Sharp solvability criteria for Dirichlet problems of mean curvature type in Riemannian manifolds: non-existence results

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February 22, 2019

Abstract

It is well known that the *Serrin condition* is a necessary condition for the solvability of the Dirichlet problem for the prescribed mean curvature equation in bounded domains of $\mathbb{R}^n$ with certain regularity. In this paper we investigate this fact for the vertical mean curvature equation in the product $M^n \times \mathbb{R}$. Precisely, given a $\mathscr{C}^2$ bounded domain Ω in M and a function $H = H(x, z)$ continuous in $\overline{\Omega} \times \mathbb{R}$ and non-decreasing in the variable z, we prove that the *strong Serrin condition* $(n-1)\mathcal{H}_{\partial\Omega}(y) \geq n \sup_{z\in\mathbb{R}} |H(y, z)| \ \forall\, y \in \partial\Omega$, is a necessary condition for the solvability of the Dirichlet problem in a large class of Riemannian manifolds within which are the Hadamard manifolds and manifolds whose sectional curvatures are bounded above by a positive constant. As a consequence of our results we deduce Jenkins-Serrin and Serrin type sharp solvability criteria.

1 Introduction

We denote by M a complete Riemannian manifold of dimension $n \geq 2$ and let Ω be a domain in M. The focus of our work is the prescribed mean curvature equation for vertical graphs in $M \times \mathbb{R}$, this is,

$$\mathcal{M}u := \operatorname{div}\left(\frac{\nabla u}{W}\right) = nH(x, u), \tag{1}$$

[*]Supported by CNPq of Brazil.

[†]Partially supported by CNPq of Brazil.

2000 AMS Subject Classification: 53C42, 49Q05, 35J25, 35J60.

Keywords and phrases: mean curvature equation; Dirichlet problems; Serrin condition; sectional curvature; Ricci curvature; radial curvature; distance functions; Laplacian comparison theorem; Hadamard manifolds; hyperbolic space.

where H is a continuous function over $\overline{\Omega} \times \mathbb{R}$ and non-decreasing in the variable z, $W = \sqrt{1 + \|\nabla u(x)\|^2}$ and the quantities involved are calculated with respect to the metric of M. In a coordinates system $(x_1, \dots, x_n)$ in M, it follows that

$$\mathcal{M}u = \frac{1}{W} \sum_{i,j=1}^{n} \left(\sigma^{ij} - \frac{u^i u^j}{W^2} \right) \nabla^2_{ij} u = nH(x, u), \tag{2}$$

where (σ^{ij}) is the inverse of the metric (σ_{ij}) of M, $u^i = \sum_{j=1}^{n} \sigma^{ij} \partial_j u$ are the coordinates of ∇u and $\nabla^2_{ij} u(x) = \nabla^2 u(x) \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right)$. We will denote by $\mathfrak{Q}$ the operator defined by

$$\mathfrak{Q}u = \mathcal{M}u - nH(x, u).$$

We notice that the matrix of the operator $\mathcal{M}$ is given by $A = \frac{1}{W} g$, where g is the induce metric on the graph of u. This implies that the eigenvalues of A are positive and depends on x and on ∇u. Hence, $\mathcal{M}$ is locally uniformly elliptic. Furthermore, if Ω is bounded and $u \in \mathscr{C}^1(\overline{\Omega})$, then $\mathcal{M}$ is uniformly elliptic in $\overline{\Omega}$ (see [19] for more details).

It has been proved in chronological order by Finn [9], Jenkins-Serrin [14] and Serrin [18], that the very well known Serrin condition is a necessary condition for the solvability of the Dirichlet problem for equation (1) in bounded domains of $\mathbb{R}^n$.

Dirichlet problems for equations whose solutions describe hypersurfaces of prescribed mean curvature has been also studied outside of the Euclidean space. Several works have considered a Serrin type condition that provides some existence theorem in their respective context (see [1], [2], [7], [8], [13], [16], [17] and [19] as examples). However, non-existence theorem has been only investigated in a few cases that we summarize below.

For instance, P.-A Nitsche [17] was concerned with graph-like prescribed mean curvature hypersurfaces in hyperbolic space $\mathbb{H}^{n+1}$. In the half-space setting, he studied radial graphs over the totally geodesic hypersurface $S = \{x \in \mathbb{R}^{n+1}; (x_0)^2 + \cdots + (x_n)^2 = 1\}$. He established an existence result if Ω is a bounded domain of S of class $\mathscr{C}^{2,\alpha}$ and $H \in \mathscr{C}^1(\overline{\Omega})$ is a function satisfying $\sup_{\overline{\Omega}} |H| \leq 1$ and $|H(y)| < \mathcal{H}_C(y)$ everywhere on $\partial\Omega$, where $\mathcal{H}_C$ denotes the hyperbolic mean curvature of the cylinder C over $\partial\Omega$. Furthermore he showed the existence of smooth boundary data such that no solution exists in case of $|H(y)| > \mathcal{H}_C(y)$ for some $y \in \partial\Omega$ under the assumption that H has a sign. We observe that his results does not provide Serrin type solvability criterion.

Also, E. M. Guio-R. Sa Earp [12, 13] considered a bounded domain Ω contained in a vertical totally geodesic hyperplane P of $\mathbb{H}^{n+1}$ and studied the Dirichlet problem for the mean curvature equation for horizontal graphs over Ω, that is, hypersurfaces which intersect at most only once the horizontal horocycles orthogonal to Ω. They considered the hyperbolic cylinder C generated by horocycles cutting ortogonally P along the boundary of Ω and the Serrin condition, $\mathcal{H}_C(y) \geq |H(y)|\ \forall\ y \in \partial\Omega$. They obtained a Serrin type solvability criterion for prescribed mean curvature $H = H(x)$ and also proved a sharp solvability criterion for constant H.

Finally, M. Telichevesky [20, Th. 6 p. 246] proved that if M is a Hadamard manifold having -1 as an upper bound for the sectional curvature, then mean convexity is a necessary condition for the existence of a vertical minimal graphs in $M\times\mathbb{R}$ over a domain Ω of M possibly unbounded. This result combined with an existence result of Aiolfi-Ripoll-Soret [1, Th. 1 p. 72] gives sharp solvability criterion for the minimal hypersurface equation in bounded domains of M.

To the best of our knowledge, no other Serrin-type solvability criterion has been proved outside of the Euclidean setting.

In this paper we generalize the aforementioned non-existence result in the $M \times \mathbb{R}$ context. More precisely, we prove the following[1]:

Theorem 1 (main theorem). *Let $\Omega \subset M$ be a bounded domain whose boundary is of class $\mathscr{C}^2$. Let $H \in \mathscr{C}^0(\overline{\Omega} \times \mathbb{R})$ be a function either non-positive or non-negative and non-decreasing in the variable z. Let us assume that there exists $y_0 \in \partial\Omega$ such that*

$$(n-1)\mathcal{H}_{\partial\Omega}(y_0) < n \sup_{z\in\mathbb{R}} |H(y_0, z)|\,.$$

Suppose also that $\operatorname{cut}(y_0)\cap\Omega = \emptyset$. Furthermore, assume that the radial curvature over the radial geodesics issuing from y_0 and intersecting Ω is bounded above by K_0, where

(a) $K_0 \leq 0$, *or*

(b) $K_0 > 0$ *and* $\operatorname{dist}(y_0, x) < \dfrac{\pi}{2\sqrt{K_0}}$ *for all* $x \in \overline{\Omega}$.

Then there exists $\varphi \in \mathscr{C}^\infty(\overline{\Omega})$ such that there is no $u \in \mathscr{C}^0(\overline{\Omega})\cap\mathscr{C}^2(\Omega)$ satisfying equation (1) *with $u = \varphi$ in $\partial\Omega$.*

The statement ensures that the *strong Serrin condition*

$$(n-1)\mathcal{H}_{\partial\Omega}(y) \geq n \sup_{z\in\mathbb{R}} |H(y,z)|\ \ \forall\ y \in \partial\Omega \tag{3}$$

is a necessary condition for the solvability of the Dirichlet problem for equation (1).

Some direct consequences inferred from our main non-existence theorem are stated as follows.

Corollary 2. *Let M be a Cartan-Hadamard manifold and $\Omega \subset M$ a bounded domain whose boundary is of class $\mathscr{C}^2$. Let $H \in \mathscr{C}^0(\overline{\Omega}\times\mathbb{R})$ be a function either*

[1] For the definition of radial curvature we refer that from Greene-Wu [11, p. 5]. We denote by $\operatorname{cut}(y_0)$ the *cut locus* of y_0.

non-negative or non-positive and non-decreasing in the variable z. Suppose there exists $y_0 \in \partial\Omega$ such that

$$(n-1)\mathcal{H}_{\partial\Omega}(y_0) < n \sup_{z\in\mathbb{R}} |H(y_0, z)| .$$

Then there exists $\varphi \in \mathscr{C}^\infty(\overline{\Omega})$ such that there is no $u \in \mathscr{C}^0(\overline{\Omega}) \cap \mathscr{C}^2(\Omega)$ satisfying equation (1) with $u = \varphi$ in $\partial\Omega$.

Corollary 3. *Let M be a simply connected and compact manifold whose sectional curvature satisfies $\frac{1}{4}K_0 < K \leq K_0$ for a positive constant K_0. Let $\Omega \subset M$ be a domain with $\operatorname{diam}(\Omega) < \frac{\pi}{2\sqrt{K_0}}$ and whose boundary is of class $\mathscr{C}^2$. Let $H \in \mathscr{C}^0(\overline{\Omega} \times \mathbb{R})$ be a function either non-negative or non-positive and non-decreasing in the variable z. Suppose there exists $y_0 \in \partial\Omega$ such that*

$$(n-1)\mathcal{H}_{\partial\Omega}(y_0) < n \sup_{z\in\mathbb{R}} |H(y_0, z)| .$$

Then there exists $\varphi \in \mathscr{C}^\infty(\overline{\Omega})$ such that there is no $u \in \mathscr{C}^0(\overline{\Omega}) \cap \mathscr{C}^2(\Omega)$ satisfying equation (1) with $u = \varphi$ in $\partial\Omega$.

We remark that the assumption in the above statement guarantees that the injectivity radius of M is greater than $\frac{\pi}{2\sqrt{K_0}}$.

2 Sharp solvability criteria

We now want to highlight Serrin type solvability criteria derived from the combination of our non-existence results with existence results obtained by others [19, 1] and by the authors [3, 4].

Firstly, we observe that the combination of corollary 2 with the existence theorem from Aiolfi-Ripoll-Soret [1, Th. 1 p. 72] for the minimal case shows that the sharp solvability criterion of Jenkins-Serrin [14, Th. 1 p. 171] also holds in Cartan-Hadamard manifolds:

Theorem 4 (Sharp Jenkins-Serrin-type solvability criterion). *Let M be a Cartan-Hadamard manifold and $\Omega \subset M$ a bounded domain whose boundary is of class $\mathscr{C}^{2,\alpha}$ for some $\alpha \in (0, 1)$. Then the Dirichlet problem for equation $\mathcal{M}u = 0$ in Ω has a unique solution for arbitrary continuous boundary data if, and only if, Ω is mean convex.*

Secondly, combining corollary 3 with the existence result of Spruck [19, Th. 1.4 p. 787] we infer the following:

Theorem 5 (Sharp Serrin-type solvability criterion). *Let M be a simply connected and compact manifold whose sectional curvature satisfies $\frac{1}{4}K_0 < K \leq K_0$ for a positive constant K_0. Let $\Omega \subset M$ be a domain with $\operatorname{diam}(\Omega) < \frac{\pi}{2\sqrt{K_0}}$ and whose boundary is of class $\mathscr{C}^{2,\alpha}$ for some $\alpha \in (0, 1)$. Then the Dirichlet problem for equation (1) in Ω has a unique solution for every constant H and arbitrary continuous boundary data if, and only if, $(n-1)\mathcal{H}_{\partial\Omega} \geq n |H|$.*

Notice that it was not established in every Cartan-Hadamard manifold a sharp Serrin type result [18, p. 416] for arbitrary constant H. For example, if $M = \mathbb{H}^n$, it follows from the existence result of Spruck [19, Th. 1.4 p. 787] that the Serrin condition is a sufficient condition if $H \geq \frac{n-1}{n}$. In the opposite case $0 < H < \frac{n-1}{n}$, Spruck noted that it was possible to establish an existence result if the strict inequality $(n-1)\mathcal{H}_{\partial\Omega} > nH$ holds. He used the entire graphs of constant mean curvature $\frac{n-1}{n}$ in $\mathbb{H}^n \times \mathbb{R}$ as barriers (see [5] for explicit formulas). However, this restriction over the Serrin condition in the last case does not allow to establish Serrin type solvability criterion for every constant H directly from the existence result of Spruck [19, Th. 5.4 p. 797] when the ambient is the hyperbolic space.

We have established an existence result [4, 3, Th. 4.4 p. 51] for prescribed $H \in \mathscr{C}^{1,\alpha}(\overline{\Omega} \times \mathbb{R})$ which extends the existence result of Spruck. We have the following Serrin type solvability criterion:

Theorem 6 (Serrin type solvability criterion 1). *Let $\Omega \subset \mathbb{H}^n$ be a bounded domain with $\partial\Omega$ of class $\mathscr{C}^{2,\alpha}$ for some $\alpha \in (0,1)$. Let $H \in \mathscr{C}^{1,\alpha}(\overline{\Omega} \times \mathbb{R})$ be a function satisfying $\partial_z H \geq 0$ e $0 \leq H \leq \frac{n-1}{n}$ em $\Omega \times \mathbb{R}$. Then the Dirichlet problem for equation* (1) *has a unique solution $u \in \mathscr{C}^{2,\alpha}(\overline{\Omega})$ for every $\varphi \in \mathscr{C}^{2,\alpha}(\overline{\Omega})$ if, and only if, the strong Serrin condition* (3) *holds.*

By combining the existence result of Spruck [19, Th. 1.4 p. 787] with corollary 2, and putting together theorem 6, we deduce that the sharp solvability criterion of Serrin [18, p. 416] for arbitrary constant H also holds in the $\mathscr{C}^{2,\alpha}$ class if we replace $\mathbb{R}^n$ for $\mathbb{H}^n$:

Theorem 7 (Sharp Serrin type solvability criterion). *Let $\Omega \subset \mathbb{H}^n$ be a bounded domain whose boundary is of class $\mathscr{C}^{2,\alpha}$. Then the Dirichlet problem for equation* (1) *has a unique solution for every constant H and for arbitrary continuous boundary data if, and only if, $(n-1)\mathcal{H}_{\partial\Omega}(y) \geq n\,|H|$.*

We have also proved [4, 3, Th. 4.1 p. 40] a generalization of an existence result of Spruck [19, Th. 1.4 p. 787] for constant mean curvature. That is, under certain regularity for $\partial\Omega$ and the boundary data φ, if the function H satisfies

$$\mathrm{Ricc}_x \geq n \sup_{z\in\mathbb{R}} \|\nabla_x H(x,z)\| - \frac{n^2}{n-1} \inf_{z\in\mathbb{R}} \left(H(x,z)\right)^2, \ \forall\, x \in \Omega, \tag{4}$$

in addition to the strong Serrin condition (3), then the Dirichlet problem for equation (1) is solvable for arbitrary boundary data sufficient smooth. This result in combination with corollary 2 yields the following generalization in the $\mathscr{C}^{2,\alpha}$ class of a theorem of Serrin [18, Th. p. 484] in the Euclidean space:

Theorem 8 (Serrin type solvability criterion 2). *Let M be a Cartan-Hadamard manifold and $\Omega \subset M$ a bounded domain whose boundary is of class $\mathscr{C}^{2,\alpha}$ for some $\alpha \in (0,1)$. Suppose that $H \in \mathscr{C}^{1,\alpha}(\overline{\Omega} \times \mathbb{R})$ is either non-negative* *or non-positive in* $\overline{\Omega}\times\mathbb{R}$*,* $\partial_z H \geq 0$ *and*

$$\mathrm{Ricc}_x \geq n \sup_{z\in\mathbb{R}} \|\nabla_x H(x,z)\| - \frac{n^2}{n-1} \inf_{z\in\mathbb{R}} \left(H(x,z)\right)^2, \ \forall\ x\in\Omega.$$

Then the Dirichlet problem for equation (1) *has a unique solution* $u\in\mathscr{C}^{2,\alpha}(\overline{\Omega})$ *for every* $\varphi\in\mathscr{C}^{2,\alpha}(\overline{\Omega})$ *if, and only if, the strong Serrin condition* (3) *holds.*

Finally, using corollary 3 we also obtain:

Theorem 9 (Serrin type solvability criterion 3). *Let* M *be a complete and compact manifold whose sectional curvature satisfies* $\frac{1}{4}K_0 < K \leq K_0$ *for a positive constant* K_0*. Let* $\Omega\subset M$ *be a domain with* $\mathrm{diam}(\Omega) < \frac{\pi}{2\sqrt{K_0}}$ *and whose boundary is of class* $\mathscr{C}^{2,\alpha}$ *for some* $\alpha\in(0,1)$*. Suppose that* $H\in\mathscr{C}^{1,\alpha}(\overline{\Omega}\times\mathbb{R})$ *is either non-negative or non-positive in* $\overline{\Omega}\times\mathbb{R}$*,* $\partial_z H\geq 0$ *and*

$$\mathrm{Ricc}_x \geq n \sup_{z\in\mathbb{R}} \|\nabla_x H(x,z)\| - \frac{n^2}{n-1} \inf_{z\in\mathbb{R}} \left(H(x,z)\right)^2, \ \forall\ x\in\Omega.$$

Then the Dirichlet problem for equation (1) *has a unique solution* $u\in\mathscr{C}^{2,\alpha}(\overline{\Omega})$ *for every* $\varphi\in\mathscr{C}^{2,\alpha}(\overline{\Omega})$ *if, and only if, the strong Serrin condition* (3) *holds.*

3 Proof of the main non-existence theorem

The proof of theorem 1 is based in two results that will be proved in the sequel. The following fundamental proposition can trace its roots back to the work of Finn [9, Lemma p. 139] when he established the theorem ensuring the non-existence of solutions for Dirichlet problems for the minimal surface equation in non-convex domain of $\mathbb{R}^2$. His lemma was extended by Jenkins-Serrin [14, Prop. III p. 182] for the minimal hypersurface equation in $\mathbb{R}^n$, and subsequently by Serrin [18, Th. 1 p. 459] for quasilinear elliptic operators (see also [10, Th. 14.10 p. 347]). Afterward M. Telichevesky [20, Lemma. 11 p. 250] extended the result for the minimal vertical equation in $M\times\mathbb{R}$. We will use some of the ideas of these works.

Proposition 10. *Let* $\Omega\in M$ *a bounded domain. Let* Γ' *be a relative open portion of* $\partial\Omega$ *of class* $\mathscr{C}^1$*. Let* $H(x,z)\in\mathscr{C}^0(\overline{\Omega}\times\mathbb{R})$ *be a function non-decreasing in the variable* z*. Let* $u\in\mathscr{C}^2(\Omega)\cap\mathscr{C}^1(\Omega\cup\Gamma')\cap\mathscr{C}^0(\overline{\Omega})$ *and* $v\in\mathscr{C}^2(\Omega)\cap\mathscr{C}^0(\overline{\Omega})$ *satisfying*

$$\begin{cases} \mathfrak{Q}u \geq \mathfrak{Q}v & \text{in } \Omega,\\ u\leq v & \text{in } \partial\Omega\setminus\Gamma',\\ \frac{\partial v}{\partial N} = -\infty & \text{in } \Gamma', \end{cases}$$

where N *is the inner normal to* Γ'*. Under these conditions* $u\leq v$ *in* Γ'*. Therefore* $u\leq v$ *em* Ω*.*

Proof. By way of contradiction, suppose that $m = \max_{\Gamma'}(u - v) > 0$. Hence, $u \le v + m$ em Γ'. Then $u \le v + m$ in $\partial\Omega$ since $u \le v$ in $\partial\Omega \setminus \Gamma'$ by hypotheses. In view of the function H is non-decreasing in z and $m > 0$, we have

$$\mathfrak{Q}(v + m) = \mathcal{M}(v + m) - nH(x, v + m) \le \mathcal{M}v - nH(x, v) = \mathfrak{Q}v \le \mathfrak{Q}u.$$

As a consequence of the maximum principle (see [10, Th. 10.1 p. 263]) $u \le v+m$ in Ω. Let $y_0 \in \Gamma'$ be such that $m = u(y_0) - v(y_0)$. Let $\gamma_{y_0} = \exp_{y_0}(tN_{y_0})$, for $t > 0$ near 0. Then

$$u(\gamma_{y_0}(t)) - u(y_0) \le (v\,(\gamma_{y_0}(t)) + m) - (v(y_0) + m) = v(\gamma_{y_0}(t)) - v(y_0).$$

Dividing the expression by t and passing to the limit as t goes to zero it follows that $\frac{\partial u}{\partial N} \le -\infty$. This is a contradiction since $u \in \mathscr{C}^1(\Gamma')$, hence, $u \le v$ in Γ'. $\square$

The next lemma plays a fundamental role in this paper. In this lemma we establish a height a priori estimate for solutions of equation $\mathcal{M}u = nH(x, u)$ in Ω in those points of $\partial\Omega$ on which the strong Serrin condition (3) fails.

Lemma 11. *Let $\Omega \subset M$ be a bounded domain whose boundary is of class $\mathscr{C}^2$. Let $H \in \mathscr{C}^0(\overline{\Omega}\times\mathbb{R})$ be a non-negative function and non-decreasing in the variable z, and $u \in \mathscr{C}^2(\Omega) \cap \mathscr{C}^0(\overline{\Omega})$ satisfying $\mathcal{M}u = nH(x, u)$. Let us assume that there exists $y_0 \in \partial\Omega$ such that*

$$(n - 1)\mathcal{H}_{\partial\Omega}(y_0) < nH(y_0, k) \tag{5}$$

for some $k \in \mathbb{R}$. Suppose also that $\mathrm{cut}(y_0) \cap \Omega = \emptyset$. Furthermore, assume that the radial curvature over the radial geodesics issuing from y_0 and intersecting Ω is bounded above by K_0, where

- *(a) $K_0 \le 0$, or*
- *(b) $K_0 > 0$ and $\mathrm{dist}(y_0, x) < \dfrac{\pi}{2\sqrt{K_0}}$ for all $x \in \overline{\Omega}$.*

Then for each $\varepsilon > 0$ there exists $a > 0$ depending only on ε, $\mathcal{H}_{\partial\Omega}(y_0)$, the geometry of Ω and the modulus of continuity of $H(x, k)$ in y_0, such that

$$u(y_0) < \max\left\{k, \sup_{\partial\Omega\setminus B_a(y_0)} u\right\} + \varepsilon. \tag{6}$$

Proof. We proceed the proof in two steps. Firstly, we will find an estimate for $u(y_0)$ depending on k and $\sup\limits_{\partial B_a(y_0)\cap\Omega} u$ for some a that does not depend on u. Secondly, we will get an upper bound for $\sup\limits_{\partial B_a(y_0)\cap\Omega} u$ in terms of $\sup\limits_{\partial\Omega\setminus B_a(y_0)} u$.

Step 1.

First of all note that from (5) there exists $\nu > 0$ such that

$$(n-1)\mathcal{H}_{\partial\Omega}(y_0) < nH(y_0,k) - 4\nu. \tag{7}$$

Let $R_1 > 0$ be such that $\partial B_{R_1}(y_0) \cap \Omega$ is connected and

$$|H(x,k) - H(y_0,k)| < \frac{\nu}{n},\ \forall\ x \in B_{R_1}(y_0) \cap \Omega. \tag{8}$$

Note also that we can construct an embedded and oriented hypersurface S, tangent to $\partial\Omega$ at y_0 and whose mean curvature with respect to the normal pointing inwards Ω at y_0 satisfies

$$\mathcal{H}_{\partial\Omega}(y_0) < \mathcal{H}_S(y_0) < \mathcal{H}_{\partial\Omega}(y_0) + \frac{\nu}{(n-1)}. \tag{9}$$

We know that for some $\tau > 0$ the map

$$\begin{array}{rccc} \Phi_t: & S & \longrightarrow & \Omega \\ & y & \longmapsto & \exp^{\perp}(y, tN_y) \end{array}$$

is a diffeomorphism for each $0 \le t < \tau$, and so $S_t := \Phi_t(S)$ is parallel to S.

Let us consider the distance function $d(x) = \mathrm{dist}(x,S)$. Let $0 < R_2 < \min\{\tau, R_1\}$ be such that

$$|\Delta d(x) - \Delta d(y_0)| < \nu\ \forall\ x \in B_{R_2}(y_0) \cap \Omega. \tag{10}$$

We now fix $a < R_2$. For $0 < \epsilon < a$ we set

$$\Omega_\epsilon = \{x \in B_a(y_0) \cap \Omega; d(x) > \epsilon\}.$$

We choose $\phi \in \mathscr{C}^2(\epsilon, a)$ satisfying

P1. $\phi(a) = 0$, P2. $\phi' \le 0$, P3. $\phi'' \ge 0$, P4. $\phi'(\epsilon) = -\infty$.

We also require that $\phi'^3\nu + \phi''$=0 in (ϵ, a). Let $v = \max\left\{k, \sup_{\partial B_a(y_0)\cap\Omega} u\right\} + \phi \circ d$. So, $v \ge u$ in $\partial\Omega_\epsilon \setminus S_\epsilon$. In addition, if N is the normal to S_ϵ inwards Ω_ϵ and $x \in S_\epsilon \cap B_a(y_0)$, then

$$\frac{\partial v}{\partial N}(x) = \langle \nabla v(x), N\rangle = \langle \phi'(d(x))\nabla d(x), \nabla d(x)\rangle = \phi'(\epsilon) = -\infty.$$

Let us fix $x \in \Omega_\epsilon$. A straightforward computation yields

$$\mathfrak{Q}v = \frac{\phi'}{(1+\phi'^2)^{1/2}}\Delta d + \frac{\phi''}{(1+\phi'^2)^{3/2}} - nH(x,v).$$

Since $v \ge k$ and H is non-decreasing in z it follows that $H(x,v) \ge H(x,k)$. Hence,

$$\mathfrak{Q}v \le \frac{\phi'}{(1+\phi'^2)^{1/2}}\Delta d + \frac{\phi''}{(1+\phi'^2)^{3/2}} - nH(x,k).$$

By means of the properties of ϕ we have

$$\frac{\phi'}{(1+\phi'^2)^{1/2}} > -1,$$

and by the assumption on the sign of H we obtain

$$-nH(x,k) < nH(x,k)\frac{\phi'}{(1+\phi'^2)^{1/2}}.$$

Therefore,

$$\mathfrak{Q}v < \frac{\phi'}{(1+\phi'^2)^{1/2}}\left(\Delta d(x) + nH(x,k)\right) + \frac{\phi''}{(1+\phi'^2)^{3/2}}. \tag{11}$$

Furthermore,

$$\begin{aligned}\Delta d(x) + nH(x,k) =& \Delta d(x) - \Delta d(y_0) + \Delta d(y_0) + nH(x,k)\\ >& -\nu - (n-1)\mathcal{H}_S(y_0) + nH(x,k) && \text{(a)}\\ >& -2\nu - (n-1)\mathcal{H}_{\partial\Omega}(y_0) + nH(x,k) && \text{(b)}\\ >& 2\nu - nH(y_0,k) + nH(x,k) && \text{(c)}\\ >& \nu, && \text{(d)}\end{aligned}$$

where (a) follows directly from (10), (b) from (9), (c) from (7) and (d) from (8). Using this estimate on (11) we have

$$\begin{aligned}\mathfrak{Q}v <& \frac{\phi'}{(1+\phi'^2)^{1/2}}\nu + \frac{\phi''}{(1+\phi'^2)^{3/2}}\\ =& \frac{1}{(1+\phi'^2)^{3/2}}\left(\phi'(1+\phi'^2)\nu + \phi''\right)\\ <& \frac{1}{(1+\phi'^2)^{3/2}}\left(\phi'^3\nu + \phi''\right).\end{aligned}$$

Let us now define ϕ explicitly by[2]

$$\phi(t) = \sqrt{\frac{2}{\nu}}\left((a-\epsilon)^{1/2} - (t-\epsilon)^{1/2}\right). \tag{12}$$

We observe that ϕ satisfies P1–P4 and that $\phi'^3\nu + \phi'' = 0$ for each $\epsilon < t < a$. Then, $\mathfrak{Q}v < 0$ in Ω_ϵ. From proposition 10 we deduce that

$$u \le v = \max\left\{k, \sup_{\partial B_a(y_0)\cap\Omega} u\right\} + \phi(\epsilon) \quad \text{in} \quad S_\epsilon \cap B_a(y_0).$$

In particular,

$$u(\gamma_{y_0}(\epsilon)) \le \max\left\{k, \sup_{\partial B_a(y_0)\cap\Omega} u\right\} + \sqrt{\frac{2}{\nu}}\left((a-\epsilon)^{1/2}\right),$$

[2] See also [10, §14.4] and [12, Th. 4.1 p. 40].

where $\gamma_{y_0}(\epsilon) = \exp_{y_0}(\epsilon N_{y_0})$. Since this estimate holds for each $0 < \epsilon < a$, we can pass to the limit as ϵ goes to zero to obtain

$$u(y_0) \leq \max\left\{k, \sup_{\partial B_a(y_0)\cap\Omega} u\right\} + \sqrt{\frac{2a}{\nu}}. \tag{13}$$

Step 2.
Let $\rho(x) = \mathrm{dist}(x, y_0)$ for $x \in \Omega' = \Omega \setminus B_a(y_0)$ and $\delta = \mathrm{diam}(\Omega)$. Choose $\psi \in \mathscr{C}^2(a,\delta)$ satisfying

P5. $\psi(\delta) = 0$, P6. $\psi' \leq 0$, P7. $\psi'' \geq 0$, P8. $\psi'(a) = -\infty$,

We also need that $\frac{c\psi'^3}{t} + \psi'' \leq 0$ in (a,δ), where c is a constant to be choose later on. Let $w = \sup_{\partial\Omega\setminus B_a(y_0)} u + \psi \circ \rho$. We remind that $\rho \in \mathscr{C}^2(M \setminus (\mathrm{cut}(y_0) \cup \{y_0\}))$, so $w \in \mathscr{C}^2(\Omega \setminus B_a(y_0))$. The idea is to use proposition 10 again. We note that $w \geq u$ in $\partial\Omega \setminus B_a(y_0)$. Also, if N is the normal to $\partial B_a(y_0) \cap \Omega$ inwards Ω', we have for each $x \in \partial B_a(y_0) \cap \Omega$ that

$$\frac{\partial w}{\partial N}(x) = \langle \nabla w(x), N\rangle = \langle \psi'(\rho(x))\nabla\rho(x), \nabla\rho(x)\rangle = \psi'(a) = -\infty.$$

For w we have

$$\mathfrak{Q}w = \frac{\psi'}{(1+\psi'^2)^{1/2}}\Delta\rho + \frac{\psi''}{(1+\psi'^2)^{3/2}} - nH(x,w).$$

Since $H \geq 0$, it follows

$$\mathfrak{Q}w \leq \frac{\psi'}{(1+\psi'^2)^{1/2}}\Delta\rho + \frac{\psi''}{(1+\psi'^2)^{3/2}}.$$

In any of the hypothesis (a) or (b), the radial geodesics issuing from y_0 and intercepting Ω do not contain conjugate points to y_0 (see [15, Th. 6.5.6 p. 151], [6, Th. p. 107]). Then the Laplacian comparison theorem [11, Th. A p. 19] can be use to estimate $\Delta\rho$.

Under the hypothesis (a) we compare M with $\mathbb{R}^n$ to obtain

$$\Delta\rho(x) \geq \frac{n-1}{\rho(x)}.$$

Under the hypothesis (b) we compare M with the sphere $S^n_{K_0}$ of sectional curvature $K_0 > 0$. In this case we have

$$\Delta\rho(x) \geq (n-1)\sqrt{K_0}\cot\left(\sqrt{K_0}\,\rho(x)\right).$$

From the second assumption on (b) there also exists $0 < \kappa < \frac{\pi}{2\sqrt{K_0}}$ such that $\mathrm{dist}(x,y_0) \leq \frac{\pi}{2\sqrt{K_0}} - \kappa$, for each $x \in \overline{\Omega}$. Thus, for each $x \in \Omega \setminus B_a(y_0)$, there exists a unique normal minimizing geodesic β such that $\beta(0) = y_0$ and $\beta(t_0) = x$, where $t_0 \leq \dfrac{\pi}{2\sqrt{K_0}} - \kappa$. Let's define the function $\xi(t) = \sqrt{K_0}\, t \cot\left(\sqrt{K_0}\, t\right)$ for $t > 0$. We note that ξ is decreasing and $\xi\left(\frac{\pi}{2\sqrt{K_0}}\right) = 0$. Then,

$$\xi(t) \geq \xi\left(\frac{\pi}{2\sqrt{K_0}} - \kappa\right) > 0,\ \forall t \in \left(0, \frac{\pi}{2\sqrt{K_0}} - \kappa\right].$$

Consequently,

$$\rho(x)\Delta\rho(x) \geq (n-1)C,$$

where

$$C = \sqrt{K_0}\left(\frac{\pi}{2\sqrt{K_0}} - \kappa\right)\cot\left(\sqrt{K_0}\left(\frac{\pi}{2\sqrt{K_0}} - \kappa\right)\right) > 0.$$

Thus $\Delta\rho(x) \geq \frac{c}{\rho}$, where $c = n-1$ in the case (a) and $c = (n-1)C$ in the case (b).

Then, we have

$$\begin{aligned}\mathfrak{Q}w \leq & \frac{\psi'}{(1+\psi'^2)^{1/2}} \cdot \frac{c}{\rho} + \frac{\psi''}{(1+\psi'^2)^{3/2}} \\ = & \frac{1}{(1+\psi'^2)^{3/2}}\left(\frac{c}{\rho}\psi'(1+\psi'^2) + \psi''\right) \\ < & \frac{1}{(1+\psi'^2)^{3/2}}\left(\frac{c}{\rho}\psi'^3 + \psi''\right).\end{aligned}$$

Let us define ψ as [3]

$$\psi(t) = \left(\frac{2}{c}\right)^{1/2}\int_t^\delta \left(\log\frac{r}{a}\right)^{-1/2} dr. \tag{14}$$

Such a function satisfies P5–P8, and also $\dfrac{c}{t}\psi'(t)^3 + \psi''(t) < 0$ for each $a < t < \delta$. Then, $\mathfrak{Q}w < 0$ em Ω'.

From proposition 10 we can conclude that $u \leq w$ in $\partial B_a(y_0) \cap \Omega$, where

$$\sup_{\partial B_a(y_0)\cap\Omega} u \leq \sup_{\partial\Omega\setminus B_a(y_0)} u + \psi(a). \tag{15}$$

We observe that, in fact, this estimate holds for each a such that $\partial B_a(y_0) \cap \Omega$ is connected.

We use (15) in (13) from step 1, so

$$u(y_0) \leq \max\left\{k, \sup_{\partial\Omega\setminus B_a(y_0)} u\right\} + \psi(a) + \sqrt{\frac{2a}{\nu}}.$$

It is easy to see that $\lim\limits_{a\to 0}\psi(a) = 0$. Hence, for each $\varepsilon > 0$, a can be choose small enough to satisfy

$$\psi(a) + \sqrt{\frac{2a}{\nu}} < \varepsilon. \qquad \square$$

[3] See also [10, §14.4]

Remark 12. *In the case where $H = H(x)$,*

$$u(y_0) < \sup_{\partial\Omega \setminus B_a(y_0)} u + \varepsilon,$$

where a is chosen as before.

At last we are able to prove theorem 1.

Proof of the main non-existence theorem. Obviously we can suppose that $H \geq 0$. Then,

$$(n-1)\mathcal{H}_{\partial\Omega}(y_0) < nH(y_0, k)$$

for some $k \in \mathbb{R}$ since H is non-decreasing in z. Let $\varepsilon > 0$ and $\varphi \in \mathscr{C}^\infty(\overline{\Omega})$ such that $\varphi = k$ in $\partial\Omega \setminus B_a(y_0)$ and $\varphi(y_0) = k + \varepsilon$. Hence, no solution of equation (1) in Ω could have φ as boundary values because such a function does not satisfy the estimate (6). □

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