# AN ASYMPTOTIC THEOREM FOR MINIMAL SURFACES AND EXISTENCE RESULTS FOR MINIMAL GRAPHS IN $\mathbb{H}^{2} \times \mathbb{R}$ 

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#### Abstract

In this paper we prove a general and sharp Asymptotic Theorem for minimal surfaces in $\mathbb{H}^{2} \times \mathbb{R}$. As a consequence, we prove that there is no properly immersed minimal surface whose asymptotic boundary $\Gamma_{\infty}$ is a Jordan curve homologous to zero in $\partial_{\infty} \mathbb{H}^{2} \times \mathbb{R}$ such that $\Gamma_{\infty}$ is contained in a slab between two horizontal circles of $\partial_{\infty} \mathbb{H}^{2} \times \mathbb{R}$ with width equal to $\pi$.

We construct vertical minimal graphs in $\mathbb{H}^{2} \times \mathbb{R}$ over certain unbounded admissible domains taking certain prescribed finite boundary data and certain prescribed asymptotic boundary data. Our admissible unbounded domains $\Omega$ in $\mathbb{H}^{2} \times\{0\}$ are non necessarily convex and non necessarily bounded by convex arcs; each component of its boundary is properly embedded with zero, one or two points on its asymptotic boundary, satisfying a further geometric condition.


## 1. Introduction

In this paper we prove an Asymptotic Theorem for minimal surfaces in $\mathbb{H}^{2} \times \mathbb{R}$. Indeed, we prove a surprising general and sharp nonexistence result. As a consequence, we deduce that there is no complete properly immersed minimal surface whose asymptotic boundary $\Gamma_{\infty}$ is a Jordan curve homologous to zero in $\partial_{\infty} \mathbb{H}^{2} \times \mathbb{R}$ contained in an open slab between two horizontal circles of $\partial_{\infty} \mathbb{H}^{2} \times \mathbb{R}$ with width equal to $\pi$. The last statement is still true in a closed slab with width equal to $\pi$ in the class of minimal surfaces continuous up to the asymptotic boundary. This result is sharp in the following sense. We show that for any $\ell>\pi$ there is a Jordan curve $\Gamma_{\infty} \subset \partial_{\infty} \mathbb{H}^{2} \times \mathbb{R}$ homologous to zero with vertical height equal to $\ell$ which is the asymptotic boundary of a

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complete minimal surface, continuous up to its asymptotic boundary $\Gamma_{\infty}$. Moreover, this surface is invariant by hyperbolic translations and consists of two vertical minimal graphs over the exterior of an equidistant curve, symmetric about the horizontal slice $\mathbb{H}^{2} \times\{0\}$. In fact, the Jordan curve $\Gamma_{\infty}$ is the union of two vertical segments with two half circles in $\partial_{\infty} \mathbb{H}^{2}$. Another consequence of our Asymptotic Theorem is that there is no complete properly immersed minimal surface contained in an open slab of width equal to $\pi$ of $\mathbb{H}^{2} \times \mathbb{R}$, such that the vertical projection of its asymptotic boundary on $\partial_{\infty} \mathbb{H}^{2} \times\{0\}$ omits an open arc.

Those results contrast with an analogous situation when the ambient space is the hyperbolic three-space $\mathbb{H}^{3}$, due to the existence of the vertical minimal graph (taking the upper half-space model) whose asymptotic boundary is any convex curve lying in $\partial_{\infty} \mathbb{H}^{3}$. Indeed, the authors have solved the Dirichlet Problem in $\mathbb{H}^{3}$ for the vertical minimal surface equation over a convex domain $\Omega$ in $\partial_{\infty} \mathbb{H}^{3}$, taking any prescribed continuous boundary data on $\partial \Omega$ ([19]). There are also the general results proved by M. Anderson [1] and [2].

We give some geometric conditions to construct vertical minimal graphs in $\mathbb{H}^{2} \times \mathbb{R}$ over certain unbounded admissible domains taking certain prescribed finite boundary data and certain prescribed asymptotic boundary data.

To obtain our existence results we establish the Perron process when the finite boundary data and the asymptotic boundary data are continuous except perhaps at a finite set.

As a consequence, we prove the following. Let $\Omega$ be a convex unbounded domain. Let $g: \partial \Omega \cup \partial_{\infty} \Omega \rightarrow \mathbb{R}$ be a bounded function everywhere continuous except at a finite set $S$. Then $g$ admits an extension $u$ satisfying the vertical minimal surface equation over $\Omega$ such that the total boundary of the graph of $u$ is the union of the graph of $g$ on $\left(\partial \Omega \cup \partial_{\infty} \Omega\right) \backslash S$ with vertical segments at the points of $S$. This result was obtained independently by M. Rodríguez and H. Rosenberg

We build barriers at each convex point of a convex finite boundary, where the boundary data are continuous and bounded and we construct barriers at each point of the asymptotic boundary where the asymptotic data is continuous. Our admissible unbounded domains $\Omega$ in $\mathbb{H}^{2} \times\{0\}$ are not necessarily convex nor necessarily bounded by convex arcs; each component of its boundary is properly embedded with zero, one or two points on its asymptotic boundary, satisfying a further geometric condition: each connected component $C_{0}$ of $\partial \Omega$ satisfies the Exterior circle of (uniform) radius $\rho$ condition. Particularly, we consider an
admissible domain $\Omega$ that is the exterior of a $C^{2}$ Jordan curve $\Gamma$ in the horizontal slice.

We obtain the existence of minimal graph $M$ over an admissible domain $\Omega$ in $\mathbb{H}^{2} \times\{0\}$ such that the finite boundary of $M$ is $\partial \Omega$ and the asymptotic boundary of $M$ is a certain Jordan curve $\Gamma_{\infty}$ consisting of the union of bounded continuous vertical graphs with the vertical segments joining the points of discontinuities, such that $\Gamma_{\infty}$ is contained inside a certain slab of $\partial_{\infty} \mathbb{H}^{2} \times \mathbb{R}$ depending on the geometry of $\Omega$.

We consider admissible domains, that we call E-admissible domains, such that each component of the boundary has two points at its asymptotic boundary and has at each point of its finite boundary an exterior equidistant curve. We obtain analogous existence results for E-admissible domains.

## 2. An asymptotic theorem

In this section we prove an Asymptotic Theorem that ensures some nonexistence results about minimal surfaces with some given asymptotic boundary.

Theorem 2.1 (Asymptotic Theorem).
Let $\gamma \subset \partial_{\infty} \mathbb{H}^{2} \times \mathbb{R}$ be an arc. Assume there exist a vertical straight line $L \subset \partial_{\infty} \mathbb{H}^{2} \times \mathbb{R}$ and a subarc $\gamma^{\prime} \subset \gamma$ such that
(1) $\gamma^{\prime} \cap L \neq \emptyset$ and $\partial \gamma^{\prime} \cap L=\emptyset$,
(2) $\gamma^{\prime}$ stays on one side of $L$,
(3) $\gamma^{\prime} \subset \partial_{\infty} \mathbb{H}^{2} \times\left(t_{0}, \pi+t_{0}\right)$, for some real number $t_{0}$.

Then there is no properly immersed minimal surface (maybe with finite boundary), $M \subset \mathbb{H}^{2} \times \mathbb{R}$, with asymptotic boundary $\gamma$ and such that $M \cup \gamma$ is a continuous surface with boundary.

Proof. By assumption there exists a point $p$ in $\gamma^{\prime} \cap L$. If there is a vertical segment in $\gamma^{\prime} \cap L$, we choose $p$ to be the midpoint of this segment. Up to a vertical translation, we can assume that $p \in \partial_{\infty} \mathbb{H}^{2} \times$ $\{0\}$. The vertical projection of $\gamma^{\prime}$ on $\partial_{\infty} \mathbb{H}^{2} \times\{0\}$ is an $\operatorname{arc} \beta$ with $p$ as one of the two end points. Let $\varepsilon>0$ be a real number to be chosen later. Let $q_{1}, q_{2} \in \partial_{\infty} \mathbb{H}^{2} \times\{0\}$ be two distinct points such that $q_{1} \in \beta$, $q_{2} \notin \beta$ and the Euclidean distance on $\partial_{\infty} \mathbb{H}^{2} \times\{0\}$ from $p$ to $q_{i}$ is $\varepsilon$, $i=1,2$. Let $c \subset \mathbb{H}^{2} \times\{0\}$ be the complete geodesic with asymptotic boundary $\left\{q_{1}, q_{2}\right\}$ and let $S=c \times \mathbb{R}$ be the vertical geodesic plane defined by $c$.

Let $M \subset \mathbb{H}^{2} \times \mathbb{R}$ be a minimal surface (if any) with asymptotic boundary $\gamma$ and such that $M \cup \gamma$ is a continuous surface with boundary. If $\varepsilon$ is small enough we have $S \cap \partial M=\emptyset$. Let $M_{0} \subset M$ be the connected
component of $M \backslash S$ containing $p$ in its asymptotic boundary. Therefore, the asymptotic boundary of $M_{0}$ is a subarc $\gamma_{0}$ of $\gamma^{\prime}$ containing $p$ in its interior: $\gamma_{0} \subset \gamma^{\prime} \subset \gamma$ and $p \in \operatorname{Int}\left(\gamma_{0}\right)$. Let $\beta^{\prime} \subset \beta \subset \partial_{\infty} \mathbb{H}^{2} \times\{0\}$ be the subarc of $\beta$ with end points $p$ and $q_{1}$. For $\varepsilon$ small enough we have $\gamma_{0} \subset \beta^{\prime} \times(-\pi / 2, \pi / 2)$. By construction there exist two real numbers $a$ and $b$ satisfying $a<0<b, b-a<\pi$ and $\partial \gamma_{0}=\left\{\left(q_{1}, a\right),\left(q_{1}, b\right)\right\}$.

Observe that, by continuity, for $\varepsilon$ small enough the whole component $M_{0}$ is inside the slab $\mathbb{H}^{2} \times(-\pi / 2, \pi / 2)$. Furthermore, the finite boundary $\partial M_{0}$ of $M_{0}$ is contained in the vertical geodesic plane $S$. Therefore, there is a complete geodesic $c_{1} \subset \mathbb{H}^{2} \times\{0\}$ with asymptotic boundary in the open $\operatorname{arc}\left(p, q_{2}\right) \subset \partial_{\infty} \mathbb{H}^{2} \times\{0\} \backslash \beta$, such that $M_{0} \cap\left(c_{1} \times \mathbb{R}\right)=\emptyset$.

Let $C \subset \mathbb{H}^{2} \times \mathbb{R}$ be a complete catenoid whose a component of the asymptotic boundary stays at height $T_{1}$ and the other component at height $T_{2}$ such that $T_{1}<a<b<T_{2}$ (such a catenoid exists since $0<b-a<\pi)$, note that $T_{2}-T_{1}<\pi$, the reader can see the geometric behaviour of the catenoids in Lemma 5.1 or [15]. By continuity, we can choose $T_{1}$ and $T_{2}$ such that $M_{0}$ is entirely contained in the open slab $\mathbb{H}^{2} \times\left(T_{1}, T_{2}\right)$. Finally, let $t_{1}, t_{2}$ be two real numbers satisfying $T_{1}<t_{1}<a<b<t_{2}<T_{2}$, such that $M_{0}$ is entirely contained in the open slab $\mathbb{H}^{2} \times\left(t_{1}, t_{2}\right)$.

Let $\bar{C}$ be the part of $C$ contained in the slab $\left\{t_{1} \leqslant t \leqslant t_{2}\right\}$, that is, $\bar{C}=C \cap\left(\mathbb{H}^{2} \times\left[t_{1}, t_{2}\right]\right)$. Observe that $\bar{C}$ is a compact surface. Up to a hyperbolic translation we can send $\bar{C}$ into the connected component of $\mathbb{H}^{2} \times \mathbb{R} \backslash\left(c_{1} \times \mathbb{R}\right)$ not containing $p$ in its asymptotic boundary, so we can assume that $\bar{C}$ has this property.

Let $c_{2} \subset \mathbb{H}^{2} \times\{0\}$ be a complete geodesic with an asymptotic boundary point in $\operatorname{Int}\left(\beta^{\prime}\right)$ and the other asymptotic boundary point in the open arc $\left(p, q_{2}\right)$. We choose $c_{2}$ such that $c_{1}$ is contained in the component of $\left(\mathbb{H}^{2} \times\{0\}\right) \backslash c_{2}$ containing $p$ in its asymptotic boundary. Consider the hyperbolic translations along $c_{2}$. Observe that all translated copies of $\bar{C}$ have a component of the finite boundary at height $t_{1}$ and the other component at height $t_{2}$. Therefore the boundary of any translated copy of $\bar{C}$ has no intersection with $\overline{M_{0}}$. Consequently some translated copy of $\bar{C}$ must achieve a first interior contact point with $M_{0}$, which contradicts the maximum principle. This concludes the proof of the Theorem.

Remark 1. Let us fix cylindrical coordinates $(\theta, t)$ in $\partial_{\infty} \mathbb{H}^{2} \times \mathbb{R}$. Notice that the assumptions on the curve $\gamma^{\prime}$ in the Statement of Theorem 2.1, ensure that there exists a point $p:=\left(\theta_{\tau}, \tau\right)$ and a continuous function $h(t)$ defined (locally) in an interval around $\tau$, say $I:=[\tau-\varepsilon, \tau+\varepsilon]$,
with $h(\tau)=\theta_{\tau}$, such that $\gamma^{\prime}$ is given by a "horizontal graph" $(h(t), t)$ for $t \in I$, and $h(t)$ reaches a local maximum or local minimum at $\tau$. Thus, $\gamma^{\prime}$ is parametrized locally by $t \mapsto(\cos (h(t)), \sin (h(t)), t), t \in I$, and is locally contained in one of the half-side $\theta \geqslant \theta_{\tau}$ or $\theta \leqslant \theta_{\tau}$ of the vertical straightline $L=\left\{\theta_{\tau}\right\} \times \mathbb{R}$ tangent to $\gamma^{\prime}$ at $p$. Notice that, if $\Gamma_{\infty}$ is a Jordan curve homologous to 0 in $\partial_{\infty} \mathbb{H}^{2} \times \mathbb{R}$, lifting to the universal cover $\mathbb{R}^{2}$, then we see that there is a vertical point $p$ and a subarc $\gamma^{\prime}$ that satisfy the assumptions (1) and (2) of Theorem 2.1. Nevertheless the assumption (3) of Theorem 2.1 may not be satisfied, see for example the surfaces given in Proposition 2.1-(1) (Figure 1(a)). Observe also that the subarc $\gamma^{\prime}$ of Theorem 2.1 can contain a vertical segment of length $<\pi$.

Of course, such Jordan curve $\Gamma_{\infty}$ is contained in a slab-closed region of $\partial_{\infty} \mathbb{H}^{2} \times \mathbb{R}$ between two copies $\partial_{\infty} \mathbb{H} \times\{a\}$ and $\partial_{\infty} \mathbb{H} \times\{b\}$ of $\partial_{\infty} \mathbb{H}^{2}-$ with width $b-a$. If $b-a \leqslant \pi$ then $\Gamma_{\infty}$ is contained in a slab of width $\pi$. If not, then considering the height function $t$ restricted to $\Gamma_{\infty}$, we have that the higher point $p_{h}$ and the lower point $p_{l}$ satisfy $t\left(p_{h}\right)-t\left(p_{l}\right)>\pi$. In this case, $\Gamma_{\infty}$ is not contained in any slab of width $\pi$.

At last, if $\Gamma_{\infty}$ is the asymptotic boundary of some minimal surface of $\mathbb{H}^{2} \times \mathbb{R}$, continuous up to $\Gamma_{\infty}$, then Theorem 2.1 shows that any subarc $\gamma^{\prime} \subset \Gamma_{\infty}$ satifying the assumptions (1) and (2) must contain a vertical segment with length $\geqslant \pi$.

Corollary 2.1. Let $\Gamma_{\infty} \subset \partial_{\infty} \mathbb{H}^{2} \times \mathbb{R}$ be a Jordan curve homologous to zero (in $\partial_{\infty} \mathbb{H}^{2} \times \mathbb{R}$ ). We have the following:
(1) Suppose that $\Gamma_{\infty}$ is strictly contained in a closed slab between two horizontal circles of $\partial_{\infty} \mathbb{H}^{2} \times \mathbb{R}$ with width equal to $\pi$. Then,
(a) there is no properly immersed minimal surface $M$ with asymptotic boundary $\Gamma_{\infty}$, possibly with finite boundary, such that $M \cup \Gamma_{\infty}$ is a continuous surface with boundary.
(b) there is no complete properly immersed minimal surface with asymptotic boundary $\Gamma_{\infty}$ (without any assumption on $\left.M \cup \Gamma_{\infty}\right)$.
(2) Suppose that $\Gamma_{\infty}$ is contained in a slab with width equal to $\pi$ but is not contained in any slab with width strictly less than $\pi$. Then, there is no complete minimal surface properly immersed in $\mathbb{H}^{2} \times \mathbb{R}$, with asymptotic boundary $\Gamma_{\infty}$ such that $M \cup \Gamma_{\infty}$ is a continuous surface with boundary.

Proof. The Statement (1a) is a direct consequence of Theorem 2.1. The Statement (1b) is a direct consequence of the proof of Theorem 2.1.

Let us prove the Statement (2). Assume there exists a properly immersed complete minimal surface $M$ with asymptotic boundary $\Gamma_{\infty} \subset$
$\partial_{\infty} \mathbb{H}^{2} \times[0, \pi]$. By the maximum principle, we deduce that $\operatorname{Int}(M) \subset$ $\mathbb{H}^{2} \times(0, \pi)$. We deduce from Theorem 2.1 that $\Gamma_{\infty}$ consists of two vertical segments of length $\pi$ : $\left\{q_{1}\right\} \times[0, \pi]$ and $\left\{q_{2}\right\} \times[0, \pi], q_{i} \in \partial_{\infty} \mathbb{H}^{2}$ (identified with $\partial_{\infty} \mathbb{H}^{2} \times\{0\}$ ), $i=1,2$, and two simple arcs $c, \gamma \subset$ $\partial_{\infty} \mathbb{H}^{2} \times[0, \pi]$, the arc $c$ joining the points $\left(q_{1}, 0\right)$ and $\left(q_{2}, 0\right)$ and the arc $\gamma$ joining the points $\left(q_{1}, \pi\right)$ and $\left(q_{2}, \pi\right)$. Observe that $c$ and $\gamma$ have the same vertical projection. Therefore we have

$$
\Gamma_{\infty}=\left(\left\{q_{1}\right\} \times[0, \pi]\right) \cup \gamma \cup\left(\left\{q_{2}\right\} \times[0, \pi]\right) \cup c
$$

Up to an ambient isometry, we can assume that $q_{1}=e^{i \pi / 4}, q_{2}=e^{-i \pi / 4}$ and that the vertical projection of $\Gamma_{\infty}$ on $\partial_{\infty} \mathbb{H}^{2} \times\{0\}$ is the arc $\left\{\left(e^{i \theta}, 0\right) \mid-\pi / 4 \leqslant \theta \leqslant \pi / 4\right\}$.

Let $H$ be the parabolic complete minimal surface (foliated by horocycles) whose asymptotic boundary is the vertical segment $\{-1\} \times[0, \pi] \subset$ $\partial_{\infty} \mathbb{H}^{2} \times \mathbb{R}$ with $\left(\partial_{\infty} \mathbb{H}^{2} \times\{0\}\right) \cup\left(\partial_{\infty} \mathbb{H}^{2} \times\{\pi\}\right)$, see [6], [10] and [18]. The "neck" of $H$ is a horocycle $N$ in the slice $\mathbb{H}^{2} \times\{\pi / 2\}$.

Claim. If $N \cap M=\emptyset$ then $H \cap M=\emptyset$.
Assume by contradiction that $N \cap M=\emptyset$ and $H^{+} \cap M \neq \emptyset$, where $H^{+}:=H \cap\left(\mathbb{H}^{2} \times[\pi / 2, \pi]\right)$. For any $\varepsilon>0$ we denote by $H_{\varepsilon}^{+}$the $\varepsilon-$ vertical translated of $H^{+}: H_{\varepsilon}^{+}=H^{+}+\varepsilon \partial / \partial t$. Observe that if $H_{\varepsilon}^{+} \cap$ $M=\emptyset$ for any $\varepsilon>0$, letting $\varepsilon \rightarrow 0$ then $M$ and $H^{+}$would have a first interior point of contact, contradicting the maximum principle. Therefore, there exists $\varepsilon>0$ such that $H_{\varepsilon}^{+} \cap M \neq \emptyset$ and $(N+\varepsilon \partial / \partial t) \cap$ $M=\emptyset$. Furthermore, the finite and asymptotic boundary of $H_{\varepsilon}^{+}$is far away from $M \cup \partial_{\infty} M$. Consider the hyperbolic translations along the geodesic $\beta$ with asymptotic boundary $\{-1,1\}$, going from 1 to -1 . Thus we would obtain a last interior contact point of $M$ and some translated copy of $H_{\varepsilon}^{+}$, which contradicts the maximum principle.

We show in the same way that $N \cap M=\emptyset$ and $H^{-} \cap M \neq \emptyset$ is not possible. This proves the Claim.

Up to a hyperbolic translation along the geodesic $\beta$ (with asymptotic boundary $\{-1,1\}$ ), we can assume that $N \cap M=\emptyset$ (since $-1 \in \partial_{\infty} \mathbb{H}^{2}$ is not in the asymptotic boundary of $M$ ), and therefore the Claim shows that $H \cap M=\emptyset$. Consider now the translated copies of $H$, along $\beta$, going from -1 to 1 . As $M$ is properly immersed, some translated copy of $H$ will have a first contact point with $M$ at a point $p \in M$, which contradicts the maximum principle. This concludes the proof of the Corollary

The following result is a direct consequence of the proof of the Asymptotic Theorem (Theorem 2.1).

Corollary 2.2. Let $S_{\infty} \subset \partial_{\infty} \mathbb{H}^{2} \times \mathbb{R}$ be a closed set strictly contained in a slab with width equal to $\pi$. Assume that the vertical projection of $S_{\infty}$ on $\partial_{\infty} \mathbb{H}^{2} \times\{0\}$ omits an open arc. Then, there is no complete properly immersed minimal surface $M$ in $\mathbb{H}^{2} \times \mathbb{R}$ with asymptotic boundary $S_{\infty}$.

Proof. By assumption there exists a complete geodesic $c \subset \mathbb{H}^{2} \times\{0\}$ such that $S_{\infty}$ is contained in the asymptotic boundary of a component of $\mathbb{H}^{2} \times \mathbb{R} \backslash c \times \mathbb{R}$. We call $U$ the other component. Let $C$ be a catenoid, observe that any compact part of $C$ may be mapped into $U$ by an ambient isometry. In this situation, we can proceed as in the proof of Theorem 2.1.

Remark 2. We will see in Proposition 2.1 that for any $t_{0}>\pi$ there exists a Jordan curve $\Gamma_{\infty} \subset \partial_{\infty} \mathbb{H}^{2} \times\left[0, t_{0}\right]$, homologous to zero, which is the asymptotic boundary of a properly embedded complete minimal surface. Therefore the results in Theorem 2.1 and Corollary 2.1 are sharp. The formulae of the generating curves in Proposition 2.1 below are the same as formulae given by the first author in [18]. The geometric description of the surfaces given in Proposition 2.1 is new. We remark that L. Hauswirth [10] has given a classification of minimal surfaces invariant by hyperbolic translations using another approach.

Proposition 2.1. Let $q_{1}, q_{2} \in \partial_{\infty} \mathbb{H}^{2}$ (identified with $\partial_{\infty} \mathbb{H}^{2} \times\{0\}$ ), be two distinct asymptotic points. Let $\gamma \subset \mathbb{H}^{2}$ (identified with $\mathbb{H}^{2} \times\{0\}$ ), be the complete geodesic with asymptotic boundary $\left\{q_{1}, q_{2}\right\}$. Let us call $c_{1}$ (resp. $c_{2}$ ) the closed arc in $\partial_{\infty} \mathbb{H}^{2}$ joining $q_{1}$ to $q_{2}$ (resp. $q_{2}$ to $q_{1}$ ) with respect to the counterclockwise orientation.

There exist a one-parameter family $M_{d}, d>0$, of complete properly embedded minimal surfaces, invariant by the hyperbolic translations along $\gamma$. The geometric behaviour of $M_{d}$ is as follows.
(1) If $d>1$, then $M_{d}$ contains the equidistant line $\gamma_{d}$ of $\gamma$ in $\mathbb{H}^{2} \times\{0\}$ staying at the distance $\cosh ^{-1}(d)$ from $\gamma$ in the connected component of $\mathbb{H}^{2} \backslash \gamma$ whose asymptotic boundary is $c_{1}$. Furthermore $M_{d}$ is symmetric with respect to the slice $\mathbb{H}^{2} \times\{0\}$ and we have (see Figure 1(a))

$$
\begin{aligned}
\partial_{\infty} M_{d}= & \left(c_{1} \times\{-H(d)\}\right) \cup\left(c_{1} \times\{H(d)\}\right) \\
& \cup\left(\left\{q_{2}\right\} \times[-H(d), H(d)]\right) \cup\left(\left\{q_{1}\right\} \times[-H(d), H(d)]\right),
\end{aligned}
$$

where

$$
\begin{equation*}
H(d):=\int_{\cosh ^{-1}(d)}^{+\infty} \frac{d}{\sqrt{\cosh ^{2} u-d^{2}}} d u, \quad d>1 \tag{1}
\end{equation*}
$$

Therefore, $\partial_{\infty} M_{d}$ is a Jordan curve homologous to zero in $\partial_{\infty} \mathbb{H}^{2} \times$ $\mathbb{R}$. Furthermore, the part $M_{d} \cap\left(\mathbb{H}^{2} \times[0, H(d)]\right.$ is a graph over the component of $\mathbb{H}^{2} \backslash \gamma_{d}$ whose asymptotic boundary is $c_{1}$. Finally, $H(d)$ is a nonincreasing function satisfying

$$
\lim _{d \rightarrow 1} H(d)=+\infty, \quad \lim _{d \rightarrow+\infty} H(d)=\frac{\pi}{2}
$$


(a)

(b)

(c)

Figure 1
(2) If $d=1$, then $M_{1}$ is the surface given by Formula 5, and its asymptotic boundary is given by (see Figure 1(b))

$$
\partial_{\infty} M_{1}=\left(\left\{q_{1}\right\} \times(-\infty, 0]\right) \cup c_{1} \cup\left(\left\{q_{2}\right\} \times(-\infty, 0]\right)
$$

(3) If $0<d<1$, then $M_{d}$ is an entire vertical graph over $\mathbb{H}^{2}$ and contains the geodesic $\gamma \times\{0\}$. The asymptotic boundary of $M_{d}$ is given by (see Figure 1(c))

$$
\begin{aligned}
\partial_{\infty} M_{d}= & \left(c_{2} \times\{-G(d)\}\right) \cup\left(c_{1} \times\{G(d)\}\right) \\
& \cup\left(\left\{q_{1}\right\} \times[-G(d), G(d)]\right) \cup\left(\left\{q_{2}\right\} \times[-G(d), G(d)]\right),
\end{aligned}
$$

where

$$
G(d):=\int_{0}^{+\infty} \frac{d}{\sqrt{\cosh ^{2} u-d^{2}}} d u, \quad 0<d<1
$$

Therefore, $\partial_{\infty} M_{d}$ is a Jordan curve non homologous to zero in $\partial_{\infty} \mathbb{H}^{2} \times \mathbb{R}$. Furthermore $G(d)$ is a nondecreasing function and we have

$$
\lim _{d \rightarrow 0} G(d)=0, \quad \lim _{d \rightarrow 1} G(d)=+\infty
$$

Proof. We work with the disk model for $\mathbb{H}^{2}$, so that

$$
\mathbb{H}^{2}=\left\{(x, y) \in \mathbb{R}^{2}, x^{2}+y^{2}<1\right\} .
$$

Therefore the product metric on $\mathbb{H}^{2} \times \mathbb{R}$ reads as follows

$$
d \tilde{s}^{2}=\left(\frac{2}{1-\left(x^{2}+y^{2}\right)}\right)^{2}\left(d x^{2}+d y^{2}\right)+d t^{2}
$$

where $(x, y) \in \mathbb{H}^{2}$ and $t \in \mathbb{R}$.
Up to an isometry, we can assume that $q_{1}=-i$ and $q_{2}=i$. Therefore we have $c_{1}=\left\{e^{i \theta} ;-\pi / 2 \leqslant \theta \leqslant \pi / 2\right\}$ and $c_{2}=\left\{e^{i \theta} ; \pi / 2 \leqslant \theta \leqslant 3 \pi / 2\right\}$.

We consider the following particular geodesic of $\mathbb{H}^{2}$

$$
\Gamma=\{(x, 0), x \in(-1,1)\} \subset \mathbb{H}^{2} .
$$

We can assume that the surfaces invariant under hyperbolic translation along $\gamma$ (called hyperbolic surfaces), are generated by curves in the vertical geodesic plane $P=\Gamma \times \mathbb{R} \subset \mathbb{H}^{2} \times \mathbb{R}$.

On the geodesic $\Gamma$ we denote by $\rho \in \mathbb{R}$ the signed distance to the origin ( 0,0 ), thus $x=\tanh (\rho / 2)$. Therefore the metric on $P$ is

$$
d s^{2}=d \rho^{2}+d t^{2}
$$

Let us consider a curve in $P$ which is a vertical graph: $c(\rho)=(\rho, \lambda(\rho))$ where $\lambda$ is a smooth real function defined on a part of $\rho \geqslant 0$. Let us call $M$ the hyperbolic surface generated by $c$. On $M$ we consider the orientation given by the upward unit normal field. With respect to this orientation the principal curvatures of $M$ are given by

$$
k_{1}(\rho)=\frac{\lambda^{\prime \prime}}{\left(1+\lambda^{\prime 2}\right)^{3 / 2}}(\rho), \quad \text { and } \quad k_{2}(\rho)=\frac{\lambda^{\prime}}{\sqrt{1+\lambda^{\prime 2}}}(\rho) \tanh (\rho)
$$

so that, $M$ is a minimal surface if and only if

$$
\lambda^{\prime 2}=\frac{d^{2}}{\cosh ^{2} \rho-d^{2}}
$$

for some $d \geqslant 0$.
Up to the isometry $(z, t) \rightarrow(z,-t)$, we can assume that $\lambda$ is a nondecreasing function, that is, $\lambda^{\prime} \geqslant 0$. Therefore, the condition for $M$ being minimal is

$$
\begin{equation*}
\lambda^{\prime}(\rho)=\frac{d}{\sqrt{\cosh ^{2} \rho-d^{2}}} \tag{2}
\end{equation*}
$$

In the case where $d>1$ we can choose, up to a vertical translation,

$$
\lambda(\rho)=\int_{\cosh ^{-1}(d)}^{\rho} \frac{d}{\sqrt{\cosh ^{2} u-d^{2}}} d u
$$

for $\rho \geqslant \cosh ^{-1}(d)$. Setting $v=\cosh u / d-1$ we obtain:

$$
H(d)=\int_{0}^{+\infty} \frac{d v}{\sqrt{(v+1)^{2}-1} \sqrt{(v+1)^{2}-1 / d^{2}}}
$$

This shows that $H(d)$ is a nonincreasing function. Furthermore

$$
\begin{aligned}
\lim _{d \rightarrow+\infty} H(d) & =\int_{0}^{+\infty} \frac{d v}{\sqrt{(v+1)^{2}-1} \sqrt{(v+1)^{2}}} \\
& =\int_{0}^{1} \frac{d x}{\sqrt{1-x^{2}}}, \quad \text { setting } x=\frac{1}{v+1} \\
& =\frac{\pi}{2}
\end{aligned}
$$

Claim. We have $\lim _{d \rightarrow 1} \lambda(\rho)=+\infty$ for any $\rho>0$.
This clearly implies that $\lim _{d \rightarrow 1} H(d)=+\infty$, since $H(d)=\lim _{\rho \rightarrow+\infty} \lambda(\rho)$ and $\lambda(\rho)$ is a nondecreasing function.

Setting $v=\cosh (u) / d-1$ we get

$$
\begin{aligned}
\lambda(\rho) & =\int_{0}^{\cosh (\rho) / d-1} \frac{1}{\sqrt{v+2} \sqrt{\left(v+1+\frac{1}{d}\right)\left(v+1-\frac{1}{d}\right)}} \frac{d v}{\sqrt{v}} \\
& \geqslant \frac{1}{\sqrt{\frac{\cosh (\rho)}{d}+1} \sqrt{\frac{\cosh (\rho)+1}{d}}} \int_{0}^{\cosh (\rho) / d-1} \frac{d v}{\sqrt{v^{2}+\left(1-\frac{1}{d}\right) v}}
\end{aligned}
$$

Denoting by $I(d)$ the last integral, we have

$$
I(d)=\frac{2}{1-\frac{1}{d}} \int_{0}^{\cosh (\rho) / d-1} \frac{d v}{\sqrt{\left(\frac{2 v}{1-\frac{1}{d}}+1\right)^{2}-1}}
$$

Setting $s=\frac{2 v}{1-\frac{1}{d}}+1$ we obtain

$$
I(d)=\int_{1}^{\frac{2 \cosh (\rho)-d-1}{d-1}} \frac{d s}{\sqrt{s^{2}-1}}=\cosh ^{-1}\left(\frac{2 \cosh (\rho)-d-1}{d-1}\right),
$$

from what we deduce that $\lim _{d \rightarrow 1} I(d)=+\infty$ for any $\rho>0$, which concludes the proof of the Claim.

In the case where $d=1$ we can choose, up to a vertical translation,

$$
\lambda(\rho)=\log \left(\frac{e^{\rho}-1}{e^{\rho}+1}\right)
$$

for $\rho>0$. The hyperbolic surface generated by $\lambda$ is a vertical graph over the connected component of $\mathbb{H}^{2} \backslash \gamma$ whose asymptotic boundary is $c_{1}$. This graph takes the value $-\infty$ on $\gamma$ and the value zero on $c_{1}$ (because $\lim _{\rho \rightarrow 0} \lambda(\rho)=-\infty$ and $\lim _{\rho \rightarrow+\infty} \lambda(\rho)=0$ ). Since this is the unique hyperbolic surface, up to isometry, with unbounded height, we deduce that $M_{1}$ is congruent to the hyperbolic surface given by Formula (5).

Finally, in the case where $0<d<1$, the function $\lambda$ is defined for any $\rho \geqslant 0$ and, up to a vertical translation, we can set

$$
\lambda(\rho)=\int_{0}^{\rho} \frac{d}{\sqrt{\cosh ^{2} u-d^{2}}} d u
$$

We can extend $\lambda$ on $\mathbb{R}$ setting $\lambda(\rho):=-\lambda(-\rho)$ for any $\rho \leqslant 0$. Therefore $\lambda$ is defined on $\mathbb{R}$ and is an odd function, and the hyperbolic surface $M_{d}$ generated by $\lambda$ is an entire vertical graph on $\mathbb{H}^{2}$ symmetric with respect to $\gamma$. We can prove in the same way as in the case where $d>1$ that for any $\rho>0$ we have $\lim _{d \rightarrow 1} \lambda(\rho)=+\infty$. This implies that $\lim _{d \rightarrow 1} G(d)=+\infty$.

The other assertions in the Statement are straightforward verifications.

## 3. VERTICAL MINIMAL GRAPHS

There are many notions of graphs in $\mathbb{H}^{2} \times \mathbb{R}$, but the notion of vertical minimal graphs has appeared in many important theorems. See, for instance [4], [7], [11], [12], [18], [23].

Consider a $C^{2}$ function $t=u(x, y)$. The vertical minimal surface equation in $\mathbb{H}^{2} \times \mathbb{R}$, is given by:

$$
\begin{equation*}
\operatorname{div}_{\mathbb{H}}\left(\frac{\nabla_{\mathbb{H}} u}{W_{u}}\right)=0, \tag{3}
\end{equation*}
$$

where $\operatorname{div}_{\mathbb{H}}$ and $\nabla_{\mathbb{H}}$ are the hyperbolic divergence and gradient respectively and $W_{u}=\sqrt{1+\left|\nabla_{\mathbb{H}} u\right|_{\mathbb{H}}^{2}}$, being $|\cdot|_{\mathbb{H}}$ the norm in $\mathbb{H}^{2}$.

Focusing on the halfspace model for $\mathbb{H}^{2}$, with Euclidean coordinates $x, y, y>0$, the vertical minimal surface equation (3) takes the following form

$$
\begin{equation*}
\left(1+y^{2} u_{x}^{2}\right) u_{y y}+\left(1+y^{2} u_{y}^{2}\right) u_{x x}-2 y^{2} u_{x} u_{y} u_{x y}-y u_{y}\left(u_{x}^{2}+u_{y}^{2}\right)=0 \tag{4}
\end{equation*}
$$

There are many explicit examples of entire and complete minimal graphs with nice geometric properties. For instance, in the half-plane model,
(1) The equation [18]

$$
t=\ell x, x \in(-\infty, \infty), y>0
$$

gives rise to an entire minimal graph (left side of Figure 2) symmetric about the geodesic $\{x=0\}$, that is constant (the constant varying in the interval $(-\infty, \infty)$ ) on each leaf of the foliation given by geodesics with a fixed common asymptotic boundary point $p$ (in this model $p=\infty$ ). Thus the asymptotic
boundary consists in the union of a vertical line with a complete embedded curve in $\partial_{\infty} \mathbb{H}^{2} \times \mathbb{R}$ asymptotic to that line.
(2) The equation [18]

$$
t=\frac{\ell}{2} \ln \left(x^{2}+y^{2}\right), \quad y>0
$$

yields an entire minimal graph (right side of Figure 2) symmetric about the geodesic $\left\{x^{2}+y^{2}=1, y>0\right\}$, that is constant on each leaf of the foliation given by (hyperbolic) translations of a fixed geodesic; hence, the asymptotic boundary consists of two embedded curves in $\partial_{\infty} \mathbb{H}^{2} \times \mathbb{R}$ with two symmetric ends, each end asymptotic to a half-vertical line.


Figure 2. Ball model for $\mathbb{H}^{2} \times\{0\}$
Of course, the previous examples give two different explicit non trivial minimal graphs over a half-plane of $\mathbb{H}^{2}$ taking zero boundary value data on a geodesic but having different asymptotic boundaries.
(3) We observe that there exists a function which takes infinite boundary value data on the positive $y$ axis and zero asymptotic value boundary data at the positive $x$ axis (halfspace model for $\mathbb{H}^{2}$ ), invariant by hyperbolic translations [18].

$$
\begin{equation*}
t=\ln \left(\frac{\sqrt{x^{2}+y^{2}}+y}{x}\right), \quad y>0, x>0 \tag{5}
\end{equation*}
$$

Notice that (5) yields a complete vertical minimal graph over a domain bounded by a geodesic in $\mathbb{H}^{2} \times\{0\}$, taking infinite boundary value data on the geodesic and zero asymptotic boundary value data on an arc $L$ of $\partial_{\infty} \mathbb{H}^{2} \times\{0\}$. The asymptotic boundary of the graph is then the
union of $L$ with the two upper half vertical lines arising from the end points of $L$. We will use this special vertical minimal graph as a barrier at an asymptotic boundary point.

We remark that the miniaml surface given by Formula (5) was used by P. Collin and H. Rosenberg [4] in the important construction of entire minimal graphs in $\mathbb{H}^{2} \times \mathbb{R}$ that are conformally the complex plane $\mathbb{C}$, disproving a conjecture by R. Schoen.

In the Poincaré disk model for $\mathbb{H}^{2}$ with Euclidean coordinates $x, y$, $x^{2}+y^{2}<1$, the vertical minimal surface equation (3) becomes

$$
\begin{align*}
& \mathcal{D}(u):=\left(1+\frac{\left(1-x^{2}-y^{2}\right)^{2}}{4} u_{x}^{2}\right) u_{y y}+\left(1+\frac{\left(1-x^{2}-y^{2}\right)^{2}}{4} u_{y}^{2}\right) u_{x x} \\
& -2 \frac{\left(1-x^{2}-y^{2}\right)^{2}}{4} u_{x} u_{y} u_{x y}+2 \frac{\left(1-x^{2}-y^{2}\right)}{4}\left(x u_{x}+y u_{y}\right)\left(u_{x}^{2}+u_{y}^{2}\right)=0 \tag{6}
\end{align*}
$$

We observe that equation (6) is a second order quasilinear strictly elliptic equation for all real values of the independent variables $x, y$. Moreover, the eigenvalue of the associate matrix are 1 and $W_{u}=1+\frac{\left(1-x^{2}-y^{2}\right)^{2}}{4}\left(u_{x}^{2}+u_{y}^{2}\right)$. The same observation holds for the equation (4), replacing $\frac{\left(1-x^{2}-y^{2}\right)^{2}}{4}$ by $y^{2}$. Hence we conclude that both equations are regular and strictly (uniformly) elliptic up to the asymptotic boundary of $\mathbb{H}^{2} \times\{0\}$. For this reason we can state the classical maximum principle and uniqueness for prescribed continuous finite and asymptotic boundary data.
Theorem 3.1 (Classical maximum principle). Let $g_{1}, g_{2}: \partial \Omega \cup \partial_{\infty} \Omega \rightarrow$ $\mathbb{R}$ be continuous functions satisfying $g_{1} \leqslant g_{2}$. Let $u_{i}: \bar{\Omega} \rightarrow \mathbb{R}$ be $a$ continuous extension of $g_{i}$ satisfying the minimal surface equation (6) on $\Omega, i=1,2$. Then $u_{1} \leqslant u_{2}$.

Proof. The proof is classical elliptic theory, since the minimal surface equation (6) is strictly elliptic up to the asymptotic boundary. A geometric approach can be done in this way. Assume that $u_{1}(p)>u_{2}(p)$ at some point $p \in \Omega$. Then, lifting the graph of $u_{2}$ vertically we obtain a last interior contact point between the graph of $u_{1}$ and the graph of $u_{2}$, which gives a contradiction by the interior maximum principle.

We will solve some Dirichlet problems over certain unbounded domains, given certain prescribed finite boundary data and given certain prescribed asymptotic boundary data.

Among such domains we will consider exterior domains $\Omega$. Of course, the classical examples of such minimal graphs over an exterior domain
are given by the one parameter family of half-catenoids, see Lemma 5.1. We will use this family as barriers. We show some generating curves in Figure 3, where $R=\tanh \rho / 2$, and $\rho$ is the hyperbolic distance from the axe $t$.


Figure 3. Ball model for $\mathbb{H}^{2} \times\{0\}$
We remark that we use also as barriers the one-parameter family of minimal surfaces invariant by hyperbolic translations given by Proposition 2.1.

## 4. The Perron process for the vertical minimal surface EQUATION

In the product $\mathbb{H}^{2} \times \mathbb{R}$, we consider the disk model for the hyperbolic plane $\mathbb{H}^{2}$. Let $\Omega \subset \mathbb{H}^{2} \times\{0\}$, be a domain. In $\overline{\mathbb{H}^{2}} \times\{0\}$, we have that $\partial \bar{\Omega}=\partial \Omega \cup \partial_{\infty} \Omega$, where $\partial \Omega \subset \mathbb{H}^{2} \times\{0\}$ and $\partial_{\infty} \Omega \subset \partial_{\infty} \mathbb{H}^{2} \times\{0\}$.
Definition 1 (Problem $(P)$ ). Let $g: \partial \Omega \cup \partial_{\infty} \Omega \rightarrow \mathbb{R}$ be a continuous function except perhaps at a finite set $S$ of points (discontinuities). We consider the Dirichlet problem, say Problem ( $P$ ), for the vertical minimal surface equation (6) taking at any point of $\partial \Omega \cup \partial_{\infty} \Omega \backslash S$, prescribed boundary (finite and asymptotic) value data $g$.

Let $u: \bar{\Omega}_{S}:=\bar{\Omega} \backslash S \rightarrow \mathbb{R}$ be a continuous function.
Let $U \subset \Omega$ be a closed round disk in $\mathbb{H}^{2} \times\{0\}$. If $u_{\mid \partial U}$ is a $C^{1}$ function then solving the Plateau problem [13] and using a standard adaptation of Rado's Theorem [17] (since $u_{\mid \partial U}$ is a vertical graph over a circle), it
follows that $u_{\mid \partial U}$ has an unique minimal extension $\tilde{u}$ on $U$, continuous up to $\partial U$. If $u_{\mid \partial U}$ is $C^{0}$, one uses an approximation argument or uses a local barrier at a boundary point of $U$. We then define the continuous function $M_{U}(u)$ on $\bar{\Omega}_{S}$ by:

$$
M_{U}(u)(x)= \begin{cases}u(x) & \text { if } x \in \bar{\Omega}_{S} \backslash U  \tag{7}\\ \tilde{u}(x) & \text { if } x \in U\end{cases}
$$

We say that $u$ is a subsolution (resp. supersolution) of $(P)$ if:
i) For any closed round disk $U \subset \Omega$ we have $u \leqslant M_{U}(u)\left(\right.$ resp. $\left.u \geqslant M_{U}(u)\right)$.
ii) $\left.u\right|_{\partial \Omega \cup \partial_{\infty} \Omega} \leqslant g$ (resp. $\left.\left.u\right|_{\partial \Omega \cup \partial_{\infty} \Omega} \geqslant g\right)$.

Remark 3. We now give some classical facts about subsolutions and supersolutions, see [5], [19].
(1) It is easily seen that if $u$ is $C^{2}$ on $\Omega$, the condition i) above is equivalent to $\mathcal{D} u \geqslant 0$ for subsolution or $\mathcal{D} u \leqslant 0$ for supersolution.
(2) As usual if $u$ and $v$ are two subsolutions (resp. supersolutions) of $(P)$ then $\sup (u, v)$ (resp. $\inf (u, v))$ again is a subsolution (resp. supersolution).
(3) Also if $u$ is a subsolution (resp. supersolution) and $U \subset \Omega$ is a closed round disk then $M_{U}(u)$ is again a subsolution (resp. supersolution).
(4) Let $\phi$ (resp. u) be a supersolution (resp. a subsolution) of Problem $(P)$ such that $u \leqslant \phi$, then we have $M_{U}(u) \leqslant M_{U}(\phi) \leqslant$ $\phi$ for any disk $U$ with $\bar{U} \subset \Omega$.

Note that if $\phi$ and $u$ are continuous on $\bar{\Omega}$ then necessarily $u \leqslant \phi$ on $\Omega$.

Note also that due to the nature of Equation (6), $\Omega$ is a bounded domain in $\overline{\mathbb{H}^{2}} \times\{0\}$.
Definition 2 (Barriers). We consider the Dirichlet Problem $(P)$, see Definition 1. Let $p \in \partial \Omega \cup \partial_{\infty} \Omega$, be a boundary point where $g$ is continuous.
(1) Suppose that for any $M>0$ and for any $k \in \mathbb{N}$ there is an open neighborhood $\mathcal{N}_{k}$ of $p$ in $\mathbb{R}^{2}$ and a function $\omega_{k}^{+}$(resp. $\omega_{k}^{-}$) in $C^{2}\left(\mathcal{N}_{k} \cap \Omega\right) \cap C^{0}\left(\overline{\mathcal{N}_{k} \cap \Omega}\right)$ such that
i) $\left.\omega_{k}^{+}(x)\right|_{\left(\partial \Omega \cup \partial_{\infty} \Omega\right) \cap \mathcal{N}_{k}} \geqslant g(x)$ and $\left.\omega_{k}^{+}(x)\right|_{\partial \mathcal{N}_{k} \cap \Omega} \geqslant M$ (resp. $\left.\omega_{k}^{-}(x)\right|_{\left(\partial \Omega \cup \partial_{\infty} \Omega\right) \cap \mathcal{N}_{k}} \leqslant g(x)$ and $\left.\left.\omega_{k}^{-}(x)\right|_{\partial \mathcal{N}_{k} \cap \Omega} \leqslant-M\right)$
ii) $\mathcal{D}\left(\omega_{k}^{+}\right) \leqslant 0$ (resp. $\mathcal{D}\left(\omega_{k}^{-}\right) \geqslant 0$ ) in $\mathcal{N}_{k} \cap \Omega$,
iii) $\lim _{k \rightarrow+\infty} \omega_{k}^{+}(p)=g(p)\left(\right.$ resp. $\left.\lim _{k \rightarrow+\infty} \omega_{k}^{-}(p)=g(p)\right)$.
(2) Suppose that there exists a supersolution $\phi$ (resp. a subsolution $\eta)$ in $C^{2}(\Omega) \cap C^{0}(\bar{\Omega})$ such that $\phi(p)=g(p)($ resp. $\varphi(p)=g(p))$. In both cases (1) or (2) we say that $p$ admits an upper barrier $\left(\omega_{k}^{+}, k \in \mathbb{N}\right.$ or $\phi$ ) (resp. lower barrier $\omega_{k}^{-}, k \in \mathbb{N}$ or $\varphi$ ) for the Problem ( $P$ ). If $p$ admits an upper and a lower barrier we say more shortly that $p$ admits a barrier.

Example 4.1. [Barrier at any convex point for any bounded continuous boundary data $g$ ]. The construction of B. Nelli and H. Rosenberg, the Scherk type minimal graph in $\mathbb{H}^{2} \times \mathbb{R}$ over a geodesic triangle, taking boundary data zero on two sides and boundary data infinite at the other side [14] is given in Example 4.3. The geodesic triangle and the boundary data are drawn in Figure 4.


Figure 4
We consider now these Scherk type surfaces when the geodesic triangle $\Delta$ is isosceles and the zero boundary data is taken on the two sides with equal length and the boundary data $-\infty$ is taken on the other side. We show that these surfaces can be used as an upper barrier (in the sense of Definition 2-(1)) at any convex point $p_{0} \in \partial \Omega$, for any boundary bounded data $g$ continuous at $p_{0}$. For the lower barrier the construction is analogous.

Let $\Delta$ be a geodesic isosceles triangle in $\mathbb{H}^{2} \times\{0\}$ with sides $A, C_{1}$ and $C_{2}$, with $\left|C_{1}\right|=\left|C_{2}\right|$. Let $\omega$ be the solution of the minimal surface equation taking zero boundary data on $C_{1}$ and $C_{2}$, and boundary data $-\infty$ on $A$. Let $S$ be the graph of $\omega$. Let $a$ be the common vertex of
$C_{1}$ and $C_{2}$. Let $\gamma$ be the axis of symmetry of $\Delta$, hence $a \in \gamma$. Let $\{b\}=\gamma \cap A$. Let $\beta$ be a geodesic intersecting $\Delta$ orthogonal to $\gamma$ at a point $d \in(a, b)$. Set $\beta \cap C_{1}=\{c\}$.

We claim the following:
(1) $\omega$ along $\gamma$ is nonincreasing in $[a, b]$.
(2) $\omega$ along $\beta$ is nonincreasing in $[c, d]$.

Assume momentarily the Claim. Let $p_{0} \in \partial \Omega$ be a convex point and let $g$ be a boundary data continuous at $p_{0}$. Let $M>0$ be any positive real number. It suffices to show that for any $k \in \mathbb{N}$ there is an open neighborhood $\mathcal{N}_{k}$ of $p_{0}$ in $\mathbb{R}^{2}$ and a function $\omega_{k}^{+}$in $C^{2}\left(\mathcal{N}_{k} \cap \Omega\right) \cap$ $C^{0}\left(\overline{\mathcal{N}_{k} \cap \Omega}\right)$ such that
i) $\left.\omega_{k}^{+}(x)\right|_{\partial \Omega \cap \mathcal{N}_{k}} \geqslant g(x)$ and $\left.\omega_{k}^{+}(x)\right|_{\partial \mathcal{N}_{k} \cap \Omega} \geqslant M$
ii) $\mathcal{D}\left(\omega_{k}^{+}\right)=0$ in $\mathcal{N}_{k} \cap \Omega$,
iii) $\omega_{k}^{+}\left(p_{0}\right)=g\left(p_{0}\right)+1 / k$.

By continuity there exists $\epsilon>0$ such that for any $p \in \partial \Omega$ such that $\operatorname{dist}\left(p, p_{0}\right)<\epsilon$ we have $g(p)<g\left(p_{0}\right)+1 / k$. By assumption there exists an open geodesic arc $\gamma^{\perp}$, through $p_{0}$ such $\gamma^{\perp} \cap \Omega=\emptyset$. We may assume that the disk $D_{\epsilon}\left(p_{0}\right)$ intersects $\gamma^{\perp}$ at two points.

We choose $\Delta$ such that $p_{0} \in \gamma,|A|<\epsilon, \bar{\Omega} \cap A=\emptyset$, and $\gamma$ orthogonal to $\gamma^{\perp}$ at $p_{0}$. Let $M_{1}>\max \left\{M, g\left(p_{0}\right)+1 / k\right\}$. We consider the Scherk surface (graph of $\omega$ ) taking $M_{1}$ boundary value data on $C_{1}, C_{2}$ and $-\infty$ on $A$. By continuity, there exists a point $p_{1}$ at $\gamma$ where $\omega\left(p_{1}\right)=$ $g\left(p_{0}\right)+1 / k$. Up to a horizontal translation along $\gamma$ sending $p_{1}$ to $p_{0}$, we may assume that $\omega\left(p_{0}\right)=g\left(p_{0}\right)+1 / k$. Therefore we set $\mathcal{N}_{k}=\Delta \cap \Omega$ and $\omega_{k}^{+}=\left.\omega\right|_{\mathcal{N}_{k}}$ is the restriction of $\omega$ to $\mathcal{N}_{k}$. The Claim shows that $\left.\omega_{k}^{+}(x)\right|_{\partial \Omega \cap \mathcal{N}_{k}} \geqslant g(x)$, as desired.

We now proceed to the proof of the Claim. Let $p_{1}, p_{2} \in[a, b)$ such that $p_{1}<p_{2}$. Let $p_{3} \in\left(p_{1}, p_{2}\right)$ be the middle point in the segment $\left[p_{1}, p_{2}\right]$ and let $\gamma_{3}$ be the geodesic orthogonal to $\gamma$ at $p_{3}$. Let $S_{3}$ be the connected component of $S \backslash\left(\gamma_{3} \times \mathbb{R}\right)$ containing $(a, 0)$. Now the maximum principle shows that the reflection of $S_{3}$ with respect to $\gamma_{3} \times \mathbb{R}$ is above $S$, since this is true on the boundary. Hence $\omega\left(p_{1}\right)>\omega\left(p_{2}\right)$, as desired. The proof of the second part of the Claim is analogous, considering the reflections about the vertical geodesic planes orthogonal to $[c, d]$. The same argument also shows that $S$ is symmetric about the vertical geodesic plane $\gamma \times \mathbb{R}$. This accomplishes the construction of the desired barrier.

Example 4.2 (Barrier at an asymptotic point). The surface given by Formula (5), may be seen as a complete vertical minimal graph over a
domain bounded by a geodesic in $\mathbb{H}^{2} \times\{0\}$, taking infinite boundary value data on the geodesic and zero asymptotic boundary value data on an arc $L$ of $\partial_{\infty} \mathbb{H}^{2} \times\{0\}$. The asymptotic boundary of the graph is then the union of $L$ with the two half vertical lines arising from the end points of $L$, see Figure 1(b). We can therefore choose the geodesic as small as we wish in the Euclidean sense, because the minimal surface equation extends smoothly to $\overline{\mathbb{H}^{2}} \times\{0\}$. Then we can put a copy of it above and below the graph of $g$ at any point $p$ where $g$ is continuous. Thus we obtain a barrier at any point $p$ of $\partial_{\infty} \Omega$ where $g$ is continuous, in the sense of Definition 2-(1).

Theorem 4.1 (Perron process). Let $\Omega \subset \mathbb{H}^{2} \times\{0\}$ be a domain and let $g: \partial \Omega \cup \partial_{\infty} \Omega \rightarrow \mathbb{R}$ be a continuous function except perhaps at a finite set $S$. Suppose that the Dirichlet Problem ( $P$ ) has a supersolution $\phi$. Set $\mathcal{S}_{\phi}=\{\varphi$, subsolution of $(P), \varphi \leqslant \phi\}$. Assume that $\mathcal{S}_{\phi} \neq \emptyset$. We define for each $x \in \bar{\Omega} \backslash S$

$$
u(x)=\sup _{\varphi \in \mathcal{S}_{\phi}} \varphi(x) .
$$

We have the following:
(1) The function $u$ is $C^{2}$ on $\Omega$ and satisfies the minimal surface equation (6).
(2) Let $p \in \partial \Omega$ be a finite boundary point where $g$ is continuous. Suppose that $p$ admits $a$ barrier in the sense of Definition 2-(1). Then the solution $u$ is continuous at $p$ and satisfies $u(p)=g(p)$. In particular, if $\partial \Omega$ is convex at $p$ then $u$ extends continuously at $p$ and $u(p)=g(p)$.
(3) Let $p \in \partial_{\infty} \Omega$ be an asymptotic boundary point where $g$ is continuous. Then $p$ admits $a$ barrier, $u$ is continuous at $p$ and satisfies $u(p)=g(p)$; that is, if $\left(x_{n}\right)$ is a sequence in $\mathbb{H}^{2} \times\{0\}$ such that $x_{n} \rightarrow p$ in the Euclidean sense then $u\left(x_{n}\right) \rightarrow g(p)$. In particular, if $g$ is continuous on $\partial_{\infty} \Omega$ then the asymptotic boundary of the graph of $u$ is the restriction of the graph of $g$ to $\partial_{\infty} \Omega$.
(4) Let $q \in \partial_{\infty} \mathbb{H}^{2}$ be an interior point of $\partial_{\infty} \Omega$ where $g$ is discontinuous. Then the vertical segment $\left\{(q, t), t \in\left[A:=\liminf _{x \rightarrow q, x \neq q} g(x), B:=\limsup _{x \rightarrow q, x \neq q} g(x)\right], x \in \partial \Omega \cup \partial_{\infty} \Omega\right\}$ belongs to the asymptotic boundary of the graph of $u$. In particular, if $A=-\infty$ and $B=+\infty$, then the whole vertical line $\{q\} \times \mathbb{R}$ belongs to the asymptotic boundary.
Proof. Observe that for any $\varphi \in \mathcal{S}_{\phi}, M_{U}(\varphi) \in \mathcal{S}_{\phi}$, for any closed disk $U \subset \Omega$. Observe also that the basic compactness theorem holds for
the vertical minimal surface equation, see [8], [22], [23] and [12]. The proof of Statements (1) and (2) follows from classical arguments as in Theorem 3.4 in [19], see also the classical reference [5]. The last assertion of Statement (2) follows from Example 4.1.

The Statement (3) follows from the previous construction of a suitable barrier, in the sense of Definition 2-(1), at any point $p$ of $\partial_{\infty} \Omega$ where $g$ is continuous, see Example 4.2.

The proof of Statement (4) follows from a continuity argument. Indeed, as $g$ is discontinuous at $q$ we have $A \neq B$. let $t_{0} \in(A, B)$. Let $\left(x_{n}\right)$ and $\left(y_{n}\right), n \in \mathbb{N}$, be two sequences in $\partial_{\infty} \Omega$, such that $x_{n}, y_{n} \rightarrow q$, $\lim g\left(x_{n}\right)=A$, and $g\left(y_{n}\right)=B$. We can assume that $g\left(x_{n}\right)<t_{0}<g\left(y_{n}\right)$, for any $n$. Let $\Gamma_{n}$ be a closed arc joining in $\bar{\Omega}$ the point $x_{n}$ to $y_{n}$, close to $q$ in the Euclidean sense and such that $\Gamma_{n} \cap \partial_{\infty} \Omega=\left\{x_{n}, y_{n}\right\}$. Notice that the restriction of the graph of $u$ to the closed arc $\Gamma_{n}$ is continuous and intersects the slice $\mathbb{H}^{2} \times\left\{t_{0}\right\}$, at some point $\left(z_{n}, t_{0}\right)$, where $z_{n}$ is an interior point of $\Gamma_{n}$, and $z_{n} \rightarrow q$ as $n \rightarrow \infty$. Hence, $\left(q, t_{0}\right)$ belongs to the asymptotic boundary of the graph of $u$, for any $t_{0} \in[A, B]$ (as the asymptotic boundary is a closed set). This completes the proof of the Theorem.

Corollary 4.1. Let $\Omega \subset \mathbb{H}^{2} \times\{0\}$ be a domain and let
$g: \partial \Omega \cup \partial_{\infty} \Omega \rightarrow \mathbb{R}$ be a bounded function everywhere continuous except perhaps at a finite set $S \subset \partial \Omega \cup \partial_{\infty} \Omega$. Assume that the finite boundary $\partial \Omega$ is convex or, alternatively, that each finite boundary point admits a barrier.

Then, $g$ admits an extension $u: \bar{\Omega} \backslash S \rightarrow \mathbb{R}$ satisfying the vertical minimal surface equation (6). Furthermore, the total boundary of the graph of $u$ (that is the finite and asymptotic boundary) is the union of the graph of $g$ on $\left(\partial \Omega \cup \partial_{\infty} \Omega\right) \backslash S$ with the vertical segments $\left\{(q, t), t \in\left[A:=\lim _{x \rightarrow q,} \inf _{x \neq q} g(x), B:=\limsup _{x \rightarrow q, x \neq q} g(x)\right], x \in \partial \Omega \cup \partial_{\infty} \Omega\right\}$ at any $q \in S$.

Proof. Since $g$ is bounded, there are some constant functions which are supersolutions and other which are subsolutions of Problem $(P)$. We consider a slight variation of Perron process taking the set $\mathcal{S}$ of continuous subsolutions of $(P)$. Let $u$ be the solution given by the Perron process (Theorem 4.1). It follows from Theorem 4.1 that the total boundary of the graph of $u$ contains the union of the graph of $g$ on $\left(\partial \Omega \cup \partial_{\infty} \Omega\right) \backslash S$ with the vertical segments given in the Statement at any $q \in S \cap \partial_{\infty} \Omega$. If $q \in S$ is on $\partial \Omega$ or is not an interior point of $\partial_{\infty} \Omega$ then, taking into account that each finite boundary point has a
barrier by assumption, we can prove in the same way that the vertical segment $[A, B]$ is contained in the total boundary of the graph of $u$.

For any $q_{i} \in S$ we set $A_{i}:=\liminf _{x \rightarrow q_{i}, x \neq q_{i}} g(x)$ and $B_{i}:=\limsup _{x \rightarrow q_{i}, x \neq q_{i}} g(x)$, $x \in \partial \Omega \cup \partial_{\infty} \Omega$.

It remains to show that for any $q_{i} \in S$ and any real number $t$ satisfying $t>B_{i}$ or $t<A_{i}$ the point $\left(q_{i}, t\right)$ is not in the total boundary of the graph of $u$.

Assume first that $t>B_{i}$. Let $\varepsilon>0$ be a real number satisfying $B_{i}+\varepsilon<t$. There exists a continuous function $g^{+}: \partial \Omega \cup \partial_{\infty} \Omega \rightarrow \mathbb{R}$ such that $g^{+}>g$ on $\left(\partial \Omega \cup \partial_{\infty} \Omega\right) \backslash S$ and $g^{+}\left(q_{j}\right)=q_{j}+\varepsilon$ for any $q_{j} \in S$. Then the minimal extension $u^{+}$of $g^{+}$given by the Perron process is continuous up to $\bar{\Omega}$. It follows that $u^{+}$is a supersolution of Problem $(P)$ for the boundary data $g$ and, consequently we have $\varphi \leqslant u^{+}$on $\bar{\Omega}$ for any $\varphi \in \mathcal{S}$. It follows that the point $\left(q_{i}, t\right)$ is not in the total boundary of the graph of $u$.

Assume now that $t<A_{i}$ and consider a continuous function $g^{-}$: $\partial \Omega \cup \partial_{\infty} \Omega \rightarrow \mathbb{R}$ such that $g^{-}<g$ on $\left(\partial \Omega \cup \partial_{\infty} \Omega\right) \backslash S$ and $g^{-}\left(q_{j}\right)=q_{j}-\varepsilon$ for any $q_{j} \in S$. Since the minimal extension of $g^{-}$is a subsolution of Problem $(P)$, we infer that the point $\left(q_{i}, t\right)$ is not in the total boundary of the graph of $u$. This concludes the proof of the Corollary.

## Remark 4.

(1) It follows from Corollary 4.1 that if $\Omega$ is a convex unbounded domain, then there exists an unique vertical minimal graph over $\Omega$ taking any prescribed bounded continuous finite and asymptotic boundary data.
(2) In the special case when $\Omega=\mathbb{H}^{2}$, consider a bounded function $g$ on $\partial_{\infty} \mathbb{H}^{2} \times\{0\}$, continuous except perhaps at a finite set of points $S$. With the aid of Corollary 4.1 we see that $g$ admits a minimal entire extension $u$. If $g$ is continuous, we remark that uniqueness of the extension follows from Theorem 3.1.

This problem when $g$ is continuous on $\partial_{\infty} \mathbb{H}^{2}$, is called Dirichlet problem at infinity and was solved by B. Nelli and H. Rosenberg [14].

Example 4.3. Let $\Delta$ be a geodesic triangle in $\mathbb{H}^{2} \times\{0\}$ with sides $A, C_{1}$ and $C_{2}$. We want to show that there exists a minimal Scherk type graph over $\Delta$ taking zero boundary value data on the interior of $C_{1} \cup C_{2}$ and taking $+\infty$ as boundary value data on $A$. This is proved by B. Nelli and H. Rosenberg in [14].

For this purpose we first show that for any $n \in \mathbb{N}$ there exists a solution $u_{n}$ of the minimal surface equation on the interior of $\Delta$ taking zero boundary value data on the interior of $C_{1} \cup C_{2}$ and taking $n$ as boundary value data on $A$. We consider the set $\mathcal{S}_{n}$ of continuous functions $\varphi$ on $\Delta$ satisfying:
(1) For any closed round disk $U \subset \operatorname{int} \Delta, \varphi \leqslant M_{U}(\varphi)$, where $M_{U}(\varphi)$ is given in Formula (7)
(2) $\varphi \leqslant 0$ on the interior of $C_{1} \cup C_{2}$
(3) $\varphi \leqslant n$ on $A$.

For any subarc $C^{\prime}$ of $C_{1} \cup C_{2}$ and any subarc $A^{\prime}$ of $A$ there is continuous subsolutions and supersolutions on $\Delta$ assuming zero boundary value data on $C^{\prime}$ and $n$ boundary value data on $A^{\prime}$. Those functions give barriers at any interior point of the sides $A, C_{1}$ and $C_{2}$. Therefore the solution $u_{n}$ given by the Perron process, Theorem 4.1-(1), assumes the desired boundary value data

Let $A_{\infty}$ be the complete geodesic containing $A$. Taking into account Formula (5), let $\phi$ be the minimal graph over the half-plane with boundary $A_{\infty}$ that contains $\Delta$, taking $+\infty$ as boundary value data on $A_{\infty}$ and zero asymptotic boundary value data. We will write down a slight variation of Perron process.

Let $\mathcal{S}_{\phi}$ be the family of continuous functions $\varphi$ defined on $\operatorname{int} \Delta \cup \operatorname{int}\left(C_{1} \cup C_{2}\right)$ satisfying:
(1) For any closed round disk $U \subset \operatorname{int} \Delta, \varphi \leqslant M_{U}(\varphi)$, where $M_{U}(\varphi)$ is given in Formula (7)
(2) $\varphi \leqslant 0$ on the interior of $C_{1} \cup C_{2}$
(3) $\varphi \leqslant \phi$

Notice that the functions $u_{n}$ constructed above belong to $\mathcal{S}_{\phi}$. Therefore we infer that the solution $u$ given by Perron process assumes infinite boundary value data on $A$. We claim that $u$ takes zero boundary value data on the interior of $C_{1} \cup C_{2}$. Actually, let $C_{3}$ be an arc of geodesic lying in $\Delta$ joining a point $c_{1}$ on $C_{1}$ to a point $c_{2}$ on $C_{2}$. Let $a=C_{1} \cap C_{2}$ and let $\Delta_{0}$ be the geodesic triangle with vertices $a, c_{1}$ and $c_{2}$.

Let $f$ be the restriction of $\phi$ to $C_{3}$. Notice that the solution of the Dirichlet problem on $\Delta_{0}$ taking zero boundary value data on the sides $\left[a, c_{1}\right],\left[a, c_{2}\right]$ and $f$ on the side $\left[c_{1}, c_{2}\right]$ gives rise to an upper barrier at any point of the interior of $C_{1} \cup C_{2}$. Of course the zero function is a lower barrier to the problem. Thus $u$ takes the desired boundary value data, as we claimed.
Example 4.4. We consider a geodesic triangle $\Delta$ in $\mathbb{H}^{2} \times\{0\}$ with two vertices $a, b$ on $\partial_{\infty} \mathbb{H}^{2} \times\{0\}$ and a third vertex $c$ on $\mathbb{H}^{2} \times\{0\}$. Doing
a similar construction as in Example 4.3, we can solve our Dirichlet problem on $\Delta$ taking infinite boundary value data on the complete geodesic $(a, b)$ and zero boundary value data on the two other sides.

Assume now that the interior angle at vertex $c$ is $\pi / k, k \in \mathbb{N}^{*}$. Using Schwarz reflection on the geodesics arcs $(c, a),(c, b)$, and successively about the geodesic boundaries as well, we obtain a complete embedded minimal surface in $\mathbb{H}^{2} \times \mathbb{R}$. Since the angle at the vertex $c$ is $\pi / k, k \in \mathbb{N}^{*}$, we get a complete graph over an ideal geodesic polygon with $2 k$ sides, taking successively boundary values $+\infty$ and $-\infty$. These minimal complete graphs can also be built combining some results on harmonic maps from the complex plane into the hyperbolic plane, done in [24], [9] and [11]. We observe that these examples are a particular case of a general result found in [4].

## 5. Minimal graphs with finite and asymptotic boundary in $\mathbb{H}^{2} \times \mathbb{R}$

Lemma 5.1. Let $\rho>0$ and let $\mathcal{C}_{\rho} \subset \mathbb{H}^{2} \times\{0\}$ be a circle of radius $\rho$. Then there exists a unique catenoid $\mathcal{M}_{\rho}$ in $\mathbb{H}^{2} \times \mathbb{R}$ orthogonal to the slice $\mathbb{H}^{2} \times\{0\}$ along $\mathcal{C}_{\rho}$. Its asymptotic boundary is $\partial_{\infty} \mathbb{H}^{2} \times\left\{ \pm t_{0}\right\}$, for some $0<t_{0}$, where $t_{0}:=f(\rho)$ is an increasing function of $\rho$ given by

$$
\begin{equation*}
f(\rho)=\int_{\rho}^{\infty} \frac{\sinh \rho}{\sqrt{\sinh ^{2} r-\sinh ^{2} \rho}} \mathrm{~d} r \tag{8}
\end{equation*}
$$

Furthermore, $\lim _{\rho \rightarrow 0} f(\rho)=0$, and $\lim _{\rho \rightarrow \infty} f(\rho)=\pi / 2$.
The proof of Lemma 5.1 follows from Proposition 5.1 of [15] and [20]. For later use we call $\mathcal{M}_{\rho}^{+}$(resp. $\mathcal{M}_{\rho}^{-}$) the part of the catenoid $\mathcal{M}_{\rho}$ in $\mathbb{H}^{2} \times[0, \infty)$ (resp. in $\left.\mathbb{H}^{2} \times(-\infty, 0]\right)$.

Proposition 5.1 (A characterization of vertical minimal graphs). Let $M$ be a minimal surface immersed in $\mathbb{H}^{2} \times \mathbb{R}$, whose finite boundary is a Jordan curve $\Gamma$ and whose asymptotic boundary is $\partial_{\infty} \mathbb{H}^{2} \times\left\{t_{0}\right\} \subset$ $\partial_{\infty} \mathbb{H}^{2} \times \mathbb{R}, t_{0} \geqslant 0$. Assume that $\Gamma$ is a vertical graph over a Jordan curve $C \subset \mathbb{H}^{2} \times\{0\}$. Assume also that the vertical projection of $M$ is contained in ext $C$.
Then $M$ is a vertical graph. Furthermore, if $\Gamma=C$, then $t_{0}<\pi / 2$ and $M$ inherits all symmetries of $\Gamma$. Particularly, if $\Gamma$ is an horizontal circle then $M$ is part of a catenoid.

Proof. The proof is somewhat straightforward. We will just sketch it as follows. The first statement is a consequence of Alexandrov Reflection

Principle on horizontal slices doing vertical reflections. The second statement $\left(t_{0}<\pi / 2\right)$ is a consequence of Lemma 5.1 using the family of catenoids, coming from the infinity towards $M$. The third statement is a consequence of Alexandrov reflection Principle on vertical geodesic planes.

The following Remark is inferred from Lemma 5.1 and maximum principle.

Remark 5. Let $C_{\rho}$ be a circle of radius $\rho$ in $\mathbb{H}^{2} \times\{0\}$, and let $\Gamma_{\infty} \subset$ $\partial_{\infty} \mathbb{H}^{2} \times \mathbb{R}$, be a Jordan curve that is a vertical graph over $\partial_{\infty} \mathbb{H}^{2} \times\{0\}$. If the height function $t$ of $\Gamma_{\infty}$ satisfies $t>f(\rho)$ there is no vertical minimal graph over $\operatorname{ext}\left(C_{\rho}\right)$ whose finite boundary is $C_{\rho}$ and whose asymptotic boundary is $\Gamma_{\infty}$.

Definition 3 (Admissible unbounded domains in $\mathbb{H}^{2}$ ). Let $\Omega$ be an unbounded domain in the slice $\mathbb{H}^{2} \times\{0\}$ and let $\partial \Omega$ be its boundary. We say that $\Omega$ is an admissible domain if each connected component $C_{0}$ of $\partial \Omega$ satisfies one of the following conditions:
(1) $C_{0}$ is a Jordan curve.
(2) $C_{0}$ is a properly embedded curve such that the asymptotic boundary is one point.
(3) $C_{0}$ is a properly embedded curve such that the asymptotic boundary is two distinct points.
Finally, each connected component $C_{0}$ of $\partial \Omega$ satisfies the Exterior circle of (uniform) radius $\rho$ condition, that is, at any point $p \in C_{0}$ there exists a circle $C_{\rho}$ of radius $\rho$ such that $p \in C_{0} \cap C_{\rho}$ and $\overline{\operatorname{int} C_{\rho}} \cap \Omega=\emptyset$.

If $\Omega$ is an unbounded admissible domain then we denote by $\rho_{\Omega}$ the supremum of the set of these $\rho$.

If the components of $\partial \Omega$ are compact, we set $C:=\partial \Omega$, hence $C=$ $C_{1} \cup \ldots \cup C_{n}$ is the union of disjoint Jordan curves $C_{j}, j=1, \ldots, n$ with pairwise disjoint interiors. We set $\operatorname{ext}(C)=\operatorname{ext}\left(C_{1}\right) \cap \cdots \cap \operatorname{ext}\left(C_{n}\right)$ and $\operatorname{int}(C)=\operatorname{int}\left(C_{1}\right) \cup \cdots \cup \operatorname{int}\left(C_{n}\right)$. In this case we set $\rho_{C}:=\rho_{\Omega}$.

In the next theorem we need the function $f(\rho)$ given in Lemma 5.1 (height of the catenoid $\mathcal{M}_{\rho}$ arising orthogonally from the slice along a circle of radius $\rho_{\Omega}$ ).

Theorem 5.1. Let $\Omega$ be an admissible unbounded domain. Let $g: \partial \Omega \cup \partial_{\infty} \Omega \rightarrow \mathbb{R}$ be a bounded function taking zero boundary value data on $\partial \Omega$, everywhere continuous except perhaps at a finite set $S \subset$ $\partial_{\infty} \Omega$. Let $\Gamma_{\infty} \subset \partial_{\infty} \mathbb{H}^{2} \times \mathbb{R}$ be the union of the graph of $g$ restricted to $\partial_{\infty} \Omega$ with the vertical segments at the points of $\partial_{\infty} \mathbb{H}^{2}$ of discontinuities
of $g$.
If the height function $t$ of $\Gamma_{\infty}$ satisfies $-f\left(\rho_{\Omega}\right) \leqslant t \leqslant f\left(\rho_{\Omega}\right)$, then there exists a vertical minimal graph over $\Omega$ with finite boundary $\partial \Omega$ and asymptotic boundary $\Gamma_{\infty}$.

Particularly, if $C \subset \mathbb{H}^{2} \times\{0\}$ is a Jordan curve satisfying the Exterior circle of radius $\rho$ condition and if $g: \partial_{\infty} \mathbb{H}^{2} \rightarrow \mathbb{R}$ is a continuous function satisfying $-f\left(\rho_{\Omega}\right) \leqslant g(p) \leqslant f\left(\rho_{\Omega}\right)$ at any point $p \in \partial_{\infty} \mathbb{H}^{2}$, then there exists a unique vertical minimal graph over $\Omega$ with finite boundary $\partial \Omega$ and asymptotic boundary $\Gamma_{\infty}$.

Finally, there is no such minimal graph, if $\partial \Omega$ is compact and the height function $t$ of $\Gamma_{\infty}$ satisfies $|t|>\pi / 2$.

Proof. Consider the family of catenoids $\mathcal{M}_{\rho}$ given by Lemma 5.1. Notice that our assumptions imply that at each point $p \in C$ there exists a circle $\mathcal{C}_{\rho_{C}}(p)$ of radius $\rho_{C}$ contained in $\operatorname{int}(C)$ with $p \in C_{\rho}(p) \cap C$. Let $\mathcal{M}_{\rho_{C}}^{+}(p)$ and $\mathcal{M}_{\rho_{C}}^{-}(p)$ be the the upper and lower half- catenoids cutting orthogonally the slice $t=0$ along the circle $\mathcal{C}_{\rho_{C}}(p)$.

Take one of these lower half-catenoids as a subsolution, and take one of these upper half-catenoids as the supersolution $\phi$ in Perron process, Theorem 4.1. It follows that this family of half- catenoids provide also a family of barriers at each point of $C$ to our problem in the sense of Definition 2-(2). Therefore our Dirichlet Problem (P), see Definition 1, can be solved using Corollary 4.1.

If $C \subset \mathbb{H}^{2} \times\{0\}$ is a Jordan curve satisfying the Exterior circle of radius $\rho$ condition, and if $g: \partial_{\infty} \mathbb{H}^{2} \rightarrow \mathbb{R}$ is continuous, then the uniqueness follows from the classical maximum principle Theorem 3.1. This proves the first assertion of the statement.

To prove the nonexistence part assume by contradiction that there exists a solution $u$ such that the height function $t$ of $\Gamma_{\infty}$ satisfies $t>$ $\pi / 2$. Notice that the graph of $u$ is above the slice $t=0$. Now choose a catenoid $\mathcal{M}_{\rho}$ with $t$ axis and large "neck" ( $\rho$ big enough) disjoint from the graph of $u$. Let $\mathcal{M}_{\rho}(\epsilon)=\mathcal{M}_{\rho}+\epsilon$ be the $\epsilon$-vertical translation of $\mathcal{M}_{\rho}$, with $\epsilon>0$ small enough. Now shrink the catenoid $\mathcal{M}_{\rho}(\epsilon)$ in the family of catenoids with the same axis making the "neck" going to zero. We will find a first interior point of contact of the graph of $u$ with one of these catenoids. This gives a contradiction by the maximum principle and completes the proof of the Theorem.

Remark 6. A computation shows that any catenoid in $\mathbb{H}^{2} \times \mathbb{R}$ has finite total extrinsic curvature. We set here a question: is it true that the same holds for any exterior minimal graph in $\mathbb{H}^{2} \times \mathbb{R}$ ?

We now restrict our attention to certain admissible domains such that each component of the boundary has two points at its asymptotic boundary and has at each point of its finite boundary an exterior equidistant curve. To be more precise:
Definition 4 (E-admissible unbounded domains in $\mathbb{H}^{2}$ ). Let $\Omega$ be an unbounded domain in the slice $\mathbb{H}^{2} \times\{0\}$ and let $\partial \Omega$ be its boundary. We say that $\Omega$ is an $E$-admissible domain if
(1) Each connected component $C_{0}$ of $\partial \Omega$ is a properly embedded curve such that the asymptotic boundary consists of two distinct points.
(2) We require that there exists $r>0$ such that each point of $\partial \Omega$ satisfies the Exterior equidistant curve of (uniform) curvature tanh $r$ condition; that is, at any point $p \in \partial \Omega$ there exists an equidistant curve $E_{r}$ of curvature $\tanh r$ (with respect to the exterior unit normal to $\Omega$ at $p$ ), with $p \in \partial \Omega \cap E_{r}$ and $E_{r} \cap \Omega=$ $\emptyset$.

Thus every E-admissible domain is an admissible domain.
If $\Omega$ is a convex domain satisfying the condition (1) of Definition 4 then $\Omega$ is an E-admissible domain.

If each connected component $C_{0}$ of $\partial \Omega$ is an equidistant curve then $\Omega$ is an E-admissible (maybe nonconvex) domain.

If $\Omega$ is an unbounded E-admissible domain then we denote by $r_{\Omega} \geqslant 0$ the infimum of the set of these $r$. If $\Omega$ is a convex E-admissible domain then $r_{\Omega}=0$.

We will use in the next result the function $H$ defined by Formula (1) in Proposition 2.1.

Theorem 5.2. Let $\Omega$ be an E-admissible unbounded domain. Let $g: \partial \Omega \cup \partial_{\infty} \Omega \rightarrow \mathbb{R}$ be a bounded function taking zero boundary value data on $\partial \Omega$, everywhere continuous except perhaps at a finite set $S \subset$ $\partial_{\infty} \Omega$. Let $\Gamma_{\infty} \subset \partial_{\infty} \mathbb{H}^{2} \times \mathbb{R}$ be the union of the graph of $g$ restricted to $\partial_{\infty} \Omega$ with the vertical segments at the points of discontinuities of $g$. If the height function $t$ of $\Gamma_{\infty}$ satisfies $-H\left(\cosh r_{\Omega}\right) \leqslant t \leqslant H\left(\cosh r_{\Omega}\right)$, then there exists a vertical minimal graph over $\Omega$ with finite boundary $\partial \Omega$ and asymptotic boundary $\Gamma_{\infty}$.

Proof. The proof is the same as in Theorem 5.1, replacing the minimal catenoids by the minimal surfaces invariant by hyperbolic translations $M_{d}, d>1$, given in Proposition 2.1. This completes the proof of the Theorem.

Notice that if $\Omega$ is convex then it is E-admissible and $r_{\Omega}=0$, thus $H\left(\cosh r_{\Omega}\right)=\infty$.

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