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VARIANTS ON ALEXANDROV REFLECTION PRINCIPLE AND OTHER APPLICATIONS OF MAXIMUM PRINCIPLE

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Abstract. In this article we discuss several derivations based on Alexandrov Reflection Principle and Maximum Principle. Particularly, we give some applications for surfaces of constant mean curvature in Euclidean and hyperbolic space. We also discuss the Perron Process for minimal vertical graphs in hyperbolic space. We infer some new related results in hyperbolic space. Namely, we infer symmetry and half-space results for properly embedded mean curvature one surfaces. Furthermore, we carry out a Molzon-Serrin type theorem for a classical overdetermined elliptic equation in hyperbolic space.

Introduction

In this article we shall focus some applications of Maximum Principle, particularly we shall carry out some results inferred from the magnificent idea due to Alexandrov called *Alexandrov Reflection Principle* (see [4]).

This paper is organized as follows: Firstly, we shall writedown a summarized exposition of some related classic theorems of both areas, Differential Geometry and Partial Differential Equations. Of course, because of our background, we shall discuss with more details the derivations on Differential Geometry that are closely related to surfaces theory. Secondly, we shall prove some new theorems when the *ambient space* is hyperbolic space, see the full statements at the end of the introduction. their proofs are writedown in section 2. We shall prove in Theorem A a symmetry result concerning compact mean curvature one surfaces in hyperbolic space whose boundary is the union of two circles. We shall infer in Theorem B a half-space type theorem. Besides, we shall give in Theorem C a Molzon-Serrin type theorem for a classical overdetermined elliptic equation in hyperbolic space. All surfaces (or hypersufaces) treated in this paper are of class C^2.

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Maybe it was the famous theorem of Hopf (see [45]) that have motivated the modern research about constant (non-zero) mean curvature surfaces. In 1951 Hopf proved that a closed genus zero surface in Euclidean space is a round sphere (the proof works in hyperbolic space as well). He then asked if the assumption about the genus of the surface could be removed from the statement of his theorem. This was called Hopf's conjecture: In 1986, Wente has given a counter-example to Hopf's conjecture. He build an immersed constant mean curvature torus in Euclidean space (see [116]). Afterwards Abresch simplified Wente construction (see [1]) and Pinkall-Stirling have obtained families of Wente tori (see [76]). Finally, Bobenko has obtained all constant mean curvature tori in R^3,S^3 and H^3, see [14]. Notice that Wente tori should be *immersed* by Alexandrov Theorem, as we will see in the sequel.

We shall begin section 1 (see Theorem 1) establishing the famous theorem proved by Alexandrov in 1956 (see [4]), characterizing the spheres as the only closed connected embedded genus g surfaces in Euclidean space with constant mean curvature. Maybe, more significant than the theorem is the procedure introduced by Alexandrov proving it: This has become customary to call either *Alexandrov Reflection Principle, Alexandrov Reflection* or *Alexandrov Method*. The idea of Alexandrov is quite simple, profound and it is based on Maximum Principle. It should be noted that Alexandrov Principle has been a source of many insights, for instance a result due to Hsiang in 1982 (see [46]) can be deduced from Alexandrov Principle as we will show later (see Theorem 4).

A first striking result (in Differential Geometry) using Alexandrov's Method is the theorem proved by Richard Schoen in 1983 (see [102]) characterizing the catenoid (see Theorem 2). In 1988-1989 held the second main contribution in this field: the theory of Meeks (see [66]) and Korevaar-Kusner-Solomon (see [50]), we shall say K-K-M-S. They have shown that a complete connected properly embedded constant mean curvature surface M in Euclidean space with two annuli ends is rotationally symmetric; namely, a Delaunay surface. We will say a few words now to show very briefly how the proof works: Firstly, Meeks has shown that each properly embedded annular end of constant mean curvature in Euclidean space is cylindrically bounded. The proof of this follows from a deep geometric insight based on Maximum Principle (comparison with spheres of same mean curvature), a priori Height Estimates and basic Topology. Applying either Height Estimates or Alexandrov Method one infers that the two ends and the whole surface M are contained inside a same cylinder. Then a subtle application of Alexandrov Principle which makes use of *tilted* planes, yields M is rotational. We remark that the argument using Alexandrov Method *via* tilted planes has been used elsewhere (see [84]). The first author and Rosenberg has pointed out that an adapted proof of this theorem hold for a wider class of f-surfaces that satisfies *height estimates* property (see [85]); one gets the *special embedded Delaunay surfaces* constructed and classified by the authors (see [91], and also [90]).

In the last decade the interest has grown considerably with many developments. K-K-M-S have proved a similar result in hyperbolic space (see [51]), using Hsiang's theorem (see [46]) cited before. Moreover, K-K-M-S have inferred that each properly embedded annulus end with constant mean curvature H converges geometrically and asymptoti-

cally to an end of a Delaunay surface: in hyperbolic space $H > 1$ (see [51]). The analogous study of the geometry of a conformally punctured disc end with mean curvature 1, finite total curvature, *regular* and embedded into the upper half-space model of hyperbolic space, have been achieved by the authors. That is, up to an isometry of the space each such an end converges geometrically and asymptotically to an end of an horosphere or an end of a Catenoid Cousin, either as Euclidean vertical graphs or as surfaces embedded into hyperbolic space. In fact, this geometric behavior follows from an asymptotic expansion we infer (see [95]). We shall give in Theorem B an application of this fact. It is worth mention that very recently the authors have presented a new approach giving meromorphic data for mean curvature one conformal immersions into hyperbolic space, see [97] and [99]: For related results , see also [86], [87], [104], [110], [111] and [112].

It turns out that in the study of compact constant mean curvature surfaces in Euclidean space (and in hyperbolic space), Alexandrov Method has also been applied as an important tool (see [16]). It has been conjectured that a connected compact embedded constant mean curvature surface with boundary a circle is spherical. This conjecture can be posed in higher dimensions, either in Euclidean space or hyperbolic space as well. It is still an open problem. We have been attracted by the geometric *non-variational* approach. Braga Brito, Meeks, Rosenberg and the first author, say B-M-R-SE, have verified the conjecture in 1991, under the assumptions that the surface is transverse, along the circle, to the plane of the circle (see Theorem 3). Braga Brito and the first author have proved in 1991 that if the radius of the circle and the mean curvature are equal to 1, then the conjecture is true, assuming only that the surface is immersed (see [15]). At the same year, Barbosa has shown that the conjecture is valid, if we assume that the surface is contained inside a cylinder of radius equal to the radius of the sphere of same mean curvature (see [6]). The hyperbolic version of the above Brito-Sa Earp result was carried out by the first author and B. Nelli (see [73]). The B-M-R-SE theorem cited above has a counterpart in hyperbolic space, as expected by the first author in 1991 (see [88], pp. 256) and proved by Nelli-Rosenberg (see [72]) in 1995. In hyperbolic space there exists a sharp result achieved by the first author and Lucas Barbosa: If a compact connected *immersed* surface with boundary a circle has constant mean curvature not greater than 1, then it is totally umbilic. This result was announced by Barbosa- Sa Earp in 1995, with a sketch of the proof (see [8]). The complete proof of Barbosa-Sa Earp result has appeared later (see [9] or [10]). We digress now to say that in the literature there are other relevant results proved by Koiso [52], Rosenberg-Sa Earp [84], López and Montiel [63]. On the other hand, it is not always true that "constant mean curvature surface inherits the symmetry of its boundary". There exist simple counter-examples for minimal surfaces in Euclidean space (see, for instance [58] or [102]). About the examples, several mathematicians were interested on the construction of constant mean curvature surfaces invariant under a subgroup of rigid motions: The reader is refer to Dajczer- Do Carmo [28], Lawson [57], Smyth [105] and Ordóñes [75]. In 1990, Kapouleas brought up many examples of complete embedded constant mean curvature surfaces in Euclidean space (see [47]). One year after he gave genus g ($g \geqslant 3$) immersed examples with boundary a circle (see [48]). An amazing fact is that Alexandrov Reflection yield at once several symmetry and uniqueness results about properly embedded constant mean curvature

H surfaces in hyperbolic space: For instance, if the asymptotic boundary is a point or a circle then one gets an horosphere or an equidistant surface (see [29]), if the asymptotic boundary is the union of two disjoint circles and $H = 0$ (embeddedness here is not necessary) then one gets a hyperbolic catenoid (see [62]), if the asymptotic boundary is the union of two disjoint circles and $H \neq 0$ then one gets a surface of revolution (see [30]). As a matter of fact, similar results hold for f-surfaces in hyperbolic space as we have remarked in a recent paper (see [92]). Recently, the authors have obtained some new symmetry results for constant mean curvature surfaces in hyperbolic space (see [92] and [93]). The authors have also inferred some general uniqueness (and existence) results for minimal vertical graphs in hyperbolic space (see [94]). We now remark that there have been carried out several derivations of Alexandrov techniques to hypersurfaces in Euclidean space (and Hyperbolic space) endowed with nice geometric structures, such as hypersurfaces with some r- mean curvature constant, since the Maximum Principle holds (see for instance [21], [22], [54], [79], [42], [43]). It should be mention now that there are a lot of interesting publications focus on the geometric non-variational aspect of the theory of constant (non-zero) mean curvature surfaces such as the works of Rosenberg-Sa Earp [83], Nelli-Spruck [74], Ros-Rosenberg [82], Semmler [100], Collin-Hauswirth-Rosenberg [26] and others. We apologize for any omission. At last, we would like to point out that there are point of view rather different than those we focus here; their methods are based on Functional Analysis, Spectral theory, Calculus of Variations or Geometric Measure Theory. These methods give rise to important analytic and geometric applications to constant mean curvature surfaces theory beyond the scope of this discussion: For readers convenience we refer to Wente [113], [114], [115], Hildebrandt [40], Gulliver [36], [37], Brezis-Coron [13], and Struwe [108], [109]. See also, Meeks-Yau [69], Bérard- Hauswirth [11], Barbosa-Bérard [7], Duzaar-Steffen [31], Steffen [107] and to the references on these articles.

* * *

Serrin has written in 1971 a very elegant paper (see [101]) where he proved that a "symmetric" overdetermined second order elliptic equation on a bounded domain Ω with "symmetric" boundary value data does imply that Ω must be a *ball*. A typical result is the following: Let Ω be a bounded domain with C^2 boundary Γ. Suppose there exists a C^2 function u satisfying the Poisson Diferential Equation $\triangle u = -1$ in Ω, together with the boundary conditions $u = 0 \,\text{and}\, \frac{\partial u}{\partial n} = \text{const} \quad \text{on}\, \Gamma$, where $\vec{n}$ is the unit inner normal vector to Γ. Then Ω must be a ball.

It is amazing that Serrin has used Alexandrov Method, to apply it back to Analysis in order to get symmetry results on Partial Differential Equations. Instead of applying it to Geometry, as did Alexandrov, he had the beautiful intuition to capture the *geometry* inside Alexandrov Method. He has improved one of its central background- the Maximum Principle at the boundary. Indeed, he has proved what we call the *boundary Maximum Principle at a corner* (see Lemma 2). Later on, Gidas, Ni and Nirenberg, have refined Serrin Method establishing in 1979 the called Method of Moving Plane to infer various symmetry and related properties of positive solutions of second order elliptic equations over bounded and unbounded domains (see [33]). C. Li has study sym-

metry and monotonicity of fully nonlinear elliptic equations in 1991 by introducting a simplified approach based soleley on Maximum Principle to carry out the the Moving Plane Method (see [60] and [61]). Caffarelli, Gidas and Spruck in 1989 have proved asymptotic radial symmetry of positive solutions for the conformal scalar curvature equation and others semilinear elliptic equations using a "measure theoretic" variation of the method of Moving Plane (see [20]). Recently, Korevaar, Mazzeo, Pacard and Schoen (see [55]) have given a more geometric argument than [20] looking deeper to the geometry of the conformal scalar equation (see [5] and [39]) to obtain refined asymptotics and a slightly stronger estimate. We would like to remark that a pioneer of the link between PDE and Differential Geometry is R. Finn (see [32]). The reader is also referred to the work of J. McCuan about symmetry via spherical reflection (see [65]). For general Maximum Principle on complete Riemannian manifolds see [117] and [27].

* * *

In section 2 we shall prove the following theorems. First, we will fix some conventions and we will recall some definitions: In what follows it is necessary knowledge of basic hyperbolic geometry developed in the following references (see [96], [106] and [78]). We will say the a set S in the n- dimensional hyperbolic space $\mathbb{H}^n$ is contained *inside* the $n-1$-dimensional horosphere $\mathscr{H}$ if S in contained in the mean convex open domain $\mathscr{B}$ in $\mathscr{H}^n$ with boundary $\mathscr{H}$ (we will also say that $\mathscr{H}$ *involves* S). This domain $\mathscr{B}$ is called the horoball bounded by $\mathscr{H}$; that is, the mean curvature vector $\vec{\boldsymbol{H}}$ of $\mathscr{H}$ points into $\mathscr{B}$. The complement of the closed domain $\overline{\mathscr{B}}$ in hyperbolic space we will call the *exterior* of $\mathscr{H}$ and if S is contained in this complement we will say that S lies *outside* $\mathscr{H}$. Of course, the conventions "S is contained inside M" or "S is contained outside M" can be extended in the case when M is a properly embedded hypersurface in hyperbolic space with non-vanishing mean curvature vector.

We will say that an horosphere $\mathscr{H}_\varepsilon$ is a *translated copy* of the horosphere $\mathscr{H}$ if there exists an orthogonal geodesic γ and a translation T_γ along γ (which is a hyperbolic isometry of hyperbolic space, see [96]) which takes $\mathscr{H}_\varepsilon$ to $\mathscr{H}$.

We now recall that the *asymptotic boundary* of a n-dimensional hypersurface immersed Σ in hyperbolic space is defined as follows (see [29]): Consider M immersed in the ball model of hyperbolic space $\mathbf{B}^{n+1} = \{\|x\| < 1\}$.

$$\partial_\infty \Sigma = \overline{\Sigma} \cap S^n(\infty)$$

where $S^n(\infty) = \{\|x\| = 1\}$ and $\overline{\Sigma}$ is the point set closure of Σ in B^{n+1}.

We also recall that the Catenoid Cousin are the noncompact constant mean curvature surfaces of revolution. This terminology is due to Robert Bryant (see [18]). Nice properties and a description of the Catenoid Cousin have been made by Bryant in his pioneer article about mean curvature one surfaces, see [18]. The reader is also referred to Ordóñes ([75]), and to the authors article (see [95]).

Next, we will extend a symmetry result deduced by the authors (see [92], Theorem 1 and [93], Theorem 4.2).

THEOREM A. — *Let M be a connected constant mean curvature one surface in hyperbolic space with boundary $\mathscr{C}_0 \cup \mathscr{C}_1$, where $\mathscr{C}_0$ and $\mathscr{C}_1$ are circles with radius r and R, respectively. Assume that $\mathscr{C}_0 \cup \mathscr{C}_1$ is invariant by rotations around a geodesic γ. Let $\mathscr{P}_0$ and $\mathscr{P}_1$ the unique geodesic "parallel" planes with $\mathscr{C}_0 \subset \mathscr{P}_0$ and $\mathscr{C}_1 \subset \mathscr{P}_1$. Let $\mathscr{H}$ be a fixed horosphere such that $\mathscr{C}_0$ lies on $\mathscr{H}$. Let us assume that $\mathscr{C}_1$ is contained inside $\mathscr{H}$, and that $R \geqslant r$. Then there exists a constant d (independent on M) such that if* $\operatorname{dist}(\mathscr{P}_0, \mathscr{P}_1) \geqslant d$, *then M is a surface of revolution.*

Theorem above has a corresponding statement in arbitrary dimensions and can be stated in the setting of f-surfaces as well. Notice that for *any* r and *any* R such that $R \geqslant r$, there always exists a piece M of an embedded Catenoid Cousin satisfying the assumptions of the theorem, *i.e.* $\partial M = \mathscr{C}_0 \cup \mathscr{C}_1$ (where r is the radius of $\mathscr{C}_0$ and R is the radius of $\mathscr{C}_1$).

THEOREM B. — *Let M be a connected properly immersed constant mean curvature one surface in $\mathbb{H}^3$. Let $\mathscr{H}$ be an horosphere. If $\partial M = \emptyset$ and M is contained inside $\mathscr{H}$, then M is equal to a translated copy of $\mathscr{H}$. If the boundary $\partial M \neq \emptyset$ (possibly non compact) is contained in $\mathscr{H}$ and $M \setminus \partial M$ is contained inside a translated copy $\mathscr{H}_\varepsilon$, then $M \setminus \partial M$ is contained inside $\mathscr{H}$. Furthemore, if M is embedded (with $M \setminus \partial M$ contained inside a horosphere) and the boundary of M is a circle lying on a horosphere, then M is part of an horosphere or part of an embedded Catenoid Cousin.*

We point out that if one replaces the word "immersed" by "embedded" in the above theorem, in the case $\partial M = \emptyset$, one gets the first part of a result of Rodriguez and Rosenberg (see [81], Theorem 1). Their proof does not extend to the immersed case: In fact, if the asymptotic boundary of M is a point and M is embedded then Do Carmo-Lawson theorem applies. The first statement can be viewed as a *half-space theorem* (see [44]) for constant mean curvature one surfaces in hyperbolic space. Let us digress for a moment to say that the Half-Space Theorem of Hoffman-Meeks for minimal surfaces in Euclidean space is one of most admired geometric application of Maximum Principle in Differential Geometry. It has an equivalent in the setting of f-surfaces of minimal type (see [90]). Another beautiful result is the Maximum Principle at Infinity inferred by Langevin-Rosenberg in 1988 (see [59]), by Meeks-Rosenberg in 1990 (see [67]) and by Soret in 1993 (see [103]).

We continue our discussion, saying that we do not know if first and second assertions in the statement of Theorem B remain true for higher dimensions. Third assertion was proved by the authors (see [93], Theorem 4.1). We have decided to restate it here for the reader's convenience. It is remarkable, in contrast with the case where the ambient is Euclidean space, we make no assumptions about the topology. Moreover, third assertion hold for constant mean curvature surfaces.

Molzon has produced the Serrin typical result (cited in the introduction) for bounded domains Ω in hyperbolic space (see [70]). We remark that Serrin general results is a project not yet undertaken in hyperbolic space.

Let Δ be the second order elliptic Laplacian operator acting on $\mathbb{H}^{n+1}$. We have the following Molzon-Serrin type result.

THEOREM C. — *Let Ω be a domain in $\mathbb{H}^{n+1}$ with boundary a properly embedded hypersurface M whose asymptotic boundary is a point. Let $\vec{n}$ be a global unit normal to M pointing into Ω. Suppose that there exists a $C^2(\overline{\Omega})$ function f nonnegative on Ω satisfying*

$$\begin{cases} \Delta f = -1 & in\,\Omega \\ f = 0 \quad and \quad \dfrac{\partial f}{\partial n} = k\ (const)\ on\ M. \end{cases}$$

Then Ω is a horoball and M is an horosphere.

1 *Some Earlier Results on Alexandrov Reflection and Maximum Principle*

We begin this section recalling the definition of f-surface. Let M be a surface either immersed into $\mathbb{R}^3$ or else immersed into $\mathbb{H}^3$, oriented by a global unit normal field N whose mean curvature $H := \dfrac{k_1 + k_2}{2}$ and extrinsic Gaussian curvature $K_e := k_1 k_2$ (k_1, k_2 are the principal curvature of M), satisfy a Weingarten relation of the form:

$$H = f(H^2 - K_e).$$

We shall require that f is a C^1 function defined on $[0, +\infty[$, satisfying:

$$\forall t \in [0, +\infty[,\ 4t(f'(t))^2 < 1.$$

It is said that f is elliptic if f satisfies the inequality above. If M satisfies the first relation above for f elliptic, then M is called either a *special Weingarten surface* or a f-surface. They have been studied by Hopf (see [45]), Hartman and Wintner (see [38]), Chern (see [24]) and by Bryant (see [19]). More recently, there has been considerable progress, as we have pointed out in the introduction. We now commence to discuss some of the *loc. cit.* theorems.

THEOREM 1 (Alexandrov). — *A compact connected embedded constant (non-zero) mean curvature surface M in Euclidean or hyperbolic space is a round sphere.*

Proof. — Briefly Alexandrov Reflection Principle works as follows: Fix a certain geodesic plane $\mathscr{P}$ and consider the *foliation* of *all* translated copies of $\mathscr{P}$ along a certain geodesic γ that cuts orthogonally $\mathscr{P}$. Then coming from the infinity towards M doing such translations, one make successive symmetries about these planes and look to the possible first point of tangent *touching* contact with M and an element of this family. We will proceed the proof as follows. Let $\mathscr{P}$ be a fixed geodesic plane far away from M. It suffices to prove that there exists a translated copy $\mathscr{P}_{t_0}$ such that M is symmetric about $\mathscr{P}_{t_0}$ and is a graph over $\mathscr{P}_{t_0}$ in each side. Let $\mathscr{P}_t$ be the 1-parameter family of translated copy of $\mathscr{P}$, where we choose the parameter t, such that $\mathscr{P}_t$, $t > 0$ is contained in the connected

component determined by $\mathscr{P}$ which contains M and $t = \operatorname{dist}(\mathscr{P}_t, \mathscr{P})$, hence $\mathscr{P}_0 = \mathscr{P}$. Translating $\mathscr{P}$ towards M one gets a first plane $\mathscr{P}_{t_1}$ that reaches M; that is $\mathscr{P}_{t_1} \cap M \neq \emptyset$, but if $t < t_1$ then $\mathscr{P}_t \cap M = \emptyset$. Thus $\mathscr{P}_{t_1}$ is tangent to M at a point, say p, and M is contained in on side of $\mathscr{P}_{t_1}$. Locally, around each such point p, M is a graph over $\mathscr{P}_{t_1}$. To continue the proof we will fix some conventions: Let Ω be the bounded domain with boundary M. Let $\mathscr{P}_t^-$ be the half-space determined by $\mathscr{P}_t$ that contains $\mathscr{P}$, let M_t be the part of M lying in $\mathscr{P}_t^-$ and let M_t^* be the symmetry of M_t about $\mathscr{P}_t$. Then there exists at least a small $\varepsilon_0 > 0$ such that $M_{t_1+\varepsilon}$ is a graph over $\mathscr{P}_{t_1}$ and $M_{t_1+\varepsilon}^*$ is contained in Ω, for $0 \leqslant \varepsilon \leqslant \varepsilon_0$.

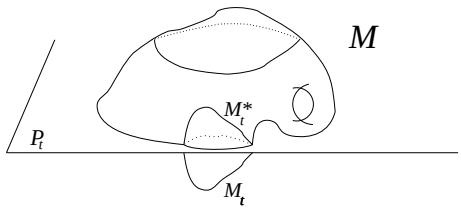

Figure 1

It is clear that for some $t_2 > t_1 + \varepsilon$, the symmetrized cap $M_{t_2}^*$ intersects the exterior of $\overline{\Omega}$ in a non empty set. Now as $t \uparrow t_2$, $t > t_1 + \varepsilon_0$ one can translate $\mathscr{P}_t$ and reflects M_t about $\mathscr{P}_t$, successively until one reachs a first point of tangent contact of the reflection of M_{t_0} about $\mathscr{P}_{t_0}$ with M, for some t_0 varying in the interval (t_1, t_2). Hence either $M_{t_0}^*$ touch M at an interior point (see Figure 2) or $M_{t_0}^*$ touch M at a boundary point (see Figure 3).

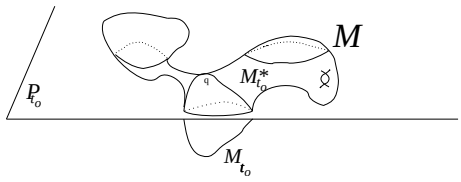

Figure 2

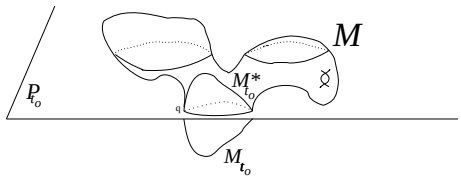

Figure 3

In any case, $M^*_{t_0}$ and M are one in a side of the other, in a neighbourhood of a first point of tangent contact q. As reflections invert normal vectors, the mean curvature vector $\vec{\boldsymbol{H}}$ of both $M^*_{t_0}$ and M at such point q are the same. At last, one can apply either Hopf Interior Maximum Principle or Hopf Boundary Maximum Principle (see for instance [77], [34] or [10]) to infer $\mathscr{P}_{t_0}$ is a plane of symmetry of M. The procedure shows that, in each step as $t \uparrow t_0$, M_t is a graph over $\mathscr{P}_t$ in both sides. This completes the proof of the theorem. □

It is worth note that there exists a same Alexandrov theorem for f-surfaces of constant (non-zero) mean curvature type, since the Maximum Principle holds (see [17]). The method of the proof is the Alexandrov Reflection Principle.

Next, we shall very briefly outline the proof of Schoen's theorem. For a proof making use of powerful tools as the *monotonicity formula* and the *maximum principle at infinity* the reader is referred to [80]. Another useful tool that has been used in minimal surface theory is the called *flux formula*. It has an important rule in the proof of Schoen's theorem bellow. It is also intensively applied to constant mean curvature theory. The reader is referred to ([102], [41] (minimal surfaces)) and ([56], [50], [16] and [9] (constant mean curvature surfaces)). Of course the main tool for minimal surfaces in Euclidean space is the so-called *Weierstrass representation*. For constant (non-zero) mean curvature surfaces in Euclidean space a Weierstrass type formula was given by Kenmotsu (see [49]). The authors have recently inferred a Weierstrass-Kenmotsu formula for prescribed mean curvature surfaces in hyperbolic space (see [98]). For related formulas see, for instance [2], [3] and [53].

THEOREM 2 (Schoen). — *The catenoid is the only complete connected properly immersed minimal surface in Euclidean space with finite total curvature and two embedded ends.*

Proof. — The main ideas in the proof of the theorem are the following: First, Schoen has derived an asymptotic expansion for minimal embedded ends on finite total curvature (each such an end is conformally equivalent to a punctured disc): namely each end is asympotic geometrically to a plane or to a catenoid (for a very clear derivation of this asymptotic expansion see [41]). Applying either Maximum Principle at infinity or Half-Space Theorem it can be shown that each end is asymptotic to a fixed catenoid. By applying Flux Formula one gets that the ends are parallel and one may suppose their limiting normals are vertical having the same logarithmic growth. In fact, Monotonicity Formula (see [34]) yields M is embedded. A beautiful *monotonicity* variation of Alexandrov Method (see [102], Theorem 1) gives rise that M has an horizontal plane of symmetry; hence the catenoids have the same axis. Then applying Alexandrov Reflection Principle again moving parallel planes to the vertical axis, one therefore infers that M is a surface of revolution □

It is amazing that the assumption *finite total curvature* can be dropped and replaced (in the setting of properly embedded surfaces) by finite topology. This is a consequence of a great theorem proved by P. Collin (see [25]) inspired by the works of Meeks-Rosenberg (see [68]) and Meeks-Yau (see [69]): *Let M be a properly embedded minimal*

surface in Euclidean space with at least two annuli ends. Then M has finite total curvature if and only if M has finite topology. Another nice characterization of the catenoid was given by Lopez and Ros (see [64]): *Among the complete embedded non-flat minimal genus zero surfaces in Euclidean space the catenoid is the only of finite total curvature.*

The first author with Braga Brito, Meeks and Rosenberg, say B-M-R-SE, have proved in 1991 the following theorem.

THEOREM 3 [B-M-R-SE]. — *Let M be a compact connected embedded constant mean curvature surface in Euclidean space with strictly convex planar boundary Γ transverse, along the boundary, to the plane P containing the boundary Then M is contained in one of the half-spaces determined by P. Under the same assumptions if Γ is a circle it follows that M is a spherical cap.*

Proof. — We will focus two central undesirable *clichés* to show how to eliminate them, rather than outline a complete proof. The first main configuration to kill is given by Figure 4. One can handle it by applying a suitable variation of Alexandrov Reflection called *graph lemma* carried out by the first author and Braga Brito (see [15]) to arrive to a contradiction. This is shown in Figure 5: Roughly speaking, let P be the horizontal plane determined by the boundary Γ. Moving vertical planes and doing Alexandrov Reflection one gets a first point of contact p, hence the surface is a graph in both sides of the plane π in Figure 5; leading to an obvious contradiction.

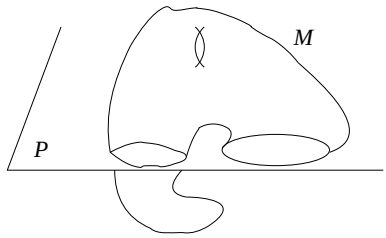

Figure 4

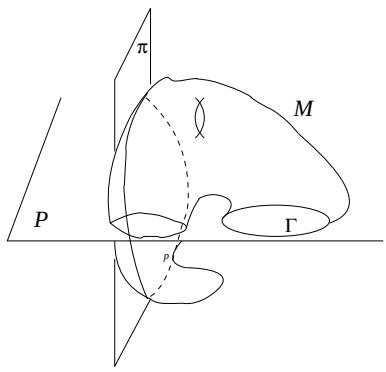

Figure 5

The second main configuration to kill is expressed in figure 6. To arrive to a contradiction one applies the flux formula to M and to M_1, see Figure 6 and Figure 7, respectively. Summarizing, let $\vec{n}$ be the interior unit conormal along ∂M, and let Y be the unit normal to P. Thus one can calculate the expression $\int_\Gamma Y \cdot \vec{n}$, where $\cdot$ denotes the standard inner product, by applying Flux Formula to the configuration given by Figure 6 and to the configuration given by Figure 7. The former gives $2H$ area (D) and the latter gives $2H$ area (D_1), where H is the mean curvature, and $D \subset P$ $D_1 \subset P$ are shown in Figures 6, 7, respectively. As area $(D_1) <$ area (D), this leads to the desired contradiction.

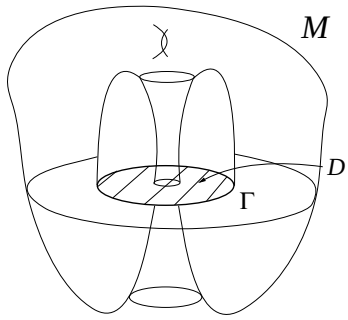

Figure 6

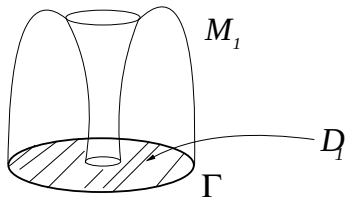

Figure 7

□

We will now sketch the proof of some results when the ambient is hyperbolic space.

THEOREM 4 (Hsiang). — *A cylindrically bounded complete properly embedded constant mean curvature surface M in hyperbolic space is rotational.*

Sketch of the proof. — The proof is a consequence of a simple observation making use of Alexandrov Method. The observation is the following: Let C be a cylinder in hyperbolic space, namely the locus of points at the same distance to a fixed geodesic γ. The intersection of *any* geodesic plane $\mathscr{P}$ such that $\partial_\infty \mathscr{P} \cap \partial_\infty \gamma = \emptyset$ with C is either *compact* or else is empty.

Now let $\mathscr{P}_{t_0}$ be a fixed geodesic plane of symmetry of C, let γ be a geodesic cutting $\mathscr{P}_{t_0}$ orthogonally and let $\mathscr{P}$ any translated copy of $\mathscr{P}_{t_0}$ along γ far away from $\mathscr{P}_{t_0}$. Then one can run Alexandrov Method, as in the Alexandrov theorem using the foliations of

geodesic planes $\mathscr{P}_{t_0}$ along γ coming from the infinity towards M, to infer that the reflection of each part of M in the two sides of $\mathscr{P}_{t_0}$ *is contained inside* M. One concludes that $\mathscr{P}_{t_0}$ is a plane of symmetry of M. □

Notice that the assumptions yield that M has non-zero mean curvature; this can be seen by applying Maximum Principle, using minimal hyperbolic catenoids as useful barriers. Of course, the above theorem is valid for f-surfaces in hyperbolic space (see [92]).

THEOREM 5 (Barbosa, Sa Earp). — *Let M be a compact surface immersed into hyperbolic space whose boundary is a circle. Assume that the mean curvature H is constant with $H^2 \leqslant 1$.Then M is part of a horosphere (if $H^2 = 1$) or part of an equidistant surface (if $H^2 < 1$).*

Sketch of the proof. — The first important observation is the following general fact: M is contained *inside* any horosphere $\mathscr{H}$ that contains the boundary of M, denoted by Γ. This assertion follows by Maximum Principle using the foliations of horospheres having the same asymptotic boundary as $\mathscr{H}$ (see Figure 8). The existence of the foliations of horospheres issuing from a given point of the asymptotic boundary is *crucial* for the proof, as we shall show in the sequel. This makes an important difference from the Euclidean situation and has allowed the proof of the theorem for $H^2 \leqslant 1$. In fact, using these foliations combined with Maximum Principle, it is not difficult to see that when Γ is a circle M can be trapped in two abstract caps: Look to the half-space model of hyperbolic space and suppose that Γ is contained in a vertical plane P, then the two abstract caps are *horizontal graphs* with respect to P. Now take two H-caps with the same boundary and same mean curvature H as M, apart from M doing horizontal Euclidean translations (which are parabolic isometries of hyperbolic space in the upper half-space model). Then moving them horizontally back towards M again it follows that it is not possible to get a first point of contact during this movement *until* the boundaries of M and the H-caps are identify. Thus we can replace the abstract caps by these H-caps (see Figure 9). We observe now that there exists a Flux Formula in hyperbolic space similar to the Flux Formula applied before in Euclidean space. We can therefore applied this Flux Formula to ensure that at least the two surfaces have a boundary point of tangent contact. Thus M is a cap of a totally umbilic surface with mean curvature H, $H^2 \leqslant 1$, as desired. This achieves the sketch of the proof of the theorem.

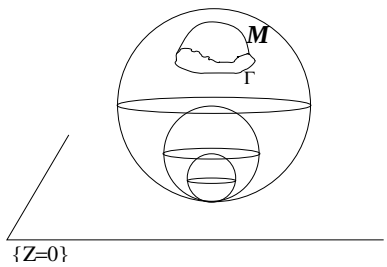

Figure 8

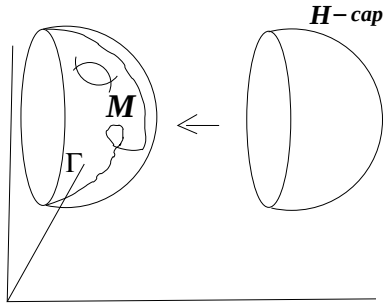

Figure 9 □

Now we shall state the *Perron Process* for minimal vertical graphs in hyperbolic space that has been established in [94] and we shall give an application based essentially on the Maximum Principle. Consider the upper half-space model of hyperbolic space $\mathbb{H}^3 = \{(x,y,z), z > 0\}$ equipped with the hyperbolic metric $\mathrm{d}s^2 = \frac{1}{z^2} \cdot (\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)$. Let $\Omega \subset \mathbb{R}$ be a domain in the asymptotic boundary and let $f : \partial\Omega \to [0,\infty[$ be a continuous function. Let us consider the Dirichlet problem (P) :

$$\begin{cases} \mathscr{D}u := \sum_{i,j=1}^{n} (\delta_{ij} - \frac{u_i u_j}{1+|Du|^2}) u_{ij} + \frac{n}{u} = & 0 \text{ on } \Omega \\ & u_{|\partial\Omega} = f \\ & u \in C^2(\Omega) \cap C^0(\overline{\Omega}) \ . \end{cases}$$

We call the equation $\mathscr{D}u = 0$ the *minimal vertical equation in hyperbolic space*. Notice that the *interior maximum principle* for minimal surfaces in hyperbolic space merely says the following: If two (connected) minimal surfaces are touching at some (common) interior point p, and one stands in a side of the other around p, then the two surfaces are equal in a neighbourhood of p. There is an analogous *boundary maximum principle*.

We shall now define the important notions of *subsolution* and *supersolution*. Consider problem (P) above where Ω is any domain of $\mathbb{R}^2$ and f is any non negative continuous function on $\partial\Omega$. Let $u : \overline{\Omega} \to [0, +\infty[$ be a continuous function. Let $U \subset \Omega$ be a closed round disc. The reader can take as a fact (see Theorem 2.3 in [94]) that $u_{|\partial U}$ has an unique minimal extension $\tilde{u}$ on U, continuous up to ∂U. We then define the continuous function $M_U(u)$ on $\overline{\Omega}$ by:

$$M_U(u)(x) = \begin{cases} u(x) & \text{if } x \in \overline{\Omega} \setminus U \\ \tilde{u}(x) & \text{if } x \in U \ . \end{cases}$$

Let now $u : \overline{\Omega} \to [0, +\infty[$ be a continuous function. We say that u is a *subsolution* (resp. *supersolution*) of (P) if:

i) $u_{|\partial\Omega} \leqslant f$ (resp. $u_{|\partial\Omega} \geqslant f$).

ii) For any closed round disc $U \subset \Omega$ we have $u \leqslant M_U(u)$ (resp. $u \geqslant M_U(u)$).

We make in the following some important remarks about subsolutions and supersolutions.

a) If u is a strictly positive C^2 function on Ω, then

$$u \text{ is a subsolution} \Leftrightarrow \mathscr{D}u \geqslant 0$$
$$u \text{ is a supersolution} \Leftrightarrow \mathscr{D}u \leqslant 0\,.$$

b) If u and v are two subsolutions (resp. supersolutions) of (P) then $\sup(u,v)$ (resp. $\inf(u,v)$) again is a subsolution (resp. supersolution).

c) Also if u is a subsolution (resp. supersolution) and $U \subset \Omega$ is a closed round disc then $M_U(u)$ is again a subsolution (resp. supersolution).

d) Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and let $u, v : \overline{\Omega} \to [0, +\infty[$ be two continuous functions such that $M_U(u) \geqslant u$ and $M_U(v) \leqslant v$ for any closed round disc $U \subset \Omega$. Suppose that $u_{|\partial\Omega} \leqslant v_{|\partial\Omega}$, then we have $u \leqslant v$ on Ω, *i.e.* "a supersolution is greater than a subsolution".

e) For any domain $\Omega \subset \mathbb{R}^2$, if f is any continuous positive function on $\partial\Omega$, then the vanishing function $u \equiv 0$ on $\overline{\Omega}$ is a subsolution for (P). Observe that for every $x \in \Omega$ there exists a subsolution u of (P) with $u(x) > 0$. Indeed let Π be any hemisphere centered at x such that $\partial_\infty \Pi \subset \Omega$. Then Π is the graph of a continuous function v defined on a closed round disc $U \subset \Omega$. Then set $u = v$ on U and $u = 0$ on $\overline{\Omega} \setminus U$. One easily verify that the function u is a subsolution.

f) Now let $\Omega \subset \mathbb{R}^2$ be any bounded domain and let $f : \partial\Omega \to [0, +\infty[$ be any continuous function. There are at least two natural ways to construct a supersolution for problem (P). Firstly, to see this just take any geodesic plane $\mathscr{P}$ (Euclidean hemispheres orthogonal to the asymptotic boundary of hyperbolic space) *involving* the graph of f in Euclidean sense. By using the foliation of geodesic planes obtained by translating (hyperbolically) $\mathscr{P}$, one can conclude that if the minimal vertical graph "escapes" from $\mathscr{P}$ one get a contradiction with the Maximum Principle. Secondly, let $C \subset \mathbb{H}^3$ be a hyperbolic catenoid such that its orthogonal projection, C_0, on $\mathbb{R}^2$ has a non empty intersection with Ω. Let $x \in \Omega$ be an interior point of C_0. Consider the homotheties $h_{\lambda,x}$ with respect to x, $\lambda > 0$. Clearly, for $\lambda > 1$ big enough, a piece of $h_{\lambda,x}(C)$ is a vertical graph v over a domain containing $\overline{\Omega}$ and $v_{|\partial\Omega} > \sup(f)$. That is, v is a supersolution for (P).

Finally, we remark that a u solution of the minimal equation in Euclidean space is a subsolution for the minimal vertical equation in hyperbolic space, *i.e.* $\mathscr{D}u \geqslant 0$.

The *Perron Process* for minimal vertical graphs in hyperbolic space is given by the following theorem

THEOREM 6 (Sa Earp, Toubiana). — *Let $\Omega \subset \mathbb{R}^2$ be a domain and let $f : \partial\Omega \to [0, +\infty[$ be a continuous function. Suppose that problem (P) has a supersolution φ. Set*

$\mathscr{S}_\varphi = \{u,$ subsolution of $(P),\ u \leqslant \varphi\}$. We define for each $x \in \overline{\Omega}$

$$v(x) = \sup_{u \in \mathscr{S}_\varphi} u(x).$$

Then the function v is C^2 on Ω and satisfies the minimal vertical equation in hyperbolic space. Furthermore, consider $p \in \partial\Omega$ and suppose that either one of the following cases holds:

i) $\partial\Omega$ is C^0-convex at p.

ii) p has a barrier and $f \geqslant \alpha > 0$ on $\partial\Omega$.

iii) p has a barrier and $f(p) = 0$.

Then v is continuous up to p and $v(p) = f(p)$. In particular if Ω is C^0 convex, the function v is continuous up to $\partial\Omega$.

As an application of *Perron Process* we shall construct complete minimal vertical graphs invariant by a discrete group of horizontal Euclidean translations:

Corollary 7 . — *Let $\Omega \subset \mathbb{R}^2$ be an unbounded domain with $\partial\Omega$ embedded. Suppose that:*

i) There exists a non null vector $\xi \in \mathbb{R}^2$ such that $\overline{\Omega}$ is invariant by the horizontal translation $T(x) = x + \xi$

ii) $\overline{\Omega}$ is contained in a band $\mathscr{B}$ invariant under T.

iii) For any $p \in \partial\Omega$, $p \neq 0$, there exists a hyperbolic catenoid C such that $\partial_\infty C \cap \Omega = \emptyset$, $p \in \partial_\infty C$ and such that the segment joining the centers of the two components of $\partial_\infty C$ intersects Ω. This last condition means that the curvature of a non convex part of the boundary of Ω is not too big with respect to the width of the band. So it can be compared with the curvature of the circles that form the asymptotic boundary of the "smallest" minimal catenoid that stands across the band. Consider problem (P) where f is the vanishing function on $\partial\Omega$. Then (P) has a solution v which is invariant under the horizontal translation T. Consequently, the graph S of v is a complete minimal surface of $\mathbb{H}^3$ invariant under the discrete group of parabolic isometries $\{T^q,\ q \in \mathbb{Z}\}$.

Sketch of the proof. — Notice first that *any* solution of the Dirichlet Problem (P) is bounded. In fact a more general result holds: Consider our domain Ω lying in the band $\mathscr{B}$ and a bounded value boundary data f on $\partial\mathscr{B}$. Take any catenoid $\mathscr{C}$ across $\mathscr{B}$ such that $\mathscr{C}$ stands above f in the Euclidean sense. There are many catenoids with this property. Of course any piece of $\mathscr{C}$ that is a graph over a part of Ω is a solution of the minimal vertical equation in hyperbolic space. What is interesting to observe is that any vertical upward translation of $\mathscr{C}$ is still a supersolution! Geometrically this means that the mean curvature vector $\vec{\boldsymbol{H}}$ of any vertical upward translation of a minimal solution is a downward pointing normal (see Figure 10). Precisely, if w is a C^2 function satisfying $\mathscr{D}w = 0$, then $\mathscr{D}(w + b) < 0$ for any $b > 0$. So using these vertical translations and maximum

principle it is now easy to conclude the assertion that any minimal extension of f to Ω must be bounded, if f is bounded. Furthermore it can be proved that if Ω is convex and f is bounded and uniformly continuous then the minimal vertical extension u to Ω is unique. Let u_B be *standard* solution in the band B of our Dirichlet Problem taking zero value boundary data. As a matter of fact this solution exists (and is unique). Let us take u_B as a supersolution for our problem. Now observe that the catenoids given by condition *iii)* are "good barriers" for our problem in the sense that any solution of our Dirichlet Problem in Ω must stands bellow such catenoids by the argument given just above (see Figure 11). This means that *any* solution must be above the subsolution $u \equiv 0$ (of course) and bellow these catenoids forcing that the solution given by *Perron Process* takes the prescribed continuous boundary value data f. At last it can be proved uniqueness, that is the solution is invariant by a discrete group of Euclidean translations.

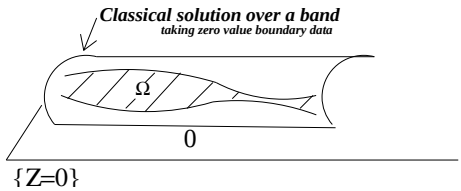

Figure 10

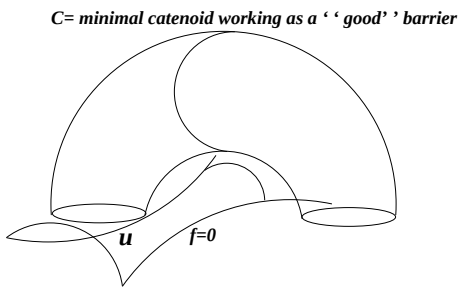

Figure 11 □

2 *Proof of the new results*

Proof of Theorem A

We will proceed the proof analogously to [92], Theorem 1 and [93], Theorem 4.2.

We now recall and fix some notations:

Let γ be a geodesic line in hyperbolic space. Let $\mathscr{C}_0$ be a circle of radius r invariant by rotations around $\boldsymbol{\gamma}$. Let $\mathscr{H}$ be a fixed (between two) horosphere such that $\mathscr{C}_0$ lies on

$\mathscr{H}$. Let $\mathscr{C}_1$ be a circle of radius R, $R \geqslant r$ such that $\mathscr{C}_0 \cup \mathscr{C}_1$ is invariant by rotations around γ. Let $\mathscr{P}_0$ and $\mathscr{P}_1$ the unique geodesic "parallel" planes with $\mathscr{C}_0 \subset \mathscr{P}_0$ and $\mathscr{C}_1 \subset \mathscr{P}_1$.

Let p be the asymptotic point of γ which is not the asymptotic point of $\mathscr{H}$. Notice that γ cuts orthogonally $\mathscr{H}$ (recall that $\mathscr{C}_0 \subset \mathscr{H}$) at a point q and that the open geodesic ray $]p,q[$ lies entirely outside $\mathscr{H}$. Take V_R the unique hyperbolic cylinder issuing from p, passing through $\mathscr{C}_1$ (note that $R = \operatorname{dist}(V_R,\gamma)$ and that V_R is invariant by rotations around γ). Let D_0 be the (closed) "disc" in $\mathscr{H}$ whose boundary is $\mathscr{C}_0$. Consider now $\mathscr{H}_t$ the 1-parameter family of horosphere such that $\mathscr{H}_t$ is the translated copy of $\mathscr{H}$ along γ. The parameter t, $-\infty < t < \infty$ is chosen such that t is the oriented distance from $\mathscr{H}_t$ to $\mathscr{H} := \mathscr{H}_0$, with the convention that γ is oriented from p to p_1, where $\partial_\infty \gamma = \{p, p_1\}$ (p_1 is the asymptotic point of $\mathscr{H}$). We can assume that $\mathscr{C}_1$ is contained in some $\mathscr{H}_{t_1}$ with $\mathscr{C}_1 = \partial D_1$, where D_1 is the closed "disc"in $\mathscr{H}_{t_1}$ whose boundary is $\mathscr{C}_1$, *i.e.* $D_1 \subset \mathscr{H}_{t_1}$ (D_1 lies in the point set closure of the interior of V_R). Our assumption implies $t_1 > 0$. Consider T, $T > 0$ bigger enough such that $\mathscr{H}_T \cap M = \emptyset$, hence M lies outside $\mathscr{H}_T$. Let D_T be the "disc" defined by the intersection of $\mathscr{H}_T$ with the point set closure of the interior of V_R.

Claim 1. — *$M \setminus \partial M$ is entirely contained inside the cylinder V_R and it is contained inside $\mathscr{H}$ as well.*

Proof of Claim 1. — Consider V_d, $d > 0$ the 1-parameter family of cylinders which are invariant by rotations around γ, where $d = \operatorname{dist}(V_d,\gamma)$ (The asymptotic boundary of the cylinders V_d are the same). A trivial compactness argument show that there exists d_1, $d_1 \geqslant R$ such that M is entirely contained inside V_d for all $d \geqslant d_1$. Now moving V_d towards V_R as $d \downarrow R$, $d \geqslant d_1$, we therefore infer that this family $\{V_d\}$ cannot fit M during this movement (for $d > R$) at an interior first point of contact, by applying Interior Maximum Principle, since the mean curvature of any cylinder calculated with respect to the inner orientation is strictly bigger than 1. Notice that a first point of contact, if any, would be necessarily point of tangent contact, since our assumption provides a guarantee that the boundary component $\mathscr{C}_0$ of M is contained inside V_R. Thus V_d does not touch M for $d > R$ and $V_R \cap M = \mathscr{C}_1$. This proves the first assertion in the statement. The idea of the proof of the second assertion is the same. Take a translated copy $\mathscr{H}_t$ of $\mathscr{H}$, choosing $t << 0$ such that M is contained inside $\mathscr{H}_t$. Thus translating $\mathscr{H}_t$ towards $\mathscr{H}$ as $t \uparrow 0$ using the same reasoning as before one completes the proof of the claim. Note that it follows from Claim 1 that $M \cap \left(\mathscr{H}_{t_1} \setminus \overline{D_1}\right) = \emptyset$ and $M \cap \mathscr{H} = \mathscr{C}_0$.

Let $\widehat{V_R}$ be the closed piece of the cylinder V_R bounded by $\mathscr{C}_1 \cup \partial D_T$. Let $\mathscr{R}$ be the closed region bounded by $\partial\mathscr{R} := D_0 \cup M \cup \widehat{V_R} \cup D_T$ (notice that $\partial\mathscr{R}$ is not smooth over $\partial M \cup \partial D_T$.) Let we now orient M by $\vec{N}$: the inner unit normal to $\partial\mathscr{R}$ induced by the mean curvature vector $\vec{\boldsymbol{H}}$ of M. Recall that $\mathscr{P}_0$ and $\mathscr{P}_1$ are the unique geodesic "parallel" planes with $\mathscr{C}_0 \subset \mathscr{P}_0$ and $\mathscr{C}_1 \subset \mathscr{P}_1$.

Claim 2. — *There exists a positive constant d such that if* $\operatorname{dist}(\mathscr{P}_0,\mathscr{P}_1) \geqslant d$, *then the mean curvature vector $\vec{\boldsymbol{H}}$ through M is pointing into the interior of $\mathscr{R}$.*

Proof of Claim 2. — It is well-known that there exists a constant d such that if $\mathrm{dist}(\mathscr{P}_0,\mathscr{P}_1) \geqslant d$ then one can find a 1-parameter family of minimal hyperbolic catenoids C_λ, $\lambda > 0$, where $\lambda = \mathrm{dist}\,(C_\lambda,\gamma)$, coming from the infinity towards γ: as $\lambda \downarrow 0$, C_λ converges to a geodesic plane $\mathscr{P}$ "between" $\mathscr{P}_0$ and $\mathscr{P}_1$ (see [46], [35]). We therefore use the family $\{C_\lambda\}$ to infer that it should be a first interior point of contact between M and some element of this family, hence $\vec{\boldsymbol{H}}$ is an interior pointing normal, by applying Interior Maximum Principle once again. This completes the proof of Claim 2.

Now, one can carry out a standard orientation argument and Maximum Principle (comparison with some horosphere $\mathscr{H}_t$, $t \geqslant t_1$) to deduce the configuration: $M \cap \mathrm{int}\,(D_1) = \emptyset$. One therefore gets that $M \setminus \partial M$ is entirely contained inside $\mathscr{H}$ and is contained outside $\mathscr{H}_{t_1}$; that is, it lies "between" $\mathscr{H}$ and $\mathscr{H}_{t_1}$. Fix now any geodesic plane P that contains γ. To accomplish the proof of the theorem, it suffices to prove that P is a plane of symmetry of M (notice that ∂M is symmetric about P).

Now taking into account basic hyperbolic geometry (see [96]) we infer that P is orthogonal to the family $\mathscr{H}_t$. Consider the family of parabolic isometries leaving invariant each horocycle that cuts P orthogonally. In particular, any such isometry leaves each horosphere $\mathscr{H}_t$, $0 \leqslant t \leqslant t_1$ invariant. We can now move P by means of this family of isometries far away from M, in each side of P. Then coming back towards P doing Alexandrov Reflection we therefore conclude that P is a plane of symmetry of M. □

Turning to Theorem B, we emphasize that we shall need some knowledge of the geometry of the Catenoids Cousin in hyperbolic space. This will be clear in the proof of the theorem. We pause momentarily to say that Ordóñes in his Doctoral Thesis [75] has given explicit parametrizations of the Catenoids Cousin (more generally: helicoidal surfaces of either constant mean or else constant Gauss curvature in space form). The reader is referred to [75] to look at the geometric behavior of the 1-parameter family of generating curves. We also refer to Castillon Thesis [23] to see some geometric and cinematic properties of Delaunay surfaces in hyperbolic space, see also [51] and [12].

Proof of Theorem B

The method of the proof of the first statement is the same as the proof of [93], Theorem 3.1, first statement. Indeed, the method of the proof has two powerful tools: Firstly, we shall need an asymptotic expansion inferred by the authors (see [95]) for vertical graphs $u = u(x,y)$ (see [89]) with constant mean curvature 1 in upper half-space model of hyperbolic space. $\mathbb{H}^3 = \{(x,y,z), z > 0\}$ (of course, equipped with the hyperbolic metric). We will need to work with a 1-parameter family of vertical graphs $\{\mathscr{A}_t\}$ which has a nice geometric behavior, as we will explain bellow: Each $\mathscr{A}_t$ is an embedded end of a non-embedded Catenoid Cousin and is a graph of a smooth function $u_t(x,y)$ which has the following asymptotic behavior: $u_t \sim cR^\alpha$ as $R^2 = x^2 + y^2 \to \infty$, where $c > 0$, $\alpha < 0$ (see [95]). In fact, each $\mathscr{A}_t$ is a graph of a function u over a exterior domain $\mathscr{D}_t$, assuming on $\partial\mathscr{D}_t$ for all t a suitable constant value $0 < \varepsilon + \delta_t < 1$, where $\delta_t > 0$ in order that the boundary of each end $\mathscr{A}_t$ lies in a fixed "horizontal" horosphere given by $\{z = \varepsilon + \delta_t\}$.

We may describe precisely the geometrical configuration in the following way: $\mathscr{D}_t$ is

the complement of an open disc B_t of radius $R(t)$, $t \in (0,t_0]$ $\mathrm{R}(t) \longrightarrow 0$ as $t \to 0$. We shall choose all such discs concentric, satisfying: if $t_1 < t_2$ then $\mathrm{B}_{t_1} \subsetneq \mathrm{B}_{t_2}$. Moreover, it follows that the family end $\mathscr{A}_t$ varies continuously on t and have growth $\alpha(t)$ with $\alpha(t) \longrightarrow 0$ when $t \to 0$. (see [75] for explicit parametrizations of the Catenoids Cousin and to look at the geometric behavior of the 1-parameter family of generating curves).

Secondly, we shall make use of Hoffman-Meeks Method to prove the Half-Space Theorem (see [44]).

We proceed the proof as follows. Suppose first that $\partial M \neq \emptyset$. We can assume that M is immersed into the upper half-space model of hyperbolic space. Up to a rigid motion of ambient space we can also assume that $\mathscr{H} = \{z = 1\}$ and that $\mathscr{H}_\varepsilon = \{z = \varepsilon\}$, with $\varepsilon \leqslant 1$, with $\partial M \subset \mathscr{H}$. We will argue by absurd. Suppose, by absurd that $\varepsilon < 1$. We can assume (for simplicity) that M is asymptotic to $\mathscr{H}_\varepsilon$ as well (notice that M is contained inside $\mathscr{H}_\varepsilon$, by our assumption). We choose δ defined above sufficiently small so that the boundary of each end does not touch M (recall that the boundary of each $\mathscr{A}_t$ lies in the fixed "horizontal" horosphere given by $\{z = \varepsilon + \delta\}$). That is, $\mathscr{A}_t$ is chosen to obtain $\partial\mathscr{A}_t \cap M = \emptyset$. Moreover , we shall choose the initial end $\mathscr{A}_{t_0}$ carefully to guarantee $\mathscr{A}_{t_0} \cap M = \emptyset$ (this choice is possible taking into account M is proper and the asymptotic expansion of $\mathscr{A}_t$). We therefore can apply Hoffman-Meeks Method starting from $\mathscr{A}_{t_0}$ moving the family $\mathscr{A}_t$ towards the horosphere $\{z = \varepsilon + \delta\}$, by making $t \downarrow 0$, until a member the family reaches M at a first tangent interior point. This leads to a contradiction with the Interior Maximum Principle, since the mean curvature vector of each member of the family $\mathscr{A}_t$ is an upward pointing unit normal. Note that when $\partial M = \emptyset$, if one supposes at the beginning of the proof that $M \neq \mathscr{H}$, one gets a contradiction in the exactly same way as before. The proof of the first part of the theorem is now complete.

We now will check the proof of the last part of the statement.

Let M be a connected properly embedded constant mean curvature 1 surface in hyperbolic space contained inside the horosphere $\mathscr{H}$ and let ∂M be a circle $\mathscr{C}$ lying on $\mathscr{H}$. If M is compact then it follows that M is part of an horosphere (see [72], [8], [92]). Thus we can assume that M is non-compact (no assumptions about the topology). Notice that the assumption "$M \setminus \partial M$ is contained inside $\mathscr{H}$" is necessary in the sense that the configuration $(M \setminus \partial M) \cap \mathscr{H} \neq \emptyset$ is forbidden by Interior Maximum Principle, since we are assuming that $M \setminus \partial M$ is contained inside a translated copy $\mathscr{H}_\varepsilon$. Moreover, let D be the closed "disc" with $D \subset \mathscr{H}$ whose boundary is $\mathscr{C}$ and let $\widehat{M} := M \cup D$ be the properly embedded surface in hyperbolic space not smooth along $\mathscr{C}$. Let us orient $\widehat{M}$ by the orientation $\vec{N}$ induced by the unit mean curvature vector $\vec{\boldsymbol{H}}$ of M. To see the configuration better assume that we focus on the upper half-space model of hyperbolic space and that $\mathscr{H} = \{z = 1\}$. Now applying Maximum Principle, using horospheres which cuts $\mathscr{H}$ transversally, as suitable barriers, one can therefore conclude that $\vec{N}$ through D is pointing into the horoball $\mathscr{B}$ whose boundary is $\mathscr{H}$. Let us give further details about this: We consider that the part of hyperbolic space outside $\mathscr{H}$ (bellow $\mathscr{H}$ in the Euclidean sense) is contained outside $\widehat{M} := M \cup D$. So the above construction using the family of foliations of horospheres issuing from a finite point of the asymptotic boundary, as barriers, coming from the infinity towards M, we conclude that the mean curvature vec-

tor $\vec{\boldsymbol{H}}$ is pointing to the interior of $\widehat{M}$. Let us give another proof of this: Just take two geodesic planes at certain distance d involving $\mathscr{C}$ in the Euclidean sense. If d is chosen properly one can use a family of minimal catenoids coming from the infinity towards M that converges to a geodesic plane in the middle of the two others to ensure again that the mean curvature vector is pointing to the interior of $\widehat{M}$. Now let P be the unique geodesic plane that contains $\mathscr{C}$, and let α be the the vertical geodesic line of symmetry of $\mathscr{C}$, cutting orthogonally P. On account of the 1-parameter family of geodesic planes P_t obtained by translations along α, coming from the outside of $\mathscr{H}$ (which lies in the outside of $\widehat{M}$) towards $\mathscr{H}$, one therefore infers by Maximum Principle that $M \setminus \partial M$ cannot intersects P, i. e $(M \setminus \partial M) \cap P = \emptyset$.

We shall show next that every geodesic plane $\mathscr{P}$ of symmetry of $\mathscr{C}$ is a plane of symmetry of M. Fix such a geodesic plane $\mathscr{P}$. Let γ be the unique (oriented) geodesic line in P cutting $\mathscr{P}$ orthogonally at a point q, with $\gamma(t = 0) = q$. Note that hyperbolic translations along γ keeps P invariant and produces a 1-parameter family $\mathscr{P}_t$, $-\infty < t < \infty$ of translated geodesic planes such that $\mathscr{P}_0 = \mathscr{P}$ and $\mathscr{P}_t$ is orthogonal to P, $\forall t$ (t is the oriented distance from $\mathscr{P}_t$ to $\mathscr{P}$). We therefore are able to apply Alexandrov Reflection on M using the foliation $\{\mathscr{P}_t\}$ coming from the infinity towards $\mathscr{P}$, to conclude that $\mathscr{P}$ is a symmetry plane of M. We remark that Alexandrov Method works because the following three conditions hold: $M \cap \mathscr{P}_t$, $t \neq 0$ is either empty or else is a compact set, $M \cap P = \mathscr{C}$, and the reflection of $\mathscr{C}$ about $\mathscr{P}_t$ lies on P, for all t. The proof of Theorem B is now complete. □

We now shall attempt to proof Theorem C. We shall need the following lemmas.

Lemma 1 (Basic hyperbolic geometry). — *Let M be a connected properly embedded surface in hyperbolic space whose asymptotic boundary consists of a single point. Let $\mathscr{P}$ be a geodesic plane such that $\partial_\infty M \subset \partial_\infty \mathscr{P}$. If M is symmetric about every such geodesic plane $\mathscr{P}$ then M is a horosphere. Furthermore, if $\mathscr{P}_t$, $\mathscr{P}_t \neq \mathscr{P}$ is an arbitrary translated copy of $\mathscr{P}$ along a geodesic γ cutting orthogonally $\mathscr{P}$, then either $\mathscr{P}_t \cap M$ is empty or else $\mathscr{P}_t \cap M$ is compact.*

The lemma above is straightforward hyperbolic geometry. We will omit the proof here. Clearly, the same statement can be formulated for hypersurfaces in $\mathbb{H}^{n+1}$.

Lemma 2 (Serrin's boundary maximum principle at a corner [101]). — *Let Ω be a domain in $\mathbb{R}^{n+1}$ with C^2 boundary and let T be a (Euclidean) hyperplane containing the normal to $\partial\Omega$ at some point q. Let Ω^+ then denote the portion of Ω lying on one particular side of T. Suppose f is of class C^2 in the closure of Ω^+ and satisfies the second order elliptic differential equation (using summation convention)*

$$L\,f = a_{ij}(x)\,f_{ij} + b(x)\,f_j \leqslant 0 \;\; in \;\; \Omega^+$$

where we denote $x = (x_1 \dots x_{n+1})$, $f_j = \dfrac{\partial f}{\partial x_j}$, $f_{ij} = \dfrac{\partial^2 f}{\partial x_i \partial x_j}$. We assume the coefficients

uniformly bounded and we assume the matrix a_{ij} positive definite satisfying

$$a_{ij}(x)\xi_i\xi_j \geqslant k|\xi|^2 \qquad k > 0$$
$$|a_{ij}(x)\xi_i\eta_j| \leqslant K(|\xi\cdot\eta| + |\xi|\cdot|d|)$$

where $\xi = (\xi_1 \dots \xi_{n+1})$ is a vector in $\mathbb{R}^{n+1}$, η is the Euclidean unit normal to the hyperplane T, and d is the distance from T. Suppose also that $f \geqslant 0$ in Ω^+ and $f = 0$ at q. Then either

$$\frac{\partial f}{\partial s} > 0 \ \ or \ \ \frac{\partial^2 f}{\partial s^2} > 0 \ \ at \ \ q$$

unless $f \equiv 0$, where $\vec{s}$ is any direction at q which enters Ω^+ non-tangentially.

Proof of Theorem C

On account of Lemma 1, we must prove that given an arbitrary geodesic hyperplane $\mathscr{P}$ such that $\partial_\infty M \subset \partial_\infty \mathscr{P}$, then M is symmetric about $\mathscr{P}$. To infer this we will make use of Alexandrov Method, following the plot due to Serrin, and carried out by Molzon when the ambient is hyperbolic space.

Notice that M is symmetric about $\mathscr{P}$, if and only if the symmetrized cap of each part of M in each side of $\mathscr{P}$ is contained inside M (it is trivial anyway). Let γ an oriented geodesic line cutting $\mathscr{P}$ orthogonally and let $\mathscr{P}_t$, $-\infty < t < \infty$ be the 1-parameter family of geodesic hyperplanes obtained by translating $\mathscr{P}$ along γ, as in the proof of the last part of Theorem B. We may suppose that $\mathscr{P}_0 = \mathscr{P}$. Denote by $\mathscr{P}^+_{t_0}$, $t_0 \in \mathbb{R}$ the connected component of $\mathbb{H}^{n+1} \setminus \mathscr{P}_{t_0}$ which contains $\mathscr{P}_t$, $t > t_0$, and denote by $\mathscr{P}^-_{t_0}$ the other component. Let Ω^-_t be the portion of Ω lying in $\mathscr{P}^-_t$ and let $\Omega^-_t(*)$ be the symmetry of Ω^-_t about $\mathscr{P}_t$ ($\Omega^-_t(*) \subset \mathscr{P}^+_t$). Similarly, let Ω^+_t be the portion of Ω lying in $\mathscr{P}^+_t$ and let $\Omega^+_t(*)$ be the symmetry of Ω^+_t about $\mathscr{P}_t$ ($\Omega^+_t(*) \subset \mathscr{P}^-_t$). Finally, if $x_t \in \Omega$ let x^*_t be the reflection of x_t across $\mathscr{P}_t$. If $\mathscr{P}_t \cap \Omega \neq \emptyset$ and $\Omega^-_t(*)$ (resp. $\Omega^+_t(*)$) is contained in Ω, we can introduce a new function g defined by

$$g(x) = f(x^*) \qquad x \in \Omega^-_t(*) \qquad \text{(in the case } \Omega^-_t(*) \text{ is contained in } \Omega)$$
or
$$g(x) = f(x^*) \qquad x \in \Omega^+_t(*) \qquad \text{(in the case } \Omega^+_t(*) \text{ is contained in } \Omega)\,.$$

Since the Laplacian operator Δ is invariant under rigid motions of ambient space and reflections are isometries of hyperbolic space it is evident that, if $\Omega^-_t(*)$ is contained in Ω, the function g satisfies the following overdetermined elliptic differential problem.

$$\begin{aligned} \Delta g &= -1 && \text{in } \Omega^-_t(*) \\ g &= f && \text{on } \partial\Omega^-_t(*) \cap \mathscr{P}_t \\ g &= 0, && \frac{\partial g}{\partial n} = k \ \text{ constant} \qquad \text{on } \partial\Omega^-_t(*) \cap \mathscr{P}^+_t\,. \end{aligned}$$

Of course, the function g satisfies an analogous problem in the case $\Omega_t^+(*)$ is contained in Ω. Notice that reflections invert normal vectors, we therefore have above a well defined problem.

We now are able to run Alexandrov Reflection on M using the 1-parameter family of geodesic hyperplanes $\mathscr{P}_t$ coming from the infinity towards $\mathscr{P}$ (as we did before), and to consider the function $f-g$ defined on $\Omega_t^-(*)$ (resp. $\Omega_t^+(*)$). If no $\partial\Omega_t^-(*)$, $t \neq 0$ and no $\partial\Omega_t^+(*)$, $t \neq 0$ *touches* M at an interior or boundary point of tangent contact then we get what we want: Indeed, it follows from this situation that the symmetry of the part of M in each side of $\mathscr{P}$ is contained inside M, we can therefore conclude M is symmetric about $\mathscr{P}$.

Henceforth, to complete the proof of the theorem it must therefore be shown that the existence of a first point, say q of either interior or boundary point of tangent contact (during the above Alexandrov-Serrin process), leads to a contradiction. Thus let us suppose by absurd that there exists a first point of contact q for some $t_0 \neq 0$, and let us denote by simplicity $\Omega(*)$ (instead of $\Omega_{t_0}^-(*)$ or $\Omega_{t_0}^+(*)$).

We first consider the configuration $\partial\Omega(*)$ is internally and smoothly tangent to M at a point q not lying on $\mathscr{P}_{t_0}$. Recall that $\Omega(*)$ is contained in Ω by construction. We can therefore consider the function $f-g$ defined in $\Omega(*)$. We get

$$\Delta(f-g) = 0 \qquad \text{in } \Omega(*) \tag{1}$$

and

$$\begin{aligned} &f-g = 0 \text{ on } \quad \partial\Omega(*) \cap \mathscr{P}_{t_0} \\ &f-g \geqslant 0 \text{ on } \quad \partial\Omega(*) \cap (\Omega \setminus \mathscr{P}_{t_0}) . \end{aligned} \tag{2}$$

Note that, as f is nonnegative by hypothesis, last inequality follows immediately, because $g = 0$ on $\partial\Omega(*) \cap (\Omega \setminus \mathscr{P}_{t_0})$. Besides, f must be stricly positive in Ω, since it satisfies the elliptic equation $\Delta f = -1$.

Now recall that $\overline{\Omega(*)}$ is compact (by Lemma 1). On account of (1) and (2) we can therefore apply Hopf Interior Maximum Principle to infer either

$$\begin{aligned} & f-g > 0 && \text{in } \Omega(*) \\ \text{or else} \quad & f-g = 0 && \text{at all points of } \Omega(*) . \end{aligned}$$

But the latter possibility imply that the symmetrized cap $\Omega(*)$ must coincide with the part of Ω on the same side of $\mathscr{P}_{t_0}$ as $\Omega(*)$, since $f>0$ in Ω. We therefore conclude that Ω is bounded which is an absurd. Now the former possibility $f>g$ on $\Omega(*)$ and the configuration $\partial\Omega(*)$ touching (internally) M at the point $q \notin \mathscr{P}_{t_0}$, yields $\dfrac{\partial(f-g)}{\partial n}(q) > 0$, applying Hopf Boundary Maximum Principle. This however leads to an obvious contradiction with the fact that $\dfrac{\partial(f-g)}{\partial n}(q) = 0$. Thus the possibility $\partial\Omega(*)$ is internally and smoothly tangent to M at a point q not lying on $\mathscr{P}_{t_0}$ cannot occur.

We next will infer that the existence of a first point $q \in \mathscr{P}_{t_0}$ of boundary point of tangent contact, *i.e.* $\mathscr{P}_{t_0}$ fitting orthogonally M at a point q, and $f > g$ on $\Omega(*)$ is also an impossible configuration. To attempt this goal, we shall use some basic hyperbolic geometry again, and we shall carry out some calculations- to enter into the spirit of Serrin and Molzon- to obtain that the first and second derivatives of $f-g$ vanish at q. Taking into account Lemma 2, we arrive to an absurd.

We proceed the details as follows. Next, we will work with the ball model $\mathbf{B}^{n+1}$ of hyperbolic space equipped with the metric $ds^2 = \lambda^2(dx_1^2 + \cdots + dx_{n+1}^2)$, where $\lambda^2(x) = \dfrac{4}{(1-|x|^2)^2}$. Up to a rigid motion (namely, a Möbius transformation), we may assume that q lies on the x_{n+1}-axis and that the hyperplane $\{x_1 = 0\}$ agrees with $\mathscr{P}_{t_0}$ at q (then the x_1 axis is normal to $\mathscr{P}_{t_0}$), and that the vector $\vec{n}$, the inner unit normal to M at q, lies in the x_{n+1}-axis with $\vec{n} \cdot e_{n+1} < 0$. Where we denote by $\cdot$ the Euclidean inner product and $e_{n+1} = (0,\ldots,1)$ is the standard vector in $\mathbb{R}^{n+1}$ along the x_{n+1}-axis. We commence to calculate the first and second derivatives of f and g. The following calculations are contained in the cited paper by Molzon. We summarize them here for the reader's convenience. Clearly, we can represent $M = \partial\Omega$ locally by the equation

$$x_{n+1} = \varphi(x_1,\ldots,x_n)$$

where φ is a C^2 function.

The condition $f = 0$ on Ω can then be written, in a neighbourhood of q, as

$$f(x_1,\ldots,x_n,\varphi(x_1,\ldots,x_n)). \tag{3}$$

Let us recall now that $\vec{n}$ is the inner unit normal to M in hyperbolic space and let $\vec{N}$ be the Euclidean unit normal to M. They satisfy the following relation

$$\vec{n} = \frac{1}{\lambda} \cdot \vec{N}.$$

Now the formula of the unit normal to a graph φ in $\mathbb{R}^{n+1}$ and the above relation yields

$$\vec{n} = \frac{1}{\lambda} \cdot \frac{1}{\sqrt{1+\varphi_1^2+\cdots+\varphi_n^2}} \cdot (-\varphi_1,\ldots,-\varphi_n,1)\,.$$

Now the condition $\frac{\partial f}{\partial n} = k$ can therefore be expressed as

$$-\varphi_1 f_1 - \cdots - \varphi_n f_n + f_{n+1} = k\lambda\sqrt{1+\varphi_1^2+\cdots+\varphi_n^2}. \tag{4}$$

Thus differentiating (3) and evaluating at q we get

$$f_j = 0 \text{ at } q \text{ for } 1 \leqslant j \leqslant n. \tag{5}$$

On account of equations (4), (5) we deduce

$$f_{n+1} = k\lambda \text{ at } q. \tag{6}$$

Next differentiating twice equation (3) with respect to x_j, $j = 1, \dots n$ and evaluating at q, taking into account (6) we obtain

$$f_{ij} = -k\lambda\,\varphi_{ij} \text{ at } q. \tag{7}$$

Now differentiating (4) with respect to x_j, $j = 1, \dots n$, and evaluating at q gives

$$f_{j\,n+1} = 0 \text{ at } q.$$

Lastly, since $\Delta f = -1$ one obtains easily that

$$f_{n+1\,n+1} = -\lambda^2 + k\lambda \cdot (\varphi_{11} + \cdots + \varphi_{nn}). \tag{8}$$

Now since the reflected cap $\Omega(*)$ is contained inside Ω, applying second order Taylor formula with remainder one gets that

$$\varphi_{1\,j} = 0 \text{ at } q, \quad j = 2, \dots, n.$$

Since g is defined in terms of f by reflection about the hyperplane $\{x_1 = 0\}$, we then infer easily that the first and second derivatives of f and g coincide at q. The proof of Theorem C is now completed.

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