# UNIQUENESS OF MINIMAL SURFACES WHOSE BOUNDARY IS A HORIZONTAL GRAPH AND SOME BERNSTEIN PROBLEMS IN $\mathbb{H}^{2} \times \mathbb{R}$ 

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#### Abstract

We deduce that a connected compact immersed minimal surface in $\mathbb{H}^{2} \times \mathbb{R}$ whose boundary has an injective horizontal projection on an admissible convex curve in $\partial_{\infty} \mathbb{H}^{2} \times \mathbb{R}$ and satisfies an admissible bounded slope condition, is the Morrey's solution of the Plateau problem and is a horizontal minimal graph. We prove that there is no entire horizontal minimal graph in $\mathbb{H}^{2} \times \mathbb{R}$.


KEY WORDS: Maximum principle, horizontal minimal equation, convex domain, Plateau problem, Bernstein problem.

## 1. Introduction

The theory of minimal and constant mean curvature surfaces in the product space $\mathbb{H}^{2} \times \mathbb{R}$ has been developed since the discovery of a holomorphic quadratic differential by Abresch and Rosenberg [1]. Recently, many results on the minimal surfaces theory were achieved for the vertical minimal graphs see, for instance, [5], [3] and [7].

However, there exists another notion of graph that also arises naturally in the theory: "horizontal graph". For the definition we follow [6] were many explicit examples were given. We choose the upper halfplane model of hyperbolic plane $\mathbb{H}^{2}=\{(x, y), y>0\}$, endowed with the hyperbolic metric $d \sigma^{2}=\frac{d x^{2}+d y^{2}}{y^{2}}$. A horizontal graph in $\mathbb{H}^{2} \times \mathbb{R}$ is the set $S=\{(x, g(x, t), t),(x, t) \in \Omega\} \subset \mathbb{H}^{2} \times \mathbb{R}$, where $\Omega \subset \partial_{\infty} \mathbb{H}^{2} \times \mathbb{R}$ is a domain and $g(x, t)>0$, for every $(x, t) \in \Omega$. This means that in any slice of $\mathbb{H}^{2} \times \mathbb{R}$ given by $t=\mathrm{cst}$, each horizontal geodesic $x=\mathrm{cst}, y>0$ intersects $S$ in one point at most. We call the positive function $g(x, t)$ the horizontal length of the graph. If $S$ is a horizontal minimal graph

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in $\mathbb{H}^{2} \times \mathbb{R}$ the positive function $g(x, t)$ satisfies:

$$
\begin{equation*}
\mathcal{M}_{\mathcal{H}}(g):=g_{x x}\left(g^{2}+g_{t}^{2}\right)+g_{t t}\left(1+g_{x}^{2}\right)-2 g_{x} g_{t} g_{x t}+g\left(1+g_{x}^{2}\right)=0 \tag{1}
\end{equation*}
$$

There are many entire vertical minimal graphs in $\mathbb{H}^{2} \times \mathbb{R}[5]$, [7], but, in this paper, we prove the following "Bernstein type result":

Theorem 1.1. There is no entire horizontal minimal graph in $\mathbb{H}^{2} \times \mathbb{R}$.
There exists a family of complete embedded minimal surfaces invariant by parabolic screw motions whose asymptotic boundary are two parallel straight lines [6]. These minimal surfaces can be seen as horizontal minimal graphs given by a a function $y=g_{\mathcal{M}}(x, t, d, \ell), d \neq 0$ over a strip in the $x t$ plane $\partial_{\infty} \mathbb{H}^{2} \times \mathbb{R}$ continuous up to the boundary taking zero boundary data [6]. The generating curves of some examples are given in Figure 1.


Figure 1
Generating curves of horizontal minimal complete graphs

We recall now the horizontal mean curvature equation in $\mathbb{H}^{2} \times \mathbb{R}$ :
$\mathcal{M}_{\mathcal{H}}(g)=\frac{2 H}{g^{2}}\left(g_{t}^{2}+g^{2}\left(1+g_{x}^{2}\right)\right)^{3 / 2}$, where $g$ is a $C^{2}$ positive function and $H$ is the mean curvature.

It turns out that there are also many examples of horizontal constant mean curvature graphs in $\mathbb{H}^{2} \times \mathbb{R}$. Indeed there exists a 2-parameter family constituted of entire horizontal graphs with positive mean curvature $H<1 / 2$ [6, Theorem 2.2]. On the other hand, there exist horizontal graphs with mean curvature $1 / 2$ in $\mathbb{H}^{2} \times \mathbb{R}$, given by explicit formulas. For example, $y=\frac{x}{\sqrt{t^{2}-1}}, x>0, t>1[6$, Equation
(31)]. Moreover, any horocylinder given by $y=c, c>0$ is an entire horizontal graph with mean curvature $H=1 / 2$.

Finally, there is no entire horizontal graph with mean curvature $H>$ $1 / 2$.

We believe that, if $S$ is an entire horizontal $H$-graph in $\mathbb{H}^{2} \times \mathbb{R}$ with constant mean $0<H \leqslant 1 / 2$, then $S$ is invariant by a 1 -parameter group of isometries of $\mathbb{H}^{2} \times \mathbb{R}$. These are the Bernstein type problems for horizontal $H$-graphs, $0<H \leqslant 1 / 2$, in $\mathbb{H}^{2} \times \mathbb{R}$.

We need the following definition to establish our uniqueness result.
Definition 1.1. We say that a $C^{0}$ Jordan domain $\Omega \subset \partial_{\infty} \mathbb{H}^{2} \times \mathbb{R}$ with boundary $\gamma$ is admissible if, up to an Euclidean translation, $\gamma=$ $\gamma^{+} \cup \gamma^{-}$, where $\gamma^{ \pm}$are graphs of functions $f^{ \pm}:[-a, a] \rightarrow \mathbb{R}, x=$ $f^{ \pm}(t), a>0$, respectively, satisfying
(1) $f^{+}>0$ and $f^{-}<0$ on $(-a, a)$.
(2) $f^{ \pm}( \pm a)=0$.

We say that a $C^{0}$ Jordan curve $\gamma$ is admissible, if it bounds an admissible domain $\Omega$.
Definition 1.2. Let $\Gamma$ be a $C^{1}$ horizontal graph in $\mathbb{H}^{2} \times \mathbb{R}$ over a $C^{1}$ convex admissible curve $\gamma$.

Let $p$ be a point of $\Gamma$ at height $c$. Let $\pi_{p}$ be the Euclidean plane passing through $p$ determined by the tangent line to $\Gamma$ at $p$ and the equidistant line to the geodesic $x=0, y>0, t=c$, issuing from the $t$ axis at height $c$ passing through $p$.

We say that $\Gamma$ satisfies an admissible bounded slope condition, if for every $p \in \Gamma, \Gamma$ is contained in one side of $\pi_{p}$.

At last, we prove the following:
Theorem 1.2. Let $M \rightarrow \mathbb{H}^{2} \times \mathbb{R}$ be a compact connected immersed minimal surface $C^{1}$ up to the boundary $\Gamma$.

Assume $\Gamma$ is a $C^{1}$ horizontal graph over a $C^{1}$ convex admissible curve $\gamma$ in $\partial_{\infty} \mathbb{H}^{2} \times \mathbb{R}$. Assume further that $\Gamma$ satisfies an admissible bounded slope condition. Then $M$ is the Morrey's solution of the Plateau problem and is a horizontal minimal graph.

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## 2. Some model horizontal minimal graphs in $\mathbb{H}^{2} \times \mathbb{R}$. Proof of the nonexistence result.

First, Let us give some remarks about the horizontal minimal equation.

Remark 2.1.
(1) Is is easy to see that the Euclidean planes in $\mathbb{H}^{2} \times \mathbb{R}$ given by $y=a x+b t+c ; a, b, c \in \mathbb{R}, y>0$ are positive subsolutions [7] of equation (1). In particular the horocylinders given by $y=c, c>0$, are subsolutions of (1).
(2) Of course, the vertical geodesic planes $\mathcal{P}=\mathcal{P}\left(R, x_{0}\right)=$ $\left\{(x, y, t) \in \mathbb{H}^{2} \times \mathbb{R} ; y=\sqrt{R^{2}-\left(x-x_{0}\right)^{2}}, t \in(-\infty, \infty)\right.$, $\left.R^{2}-\left(x-x_{0}\right)^{2}>0\right\}$, where $R>0$ and $x_{0} \in \mathbb{R}$, are solutions of equation (1).
(3) Minimality is invariant by a positive isometry of $\mathbb{H}^{2} \times \mathbb{R}$ given by a hyperbolic translation of $\mathbb{H}^{2} \times\{0\}$ along a geodesic $L$ [8]. In particular, the homothety in $\mathbb{H}^{2}$ with center $x_{0} \in \mathbb{R}$, ratio $\lambda>0$, keeping $\mathbb{H}^{2}$ invariant gives rise to an isometry of the product space $\mathbb{H}^{2} \times \mathbb{R}$ given by $(x, y, t) \mapsto\left(\lambda\left(x-x_{0}\right), \lambda y, t\right)+$ $\left(x_{0}, 0,0\right)$. Thus, the family of equidistant curves to the geodesic $x=0, y>0$, at any slice $\{t=\mathrm{cst}\}$, provides a Killing system of coordinates of the ambient space. Explicit formulas are given in [8, Exercise 2.6.1].

In view of this observation, we deduce that if $y=g(x, t)$ is a solution of (1) on a domain $\Omega$ then $y=\lambda g\left(\frac{x}{\lambda}, t\right), \lambda>0$ is also a solution of (1) on $T_{\lambda}(\Omega)$, where $T_{\lambda}$ is the linear map given by the matrix $\left(\begin{array}{ll}\lambda & 0 \\ 0 & 1\end{array}\right)$.
An entire horizontal graph in $\mathbb{H}^{2} \times \mathbb{R}$ is a graph given by a function $y=g(x, t)$ defined for all values of the independent variables $x, t$. We use the geometry of the hyperbolic minimal surfaces described in [7] to prove the non existence of an entire horizontal minimal graph in $\mathbb{H}^{2} \times \mathbb{R}$. We remark that Laurent Hauswirth [2] has given a classification of minimal surfaces invariant by hyperbolic translations.
Example 2.1.
Keeping the notations of [7], we summarize the properties of the family $M_{d}, d>1$, as follows: First, we consider the disk model for $\mathbb{H}^{2}$. Each $M_{d}, d>1$ is invariant by hyperbolic translations along a geodesic $\gamma$ in $\mathbb{H}^{2} \times\{0\}$. Furthermore, $M_{d}$ is symmetric with respect to the slice $t=0$, it contains the equidistant curve $\gamma_{d}$ of $\gamma$ in $\mathbb{H}^{2} \times\{0\}$ and is contained in a slab $-H(d) \leqslant t \leqslant H(d)$ of $\mathbb{H}^{2} \times \mathbb{R}$. Each slice
$t=c, c \in(-H(d), H(d)), c \neq 0$, cuts $M_{d}$ along an equidistant curve of $\gamma$ contained in the non mean convex side of $\gamma_{d}$. Let $\Gamma$ be the geodesic orthogonal to $\gamma$ passing through the origin of $\mathbb{H}^{2} \times\{0\}$ and let us denote by $\infty$ the point in the asymptotic boundary of $\Gamma$ lying in the component of $\mathbb{H}^{2} \times\{0\} \backslash \gamma$ not intersecting $M_{d}$. It follows that each $M_{d}$ is transverse to the family of geodesics with same asymptotic boundary $\infty$. Let $c_{1}$ be the closed arc of $\partial_{\infty} \mathbb{H}^{2} \times\{0\}$, given by the asymptotic boundary of the component of $\mathbb{H}^{2} \times\{0\} \backslash \gamma$ that contains $\gamma_{d}$. The asymptotic boundary of $M_{d}$ consists of two copies of $c_{1}$ in the slices $t= \pm H(d)$, respectively, and two vertical segments joining these arcs.

Now turning to our model of $\mathbb{H}^{2} \times \mathbb{R}$, we infer that there exists a family $M_{\widehat{d}}$ of complete horizontal minimal graph, given by a function $h_{\widehat{d}} \in C^{2}(\mathcal{R}) \cap C^{0}(\overline{\mathcal{R}})$ defined in a rectangle $\mathcal{R}=\mathcal{R}(a, \widehat{d})=\{(x, t) ;-a \leqslant$ $x \leqslant a,-(\widehat{d}+\pi) / 2 \leqslant t \leqslant(\widehat{d}+\pi) / 2, a, \widehat{d}>0\}$, taking zero value asymptotic boundary data on $\partial \mathcal{R}$. Notice that the vertical width of $\mathcal{R}$ is greater than $\pi$. For any $a>0$ and $\widehat{d}>0$, there exists such a model minimal surface. Furthermore, letting $a \downarrow 0$ we have that the horizontal length $h_{\widehat{d}}(x, t)$ goes to zero uniformly.

## Proof of the Theorem 1.1.

Proof. We argue by contradiction. Were the statement false, there would exist an entire horizontal minimal graph $S$, given by a function $y=g(x, t)$ defined for all value of the independent variables $x, t$. Recalling Example 2.1, let $M_{\widehat{d}}$ be a fixed complete horizontal minimal graph, given by a function $h_{\widehat{d}} \in C^{2}(\mathcal{R}) \cap C^{0}(\overline{\mathcal{R}})$ defined in the fixed rectangle $\mathcal{R}=\{(x, t) ;-a \leqslant x \leqslant a,-(\widehat{d}+\pi) / 2 \leqslant t \leqslant(\widehat{d}+\pi) / 2 ; a, \widehat{d}>0\}$, taking zero value boundary data on $\partial \mathcal{R}$. Consider that 1-parameter family of complete graphs given by $y=\lambda h_{\widehat{d}}\left(\frac{x}{\lambda}, t\right), \lambda>0,(x, t) \in T_{\lambda}(\mathcal{R})$. This is still a complete graph of the family $M_{\widehat{d}}$. Note that if $\lambda<1$ then $T_{\lambda}(\mathcal{R})$ is a thinner rectangle with the same vertical height as $\mathcal{R}$ and $T_{\lambda}(\mathcal{R}) \subset \mathcal{R}$. If $\lambda>1$ then $T_{\lambda}(\mathcal{R})$ is a larger rectangle with vertical height the same as $\mathcal{R}$ and $T_{\lambda}(\mathcal{R}) \supset \mathcal{R}$.

Now, let $c:=\min _{\mathcal{R}} g(x, t)$. We may choose $\lambda=\lambda_{0} \lll 1$ so small such that $\lambda_{0} h_{\widehat{d}}\left(\frac{x}{\lambda_{0}}, t\right)<c, \forall(x, t) \in T_{\lambda_{0}}(\mathcal{R})$. We may also choose $\lambda=\lambda_{1} \operatorname{big}$ enough such that $\lambda_{1} h_{\widehat{d}}(0,0) \geqslant \max _{\mathcal{R}} g(x, t)$. These choices of $\lambda$ are given geometrically by applying hyperbolic translations to our fixed model surface $M_{\widehat{d}}$, in each slice of $\mathbb{H}^{2} \times \mathbb{R}$. Thus, we may assume that our initial model minimal graph given by the function $\lambda_{0} h_{\widehat{d}}\left(\frac{x}{\lambda_{0}}, t\right)$ is below (horizontally) the entire graph $S$ restricted to the thin rectangle $T_{\lambda_{0}}(\mathcal{R})$. Moreover, we also may assume that if $\lambda=\lambda_{1}$ then, at the
origin, the horizontal length of corresponding model minimal surface is above (horizontally) the entire graph $S$ restricted to $\mathcal{R}$. Therefore, we may move the family $M_{\widehat{d}}$ toward $S$, coming from the infinity by doing hyperbolic translations. We must then find a first point of tangent contact with $S$, during the movement of the family $M_{\widehat{d}}$. So some model minimal surface of the family $M_{\widehat{d}}$ touches $S$ and is below (horizontally) $S$, for some $\lambda_{0}<\lambda<\lambda_{1}$. We conclude therefore that $S$ is a minimal model surface, by the maximum principle. We thus arrive to a contradiction, since $S$ is entire. This completes the proof of Theorem 1.1.

Remark 2.2. We observe that the proof of Theorem 1.1 shows the following:

There is no complete properly immersed minimal surface $M$ in $\mathbb{H}^{2} \times$ $\mathbb{R}$, such that the part of the asymptotic boundary of $M$ in $\partial_{\infty} \mathbb{H}^{2} \times \mathbb{R}$ is contained in some vertical straight line.

## 3. Proof of the uniqueness result

We now prove our uniqueness result:

## Proof of the Theorem 1.2:

Proof. Let $\Omega$ be the admissible domain with boundary $\gamma$. Let $L$ be the $t$-axis $\{x=0\}$ in $\partial_{\infty} \mathbb{H}^{2} \times \mathbb{R}$.

Without loss of generality, we assume that the two connected components of $\gamma \backslash L$ are graphs over the same interval of $L$, each one is contained in one side of $L$, as in Definition 1.1. We prove the Theorem by establishing several claims.
Claim 1. The "horizontal cylinder" $\mathcal{C}:=\gamma \times\{y ; y>0\}$ has "nonnegative mean curvature" with respect to the unit inner normal in the following sense: We recall that the Euclidean planes $t=\ell x+\mathrm{cst}, y>0$ are minimal surfaces [6]. Of course, the slices $t=\mathrm{cst}, y>0$ and the vertical planes $x=\operatorname{cst}, y>0$, are minimal surfaces, because they are geodesic surfaces. At any point $p$ belonging to the horizontal cylinder $\mathcal{C}$, there is one (and only one) of those minimal surfaces tangent o $\mathcal{C}$ at $p$, staying in $\mathbb{H}^{2} \times \mathbb{R} \backslash \Omega \times \mathbb{R}$.
Claim 2. The horizontal projection of $M \backslash \partial M$ into $\partial_{\infty} \mathbb{H}^{2} \times \mathbb{R}$ lies in $\Omega$. In other words, $M \backslash \partial M$ in strictly contained in $\Omega \times\{y ; y>0\}$, that we call the interior of the mean convex side of $\mathcal{C}$.
Since $\partial M \subset \mathcal{C}$, by comparing with the Euclidean planes of Claim 1, by maximum principle, one obtains that $M \backslash \partial M$ is contained in $\Omega \times\{y ; y>0\}$. This proves the assertion.

Claim 3. $M$ is a Killing-graph with respect to the 1-parameter group $\mathcal{G}$ of isometries given by $(x, y, t) \mapsto(\lambda x, \lambda y, t), \lambda>0$ (hyperbolic translations with respect to the $y$-axis in each slice of $\mathbb{H}^{2} \times \mathbb{R}$ ). In particular, $M$ is embedded. Let $L_{2}$ be the top horizontal line $\{(a, f(a), y), y>0\}$ of $\mathcal{C}$ and $L_{1}$ the bottom horizontal line $\{(-a, f(-a), y), y>0\}$ of $\mathcal{C}$. First, by the geometry of our admissible domains, we have that $\mathcal{C} \backslash L_{1} \cup L_{2}$, is a Killing graph. Secondly, as $\Gamma$ is a horizontal graph, we conclude therefore that the boundary $\Gamma$ of $M$ is a graph in the Killing system of coordinates Furthermore, in view of the geometry of the admissible domains, doing the translations (homotheties in each slice of $\mathbb{H}^{2} \times \mathbb{R}$ ) with ratio $\lambda>1$, we see that the boundary of the translated surface does not touch the original surface during the movement. That is, the part of any orbit of the Killing field contained in the exterior of the mean convex side of $\mathcal{C}$, passing through a point of $\Gamma$, does not intersect $M \backslash \partial M$. In fact, for $\lambda>1$ the two points of the boundary of the translated surface lying in $\mathcal{C}$ have horizontal length strictly bigger than the corresponding length of the original surface. The other part of the boundary of the translated surface stays in the interior of the non mean convex side of $\mathcal{C}$. On the other hand, the admissible bounded slope assumption, using the Euclidean planes as suitable barriers, ensures that the part of any orbit of the Killing field contained in the interior of the mean convex side of $\mathcal{C}$, passing through a point of $\Gamma$, does not intersect $M \backslash \partial M$. Taking into account this geometric phenomenon, we will argue by absurd.
Suppose, to the contrary, that there are two points $p, q \in M$ in the same orbit of $\mathcal{G}$ (equidistant curves in each slice). As we proved in the last paragraph, $p, q$ are interior points of $M$,i.e $p, q \in M \backslash \partial M$. Let us perform now hyperbolic translations, making $\lambda \uparrow \infty, \lambda>1$. Recall that during this movement, the boundary of the translated surfaces keeps away from the interior of the mean convex side of $\mathcal{C}$. We would then find a translated copy of $M$ touching $M$ at some first interior point. Hence $M$ would be equal to some translated copy, for $\lambda>1$, by the maximum principle. This contradiction proves the assertion.

Claim 4. There exists only one Killing-graph $M$ with boundary $\Gamma$. That is, $M$ is the Morrey's solution of the Plateau problem [4]. Mimicking the proof of the Claim 3, we can deduce the uniqueness of the Killinggraph with the same boundary, that is $M=S$. To see this carefully, we do hyperbolic translations on $S$ towards the infinity $(\lambda \uparrow \infty, \lambda>$ 1), until $S$ is above $M$, then moving back $S$ towards $M$, we infer, by maximum principle, that $S$ does not touch $M$ before the copy of the boundary of $S$ reaches the original position, identifying with the
boundary of $M$. Hence $S$ is above $M$, in the Killing system. Conversely, doing hyperbolic translations on $M$, using the preceding reasoning, we conclude that $M$ is above $S$. We have therefore $M=S$, as desired.

Claim 5. $M$ is a horizontal minimal graph. Being $M$ a Killing graph, each horizontal line $x=0, t=t_{0}$ starting from a point of $L$ at height $t_{0}, t_{0} \in[-a, a]$ intersects $M$ at exactly one point. Now observe that $\Gamma$ is a $C^{1}$ graph over $\gamma$, by our assumptions. Hence, the vertical geodesic plane $\{x=0, y>0\}$ passing through $L$ is transverse to $M$, by the maximum principle. We find a neighborhood $U$ of $L \cap \bar{\Omega}$, say $U=$ $\{-\eta<x<\eta\} \cap \bar{\Omega}$, for some $\eta>0$, such that the restriction of $M$ to $U \times\{y ; y>0\}$ is a horizontal graph, as well.

Now we focus the restriction of $M$ to the right side of $L$, i.e, the points of $M$ such that the coordinate $x$ is positive. The argument about the restriction of $M$ to the left side of $L$, i.e, the points of $M$ satisfying $x<0$ is the same. Let us denote by $L_{\bar{x}}$ the intersection of the vertical line $y=0, x=\bar{x}$, in the $x t$ plane, with $\bar{\Omega}$. Let us set $x_{0}:=\sup x, x>0$, such that the restriction of $M$ to $L_{x} \times\{y ; y>0\}, 0 \leqslant x<x_{0}$ is a horizontal graph. Let us denote $L_{x_{0}}=[p, q]$, where $p=\left(x_{0}, a_{1}\right)$ and $q=\left(x_{0}, b_{1}\right)$ are the points of $L_{x_{0}}$ in $\gamma$. Noticing that $x_{0} \geqslant \eta$, by the previous argument, let ( $d, l$ ) be the vertical point of $\gamma$ in the halfplane $x>0$, i.e $x=d$ is the maximum of the coordinate $x$ of $\gamma$ restricted to the halfplane $\{x>0\}$. We deduce that if $x_{0}=d$, then the restriction of $M$ to the right side of $L$ is a horizontal graph. If $x_{0}<d$, we argue by contradiction. Let $\widetilde{\gamma}$ be the arc of $\gamma$ in the right side of $L_{x_{0}}$ joining the points $\left(x_{0}, a_{1}\right)$ and $\left(x_{0}, b_{1}\right)$ of $\gamma$. Now, we may choose a convex arc $C$ in the intersection of $\Omega$ with the vertical strip $0<x<x_{0}$, joining the points $\left(x_{0}, a_{1}\right)$ and $\left(x_{0}, b_{1}\right)$ of $\gamma$, such that $\widetilde{\gamma} \cup C$ bounds a convex admissible domain $\Omega^{\prime}$ contained in $\Omega$. The construction yields that the restriction of $M$ to $\widetilde{\gamma} \cup C \times\{y ; y>0\}$ is a horizontal graph. But then, by employing the translations $(x, y, t) \rightarrow\left(\lambda\left(x-x_{0}\right), \lambda y, t\right)+\left(x_{0}, 0,0\right)$ (homotheties in each slice starting at a point of $L_{x_{0}}$ ), working in the same way as before, we derive that $M$ (restricted to $\Omega^{\prime} \times\{y ; y>0\}$ ) is a Killing graph with respect to this coordinate system. Thus each horizontal line $x=x_{0}, t \in\left[a_{1}, b_{1}\right], y>0$, starting from $L_{x_{0}}$, intersects $M$ at exactly one point. We observe that $L_{x_{0}}$ is transverse to $\gamma$ at the two points $\left(x_{0}, a_{1}\right)$ and $\left(x_{0}, b_{1}\right)$ of intersection, by the convexity of $\gamma$. Thus, by the maximum principle again, as the restriction of $M$ to $L_{x_{0}} \times\{y, y>0\}$ is a horizontal graph over $L_{x_{0}}$, then the vertical geodesic plane $\left\{x=x_{0}, y>0\right\}$ passing through $L_{x_{0}}$ is transverse to $M$. Thus, we find a neighborhood $U^{\prime}$ of $L_{x_{0}}$ in the $x t$ plane such that the restriction of $M$ to $U^{\prime} \times\{y ; y>0\}$ is a horizontal graph, a contradiction
with the assumption on $x_{0}$. Therefore, $x_{0}=d$ and we conclude that $M$ is a horizontal minimal graph, as desired. This completes the proof of Theorem 1.2.

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