\frac{1}{2})=-\beta(n+\frac{1}{2})$, thus proving that $\mu(2n+\frac{1}{2})=\mu(1-\frac{1}{2})$.

Now let

$$\tilde{N}(i+\tfrac{1}{2})=\frac{1}{2}\left(Q(i+\frac{1}{2})+Q(i+n+\frac{1}{2})\right)=M_i+\frac{1}{2}\gamma(i+\frac{1}{2})v(i+\frac{1}{2}),$$

where $\gamma(i+\frac{1}{2})=\mu(i+\frac{1}{2})+\mu(i+n+\frac{1}{2})$ denote the vertices of the AE of $\mathcal{Q}$. We have

$$\left[\tilde{N}(i+\tfrac{1}{2})-M_i,P_{i+n}-P_i\right]=\gamma(i+\frac{1}{2})[v(i+\tfrac{1}{2}),u_i].$$

We claim that $\tilde{N}(i+\frac{1}{2})=N(i+\frac{1}{2})$, for any $0\leq i\leq n-1$. The proof of this claim will now be given for $i=0$, the other cases being similar. It follows from (4.3) that

$$\gamma(i+\frac{1}{2})[v(i+\tfrac{1}{2}),u_i]=\gamma(i-\frac{1}{2})[v(i-\tfrac{1}{2}),u_i]+2[v(i-\tfrac{1}{2}),u_i].$$

Summing from $i=1$ to $i=n$ we obtain

$$\gamma(\frac{1}{2})[v(\tfrac{1}{2}),u_1]=\sum_{j=1}^{n}[v(j+\tfrac{1}{2}),u_j]=\frac{1}{4}\left(A^2-A^1\right),$$

where the last equality follows from lemma 4.1. Thus the claim is proved.

It remains to show that $\mathcal{Q}_{\mu_0}$ is convex for some μ_0. We can choose μ_0 sufficiently large such that $\mathcal{Q}_{\mu_0}$ contains the AE of $\mathcal{P}$ in its interior. For such μ_0, one easily verify that $\mathcal{Q}_{\mu_0}$ has no self-intersections. Since its sides are parallel to u_i, $1\leq i\leq n$, which are cyclically ordered, we conclude that $\mathcal{Q}_{\mu_0}$ is convex. □

Corollary 4.3. *Suppose that the line $P_iN(i+\frac{1}{2})$ intersects the line $P_{n+i}P_{n+i+1}$ at a point P_i' at the segment $P_{n+i}P_{n+i+1}$. Then $N(i+\frac{1}{2})$ is the mid-point of the chord P_iP_i' which divides the polygon $\mathcal{P}$ in two regions of equal areas.*

Corollary 4.4. *For any CPOS polygon $\mathcal{P}$, the number of cusps of the AE is odd and ≥ 3.*

Proof. Apply proposition 2.3 to $\mathcal{Q}_\lambda$. □

5. Area parallels

For x inside $\mathcal{P}$, there may exist more than one chord c with x as midpoint. Each chord c divide $\mathcal{P}$ in two regions, and we shall denote by $A(x,c)$ the smallest area among these regions. For any $0 \leq \lambda \leq \frac{A(\mathcal{P})}{2}$, the area parallel of level λ is the set of points x for which there exists c with x as midpoint and $A(x,c) = \lambda$.

5.1. Structure of area parallels

Inside a parallelogram the area parallels are arcs of hyperbolas. To represent them, we shall consider segments connecting the intersections of these hyperbolas with the boundary of the parallelogram (see figure 8).

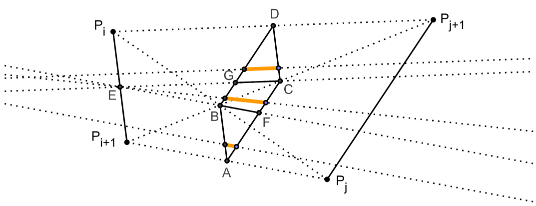


FIGURE 8. Area parallels inside a parallelogram.

Below A, B, C, D and E denote the midpoints of $P_{i+1}P_j$, P_iP_j, $P_{i+1}P_{j+1}$, P_iP_{j+1} and P_iP_{i+1}, respectively. Assuming that the area parallel through B intersects the segment AC, denote by F this intersection. Finally denote by G the intersection of the line EC with the segment BD.

Lemma 5.1. *For any point in the segment BG, the area parallel is a segment whose support line pass through E. For a point in the segment GD, the area parallel is a segment parallel to P_iP_{j+1}, while for a point in the segment AF, the area parallel is a segment parallel to $P_{i+1}P_j$.*

Proof. We may assume that the segment P_iP_{i+1} is contained in the y-axis and the segment P_jP_{j+1} is contained in the x-axis. In this case the area at a point (x,y) is xy. Then straightforward calculations prove the lemma. □

In figure 9 we can see an area parallel of a parallel opposite sides octogon.

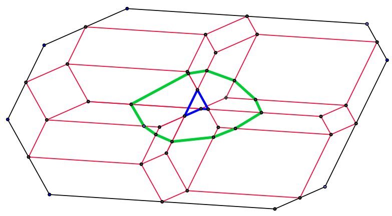


FIGURE 9. An area parallel of a parallel opposite sides octogon.

5.2. Cusps of the area parallels

A polygonal line representing an area parallel has its vertices at the boundaries of the parallelograms $M(i-\frac{1}{2}, j-\frac{1}{2})$. Consider a vertex x at the boundary $(i-\frac{1}{2}, j)$. The area parallel passing through x at the parallelogram $M(i-\frac{1}{2}, j+\frac{1}{2})$ will continue the initial polygonal line unless, $j = i-1+n$, i.e., $e(i-\frac{1}{2})$ is parallel to $e(j+\frac{1}{2})$.

In the latter case, the continuation of the area parallel may occur at the same side of $m_{i-\frac{1}{2}}$ or on the other side. We call x a cusp when the continuation of the area parallel occurs at the same side of $m_{i-\frac{1}{2}}$. We have the following proposition:

Proposition 5.2. *A vertex of an area parallel is a cusp if and only if it belongs to the area evolute.*

Proof. There are two cases to consider: If x is not in the area evolute, then the area parallel will continue in the parallelogram $M(i+\frac{1}{2}, i+n-\frac{1}{2})$ and thus x is not a cusp (see figure 2). If x is in the area evolute, the area parallel will continue in the parallelogram $M(i-\frac{1}{2}, i+n+\frac{1}{2})$ (see figure 3). Thus, close to x, the area parallel is contained in the same side of $m(i-\frac{1}{2})$ and is thus a cusp point. □

5.3. Affine area symmetry set

The *affine area symmetry set* (AASS) is the locus of self-intersections of the area parallels. It is not easy to describe this set for a general CPOS polygon, so we shall make a simplifying assumption. We say that $\mathcal{P}$ is *almost symmetric* if there exists μ_0 such that $\mathcal{Q}_{\mu_0}$ contains the area evolute in its interior and every mid-point of 1-diagonals are outside it (see figure 10).

Under the almost symmetry hypothesis, the dual polygons $\mathcal{Q}_\mu$, $\mu \geq \mu_0$, are area parallels. In fact, it follows from 5.1 that each edge of an area parallel is parallel to the corresponding great diagonal. In particular, using corollary 4.3, we have that the area parallel of level $\frac{A(\mathcal{P})}{2}$ is exactly the AE of any $\mathcal{Q}_\mu$.

Proposition 5.3. *Under the almost symmetry hypothesis, the AASS of $\mathcal{P}$ coincides with the ESS of $\mathcal{Q}$.*

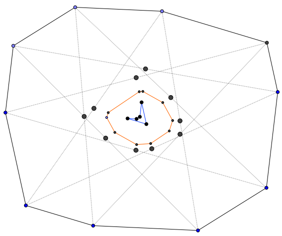


FIGURE 10. Area evolute, an equidistant of $\mathcal{Q}$ and the 10 midpoints of 1-diagonals (circles). The polygon $\mathcal{P}$ is almost symmetric.

Proof. We first observe that, as a consequence of proposition 5.2, if an area parallel contains the AE in its interior, then it has no self-intersections. Thus, by the almost symmetry hypothesis, the self-intersections of area parallels occur only for those levels corresponding to $\mathcal{Q}_\mu$, $\mu \geq \mu_0$. We conclude that the AASS of $\mathcal{P}$ coincides with the ESS of $\mathcal{Q}$. □

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