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SOME PROPERTIES OF HYPERSURFACES OF PRESCRIBED MEAN CURVATURE IN $\mathbf{H}^{n+1}$

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1. Introduction

In this paper we study the behaviour of graphs with prescribed mean curvature in hyperbolic space of dimension n+1.

There are different possibilities in choosing coordinates to define a graph in hyperbolic space and the form of the mean curvature equation obtained depends on this choice.

We consider $\mathbb{H}^{n+1}$ in the half-space model i.e.

$$\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_{n+1} > 0\}$$

with the metric $ds^2 = {x_{n+1}}^{-2}(dx_1^2 + \ldots + dx_{n+1}^2)$.

Above all, we deal with the following system of coordinates.

1. Let Ω be a domain on a totally geodesic hyperplane $x_j = c$ $(j \leq n)$, and f a real function that at each $p \in \Omega$ associates a point on the horocycle passing by p and orthogonal to the hyperplane $\{x_j = c\}$.

This system of coordinates is treated in [BaS], where the following existence result is proved. Let Ω be a domain in a hyperplane whose boundary is a closed submanifold with principal curvatures greater than one and let $H : \overline{\Omega} \longrightarrow \mathbb{R}$ be a C^k function with $|H(x)| < 1$ for each $x \in \overline{\Omega}$; then there exists a C^{k+1} function on Ω that is zero on $\partial\Omega$, whose graph, in this system of coordinates, is a hypersurface of mean curvature H.
The paper is organized as follows.

In section 2 we introduce another system of coordinates and discuss some differences. Then we come back to system 1: in section 3 we prove a removable singularity theorem, in section 4 we prove a flux formula and two nice applications of it and in section 5 we give an estimate of the height of our graphs.

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2. The two equations

Consider the following system of coordinates.

2. Let Ω be a domain on a horosphere $x_{n+1} = c$, $c > 0$ and f a real function that at each $p \in \Omega$ associates a point on the geodesic passing by p and orthogonal to the horosphere.

In [RoS] Rosenberg and Spruck resolve the following Plateau problem. Given a constant $K \in (-1, 0)$ and a codimension one embedded submanifold Γ of the boundary at infinity of $\mathbb{H}^{n+1}$, there exists a hypersurface M of $\mathbb{H}^{n+1}$ with constant Gauss curvature K and asymptotic boundary Γ. An important part of their study is an existence theory for K-hypersurfaces which are graphs in this system of coordinates over a bounded domain in a horosphere; the desired M is constructed as the limit of such graphs.

Recently Rosenberg proved that if Γ is a codimension one, convex, compact embedded submanifold of a horosphere $\{x_{n+1} = c\}$ of $\mathbb{H}^{n+1}$ the only minimal hypersurface bounded by Γ is a graph in this system of coordinates. Furthermore, if Γ

is a codimension one, convex embedded submanifold of the asymptotic boundary of $\mathbb{H}^{n+1}$, he constructs a minimal graph M with asymptotic boundary Γ as a limit of such graphs (personal comunication).

We now write the mean curvature equations for both system 1 and 2 and discuss some differences.

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the tangent space to the hypersurface, $\mathbf{N}$ a normal unitary vector and $\overline{\nabla}$ the Riemannian connection of $\mathbb{H}^{n+1}$; we recall that the mean curvature vector of a hypersurface is $\mathbf{H} = \frac{1}{n} < \overline{\nabla}_{e_i} e_i, \mathbf{N} > \mathbf{N}$ and

it does not depend on the choice of $\mathbf{N}$, while the mean curvature function of a hypersurface is $H = \frac{1}{n} < \overline{\nabla}_{e_i} e_i, \mathbf{N} >$ and its sign depends on the choice of $\mathbf{N}$.

Let Ω be a domain in the plane $\{x_j = 0\}$ $(j \leq n)$, $f : \Omega \longrightarrow \mathbb{R}$ a C^2 function and $H : \Omega \longrightarrow \mathbb{R}$ a continuous function; let $\mathbf{N}$ be the unitary exterior normal vector to the graph i.e. $\mathbf{N} = x_{n+1} {W_f}^{-1}(-f_1, \ldots, 1, \ldots, -f_{n+1})$ where 1 is in the jth place, $f_i = \frac{\partial f}{\partial x_i}$ for $i = 1, \ldots, n+1$, $i \neq j$, $\nabla f = (f_1, \ldots, \hat{f}_j, \ldots, f_{n+1})$ and $W_f = \sqrt{1 + |\nabla f|^2}$.

If the graph of f (in one of the two senses) is a surface of mean curvature H with respect to the unitary exterior normal vector to the graph, then f satisfies one of the following equations.

1. $\operatorname{div}\left(\dfrac{\nabla f}{W_f}\right) = \dfrac{n}{x_{n+1}}\left(H(x) + \dfrac{f_{n+1}}{W_f}\right)$,

2. $\operatorname{div}\left(\dfrac{\nabla f}{W_f}\right) = \dfrac{n}{f}\left(H(x) - \dfrac{1}{W_f}\right)$,

where div is the divergence in $\mathbb{R}^n$.

Remark 2.1. The equations we have obtained are quasi-linear elliptic equations and they satisfy a general maximum principle [GT].

Remark 2.2. The first term of both equations is the mean curvature function of the graph in euclidean space; we denote it by $\widetilde{H}$.

We will prove that a solution of equation 1 in a pointed domain extends to the point; on the contrary here is an example that shows that a solution of equation 2 in a pointed domain doesn't extend necessarily to the point, at least if $|H| > 1$. Take a cylinder in hyperbolic space, for example the locus of points with equal hyperbolic distance from the x_3 axis i.e. $C = \{(x_1, x_2, x_3) \mid x_3 = (x_1^2 + x_2^2)\tan\theta\}$ and consider the part of it contained in the slab $0 < x_3 \leq 1$. It has mean curvature $H = -\frac{1}{2}((\sin\theta)^{-1} + \sin\theta) < -1$ with respect to the exterior normal unitary vector and it is a graph on $D^*_{\tan\theta}$ that doesn't extend to the puncture.

3. A removable singularity theorem

In this section we prove a removable singularity theorem that, in the case of euclidean space, is proved in [RS].

First we prove a lemma by using techniques developed in [CNS].

Lemma 3.1. *Let Ω be a compact domain in the plane $\{x_j = 0\}$ $(j \leq n)$, such that $\inf_{x\in\Omega} x_{n+1} > 0$. Let $H : \Omega \longrightarrow \mathbb{R}$ be a C^1 function and let $u : \Omega \longrightarrow \mathbb{R}$ be a C^3 function that satisfies the following partial differential equation in Ω*

$$\operatorname{div}\left(\frac{\nabla u}{W_u}\right) = \frac{n}{x_{n+1}}\left(H(x) + \frac{u_{n+1}}{W_u}\right)$$

Assume further that u is bounded in Ω and $|\nabla u|$ is bounded in $\partial\Omega$. Then $|\nabla u|$ is bounded in $\operatorname{int}(\Omega)$.

Proof. Let $j = 1$. To estimate $|\nabla u|$ in $\operatorname{int}(\Omega)$ we shall obtain a bound for $z = |\nabla u| \mathrm{e}^{Au}$ where A is a positive constant to be chosen later. If z achieves its maximum on $\partial\Omega$ then, by the estimates in the hypothesis, we are through. If it is not the case, z assumes its maximum at a point $x \in \operatorname{int}(\Omega)$. Up to a rotation of coordinates we can assume that $|\nabla u(x)| = u_2(x) > 0$, $u_k(x) = 0$, $k \geq 3$. As x is a point of maximum for z, it is a maximum for the function $\ln(z) = Au + \ln|\nabla u|$.

It follows that at x

$$\frac{u_{2k}}{u_2} + Au_k = 0, \quad k = 2, \ldots, n+1$$

so

$$u_{22} = -Au_2^2, \; u_{2k} = 0 \;\; k = 3, \ldots, n+1 \tag{1}$$

Further at x, we have $\frac{\partial}{\partial x_k}(u_2^{-1}u_{2k} + Au_k) \leq 0$ for $k = 2, \ldots, n+1$ and this gives

$$u_{222} \leq 2A^2u_2^3, \; u_{2kk} \leq -Au_2u_{kk} \;\; k = 3, \ldots, n+1 \tag{2}$$

We remark that u, $|\nabla u|$ and $\operatorname{div}(\frac{\nabla u}{W_u})$ are invariant by rotations in the plane $\{x_1 = 0\}$ but ∇u is not invariant, hence when we rotate coordinates as above we have to take care of the fact that the mean curvature equation changes. Let $O(n)$ be the matrix of the rotation and let $\alpha_2, \ldots, \alpha_{n+1}$ be the coeficients of the last line of $O(n)$ $(\alpha_k \leq 1, \; k = 2, \ldots, n+1)$; the mean curvature equation in the rotated coordinates (that we still denote by $(x_2, \ldots, x_{n+1})$) is

$$\operatorname{div}\left(\frac{\nabla u}{W_u}\right) = \frac{n}{\alpha_k x_k}\left(H(x) + \frac{\alpha_k u_k}{W_u}\right)$$

where summation convention is used.

Denote by $\Psi \in C^1(\Omega \times \mathbb{R} \times \mathbb{R}^2)$ the second term of the previous equation, then it is equivalent to

$$\sum_{i,j=2}^{n+1} a_{ij} u_{ij} = \Psi W_u^3 \tag{3}$$

where $a_{ij} = W_u^2 \delta_{ij} - u_i u_j$ for $i, j = 2, \ldots, n+1$.

By differentiating (3) with respect to x_2 and calculating at x we have

$$u_{222} + (1 + u_2^2) u_{2kk} + 2 u_2 u_{22} u_{kk} = 3 W_u u_2 u_{22} \Psi + W_u^3 \frac{\partial \Psi}{\partial x_2}$$

By substituting (1) and (2) in the obtained equation, we have at x

$$A^2 \left(\frac{u_2^3 (u_2^2 - 1)}{(u_2^2 + 1)^{\frac{5}{2}}} \right) + A \Psi u_2 W_u^{-2} \leq - \frac{\partial \Psi}{\partial x_2} \tag{4}$$

The derivative of Ψ with respect to x_2, calculated at x is

$$\frac{\partial \Psi}{\partial x_2} = - \frac{n \alpha_2}{x_{n+1}^2} \left(H + \frac{u_2 \alpha_2}{W_u} \right) + \frac{n}{x_{n+1}} \left(H_2 - \frac{A u_2^2 \alpha_2}{W_u^3} \right)$$

where H and H_2 are the values at x of the curvature function and its derivative respectively and, by abuse of notation, we denote by x_{n+1} the third coordinate before the rotation. Now, by substituting the value of $\frac{\partial \Psi}{\partial x_2}$ in (4) we obtain

$$A^2 \left(\frac{u_2^3 (u_2^2 - 1)}{(u_2^2 + 1)^{\frac{5}{2}}} \right) + \frac{n A H u_2}{x_{n+1} W_u^2} \leq \frac{n H \alpha_2}{x_{n+1}^2} - \frac{n H_2}{x_{n+1}} + \frac{n u_2 {\alpha_2}^2}{x_{n+1}^2 W_u} \tag{5}$$

We remark that the following inequality

$$\frac{u_2^3 (u_2^2 - 1)}{(u_2^2 + 1)^{\frac{5}{2}}} \leq \frac{1}{2} \tag{6}$$

yields a bound for u_2, and hence for $\max |\nabla u| \mathrm{e}^{Au}$.

By (5), inequality (6) is implied by

$$\frac{1}{A^2} \left(- \frac{n A H u_2}{x_{n+1} W_u^2} + \frac{n H \alpha_2}{x_{n+1}^2} - \frac{n H_2}{x_{n+1}} + \frac{n u_2 {\alpha_2}^2}{x_{n+1}^2 W_u} \right) \leq \frac{1}{2} \tag{7}$$

thus we are looking for a constant A such that (7) holds.

Now let $\lambda = \inf_{x\in\Omega} x$ and

$$K = \max\left\{\frac{n}{\lambda^2}\sup_{\Omega}|H| + \frac{n}{\lambda}\sup_{\Omega}|H_2| + \frac{n}{\lambda^2},\ \frac{n}{\lambda}\sup_{\Omega}|H|\right\}$$

By a straightforward computation we have that if $A > K + \sqrt{K^2 + 2K}$ then (7) and so (6) holds. We remark that A does not depend on u. □

Remark 3.2. In the case n=2 Leon Simon establishes interior *a priori* gradient estimates for solutions of equations of type 1, assuming *a priori* C^o-bounds. The reasons equation 1 satisfies the hypothesis of theorem $2'$ of [S] are:
1) the second term of equation 1 (b^* in [S]) does not depends on x_1, x_2 and u (considering u as an independent variable), it is bounded and its derivatives with respect to u_i, u_j (considering u_i, u_j as independent variables) are bounded;
2) the coefficients a_{ij} are the same as the coefficients of the mean curvature equation in $\mathbb{R}^3$.

Theorem 3.3. *Let Ω be a domain in the hyperplane $\{x_j = 0\}$ $(j \leq n)$, $p \in \Omega$, $H : \Omega \longrightarrow \mathbb{R}$ a C^1 function. Let $f : \Omega \setminus \{p\} \longrightarrow \mathbb{R}$ be a C^2 function such that the graph of f has mean curvature function H (with respect to the exterior normal vector). Then f extends C^2 to p.*

Proof. We will prove this theorem by the following three steps.

1. f is bounded.

2. There exists $R > 0$ such that the closed euclidean n-ball D_R with center at p and euclidean radius R is contained in Ω and the Dirichlet problem

$$\begin{cases} \operatorname{div}\left(\dfrac{\nabla u}{W_u}\right) = \dfrac{n}{x_{n+1}}\left(H(x) + \dfrac{u_{n+1}}{W_u}\right) & \text{in } D_R \\ u = f \quad \text{on } \partial D_R \end{cases}$$

has a solution in $C^{2,\alpha}(D_R)$.

3. $f = u$ on $D_R \setminus \{p\}$, i.e. u is an extension of f.

1^{st}STEP. We can assume that p is the origin; we look at the graph of f as a hypersurface in euclidean space with mean curvature $\widetilde{H}$; we want to use a generalized real Delaunay hypersurface as a barrier [HY].

As $\widetilde{H}$ is bounded in a compact set $K \subset \Omega$, there exists a constant $a > 0$ which depends on $\sup_K H$ and $\inf_K x_{n+1}$ such that $|\widetilde{H}| \leq \frac{n-1}{na}$; let $R < a$ be such that a ball D_R centered at the origin, of radius R is contained in K and let $\epsilon < R$. We consider a portion Del_ϵ of a Delaunay hypersurface such that:

(A) Del_ϵ has mean curvature $H_{Del_\epsilon} = \frac{n-1}{na}$;

(B) Del_ϵ is a graph over the annulus $A_\epsilon = D_R \setminus D_\epsilon$.

(C) Del_ϵ is tangent to the hypercylinder $\{x \in \mathbb{R}^{n+1} \mid x_1^2 + \ldots + \hat{x}_j^2 + \ldots + x_{n+1}^2 = \epsilon^2\}$.

f is bounded on A_ϵ, then by a translation in the direction of the x_j axis we can place Del_ϵ at the right side of the graph of f, to be disjoint from it. Now move Del_ϵ horizontally to the left in order to find a first point of contact between the two hypersurfaces. As the mean curvature vector of Del_ϵ points into the interior of the hypersurface, by the interior maximum principle (see [RS]) the first point of contact cannot be interior, hence it is on the boundary. If it is on the internal cylinder, by (C), the graph of f must be vertical there and this is a contradiction with the fact that f is a C^2 graph on the pointed disc. So the first point of contact is on the hypersphere $\{x_1^2 + \ldots + \hat{x}_j^2 + \ldots + x_{n+1}^2 = R^2,\ x_j = \sup_{\delta D_R} f\}$.

Letting $\epsilon \longrightarrow 0$, the height of Del_ϵ tends to $\frac{na}{n-1}$, thus we have

$$f < \sup_{\delta D_R} f + \frac{na}{n-1}$$

on D_R^*. By the same argument, using $Del_{-\epsilon}$ instead of Del_ϵ we obtain $\inf_{\delta D_R} f - \frac{na}{n-1} < f$ on D_R^*.

2^{nd}STEP. We remark that, by theorem 3.56 [A] the function f is $C^{2,\alpha}(\Omega \setminus \{p\})$, hence by theorem 13.8 [GT] we have only to prove that for each u that is a $C^{2,\alpha}(D_R)$ solution of the Dirichlet problem

$$\begin{cases} \operatorname{div}\left(\dfrac{\nabla u}{W_u}\right) = \dfrac{n\sigma}{x_{n+1}}\left(H(x) + \dfrac{u_{n+1}}{W_u}\right) & \text{in } D_R \\ u = \sigma f \quad \text{on } \partial D_R \end{cases}$$

where $\sigma \in [0,1]$, u is a *priori* bounded in $C^1(D_R)$.

It will be evident that it is sufficient to prove it for $\sigma = 1$.

The two Dirichlet problems

$$\begin{cases} \operatorname{div}\left(\dfrac{\nabla u}{W_u}\right) = \pm\dfrac{(n-1)}{na} & \text{in } D_R \\ u = f \quad \text{on } \partial D_R \end{cases}$$

are solvable in $C^{2,\alpha}(D_R)$ by theorem 16.11[GT]. So, we have a subsolution and a supersolution of our previous Dirichlet problem and, by the maximum principle, we have an *a priori* estimate for u in D_R and for $|\nabla u|$ on ∂D_R.

Now, we can apply lemma 3.1 to have an *a priori* estimate of $|\nabla u|$ in $\operatorname{int}(D_R)$.

<u>3rd STEP</u>. Let R be as above and let u be a solution of the Dirichlet problem of the preceding step ($\sigma = 1$).

Consider the form θ defined in D_R^* by

$$\begin{aligned} \theta = \frac{f-u}{x_{n+1}^n}\Bigg\{ &\sum_{i=1}^{j-1}(-1)^{i-1}\left(\frac{f_i}{W_f} - \frac{u_i}{W_u}\right) dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_{n+1} \\ &+ \sum_{i=j+1}^{n+1}(-1)^{i}\left(\frac{f_i}{W_f} - \frac{u_i}{W_u}\right) dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_{n+1} \Bigg\} \end{aligned}$$

The form θ is bounded on D_R^* because f is bounded and $\frac{|f_i|}{W_f}$, $\frac{|u_i|}{W_u} \leq 1$, $\forall i$.

Let $A_\epsilon = D_R \setminus D_\epsilon$; since $f \equiv u$ in ∂D_R we have

$$\int_{\partial A_\epsilon} \theta \longrightarrow 0 \quad \text{as} \quad \epsilon \longrightarrow 0.$$

By Stokes' theorem we have

$$\int_{\partial A_\epsilon} \theta = \int_{A_\epsilon} d\theta.$$

As

$$nH = (x_{n+1})^{n+1}\operatorname{div}\left(\frac{\nabla f}{x_{n+1}^n W_f}\right) = (x_{n+1})^{n+1}\operatorname{div}\left(\frac{\nabla u}{x_{n+1}^n W_u}\right)$$

we obtain that

$$d\theta = \frac{1}{x_{n+1}^n}\Big\{\sum_{i=1\ i\neq j}^{n+1}(f_i-u_i)\Big(\frac{f_i}{W_f}-\frac{u_i}{W_u}\Big)\Big\}dx_1\dots\widehat{dx_j}\dots dx_{n+1}$$
$$= \Big(\frac{W_f+W_u}{2x_{n+1}^n}\Big)\Big\{\sum_{i=1,i\neq j}^{n+1}\Big(\frac{f_i}{W_f}-\frac{u_i}{W_u}\Big)^2+\Big(\frac{1}{W_f}-\frac{1}{W_u}\Big)^2\Big\}dx_1\dots\widehat{dx_j}\dots dx_{n+1}$$

Thus $d\theta$ is non negative and it is 0 if and only if $f_i = u_i, \forall i$; letting $\epsilon \longrightarrow 0$ we obtain $\nabla f \equiv \nabla u$ and so $f \equiv u$ on D_R^*. □

Remark 3.4. In the second step of the preceeding theorem, we prove the existence of solutions of Dirichlet problem for equation 1 on small domains and with arbitrary C^2 boundary data.

4. A flux formula and applications

Let M be an immersed surface in $\mathbb{H}^3$ with constant mean curvature and such that its boundary ∂M is contained in the plane $\{x_2 = 0\}$. Let $\mathbf{H}$ be the mean curvature vector as defined in section 2, let $H = |\mathbf{H}|$ and orient M by the unit normal vector defined by $\mathbf{N} = H^{-1}\mathbf{H}$.

Let $\Omega \subset \{x_2 = 0\}$ be such that $\partial M = \partial\Omega$ and let $\mathbf{n}$ be the unitary interior conormal vector to ∂M; orient ∂M by the counterclockwise orientation of the plane x_1-x_3 and let $\mathbf{v}$ be the tangent vector to ∂M with this orientation.

Then let:

1. $\mathbf{N_\Omega} = (0, x_3, 0)$ if $< \mathbf{H}, \mathbf{v}\wedge\mathbf{n} >> 0$ i.e. if the orientation on M induces the counterclockwise orientation on ∂M;

2.$\mathbf{N_\Omega} = (0, -x_3, 0)$ if $< \mathbf{H}, \mathbf{v}\wedge\mathbf{n} >< 0$ i.e. if the orientation on M induces the clockwise orientation on ∂M.

¿From now on by $< , >$ we mean the scalar product in H^3 and for indicating the scalar product in $\mathbb{R}^3$ we will use a subscript; even for integrals, when we don't specify the form, we means integrals in hyperbolic space.

In the next theorem we prove a Flux Formula; the corresponding result in the euclidean case is proved in [BS]. For further considerations on this kind of formula, see for example [KKMS] and [R].

Theorem 4.1. *Let* $\mathbf{j} = (0, 1, 0)$, *then in the above notation*

$$\int_{\partial M} < \mathbf{j}, \mathbf{n} >= 2H \int_{\Omega} < \mathbf{N}_{\Omega}, \mathbf{j} >$$

Proof. Consider the 1 parameter family of surfaces $\{M_t\}$, obtained from M by the translations defined by $(p, t) \longrightarrow p + t\mathbf{j}$. Denote by $A(t)$ the area of the surface M_t. As $\mathbf{j}$ is a killing vector field, the area $A(t)$ is constant, so we have

$$0 = A'(0) = \int_M \mathrm{div}_M \mathbf{j}.$$

Let $\mathbf{j^T}$ be the component of $\mathbf{j}$ on the tangent space to M and $\mathbf{j^N} =< \mathbf{j}, \mathbf{N} > \mathbf{N}$ the normal component. We can write the previous formula as

$$\int_M \mathrm{div}_M(\mathbf{j^T}) + \int_M \mathrm{div}_M(\mathbf{j^N}) = 0$$

and by Stokes' theorem

$$-\int_{\partial M} < (\mathbf{j^T}), \mathbf{n} > + \int_M \mathrm{div}_M(\mathbf{j^N}) = 0 \tag{1}$$

where on ∂M we take the orientation induced by the orientation on M.

Let $X_1,\ X_2$ be an orthonormal basis of the tangent space to M, then

$$\begin{aligned}\mathrm{div}_M(\mathbf{j^N}) &= \sum_{i=1}^{2} < \nabla_{X_i}\mathbf{j^N}, X_i >= -\sum_{i=1}^{2} < \mathbf{j^N}, \nabla_{X_i} X_i > \\ &= -\sum_{i=1}^{2} < \mathbf{j}, \mathbf{N} >< \mathbf{N}, \nabla_{X_i} X_i >= -2H < \mathbf{j}, \mathbf{N} >\end{aligned}$$

where 2nd and 4th equalities are given respectively by $< \mathbf{j^N}, X_i >= 0$ and the definition of mean curvature. By substituting in (1) and using $< \mathbf{j}, \mathbf{n} >=< \mathbf{j^T}, \mathbf{n} >$ we obtain

$$\int_{\partial M} < \mathbf{j}, \mathbf{n} >= -2H \int_M < \mathbf{j}, \mathbf{N} > \tag{2}$$

In the halfspace model the relation between the area forms of hyperbolic and euclidean spaces is $d\omega_{H^3} = x_3^{-2} d\omega_{\mathbb{R}^3}$, hence

$$\int_M < \mathbf{j}, \mathbf{N} >= \int_M < x_3^{-3}\mathbf{j}, x_3^{-1}\mathbf{N} >_{\mathbb{R}^3} d\omega_{\mathbb{R}^3} = -\int_{\partial M} x_1 {x_3}^{-3} dx_3 \tag{3}$$

where the orientation on ∂M is induced by the orientation on M and the last equality is given by Stokes' theorem (see theorem 5.9 [S]). By the Gauss-Green formula in the plane (see theorem 5.7 [S])

$$-\int_{\partial M} x_1 x_3^{-3} dx_3 = \pm \int_{\Omega} x_3^{-3} dx_1 dx_3$$

where there is $+$ in the case that the orientation induced on ∂M is clockwise and there is $-$ in the case that the orientation induced on ∂M is counterclockwise. Then we have

$$\int_M < \mathbf{j}, \mathbf{N} >= \pm \int_{\Omega} x_3^{-3} dx_1 dx_3. \tag{4}$$

By definition of $\mathbf{N_\Omega}$

$$\int_{\Omega} x_3^{-3} dx_1 dx_3 = \pm \int_{\Omega} x_3^{-4} < \mathbf{N_\Omega}, \mathbf{j} >_{\mathbb{R}^3} dx_1 dx_3 = \pm \int_{\Omega} < \mathbf{N_\Omega}, \mathbf{j} > \tag{5}$$

(we remark that last integral is calculated with respect to the hyperbolic metric).

By substituting (5) in (4) we have

$$\int_M < \mathbf{j}, \mathbf{N} >= -\int_{\Omega} < \mathbf{N_\Omega}, \mathbf{j} >$$

and by substituting this last equality in (2) we obtain

$$\int_{\partial M} < \mathbf{j}, \mathbf{n} >= 2H \int_{\Omega} < \mathbf{N_\Omega}, \mathbf{j} > .$$

□

Remark 4.2. Theorem 4.1 holds under slightly more general hypothesis, that is: M an immersed surface in $\mathbb{H}^3$, with constant mean curvature and such that its boundary is a graph over the plane $\{x_2 = 0\}$.

In this case we take $\Omega \subset \{x_2 = 0\}$ such that ∂M is a graph over $\partial\Omega$. We observe that

$$\int_{\partial M} x_1 x_3^{-3} dx_3 = \pm \int_{\partial\Omega} x_1 x_3^{-3} dx_3$$

because the integrand does not depend on x_2 (here the sign depends on the orientation induced on ∂M by the orientation of M). Then we can use the Gauss-Green formula in the plane and proceed as in the proof of theorem 4.1.

We recall some elementary facts of hyperbolic geometry, useful to show two nice applicatations of the flux formula. A circle in the plane $\{x_2 = 0\}$, with hyperbolic center at (0,0,1) and hyperbolic radius ρ is the euclidean circle

$$S_\rho = \{(x_1, x_2, x_3) \mid x_2 = 0,\ x_1^2 + (x_3 - \cosh\rho)^2 = \sinh^2\rho\}$$

and the curvature of S_ρ is cotanghρ. Further, the mean curvature of a sphere of hyperbolic radius ρ is $H = \text{cotangh}\rho$.

We now prove a result that in the euclidean case is proved in [BS].

Theorem 4.3. *Let M be an immersed surface in $\mathbb{H}^3$ such that the boundary of M is a circle of hyperbolic radius ρ and the mean curvature of M is $H = \text{cotangh}\rho$. Then M is a half-sphere of hyperbolic radius ρ.*

Proof. We use the notations of the beginning of this section.

Up to an isometry of $\mathbb{H}^3$ we can assume that ∂M is contained in the totally geodesic plane $\{x_2 = 0\}$ and that the hyperbolic center of ∂M is the point (0,0,1), so $\Omega = D = \{(x_1, x_2, x_3) \mid x_2 = 0,\ x_1^2 + (x_3 - \cosh\rho)^2 \leq \sinh^2\rho\}$.

First of all we prove the following equality:

$$\int_{\partial D} x_3^{-2} ds = 2\text{cotangh}\rho \int_D x_3^{-3} dx_1 dx_3 \tag{6}$$

where s is the euclidean arc on ∂D.

Let $(r, \theta) \in [0, \sinh\rho] \times [0, 2\pi]$ be parameters for D such that

$$\begin{cases} x_1 = r\cos\theta \\ x_3 = r\sin\theta + \cosh\rho \end{cases}$$

then

$$\int_D x_3^{-3} dx_1 dx_3 = \int_0^{2\pi} \left(\int_0^{\sinh\rho} \frac{r dr}{(r\sin\theta + \cosh\rho)^3} \right) d\theta.$$

By integration with respect to r (Hermite formula for rational integrals) we obtain

$$\int_0^{\sinh\rho} \frac{r dr}{(r\sin\theta + \cosh\rho)^3} = \frac{\sinh\rho}{2\text{cotangh}\rho(r\sin\theta + \cosh\rho)^2}.$$

Further, as $ds = \sqrt{dx_1^2 + dx_3^2} = \sinh\rho d\theta$

$$\int_{\partial D} x_3^{-2} ds = \int_0^{2\pi} \frac{\sinh \rho d\theta}{(r \sin\theta + \cosh\rho)^2}.$$

The last three equalities imply (6).

By the proof of the flux formula we have (with sign depending on orientation induced by M on ∂D)

$$\int_{\partial D} < \mathbf{j}, \mathbf{n} >= \pm 2H \int_D x_3^{-3} dx_1 dx_3. \tag{7}$$

As $H = \text{cotangh}\rho$, by substituting (6) in (7) we have

$$\int_{\partial D} < \mathbf{j}, \mathbf{n} >= \pm \int_{\partial D} x_3^{-2} ds. \tag{8}$$

The scalar product and the first integral in (8) are calculated with respect to the hyperbolic metric, so

$$\int_{\partial D} < \mathbf{j}, \mathbf{n} >= \pm \int_{\partial D} x_3^{-2} < \mathbf{j}, x_3^{-1}\mathbf{n} >_{\mathbb{R}^3} ds; \tag{9}$$

by (8), (9) and $|\mathbf{j}|_{\mathbb{R}^3} = |x_3^{-1}\mathbf{n}|_{\mathbb{R}^3} = 1$ we obtain $< \mathbf{j}, x_3\mathbf{n} >_{\mathbb{R}^3} = \pm 1$.

This means that the boundary of M is orthogonal to the plane $\{x_2 = 0\}$ and it is a line of curvature and a geodesic. So, we may extend M by reflection along the boundary (i.e. with respect to the plane $\{x_2 = 0\}$); denote by $\widetilde{M}$ the union of M and its reflection. At a point $p \in \partial M$, the two principal directions are determined by the conormal vector $\mathbf{n}$ at p and the tangent vector to ∂M at p; as the curvature of ∂M and the mean curvature of M are both cotanghρ, the principal curvature of M on the boundary are both equal to cotanghρ then the boundary of M is composed of umbilical points.

Following the method of [H] (section VI) by using Codazzi-Mainardi equations of M in $\mathbb{H}^3$ one may derive that an umbilical point of a constant mean curvature surface in $\mathbb{H}^3$ is either isolated, or the surface is totally umbilic.

As $\partial M \subset \widetilde{M}$ is composed of umbilical points, then $\widetilde{M}$ is totally umbilic; by the classification of totally umbilic surfaces in $\mathbb{H}^3$ ([S1] theorem 29) $\widetilde{M}$ is a hyperbolic sphere.

□

We now use the flux formula to obtain an estimate of the mean curvature of a surface.

Theorem 4.4. *Let $D_\rho = \{(x_1, x_2, x_3) \mid x_2 = 0,\ x_1^2 + (x_3 - \cosh\rho)^2 \leq \sinh^2\rho\}$; let M be an immersed surface with constant mean curvature $H = |\mathbf{H}|$, such that ∂M is a graph of a C^2 function $f : \partial D_\rho \longrightarrow \mathbb{R}$. Then*

$$H \leq \frac{\text{cotangh}\rho}{\sinh\rho}\sqrt{\sinh^2\rho + \sup_{\partial D_\rho}(f')^2}$$

In particular if $f \equiv 0$ and H takes the maximum value $H = \text{cotangh}\rho$ then M is a part of a sphere.

Proof. We continue to use the notation of the beginning of this section; by (2) and (3) of the proof of theorem 4.1 (we remark that up to that point the proof of 4.1 does not involve the fact that $\partial M \subset \{x_2 = 0\}$) we have

$$2H = \frac{\int_{\partial M} < \mathbf{j}, \mathbf{n} >}{\int_{\partial M} x_1 {x_3}^{-3} dx_3}. \tag{10}$$

The sign of the second term does not depend on the orientation on ∂M as we integrate on ∂M twice. On ∂M we choose the orientation that gives the sign + in the remark 4.2. We have

$$\int_{\partial M} x_1 {x_3}^{-3} dx_3 = \int_{\partial D_\rho} x_1 {x_3}^{-3} dx_3 = \int_{D_\rho} {x_3}^{-3} dx_1 dx_3 = \frac{1}{2\text{cotangh}\rho}\int_{\partial D_\rho} x_3^{-2} ds$$

where s is the euclidean arc on ∂D_ρ, and equalities are given respectively by remark 4.2, the Gauss-Green formula in the plane and (6).

Now we transform the numerator of (10).

A parametrization of ∂M is given by $\alpha : [0, 2\pi] \longrightarrow H^3$ defined by

$$\alpha(\theta) = (R\cos\theta, f(\theta), R\sin\theta + \cosh\rho)$$

where $R = \sinh\rho$. This gives $|\alpha'| = \sqrt{R^2 + (f')^2}$, then

$$\int_{\partial M} < \mathbf{j}, \mathbf{n} > = \int_0^{2\pi} \frac{< \mathbf{j}, \mathbf{n} > \sqrt{R^2 + (f')^2}}{(R\sin\theta + \cosh\rho)} d\theta$$

As $< \mathbf{j}, \mathbf{n} > \leq |\mathbf{j}| = x_3^{-1}$, we have

$$\begin{aligned}\int_{\partial M} <\mathbf{j},\mathbf{n}> &\leq \int_0^{2\pi} \frac{\sqrt{R^2+(f')^2}}{(R\sin\theta+\cosh\rho)^2}d\theta \\ &= \sup_{[0,2\pi]} \sqrt{R^2+(f')^2}\int_0^{2\pi} \frac{d\theta}{(R\sin\theta+\cosh\rho)^2} \\ &\leq \frac{\sqrt{(\sinh\rho)^2+\sup_{[0,2\pi]}(f')^2}}{\sinh\rho}\int_{\partial D_\rho} {x_3}^{-2}ds\end{aligned}$$

By substituting (11) and this last formula in (10), we obtain

$$2H \leq \frac{2\text{cotangh}\rho\sqrt{(\sinh\rho)^2+\sup_{[0,2\pi]}(f')^2}}{\sinh\rho}$$

In the case $f \equiv 0$ this inequality gives $H \leq \text{cotangh}\rho$ and the proposition follows from theorem 4.3.

□

5. A further property

Let $\Omega \subset \{x_2 = 0\}$ be a compact domain and let $f : \Omega \longrightarrow \mathbb{R}$ be a C^2 function such that $f_{|\partial\Omega} \equiv 0$; let G_f denote the graph of f. Let $H_f : \Omega \longrightarrow \mathbb{R}$ be a continuous function such that $0 < |H_f| < 1$ and that is the mean curvature function of G_f (as in section 2).

Theorem 5.1. *In the notation above, there exists a constant C which depends on $\sup_\Omega |H_f|$ and $\sup_\Omega\{x_3 \mid (x_1,x_3) \in \Omega\}$ such that*

$$\sup_\Omega |f(x_1,x_3)| \leq C.$$

Proof. Let $\theta \in (0,\pi)$, $c \in \mathbb{R}$ and consider the plane P_θ^c which is parallel to the x_1 axis, forms an angle θ with the x_2 axis and passes by the point $(0,0,-c)$.

The mean curvature vector of P_θ^c is the constant vector

$$\mathbf{H}_\theta = \cos\theta(\cos\theta e_3 - \sin\theta e_2)$$

where $\{e_1,e_2,e_3\}$ is the standard basis of $\mathbb{R}^3$.

The mean curvature vector of P_θ^c points upwards and depends on θ only; further, we remark that changing c means translating P_θ^c parallel to itself in the x_2 direction. From now on we omit the superscript c.

As $0 < |H_f| < 1$ there exists θ such that $\sup_\Omega |H_f| = |\mathbf{H}_\theta| = |\cos\theta|$; by choosing either $\theta \in (0, \frac{\pi}{2})$ or $\theta \in (\frac{\pi}{2}, \pi)$ we have that the mean cuvature vectors of G_f and P_θ point in the same direction. Then, without loss of generality, we can restrict to the case $\theta \in (0, \frac{\pi}{2})$ and $\sup_\Omega H_f = \cos\theta$.

As G_f is compact, it is possible to translate P_θ along the x_2 axis such that $P_\theta \cap G_f = \emptyset$. Then translate P_θ towards G_f; if the first point of contact between the plane and the graph of f is interior to the graph, we have a contradiction by the maximum principle, hence it must be on ∂G_f. This means that

$$\sup_\Omega |f| \leq \frac{\max_\Omega |H_f|}{\sqrt{1 - (\max_\Omega |H_f|)^2}} \max_\Omega \{x_3 \mid (x_1, x_3) \in \Omega\}.$$

□

For further considerations on Height Estimates see [R].

References

[A] T. Aubin: *Non Linear Analysis on Manifolds. Monge Ampère equations,* Springer-Verlag, (1982).

[BaS] J.L. Barbosa, R. Sa Earp: *Compact Constant Mean Curvature Hypersurfaces in $\mathbb{H}^{n+1}$ with Convex Planar Boundary, I,* in Preparation.

[BS] F. Brito, R. Sa Earp: *Geometric configurations of constant mean curvature surfaces with planar boundary,* An. Acad. Bras. Ci. 63-1 (1991), 5-19.

[CNS] L. Caffarelli, L. Nirenberg, J. Spruck: *Nonlinear Second-Order Elliptic Equations V. The Dirichlet Problem for Weingarten Hypersurfaces,* Comm. on Pure and Applied Math. 61 (1988), 47-70.

[GT] D. Gilbarg, N.S. Trudinger: *Elliptic Partial Differential Equations of Second Order,* Springer-Verlag, (1983).

[H] H. Hopf: *Differential Geometry in the Large,* Lecture Notes in Mathematics 1000, Springer-Verlag, (1983).

[HY] W.Y. Hsiang, W. Yu: *A Generalization of a theorem of Delaunay,* J. of Diff. Geom. 16 (1981), 161-177.

[KKMS] N. Korevaar, R. Kusner, W.H. Meeks III, B. Solomon, *Constant Mean Curvature Surfaces in Hyperbolic Space,* American Journal of Math. 114 (1992), 1-143.

[R] H. Rosenberg: *Hypersurfaces of Constant Curvature in Space Forms,* Bull. Sc. Math. 2e serie, 117 (1993), 211-239.

[RS] H. Rosenberg, R. Sa Earp: *Some Remarks on Surfaces of Prescribed Mean Curvature,* Pitman Monographs and Surveys in Pure and Applied Mathematics 52 (1991), 123-148.

[RoS] H. Rosenberg, J. Spruck: *On the Existence of Convex Hypersurfaces of Constant Gauss Curvature in Hyperbolic Space,* to appear in J. of Diff. Geometry.

[S] Leon Simon: *Equations of Mean Curvature Type in 2 Indipendent Variables,* Pac. J. of Math. 69, n.1 (1977), 245-268.

[Sp] M. Spivak: *Calculus on Manifolds,* W. A. Benjamin Inc., New York (1965).

[Sp1] M. Spivak: *A Comprehensive Introduction to Differential Geometry IV,* Publish or Perish Inc., Berkley, (1979).

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