# UNIQUENESS OF $H$-SURFACES IN $\mathbb{H}^{2} \times \mathbb{R},|H| \leqslant 1 / 2$, WITH BOUNDARY ONE OR TWO PARALLEL HORIZONTAL CIRCLES 

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#### Abstract

We prove that a $H$-surface $M$ in $\mathbb{H}^{2} \times \mathbb{R}, H \leqslant \frac{1}{2}$, inherits the symmetries of its boundary $\partial M$, when $\partial M$ is either a horizontal curve with curvature greater than one or two parallel horizontal curves with curvature greater than one, whose distance is greater or equal to $\pi$. Furthermore we prove that the asymptotic boundary of a surface with mean curvature bounded away from zero consists of parts of straight lines, provided it is sufficiently regular.


## 1. Introduction

An old question in classical Differential Geometry in Euclidean space concerns the influence of the boundary on the behavior of a $H$-surface. A similar question can also be asked when the ambient space is a homogeneous 3-manifold.

In this paper we consider this problem when the ambient space is the product $\mathbb{H}^{2} \times \mathbb{R}$ and the boundary of the $H$-surface $M,|H| \leqslant 1 / 2$, is either a horizontal curve (Theorem 2.2) or two parallel horizontal curves (Theorem 3.1) with curvature greater than one. By parallel curves we mean congruent up to a vertical translation. The main result is that $M$ inherits the symmetries of its boundary. In particular, if the boundary curve is a horizontal circle or two parallel horizontal circles with distance greater or equal to $\pi$, then $M$ is rotational.

Our results strongly depend on the geometry of rotational $H$-surfaces in $\mathbb{H}^{2} \times \mathbb{R}$ for $H \leqslant 1 / 2$. We point out that there are no rotational $H$-surfaces in $\mathbb{R}^{3}$ with geometric behavior analogous to the rotational surfaces in $\mathbb{H}^{2} \times \mathbb{R}$ with $H \leqslant 1 / 2$. In fact, the geometry of rotational $H$-surfaces in $\mathbb{R}^{3}$ (Delaunay surfaces) is similar to that of rotational surfaces in $\mathbb{H}^{2} \times \mathbb{R}$ with $H>1 / 2$ (see the Appendix).

[^0]We recall the principal related results in Euclidean and hyperbolic 3 -space.

It has been conjectured that a connected compact embedded $H$ surface in $\mathbb{R}^{3}$ with boundary a round circle is spherical. Of course, this conjecture can be posed in hyperbolic space $\mathbb{H}^{3}$ and in the product space $\mathbb{H}^{2} \times \mathbb{R}$. In $\mathbb{R}^{3}$ this is still an open problem. F. Braga Brito, W. Meeks, H. Rosenberg and R. Sa Earp proved the conjecture provided the surface is transverse to the plane containing the circle, along the circle ([10]). F. Braga Brito and R. Sa Earp proved that, if the radius of the circle and the mean curvature are equal to one, then the conjecture is true assuming only that the surface is immersed ([8]). In fact they also proved analogous characterizations of a spherical cap for $f$-surfaces (special Weingarten surfaces) of disk type ([9]). We remark that N. Kapouleas announced examples of immersed $H$-surfaces with genus $g \geqslant 3$, with boundary a circle ([17]).

In hyperbolic space L. Barbosa and R. Sa Earp proved the following sharp result: if a compact connected immersed surface in $\mathbb{H}^{3}$, with boundary a round circle, has constant mean curvature smaller or equal to one, then it is totally umbilical ([4], [5] and [6]). B. Nelli and H. Rosenberg proved the same result in the embedded case ([22]). In [32], R. Sa Earp and E. Toubiana generalized the above result of [22] to $f$-surfaces in hyperbolic 3 -space satisfying $f^{2} \leqslant 1$. They also proved the following: if $M$ has constant mean curvature one, is embedded into $\mathbb{H}^{3}$ with $\partial M=C_{1} \cup C_{2}$, where $C_{1}$ and $C_{2}$ are two parallel circles and $\operatorname{dist}\left(C_{1}, C_{2}\right)$ great enough, then $M$ is a piece of a catenoid cousin ([32]).

For further results, the reader is referred to [3], [18] and [28] for the Euclidean space and [25] and [31] for the hyperbolic space.

Now, we describe our results for $H$-surfaces in $\mathbb{H}^{2} \times \mathbb{R}$.
First, assume $M$ is compact, immersed into $\mathbb{H}^{2} \times \mathbb{R}$, with mean curvature $1 / 2$, with boundary a round circle in a horizontal slice. Then $M$ is part of the rotational surface with vanishing Abresch-Rosenberg holomorphic quadratic differential $Q$. An analogous result holds if $M$ has constant mean curvature less than $1 / 2$ (Theorem 2.2 and 5.1). In both situations $M$ is part of an entire rotational vertical graph. More generally, we prove that if $\partial M$ is a horizontal curve with curvature greater than one, then $M$ is a vertical graph, in particular $M$ has genus zero and inherits the symmetries of its boundary (Theorem 2.2). Assume now $M$ is compact, embedded and has constant mean curvature $\leqslant 1 / 2$. Assume also that $\partial M=C_{1} \cup C_{2}$, where $C_{1}$ and $C_{2}$ are parallel horizontal curves with curvature greater than one and $\operatorname{dist}\left(C_{1}, C_{2}\right) \geqslant \pi$. Then $M$ inherits the symmetries of $C_{1} \cup C_{2}$. Consequently, if $C_{1}$ and $C_{2}$ are two parallel circles then $M$ is part of an embedded complete rotational
$H$-annulus. The last assertion follows from the geometric classification of rotational $H$-surfaces (Appendix).

Finally, we consider surfaces $M$ in $\mathbb{H}^{2} \times \mathbb{R}$ which are regular up to the asymptotic boundary and whose mean curvature is bounded away from zero. In [30] R. Sa Earp gives examples suggesting that the asymptotic boundary of such surfaces must lie on a vertical line in $\partial_{\infty}\left(\mathbb{H}^{2}\right) \times \mathbb{R}$. We prove that if the asymptotic boundary $\partial_{\infty} M$ in $\partial_{\infty}\left(\mathbb{H}^{2}\right) \times \mathbb{R}$ is $C^{1}$ and the surface is $C^{1}$ up to $\partial_{\infty} M$, then each component of $\partial_{\infty} M$ is part of a vertical line. As far as we know, the same question is open, if one relaxes the regularity up to the asymptotic boundary. Observe that in the minimal case many other possibilities can occur ([12], [23], [30] and [34]). In particular, in [23] B. Nelli and H. Rosenberg solve the Dirichlet Problem in $\mathbb{H}^{2} \times \mathbb{R}$ for any Jordan curve $\gamma$ in the asymptotic boundary $\partial_{\infty}\left(\mathbb{H}^{2}\right) \times \mathbb{R}$ that is a vertical graph.

The paper is organized as follows. Section 2 and 3 deal with compact $H$-surfaces with boundary either one curve or two curves, with curvature greater than one. In Section 4, we study the behavior of the asymptotic boundary of a $H$-surface with strictly positive mean curvature. Finally, in the Appendix we discuss the geometry of rotational $H$-surfaces for any $H \in \mathbb{R}$, since we will need to use these surfaces throughout the whole paper. Rotational surfaces in $\mathbb{H}^{2} \times \mathbb{R}$ have been studied in [1], [16], [21], [27] and [34]. Further results on $H$-surfaces in $\mathbb{H}^{2} \times \mathbb{R}$ are in [14], [24].

## 2. H-Surfaces with boundary a curve with curvature GREATER THAN ONE

In this section we discuss existence and uniqueness of compact $H$-surfaces, $|H| \leqslant 1 / 2$, with boundary a curve on a horizontal slice, with curvature greater than one. Then, we study under which conditions a compact $H$-surface with boundary a planar curve inherits the symmetries of its boundary. In particular we deal with the circle boundary case.

Let $H \in(0,1 / 2]$. Denote by $t$ the third coordinate in $\mathbb{H}^{2} \times \mathbb{R}$ and by $\sigma$ the origin in $\mathbb{H}^{2}$ (in the disk model of $\mathbb{H}^{2}$ we have $\sigma=0$ ). We denote by $S^{H}$ the simply connected embedded surface in $\mathbb{H}^{2} \times \mathbb{R}$ with constant mean curvature $H$, invariant by rotation about the axis $\{\sigma\} \times \mathbb{R}$, tangent to $\mathbb{H}^{2} \times\{0\}$ at $\sigma$. Recall that $S^{H}$ is an entire vertical graph (Appendix).

Let us recall the Convex Hull Lemma ([26]). Let $K$ be a compact set in $\mathbb{H}^{2} \times \mathbb{R}$. For any $H \in\left(0, \frac{1}{2}\right]$, we define $\mathcal{F}_{K}^{H}$ as follows. A surface $B$ belongs to $\mathcal{F}_{K}^{H}$ if $K$ is contained in the mean convex side of $B$ and
if it is obtained from $S^{H}$ either by vertical and horizontal translations or by symmetry with respect to a horizontal slice.

Using the maximum principle one can prove the following.
Lemma 2.1. (Convex Hull Lemma) [26] Let $M$ be a compact surface immersed in $\mathbb{H}^{2} \times \mathbb{R}$ with constant mean curvature $H \in\left(0, \frac{1}{2}\right]$. Then $M$ is contained in the convex hull of the family $\mathcal{F}_{\partial M}^{H}$.

Now we recall the Flux Formula in our context. Let $\Omega$ be Jordan domain lying in the slice $\mathbb{H}^{2}:=\mathbb{H}^{2} \times\{0\}$ of $\mathbb{H}^{2} \times \mathbb{R}$. Let $Y=\frac{\partial}{\partial t}$ be the vertical Killing vector field in $\mathbb{H}^{2} \times \mathbb{R}$. Let $n=n_{\Omega}$ be a unit normal field on $\Omega$. The flux $\Phi_{\Omega}$ of $Y$ through $\Omega$ is defined by $\Phi_{\Omega}=\int_{\Omega}\langle Y, n\rangle \mathrm{d} A$. Let $\Omega_{t} \subset \mathbb{H}^{2} \times\{t\}$, be any vertical copy of $\Omega$. Then it is clear that the Flux of $Y$ through $\Omega_{t}$ (with the same orientation) is the same that the Flux through $\Omega$, i.e $\Phi_{\Omega_{t}}=\Phi_{\Omega}$, since $Y$ is vertical. Let $M$ be a compact $H$ surface in $\mathbb{H}^{2} \times \mathbb{R}$ with boundary $C=\partial \Omega$. Assuming that $0<H \leqslant 1 / 2$ and $C$ has curvature greater than one, the Convex Hull Lemma implies that the intersection of the vertical cylinder over $\partial \Omega$ with $M \backslash \partial M$ is empty. Let $\Omega_{t}$ be any fixed vertical copy of $\Omega$ such that $\Omega_{t} \cap M=\emptyset$. Let $\Sigma$ be the piece of the vertical cylinder over $C$ bounded by $C$ and $\partial \Omega_{t}$. The closed surface $M \cup \Sigma \cup \Omega_{t}$, with the orientation induced by the (nonvanishing) mean curvature vector of $M$ is an oriented homological boundary of a three-dimensional chain in $\mathbb{H}^{2} \times \mathbb{R}$. The following Flux Formula holds ([19], [20],[4], Appendix B in [6], or [15]).

$$
\begin{equation*}
\int_{\partial M}\langle Y, \nu\rangle \mathrm{d} s=2 H \int_{\Omega}\langle Y, n\rangle \mathrm{d} A \tag{1}
\end{equation*}
$$

In $\mathbb{H}^{2} \times \mathbb{R}$ there is a natural notion of vertical graph ( [14], [26] or [30]): let $\Omega$ be a subset of $\mathbb{H}^{2}$ and let $u: \Omega \longrightarrow \mathbb{R}$ be a $C^{2}$ function. The vertical graph of $u$ is the subset of $\mathbb{H}^{2} \times \mathbb{R}$ given by

$$
\{(x, y, t) \in \Omega \times \mathbb{R} \mid t=u(x, y)\}
$$

We choose the unit normal vector field to the graph of $u$ with positive third component and we compute the mean curvature (of the graph) with respect to it.

The graph of a function $u: \mathbb{H}^{2} \longrightarrow \mathbb{R}$ has the function $H$ as mean curvature if and only if $u$ satisfies the following partial differential equation.

$$
\begin{equation*}
\operatorname{div}_{\mathbb{H}}\left(\frac{\nabla_{\mathbb{H}} u}{W_{u}}\right)=2 H, \tag{2}
\end{equation*}
$$

where $\operatorname{div}_{\mathbb{H}}, \nabla_{\mathbb{H}}$ are the hyperbolic divergence and gradient respectively and $W_{u}=\sqrt{1+\left|\nabla_{\mathbb{H}} u\right|_{\mathbb{H}}^{2}}$, being $|\cdot|_{\mathbb{H}}$ the norm in $\mathbb{H}^{2}$.

Consider the halfspace model for $\mathbb{H}^{2}$, with Euclidean coordinates $x, y, y>0$. In this model, equation (2) takes the following form

$$
\begin{equation*}
\operatorname{div}\left(\frac{\nabla u}{W_{u}}\right)=\frac{2 H}{y^{2}} \tag{3}
\end{equation*}
$$

where div is the Euclidean divergence and $W_{u}=\sqrt{1+y^{2}\left(u_{x}^{2}+u_{y}^{2}\right)}$.
The following Theorem is a consequence of a result proved by L. Hauswirth, H. Rosenberg and J. Spruck ([13], Th. 3.2). We sketch a proof for completeness.

Theorem 2.1. Let $\Omega$ be a domain in $\mathbb{H}^{2}$ such that $\partial \Omega$ is $C^{2, \alpha}$ and has curvature greater than one. For any $H \in\left[-\frac{1}{2}, \frac{1}{2}\right]$, there exists a vertical graph $G_{H}$ with constant mean curvature $H$ and boundary $\partial \Omega$.

## Proof.

By formula (3), one has to solve the following Dirichlet problem

$$
\left\{\begin{array}{l}
F[u]=\operatorname{div}\left(\frac{\nabla u}{W_{u}}\right)-\frac{2 H}{y^{2}}=0 \text { in } \Omega \\
u=0 \text { on } \partial \Omega
\end{array}\right.
$$

In order to prove existence, we use the continuity method. For every $t \in[0,1]$, consider the Dirichlet problem

$$
\left\{\begin{array}{l}
F^{t}[u]=\operatorname{div}\left(\frac{\nabla u}{W_{u}}\right)-\frac{2 H t}{y^{2}}=0 \text { in } \Omega  \tag{4}\\
u=0 \text { on } \partial \Omega
\end{array}\right.
$$

Let $S=\{t \in[0,1] \mid$ there exists a solution of (4) $\}$. Observe that $u \equiv 0$ is a solution of (4) for $t=0$, hence $0 \in S$. If one proves that $S$ is open and closed, then $1 \in S$ and the desired solution is a solution of (4) for $t=1$.

That $S$ is open follows from the Implicit Function Theorem. $S$ closed follows from $C^{2, \alpha}$ a-priori estimates for solutions of (4). By Schauder's theory ( $[11]$ ), $C^{2, \alpha}$ a-priori estimates follow from $C^{1} a$-priori estimates. The Convex Hull Lemma guarantees $C^{0}$ estimates and boundary gradient bounds on solutions of (4). Therefore, we infer with Theorem 3.1 in [35] that boundary gradient bounds imply interior gradient bounds.

Theorem 2.2. Let $M$ be a compact surface immersed in $\mathbb{H}^{2} \times \mathbb{R}$ with boundary a $C^{2, \alpha}$ Jordan curve $C$ with curvature greater than one, contained in the slice $\mathbb{H}^{2} \times\{0\}$. Assume that $M$ has constant mean curvature $H \in\left(0, \frac{1}{2}\right]$. Then $M$ is a vertical graph (given by Theorem 2.1). In particular $M$ has genus zero and inherits the same symmetries of its boundary. If $C$ is a circle, then $M$ is a part of the simply connected rotational surface containing $C$ with constant mean curvature $H$.

## Proof.

Denote by $P=\{t=0\}$ the slice containing the boundary curve $C$ and denote by $\Omega$ the domain in $P$, bounded by $C$. Consider the convex hull of the family $\mathcal{F}_{C}^{H}$. As $C$ has curvature greater than one, for any point of $p \in C$ there is a surface of the family $\mathcal{F}_{C}^{H}$ tangent to $C$ at $p$. By Lemma 2.1, $M$ is contained in the convex hull of the family $\mathcal{F}_{C}^{H}$, hence $M \backslash \partial M$ is entirely contained in the vertical cylinder over $C$ and $M$ does not meet $P$ outside $\Omega$.

By Theorem 2.1 there exists a graph $G_{H}$ on $\Omega$, with boundary $C$ and constant mean curvature $H$. We can choose $G_{H}$ to be contained in the halfspace $t \geqslant 0$ with mean curvature vector pointing downwards. We will prove that $M$ is contained in one of the two halfspaces determined by the slice $P$.

First assume that $M$ is embedded. Lift up $G_{H}$ to be above $M$, then move $G_{H}$ down towards $M$ : by the maximum principle, one can not touch $M$ till the boundary of $M$ and the boundary of $G_{H}$ coincide. Hence $M$ lies below $G_{H}$. Then $G_{H} \cup M$ bounds a domain $U$ in $\mathbb{H}^{2} \times \mathbb{R}$ and the mean curvature vector of $M$ points either inside $U$ or outside $U$.

Assume, by contradiction, that $M$ has points in both halfspaces $\{t>0\},\{t<0\}$. By the maximum principle, the mean curvature vector at a highest point of $M$ points downwards, i.e. outside $U$, while the mean curvature vector at a lowest point of $M$ points upwards, i.e. inside $U$. This is a contradiction. Then $M$ is contained in a halfspace, say $\{t>0\}, M \cup \Omega$ bounds a domain $W$ in $\mathbb{H}^{2} \times \mathbb{R}$ and the mean curvature vector of $M$ points inside $W$. We already know that $M$ lies below $G_{H}$. Now move down $G_{H}$ to be disjoint from $M$, then lift $G_{H}$ up towards $M$. At a first interior contact point the mean curvature vectors of $M$ and $G_{H}$ coincide. Contradiction by the maximum principle. Then one can lift $G_{H}$ till the boundaries of $M$ and $G_{H}$ coincide, that is $M$ is above $G_{H}$. This implies that $G_{H} \equiv M$, as desired.

Now, by applying Alexandrov's reflection method with vertical geodesic planes $([2],[29]$ or $[33])$, one obtains that $M$ has all the symmetries of $C$. In particular, if $C$ is a circle, then $M$ is part of the simply
connected rotational surface $S^{H}$ containing $C$. Observe that if $M$ is embedded a simpler alternative argument based on Alexandrov Reflection Principle, using the horizontal slices, yields that $M$ is a vertical graph.

Now let $M$ be an immersed surface (not necessarily embedded).
First assume that $C$ is a circle.
By Lemma 2.1, $M$ is contained in the convex hull of the family $\mathcal{F}_{C}^{H}$. As $C$ is a circle, this convex hull is the domain bounded by the compact part of $S^{H}$ containing $C$, say $B_{C}^{1}$, and its symmetry with respect to the slice $P$, say $B_{C}^{2}$.

Let $\nu_{3}^{1}, \nu_{3}, \nu_{3}^{2}$ the third components of the inward unit conormal along $C$ of $B_{C}^{1}, M, B_{C}^{2}$ respectively. As $M$ is between $B_{C}^{1}$ and $B_{C}^{2}$, then at any point of $C$

$$
\begin{equation*}
\nu_{3}^{2} \leqslant \nu_{3} \leqslant \nu_{3}^{1} \tag{5}
\end{equation*}
$$

Consider the Flux Formula for $M$, with $\Omega$ equal to the planar domain bounded by $C, Y=(0,0,1)$ and $n_{\Omega}=(0,0, \pm 1)$ according to the orientation given by $M$. In order to fix ideas, assume that $n_{\Omega}=(0,0,1)$. Formula (1) yields

$$
\begin{equation*}
\int_{C} \nu_{3}=2 H \operatorname{Area}(\Omega) \tag{6}
\end{equation*}
$$

Now, consider the Flux Formula for $B_{C}^{1}$ and $B_{C}^{2}$, with $\Omega$ equal to the planar domain bounded by $C, Y=(0,0,1)$. By formula (1)

$$
\begin{equation*}
\int_{C} \nu_{3}^{1}=2 \operatorname{HArea}(\Omega)=-\int_{C} \nu_{3}^{2} \tag{7}
\end{equation*}
$$

Then, equalities (6) and (7) yield

$$
\begin{equation*}
-\int_{C} \nu_{3}^{2}=\int_{C} \nu_{3}=\int_{C} \nu_{3}^{1} \tag{8}
\end{equation*}
$$

If, in equation (5), the inequalities are strict at every point of $C$ then, one has a contradiction by equation (8).

Then, there is at least one point $p$ in $C$ such that $\nu_{3}$ agrees with either $\nu_{3}^{2}$ or $\nu_{3}^{1}$ at $p$. Therefore, by the boundary maximum principle, $M$ coincides with either $B_{C}^{2}$ or $B_{C}^{1}$.

If the boundary of $M$ is an embedded curve $C$ with curvature greater than one the proof is analogous. It is enough to replace the caps $B_{C}^{i}$, $i=1,2$ by the graph $G_{H}$ of mean curvature $H$ and boundary $C$ and its symmetry with respect to the slice $P$.

Remark 1. Notice that one can prove the analogous results for a surface $M$, whose mean curvature function $H(x, y)$ satisfies for any $(x, y) \in \bar{\Omega}: 0<|H(x, y)| \leqslant \frac{1}{2}$.

## 3. H-SURFACES WITH BOUNDARY TWO PARALLEL CURVES WITH CURVATURE GREATER THAN ONE

We say that $C_{1}$ and $C_{2}$ are parallel curves, if they are congruent up to a vertical translation.

Theorem 3.1. Let $M$ be a compact embedded surface in $\mathbb{H}^{2} \times \mathbb{R}$, with boundary two parallel, embedded $C^{2, \alpha}$ curves $C_{a} \subset P_{a}=\{t=a\}$, and $C_{-a} \subset P_{-a}=\{t=-a\}$, with curvature greater than one. Assume that $M$ has constant mean curvature $H$, with $|H| \leqslant 1 / 2$. Then $M$ is symmetric with respect to the horizontal slice $\{t=0\}$. If $2 a \geqslant \pi$, then $M$ is contained in the closed slab $\{-a \leqslant t \leqslant a\}$, with $M \cap\left(P_{a} \cup P_{-a}\right)=$ $C_{a} \cup C_{-a}$. Furthermore, $M$ inherits the symmetries of $C_{a} \cup C_{-a}$.

## Proof.

Let $D_{a}$ be the bounded domain in $P_{a}$ with boundary $C_{a}$. Let $\operatorname{ext}\left(D_{a}\right)=$ $P_{a} \backslash D_{a}$. By the Convex Hull Lemma, $M \cap \operatorname{ext}\left(D_{a}\right)=C_{a}$ and $M \cap \operatorname{ext}\left(D_{-a}\right)=C_{-a}$. For any point $p \in C_{a}$, there exists a circle containing $D_{a}$ in its interior, tangent to $C_{a}$ at $p$. Let $Z_{p}$ be the cylinder over such a circle. Denote by $Z_{p}^{+}$the mean convex open domain of $\mathbb{H}^{2} \times \mathbb{R}$ bounded by $Z_{p}$. Each $Z_{p}$ has mean curvature greater than $\frac{1}{2}$ hence, by the maximum principle, $M$ is contained in $\cap_{p \in C_{a}} Z_{p}^{+}$. Notice that $\partial\left(\cap_{p \in C_{a}} Z_{p}^{+}\right)$is the cylinder over $C_{a}$. We call it $Z$ and we have that $M \cap Z=C_{a} \cap C_{-a}$. Then, we apply Alexandrov reflection with horizontal slices to infer that the slice $\{t=0\}$ is a plane of symmetry for $M$.

Now, assume that $2 a \geqslant \pi$.
By Theorem 2.1, there exists a graph $S_{1}$ with boundary $C_{a}$ with mean curvature $H$ and mean curvature pointing downward, and a graph $S_{2}$ with boundary $C_{-a}$, with mean curvature $H$ and mean curvature vector pointing upward. By the maximum principle $M$ lies below $S_{1}$ and above $S_{2}$, hence $M \cup S_{1} \cup S_{2}$ is a closed embedded surface, not smooth along $C_{a} \cup C_{-a}$. Let $U$ be the domain in $\mathbb{H}^{2} \times \mathbb{R}$ bounded by $M \cup S_{1} \cup S_{2}$. We claim that the mean curvature vector at any point of $M$ points towards $U$. Consider the family of minimal catenoids (Appendix). Let us recall their shape. For any $t \in\left(0, \frac{\pi}{2}\right)$ there exists a catenoid bounded by two circles at infinity at height $\pm t$. When $t \rightarrow 0$, the catenoids tend to the double covering of the slice $\{t=0\}$ and when $t \rightarrow \frac{\pi}{2}$, the catenoids tend to infinity. Then, one comes with catenoids from infinity towards $M$.

Let $p$ be the first contact point between $M$ and one of the catenoids. By the maximum principle, the mean curvature vector of $M$ at $p$ points inside $U$, then it points inside $U$ at any point of $M$.

Now we prove that $M$ is contained in the closed slab $\{-a \leqslant t \leqslant a\}$, with $M \cap\{t= \pm a\}=C_{a} \cup C_{-a}$.

Assume by contradiction that $M$ has some points above the slice $t=a$, and let $q$ be a highest point of $M$. As $S_{1}$ is above $M$, the mean curvature vector at $q$ points upwards. This gives a contradiction by comparing $M$ with a horizontal slice. Now, assume by contradiction that $M$ has some points below the slice $t=-a$, and let $q$ be a lowest point of $M$. As $S_{2}$ is below $M$, the mean curvature vector at $q$ points downwards. And this gives a contradiction by comparing $M$ with a horizontal slice.

By applying the Alexandrov reflection method with vertical geodesic planes, we obtain that $M$ inherits the symmetries of $C_{a} \cup C_{-a}$.
Corollary 3.1. Let $M$ be a compact embedded surface in $\mathbb{H}^{2} \times \mathbb{R}$ with boundary two parallel circles. Assume that $M$ has constant mean curvature $H$, with $|H| \leqslant 1 / 2$. Let d be the distance of the two boundary curves. If $d \geqslant \pi$, then $M$ is part of an embedded complete rotational surface of constant mean curvature $H$.

## Proof.

By the previous Theorem, $M$ is part of a rotational surface with mean curvature $H$. From the proof of Lemma 3.1, we infer that the mean curvature vector of $M$ points towards the interior of $M \cup D_{a} \cup D_{-a}$, where $D_{a} \subset P_{a}, D_{-a} \subset P_{-a}$ are the domains bounded by $C_{a}$ and $C_{-a}$ respectively. By the geometric classification of the rotational surface with constant mean curvature $H, M$ must be a part of an embedded complete rotational surface.

## 4. Surfaces with nonempty asymptotic boundary

In [30] R. Sa Earp describes many examples of complete $H$-surfaces with $H>0$. When the asymptotic boundary is nonempty, then it consists of parts of straight lines.
Theorem 4.1. Let $M$ be a surface in $\mathbb{H}^{2} \times \mathbb{R}$ with mean curvature satisfying $0<\delta \leqslant H(p)$ at any point $p \in M$. Assume that the asymptotic boundary of $M$ in $\partial_{\infty}\left(\mathbb{H}^{2}\right) \times \mathbb{R}$ is a $C^{1}$ curve and that $M$ is $C^{1}$ up to the asymptotic boundary (eventually $M$ has nonempty finite boundary).

Then, each connected component of the asymptotic boundary of $M$ is part of a vertical straight line in $\partial_{\infty}\left(\mathbb{H}^{2}\right) \times \mathbb{R}$.

## Proof.

Let $C$ be a connected component of the asymptotic boundary of $M$, $C \subset \partial_{\infty} M \subset \partial_{\infty}\left(\mathbb{H}^{2}\right) \times \mathbb{R}$. We show that $C$ is vertical at the point $p$ for any $p \in C$. We can assume that $p=1 \in \partial_{\infty}\left(\mathbb{H}^{2}\right) \times\{0\}$.

Suppose that $C$ is not vertical at $p$. Then, an open neighborhood of $p$ in $C$ is a graph $\left(e^{i \theta}, t(\theta)\right)$ for $\theta$ in an open interval around $0\left(p=1=e^{i 0}\right)$ with $t(0)=0$. Let $\varepsilon>0$ be a small number. There exists a real number $\nu=\nu(\varepsilon)>0$ such that $|t(\theta)|<\varepsilon / 2$ for any $\theta \in[-\nu, \nu]$. We set $p_{1}:=\left(e^{i \nu}, t(\nu)\right)$ and $p_{2}:=\left(e^{-i \nu}, t(-\nu)\right)$ and we call $C_{\varepsilon} \subset C$ the closed subarc of $C$ bounded by $p_{1}$ and $p_{2}$ and containing $p$

$$
C_{\varepsilon} \subset \partial_{\infty}\left(\mathbb{H}^{2}\right) \times(-\varepsilon / 2, \varepsilon / 2), \partial C_{\varepsilon}=\left\{p_{1}, p_{2}\right\} .
$$

Since $M$ is $C^{1}$ up to the asymptotic boundary, there exists a simple arc $C_{\varepsilon}^{\prime} \subset M$ with asymptotic boundary $p_{1}$ and $p_{2}$ such that the connected component $M_{\varepsilon}$ of $M \backslash C_{\varepsilon}^{\prime}$ containing $C_{\varepsilon}$ in its asymptotic boundary satisfies $M_{\varepsilon} \subset \mathbb{H}^{2} \times(-\varepsilon, \varepsilon)$. Summarizing, we have
$M_{\varepsilon} \subset M \cap \mathbb{H}^{2} \times(-\varepsilon, \varepsilon), \partial M_{\varepsilon}=C_{\varepsilon}^{\prime}, \partial_{\infty} M_{\varepsilon}=C_{\varepsilon} \subset \partial_{\infty}\left(\mathbb{H}^{2}\right) \times(-\varepsilon / 2, \varepsilon / 2)$.
Let $\Pi: \overline{\mathbb{H}^{2}} \times \mathbb{R} \rightarrow \overline{\mathbb{H}^{2}}$ be the projection on the first two coordinates.
For $\varepsilon$ small enough, $\Pi\left(C_{\varepsilon}\right) \subset \partial_{\infty} \mathbb{H}^{2}$ is an open arc with end points $\Pi\left(p_{1}\right)$ and $\Pi\left(p_{2}\right)$ and containing $1=p$ in its interior. Moreover $\Pi\left(M_{\varepsilon}\right)$ is an open subset of $\mathbb{H}^{2}$ with asymptotic boundary $\Pi\left(C_{\varepsilon}\right)$.

Let $\Gamma \in \mathbb{H}^{2}$ be the geodesic $(-1,1)$ and let $T_{s}, s>0$, be the hyperbolic translation along $\Gamma$ defined as follows: any point $x \in \Gamma$ is sent to the point of $\Gamma$ between $x$ and -1 whose hyperbolic distance from $x$ is $s$. Then, extend $T_{s}$ to $\mathbb{H}^{2} \times \mathbb{R}$ by vertical translation.

Let $\eta>0$ be a small number. For $s$ great enough and $\varepsilon$ small enough, the curve $T_{s}\left(\Pi\left(C_{\varepsilon}^{\prime}\right)\right)$ is inside the open euclidean ball centered at -1 with radius $\eta$. Thus $T_{s}\left(\Pi\left(M_{\varepsilon}\right)\right)$ is the connected component of $\mathbb{H}^{2} \backslash T_{s}\left(\Pi\left(C_{\varepsilon}^{\prime}\right)\right)$ containing 1 in the asymptotic boundary.

The vertical projection of the surface $T_{s}\left(M_{\varepsilon}\right)$ covers a large part of $\mathbb{H}^{2}$ in the euclidean sense. Its boundary is the simple arc $T_{s}\left(C_{\varepsilon}^{\prime}\right)$. Let $\gamma_{s} \subset \partial_{\infty}\left(\mathbb{H}^{2}\right) \times(-\varepsilon, \varepsilon)$ be a $C^{1}$ arc with end points $T_{s}\left(p_{1}\right)$ and $T_{s}\left(p_{2}\right)$ such that $\gamma_{s} \cup T_{s}\left(C_{\varepsilon}\right)$ is a Jordan curve which projects one to one onto $\partial_{\infty}\left(\mathbb{H}^{2}\right) \times\{0\}$.

Finally, let $R_{s} \subset \mathbb{H}^{2} \times \mathbb{R}$ be any embedded smooth disk, disjoint from the interior of $T_{s}\left(M_{\varepsilon}\right)$, with finite boundary $T_{s}\left(C_{\varepsilon}^{\prime}\right)$ and asymptotic boundary $\gamma_{s}$ :

$$
\begin{aligned}
R_{s} \cap T_{s}\left(M_{\varepsilon}\right) & =T_{s}\left(C_{\varepsilon}^{\prime}\right), \\
\partial R_{s} & =T_{s}\left(C_{\varepsilon}^{\prime}\right), \\
\partial_{\infty} R_{s} & =\gamma_{s} .
\end{aligned}
$$

Then $R_{s} \cup T_{s}\left(M_{\varepsilon}\right)$ is an embedded simply connected surface, it is $C^{0}$ along $T_{s}\left(C_{\varepsilon}^{\prime}\right)$ and smooth everywhere else. The surface $R_{s} \cup T_{s}\left(M_{\varepsilon}\right)$ separates $\mathbb{H}^{2} \times \mathbb{R}$ in two connected components and its asymptotic boundary is the Jordan curve $\gamma_{s} \cup T_{s}\left(C_{\varepsilon}\right) \subset \partial_{\infty}\left(\mathbb{H}^{2}\right) \times[-\varepsilon, \varepsilon]$.

First, we assume that the mean curvature vector of $T_{s}\left(M_{\varepsilon}\right)$ points towards the connected component containing $\mathbb{H}^{2} \times(\varepsilon,+\infty)$.

We can assume that $\delta<1 / 2$. Let us consider the simply connected $H$-surface $S^{\delta}$ given in Proposition 5.2 with $H=\delta$ and with rotational axis equals to the vertical geodesic $\{0\} \times \mathbb{R}$. Now, lift up $S^{\delta}$ to be above $R_{s} \cup T_{s}\left(M_{\varepsilon}\right)$, then move $S^{\delta}$ down. By our construction and the geometry of $S^{\delta}$ the first contact with $R_{s} \cup T_{s}\left(M_{\varepsilon}\right)$ will be at an interior point of $T_{s}\left(M_{\varepsilon}\right)$. This gives a contradiction with the maximum principle. If the mean curvature vector of $T_{s}\left(M_{\varepsilon}\right)$ points towards the other component, one does the same reasoning with the surface obtained from $S^{\delta}$ by symmetry with respect to the slice $\mathbb{H}^{2} \times\{0\}$. Therefore, the asymptotic boundary $C$ is vertical at any point $p \in C$.

## 5. Appendix: Geometric behavior of the rotational $H$-SURFACES IN $\mathbb{H}^{2} \times \mathbb{R}$

In the Appendix we describe in details the geometric behavior of rotational $H$-surfaces. Our discussion is based on formulae founded in [34]. Rotational surfaces in $\mathbb{H}^{2} \times \mathbb{R}$ have been studied in [1], [16], [21], [27] and [34]. Recall that the plane and the catenoid are the unique rotational minimal surfaces in $\mathbb{R}^{3}$. In 1841, Delaunay ([7]) determined all rotational $H$-surfaces in $\mathbb{R}^{3}$ with $H \neq 0$, by a geometric construction. They are called Delaunay's surfaces. Namely, the spheres, the circular cylinders, the undoloids and the nodoids. We will see in Proposition 5.3 that for $H>1 / 2$ the geometric behavior of rotational $H$-surfaces in $\mathbb{H}^{2} \times \mathbb{R}$ is analogous to the geometric behavior of the Delaunay's surfaces.

We work with the disk model for $\mathbb{H}^{2}$, so that

$$
\mathbb{H}^{2}=\left\{(x, y) \in \mathbb{R}^{2}, x^{2}+y^{2}<1\right\},
$$

and the metric is

$$
\mathrm{d} s_{\mathbb{H}}^{2}=\left(\frac{2}{1-\left(x^{2}+y^{2}\right)}\right)^{2}\left(\mathrm{~d} x^{2}+\mathrm{d} y^{2}\right)
$$

Therefore the product metric on $\mathbb{H}^{2} \times \mathbb{R}$ reads as follows

$$
\mathrm{d} \tilde{s}^{2}=\left(\frac{2}{1-\left(x^{2}+y^{2}\right)}\right)^{2}\left(\mathrm{~d} x^{2}+\mathrm{d} y^{2}\right)+\mathrm{d} t^{2}
$$

where $(x, y) \in \mathbb{H}^{2}$ and $t \in \mathbb{R}$. We consider the following particular geodesic of $\mathbb{H}^{2}$

$$
\Gamma=\{(x, 0), x \in(-1,1)\} \subset \mathbb{H}^{2}
$$

Up to ambient isometry, we can assume the rotational surfaces are generated by curves in the vertical geodesic plane $P=\Gamma \times \mathbb{R} \subset \mathbb{H}^{2} \times \mathbb{R}$ and that the rotational axis is the vertical geodesic $R:=\{(0,0)\} \times \mathbb{R}$.

On the geodesic $\Gamma$ we denote by $\rho \in \mathbb{R}$ the signed distance to the origin $(0,0)$, thus $x=\tanh \rho / 2$. Therefore the metric on $P$ is

$$
\mathrm{d} s^{2}=\left(\frac{2}{1-x^{2}}\right)^{2} \mathrm{~d} x^{2}+\mathrm{d} t^{2}=\mathrm{d} \rho^{2}+\mathrm{d} t^{2}
$$

Let us consider a curve in $P$ which is a vertical graph: $c(\rho)=(\rho, \lambda(\rho))$ where $\lambda$ is a smooth real function defined for $\rho \geqslant 0$. On the rotational surface generated by $c$ we consider the orientation given by the unit normal field pointing up. It is shown in [34] (formula (21)) that the curve $c$ generates a rotational surface with constant mean curvature $H$ if and only if the function $\lambda$ is given by

$$
\begin{equation*}
\lambda(\rho)=\int_{*}^{\rho} \frac{d+2 H \cosh r}{\sqrt{\sinh ^{2} r-(d+2 H \cosh r)^{2}}} d r \tag{9}
\end{equation*}
$$

where $d$ is a real parameter and $*$ is the minimum such that the condition $\sinh ^{2} r-(d+2 H \cosh r)^{2} \geqslant 0$ is satisfied. Using the isometry $(x, y, t) \mapsto(x, y,-t)$ we can assume that $H \geqslant 0$. We will analyze consecutively the cases $H=0, H \in(0,1 / 2]$ and $H>1 / 2$.

Proposition 5.1. (Minimal rotational surfaces) For each $d \geqslant 0$ there exists a complete minimal rotational surface $\mathcal{M}_{d}$ (Figure 1). The surface $\mathcal{M}_{0}$ is the horizontal slice $\{t=0\}$. For $d>0$ the rotational surface $\mathcal{M}_{d}$ (called catenoid) is embedded and homeomorphic to an annulus. The distance between the rotational axis and the "neck" of $\mathcal{M}_{d}$ is $\operatorname{arcsinh} d$. The asymptotic boundary of $\mathcal{M}_{d}$ is two horizontal circles in $\partial_{\infty}\left(\mathbb{H}^{2}\right) \times \mathbb{R}$ and the vertical distance between them is a nondecreasing function $h(d)$ satisfying $\lim _{d \rightarrow 0} h(d)=0$ and $\lim _{d \rightarrow+\infty} h(d)=\pi$. Therefore $\mathcal{M}_{d}$ converges to the double covering of the slice $\{t=0\}$ when d goes to 0 .

Moreover any minimal rotational surface is, up to an ambient isometry, a part of a complete surface $\mathcal{M}_{d}$.


Figure 1

## Proof.

If the graph of a function $\lambda$ generates a minimal surface we deduce from formula (9) that

$$
\lambda(\rho)=\int_{\operatorname{arcsinh}(d)}^{\rho} \frac{d}{\sqrt{\sinh ^{2} r-d^{2}}} d r .
$$

Thus we have $\lambda \equiv 0$ for $d=0$. For $d>0$ the function $\lambda$ is defined for $\rho \geqslant \operatorname{arcsinh}(d)>0$ and the graph $c$ has a vertical tangent at $\rho=\operatorname{arcsinh} d$. It is clear that $\lim _{\rho \rightarrow+\infty} \lambda(\rho)$ exists and is finite. Let us call $\widetilde{c}$ the union of the curve $c$ with its symmetry with respect to the horizontal geodesic $\{y=0\}$ of $P$. Therefore $\widetilde{c}$ is a complete curve which generates a rotational complete and embedded minimal surface homeomorphic to an annulus.

As in formula (41) of [34] we introduce the new coordinate $s$ setting $d s=\sqrt{1+\lambda^{\prime 2}} d \rho(s$ is the arclength of the graph $c(\rho)=(\rho, \lambda(\rho)))$. Then using the formulae (49), (36) and (37) of [34] we get

$$
\begin{aligned}
\rho(s) & =\int_{0}^{s} \frac{\sinh t}{\sqrt{\left(1+d^{2}\right) \cosh ^{2} t-1}} d t+\operatorname{arcosh} \sqrt{1+d^{2}}, \\
& =\operatorname{arcosh}\left(\sqrt{1+d^{2}} \cosh s\right) .
\end{aligned}
$$

and

$$
\lambda \circ \rho(s)=\int_{0}^{s} \frac{d}{\sqrt{\left(1+d^{2}\right) \cosh ^{2} t-1}} d t
$$

for $s \geqslant 0$. Therefore we have

$$
h(d)=2 \int_{0}^{+\infty} \frac{d}{\sqrt{\left(1+d^{2}\right) \cosh ^{2} t-1}} d t .
$$

Consider the positive function defined by $f(t, d)=\frac{d}{\sqrt{\left(1+d^{2}\right) \cosh ^{2} t-1}}$ for $t, d \geqslant 0$. Let $d_{0}>0$ be any positive real number. Clearly the integral $h(d)$ is convergent for any $d \in\left[d_{0},+\infty\right)$. Moreover we have

$$
\frac{\partial}{\partial d} f(t, d)=\frac{\sinh t}{\left(\left(1+d^{2}\right) \cosh ^{2} t-1\right)^{3 / 2}}
$$

We deduce that the integral

$$
\int_{0}^{+\infty} \frac{\partial}{\partial d} f(t, d) d t
$$

is uniformly convergent for $d \geqslant d_{0}>0$. Consequently the function $h(d)$ is differentiable on $\left[d_{0},+\infty\right)$ and

$$
\begin{align*}
h^{\prime}(d) & =2 \int_{0}^{+\infty} \frac{\partial}{\partial d} f(t, d) d t  \tag{10}\\
& =2 \int_{0}^{+\infty} \frac{\sinh t}{\left(\left(1+d^{2}\right) \cosh ^{2} t-1\right)^{3 / 2}} d t \tag{11}
\end{align*}
$$

As this is true for $d \geqslant d_{0}>0$ for any $d_{0}>0$, then $h$ is differentiable for $d>0$ and its derivative is given by (11). We deduce that $h(d)$ is nondecreasing and we have $\lim _{d \rightarrow 0} h(d)=0$.

Finally, from the inequalities

$$
\frac{d}{\sqrt{\left(1+d^{2}\right)}} \frac{1}{\cosh t} \leqslant \frac{d}{\sqrt{\left(1+d^{2}\right) \cosh ^{2} t-1}} \leqslant \frac{1}{\cosh t}
$$

for any $d, t>0$, we get

$$
\begin{aligned}
\lim _{d \rightarrow+\infty} h(d) & =2 \int_{0}^{+\infty} \lim _{d \rightarrow+\infty} \frac{d}{\sqrt{\left(1+d^{2}\right) \cosh ^{2} t-1}} d t \\
& =2 \int_{0}^{+\infty} \frac{d t}{\cosh t}, \\
& =2 \int_{0}^{+\infty} \frac{d u}{u^{2}+1}, \quad(u=\sinh t) \\
& =\pi
\end{aligned}
$$

This concludes the proof.

For later use we define the functions $g(\rho)$ and $f(\rho)$ setting for $d \in \mathbb{R}$ and $H>0$,

$$
\begin{aligned}
g(\rho) & =d+2 H \cosh \rho \\
f(\rho) & =\sinh ^{2} \rho-(d+2 H \cosh \rho)^{2} \\
& =\left(1-4 H^{2}\right) \cosh ^{2} \rho-4 d H \cosh \rho-1-d^{2}
\end{aligned}
$$

so that $\lambda^{\prime}(\rho)=g(\rho) / \sqrt{f(\rho)}$.
Lemma 5.1. Assume $0<H<1 / 2$. We have $f(\rho) \geqslant 0$ if and only if $\cosh \rho \geqslant \frac{2 d H+\sqrt{1-4 H^{2}+d^{2}}}{1-4 H^{2}}$. Let $\rho_{1} \geqslant 0$ such that $\cosh \rho_{1}=$ $\frac{2 d H+\sqrt{1-4 H^{2}+d^{2}}}{1-4 H^{2}}$, then $f\left(\rho_{1}\right)=0$ and $\rho_{1}=0$ if and only if $d=-2 H$.
(1) If $d>-2 H$, then $\frac{-d}{2 H}<\cosh \rho_{1}$. Consequently the function $\lambda$ is nondecreasing for $\rho \geqslant \rho_{1}>0$ and has a nonfinite derivative at $\rho_{1}$.
(2) If $d=-2 H$, then $\lambda^{\prime}(\rho)=\frac{2 H \sqrt{\cosh \rho-1}}{\sqrt{\left(1-4 H^{2}\right) \cosh \rho+4 H^{2}+1}}$. Therefore the function $\lambda$ is defined for $\rho \geqslant 0$, it has a zero derivative at 0 and is nondecreasing for $\rho>0$.
(3) If $d<-2 H$, then there exists $\rho_{0}>\rho_{1}>0$ such that $\frac{-d}{2 H}=$ $\cosh \rho_{0}$. Consequently the function $\lambda$ is defined for $\rho \geqslant \rho_{1}>0$ with a nonfinite derivative at $\rho_{1}$, it is nonincreasing for $\rho_{1}<\rho<\rho_{0}$, has a zero derivative at $\rho_{0}$ and it is nondecreasing for $\rho>\rho_{0}$.
(4) For any d we have $\lim _{\rho \rightarrow+\infty} \lambda(\rho)=+\infty$.

Next Lemma, is analogous to Lemma 5.1 in the case $H=1 / 2$. We observe that, in this case, the set $\{\rho>0 \mid f(\rho)>0\}$ is nonempty if and only if $d<0$.

Lemma 5.2. Assume $H=1 / 2$ and $d<0$. Then $f(\rho) \geqslant 0$ if and only if $\cosh \rho \geqslant \frac{1+d^{2}}{-2 d}$. Let $\rho_{1} \geqslant 0$ such that $\cosh \rho_{1}=\frac{1+d^{2}}{-2 d}$, then $f\left(\rho_{1}\right)=0$ and $\rho_{1}=0$ if and only if $d=-1$.
(1) If $d \in(-1,0)$, then $\frac{-d}{2 H}<\cosh \rho_{1}$. Consequently the function $\lambda$ is nondecreasing for $\rho \geqslant \rho_{1}>0$ and has a nonfinite derivative at $\rho_{1}$.
(2) If $d=-1$, then $\lambda^{\prime}(\rho)=\frac{1}{\sqrt{2}} \sqrt{\cosh \rho-1}$. Therefore the function $\lambda$ is defined for $\rho \geqslant 0$, it has a zero derivative at 0 and is nondecreasing for $\rho>0$.
(3) If $d<-1$ there exists $\rho_{0}>\rho_{1}>0$ such that $\frac{-d}{2 H}=\cosh \rho_{0}$. Consequently the function $\lambda$ is defined for $\rho \geqslant \rho_{1}>0$ with a nonfinite derivative at $\rho_{1}$, it is nonincreasing for $\rho_{1}<\rho<\rho_{0}$, has a zero derivative at $\rho_{0}$ and it is nondecreasing for $\rho>\rho_{0}$.
(4) For any $d$ we have $\lim _{\rho \rightarrow+\infty} \lambda(\rho)=+\infty$.

The proof of Lemma 5.1 and 5.2 is a straightforward computation taking into account Formula (9). As a consequence of Lemma 5.1 and 5.2 we have the following results.

Proposition 5.2. (Rotational $H$-surfaces with $|H| \leqslant 1 / 2$ )
Assume $0<H \leqslant 1 / 2$. There exists a one-parameter family $\mathcal{H}_{d}$, $d \in \mathbb{R}$ for $H<1 / 2$ and $d<0$ for $H=1 / 2$, of complete rotational H-surfaces.
(1) For $d>-2 H$, the surface $\mathcal{H}_{d}$ is a properly embedded annulus (Figure 2-a), symmetric with respect to the slice $\{t=0\}$, the distance between the "neck" and the rotational axis $R=$ $\{(0,0)\} \times \mathbb{R}$ is $\operatorname{arcosh}\left(\frac{2 d H+\sqrt{1-4 H^{2}+d^{2}}}{1-4 H^{2}}\right)$ for $H<1 / 2$ and $\operatorname{arcosh}\left(\frac{1+d^{2}}{-2 d}\right)$ for $H=1 / 2$.
(2) For $d=-2 H$, the surface $\mathcal{H}_{-2 H}$ is an entire vertical graph, denoted by $S^{H}$ (Figure 2-b). Moreover $S^{H}$ is contained in the halfspace $\{t \geqslant 0\}$ and it is tangent to the slice $\mathbb{H}^{2} \times\{0\}$ at the point $(0,0,0)$.
(3) For $d<-2 H$, the surface $\mathcal{H}_{d}$ is a properly immersed (and nonembedded) annulus (Figure 2-c), it is symmetric with respect to the slice $\{t=0\}$, the distance between the "neck" and the rotational axis $R$ is $\operatorname{arcosh}\left(\frac{2 d H+\sqrt{1-4 H^{2}+d^{2}}}{1-4 H^{2}}\right)$ for $H<1 / 2$ and $\operatorname{arcosh}\left(\frac{1+d^{2}}{-2 d}\right)$ for $H=1 / 2$.
(4) In each of the previous case the surface is unbounded in the $t$-coordinate. When $d$ tends to $-2 H$ with either $d>-2 H$ or $d<-2 H$, then the surfaces $\mathcal{H}_{d}$ tends towards the union of $S^{H}$ and its symmetry with respect to the slice $\{t=0\}$. Furthermore, any rotational $H$-surface with $0<H \leqslant 1 / 2$ is, up to an ambient isometry, a part of a surface of the family $\mathcal{H}_{d}$.


Figure 2-a
Figure 2-b


Figure 2-c

## Proof.

The result is a straightforward consequence of Lemma 5.1 and 5.2. For $d=-2 H, \mathcal{H}_{-2 H}$ is the rotational surface generated by the graph of the function $\lambda$.

For $d \neq-2 H$, let $\gamma$ be the union of the graph of $\lambda$ joint with its symmetry with respect to the slice $\{t=0\}$. Then $\mathcal{H}_{d}$ is the rotational surface generated by the curve $\gamma$.

Observe that, for $H>1 / 2$, the set $\{\rho>0 \mid f(\rho)>0\}$ is nonempty if and only if $d<-\sqrt{4 H^{2}-1}$.

Lemma 5.3. Let $H$ and $d$ satisfying $H>1 / 2$ and $d<-\sqrt{4 H^{2}-1}$. Then, there exist two numbers $0 \leqslant \rho_{1}<\rho_{2}$ such that $\cosh \rho_{1}=$ $\frac{2 d H+\sqrt{1-4 H^{2}+d^{2}}}{1-4 H^{2}}$ and $\cosh \rho_{2}=\frac{2 d H-\sqrt{1-4 H^{2}+d^{2}}}{1-4 H^{2}}$. Therefore, $f(\rho)>0$ if and only if $\rho_{1}<\rho<\rho_{2}$ and $f\left(\rho_{1}\right)=f\left(\rho_{2}\right)=0$.
(1) If $d<-2 H$, then $\rho_{1}>0$ and there exists a unique number $\rho_{0} \in\left(\rho_{1}, \rho_{2}\right)$ satisfying $g\left(\rho_{0}\right)=0$. Furthermore $g \leqslant 0$ on $\left[\rho_{1}, \rho_{0}\right)$ and $g \geqslant 0$ on $\left(\rho_{0}, \rho_{2}\right]$. Consequently, the function $\lambda$ is defined on $\left[\rho_{1}, \rho_{2}\right]$, has a nonfinite derivative at $\rho_{1}$ and $\rho_{2}$, has a zero derivative at $\rho_{0}$, is nonincreasing on $\left(\rho_{1}, \rho_{0}\right)$ and nondecreasing on ( $\rho_{0}, \rho_{2}$ ).
(2) If $d=-2 H$, then $\rho_{1}=0$ and $\lambda^{\prime}(\rho)=\frac{2 H \sqrt{\cosh \rho-1}}{\sqrt{\left(1-4 H^{2}\right) \cosh \rho+4 H^{2}+1}}$. Consequently, the function $\lambda$ is defined on $\left[0, \rho_{2}\right]$, is nondecreasing, has a zero derivative at 0 and a nonfinite derivative at $\rho_{2}$.
(3) If $-2 H<d<-\sqrt{4 H^{2}-1}$, then $\rho_{1}>0$ and $g \geqslant 0$ on $\left[\rho_{1}, \rho_{2}\right]$. Therefore the function $\lambda$ is defined on $\left[\rho_{1}, \rho_{2}\right]$, is nondecreasing and has nonfinite derivative at $\rho_{1}$ and $\rho_{2}$.

The proof of Lemma 5.3 follows from formula (9) by a computation. In the following Proposition we assume the notations of Lemma 5.3

Proposition 5.3. (Rotational surfaces with $H>1 / 2$ )
Assume $H>1 / 2$. There exists a one-parameter family $\mathcal{D}_{d}$ of complete rotational $H$-surfaces, $d \leqslant-\sqrt{4 H^{2}-1}$.
(1) For $d<-2 H$, the surface $\mathcal{D}_{d}$ is an immersed (and nonembedded) annulus, invariant by a vertical translation and is contained in the closed region bounded by the two vertical cylinders $\rho=\rho_{1}$ and $\rho=\rho_{2}$. Furthermore $\rho_{1} \rightarrow+\infty$ and $\rho_{2} \rightarrow+\infty$ when $d \rightarrow-\infty$ and $\rho_{1} \rightarrow 0$ and $\rho_{2} \rightarrow \operatorname{arcosh}\left(\frac{4 H^{2}+1}{4 H^{2}-1}\right)$ when $d \rightarrow-2 H$. Such surfaces are analogous to the nodoids of Delaunay in $\mathbb{R}^{3}$ (Figure 3-a).
(2) For $d=-2 H$, the surface $\mathcal{D}_{-2 H}$ is an embedded sphere and the maximal distance from the rotational axis is $\rho_{2}=\operatorname{arcosh}\left(\frac{4 H^{2}+1}{4 H^{2}-1}\right)$ (Figure 3-b).
(3) For $-2 H<d<-\sqrt{4 H^{2}-1}$, the surface $\mathcal{D}_{d}$ is an embedded annulus, invariant by a vertical translation and is contained in the closed region bounded by the two vertical cylinders $\rho=\rho_{1}$ and $\rho=\rho_{2}$. Furthermore $\rho_{1} \rightarrow 0$ and $\rho_{2} \rightarrow \operatorname{arcosh}\left(\frac{4 H^{2}+1}{4 H^{2}-1}\right)$ when $d \rightarrow-2 H$ and both $\rho_{1}, \rho_{2} \rightarrow \operatorname{arcosh}\left(\frac{2 H}{\sqrt{4 H^{2}-1}}\right)$ when $d \rightarrow$ $-\sqrt{4 H^{2}-1}$. Moreover $\rho_{1}<\operatorname{arcosh}\left(\frac{2 H}{\sqrt{4 H^{2}-1}}\right)<\rho_{2}$. Such surfaces are analogous to the undoloids of Delaunay in $\mathbb{R}^{3}$ (Figure 3-c).
(4) For $d=-\sqrt{4 H^{2}-1}$, the surface $\mathcal{D}_{-\sqrt{4 H^{2}-1}}$ is the vertical cylinder over the circle with hyperbolic radius $\operatorname{arcosh}\left(\frac{2 H}{\sqrt{4 H^{2}-1}}\right)$.

## Proof.

For $d=-\sqrt{4 H^{2}-1}$ we get the limit case of a vertical cylinder given by $\cosh \rho=\frac{2 H}{\sqrt{4 H^{2}-1}}$. A straightforward computation shows the mean curvature of such a cylinder is $H$.

For the other cases the proof is a straightforward consequence of Lemma 5.3. Let $\gamma$ be the union of the graph of $\lambda$ joint with its symmetries with respect to the horizontal slices on which $\lambda$ is vertical. When $d=-2 H, \gamma$ is a compact arc and for $d \neq-2 H, \gamma$ is periodic and complete, embedded only when $d>-2 H$.


Figure 3-a


Figure 3-b


Figure 3-c

From Proposition 5.1, 5.2, 5.3 and Proposition 26 in [34], we infer the following classification of rotational $H$-surfaces with vanishing Abresch-Rosenberg holomorphic quadratic differential. The classification of $H$-surfaces with vanishing Abresch-Rosenberg holomorphic quadratic differential is established in [1].

Theorem 5.1. Let $M$ be a rotational $H$-surface, $H \geqslant 0$, with vanishing Abresch-Rosenberg holomorphic quadratic differential. We have up to congruence:
(1) If $H=0$ then $M$ is a slice $\mathbb{H}^{2} \times\{t\}$.
(2) If $H>1 / 2$ then $M$ is an embedded two-sphere.
(3) If $H=1 / 2$ then $M$ is the entire vertical graph $S^{1 / 2}$.
(4) If $H<1 / 2$ then $M$ is either the entire vertical graph $S^{H}$ or the embedded annulus $\mathcal{H}_{2 H}$.

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