CONCENTRATION OF TOTAL CURVATURE OF MINIMAL SURFACES IN $\mathbb{H}^2 \times \mathbb{R}$

RICARDO SA EARP AND ERIC TOUBIANA

Abstract. We prove a phenomenon of concentration of total curvature for stable minimal surfaces in the product space $\mathbb{H}^2 \times \mathbb{R}$, where $\mathbb{H}^2$ is the hyperbolic plane. Under some geometric conditions on its asymptotic boundary, an oriented stable minimal surface immersed in $\mathbb{H}^2 \times \mathbb{R}$ has infinite total curvature.

We exhibit an example of a minimal graph such that in a domain whose asymptotic boundary is a vertical segment the total curvature is finite, but the total curvature of the graph is infinite, by the theorem cited before. We also present some simple and peculiar examples of infinite total curvature minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and their asymptotic boundaries.

1. Introduction

In this paper, we prove a phenomenon of concentration of total curvature for stable minimal surfaces in the product space $\mathbb{H}^2 \times \mathbb{R}$, where $\mathbb{H}^2$ is the hyperbolic plane.
We recall that a minimal surface M immersed in $\mathbb{H}^2 \times \mathbb{R}$ has *finite intrinsic total curvature*, or simply *finite total curvature*, if $\int_M K\, dA$ is finite, where K is the (intrinsic) Gaussian curvature of M.
Our main theorem (Theorem 1.1), ensures that under some geometric conditions on the asymptotic boundary, an oriented stable minimal surface immersed in $\mathbb{H}^2 \times \mathbb{R}$ (but not necessarily properly immersed) has infinite total curvature.
Our main result is the following.

Theorem 1.1 (Main Theorem)**.** *Let M be a connected and oriented minimal surface immersed in $\mathbb{H}^2 \times \mathbb{R}$ (not necessarily complete). We assume that M is stable and moreover*

1. *the finite asymptotic boundary of M is composed of an arc α properly embedded in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$,*
2. *there exists an open and simple arc $\alpha_0 \subset \alpha$ in $\partial^f_\infty M \setminus \partial_\infty(\partial M)$ which is not contained in a vertical line.*

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Then M has infinite total curvature.
Moreover, let $p_\infty := (x_\infty, t_0) \in \partial_\infty \mathbb{H}^2 \times \mathbb{R}$, be any point of α_0 which does not belong to a vertical segment contained in α_0. Then $|n_3(p)| \to 1$ if $p \to p_\infty$, $p \in M$, where n_3 is the third coordinate of the Gauss map of M.

We emphasize that when the finite asymptotic boundary of a minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ is a vertical segment there are examples either of finite or infinite total curvature:

First, we summarize the example given by F. Morabito and M. Rodriguez [19]: It consists of a complete minimal surface M in $\mathbb{H}^2 \times \mathbb{R}$, invariant by a discrete group of vertical translations. The finite asymptotic boundary is the finite union of vertical lines in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$. It is interesting to note that *any* nonempty domain $S \subset M$ of finite vertical height has *finite total curvature*. One can choose such S to be a vertical graph. Of course, the surface M has infinite total curvature, but the total curvature *does not concentrates* in a subset $S \subset M$ of bounded vertical height whose asymptotic boundary is a vertical segment. The reader is referred to the constructions due to L. Hauswirth and A. Menezes [11], to find other related results.

Secondly, we consider a classical minimal surface M_1 in $\mathbb{H}^2 \times \mathbb{R}$ which has been useful as barrier in many papers about minimal graphs theory, see P. Collin and H. Rosenberg [4, Lemma 1], B. Nelli and H. Rosenberg [21, Errata corrige 4 (b)] and the authors [28, Theorem 4.1 (3)]. The surface M_1 is globally a vertical graph, see [26, Equation (32)] for an explicit formula. Thus, M_1 is stable. The surface M_1 has been characterized by I. Fernández and P. Mira [6]. A generalization of M_1 was carried out by the work of J. A. Gálvez and H. Rosenberg [9, Proposition 3.1], by J. Plehnert [24, Section 3.2], and by P. Bérard and the first author in [1].
As a matter of fact, M_1 is invariant under an one parameter group of hyperbolic translations along a horizontal geodesic $\gamma \subset \mathbb{H}^2 \times \{0\}$. Let us denote by p_∞, q_∞ the asymptotic boundary of γ. Then the finite asymptotic boundary of M_1 is composed of two vertical half-lines in $\partial_\infty \mathbb{H}^2 \times (0, +\infty)$, issuing from p_∞ and q_∞, and one of the two arcs of $\partial_\infty \mathbb{H}^2 \times \{0\} \setminus \{p_\infty, q_\infty\}$, see [28, Proposition 2.1]. Let S be a nonempty open subset of M_1 whose asymptotic boundary is *a vertical segment* of the finite asymptotic boundary of M_1. Since M_1 is invariant by horizontal translations, it follows that S has infinite total curvature.

We observe that in $\mathbb{H}^2 \times \mathbb{R}$, finite total curvature of a complete oriented minimal surface implies finite index. This is a theorem by P. Bérard and the first author in [1]. Notice that in Euclidean space a famous result of D. Fisher-Colbrie [7] states that a complete oriented minimal surface has finite index if and only if it has finite total curvature. Notice also that finite index does not imply finite total curvature in $\mathbb{H}^2 \times \mathbb{R}$, as the preceding example M_1 shows. The catenoid in $\mathbb{H}^2 \times \mathbb{R}$ is another counter-example: It has infinite total curvature and index one [1, Proposition 3.3 and Theorem 3.5].

We pause momentarily to ask here if the assumption "complete" can be removed from the Bérard-Sa Earp theorem? If this assertion is true, then a minimal surface in

$\mathbb{H}^2 \times \mathbb{R}$ with finite total curvature is stable, outside a compact subset, see [7, proof of Proposition 1].
In $\mathbb{R}^3$, S.-Y. Cheng and J. Tysk showed that a complete minimal surface with boundary and with finite total curvature has finite index [2, Theorem 5]. Afterward, A. Grigor'yan and S.-T. Yau generalized this result assuming only that M is a minimal surface with finite total curvature, that is they dropped the assumption of completness [10, Theorem 4.9].

We observe also that the hypothesis (2) of the Main Theorem 1.1 is crucial since there exist many minimal surfaces with finite total curvature whose the asymptotic boundary is equal to the asymptotic boundary of the boundary, see Remark 3.1.

In Example 5.1 we construct a minimal graph such that in a domain whose asymptotic boundary is a vertical segment the total curvature is finite, but any neighborhood of another part of the asymptotic boundary has infinite total curvature. This phenomenon of concentration of total curvature follows from the Main Theorem 1.1.

We also present in Section 5 some simple and peculiar examples of infinite total curvature minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and their asymptotic boundaries.

We point out now an important property of a complete minimal surface immersed in $\mathbb{H}^2 \times \mathbb{R}$: The finite asymptotic boundary of a complete minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ with finite total curvature is constituted of vertical lines, see [12, Theorem 2.1 and Proposition 2.4].

Loosely speaking, we note that the main result in our paper [31] is a counterpart of the Main Theorem in the present paper. To see that, we recall some results from [31] about oriented stable minimal annular ends with compact boundary, properly immersed into $\mathbb{H}^2 \times \mathbb{R}$.
In [31] we introduced a suitable notion of asymptotic boundary of a surface M. We call the set of points of the asymptotic boundary of M with finite (vertical) height, the *finite asymptotic boundary* (see Definition 2.2 below). The main result in [31] ensures that if the end M converges to a vertical plane and the finite asymptotic boundary of M is contained in two vertical lines, then M has finite total curvature. If the end of M is embedded we showed that it is a horizontal graph with respect to a horizontal geodesic γ, or simply a horizontal graph. For a definition of a horizontal graph, see, for instance, [12].
Summarizing, we proved in [31] the following geometric behavior, up to a compact part of M:

- Any equidistant curve of γ intersects the end M at most at one point and it intersects it transversally ([12, Proposition A.3] and [31, Step 7].
- Let n_3 be the third coordinate of the unit normal field on M with respect to the product metric on $\mathbb{H}^2 \times \mathbb{R}$. We have that $n_3(p) \to 0$ uniformly when $p \to \partial_\infty E$

[31, Step 3]. Consequently, the tangent plane throughout the end is nowhere horizontal.

It will be very interesting to investigate a similar result as in [31], for minimal surfaces immersed into $\mathbb{H}^2 \times \mathbb{R}$ with nonempty non compact boundary. The example 5.1 suggests the following problem: Find geometric conditions on a minimal surface S immersed into $\mathbb{H}^2 \times \mathbb{R}$, with nonempty non compact boundary and whose finite asymptotic boundary is contained in a vertical line, to have finite total curvature.

We remark that a horizontal graph with respect a geodesic γ which is transverse to the intersecting equidistant curves to γ is stable. A vertical graph (see the definition, for instance, in [28]), which is transverse to the intersecting vertical geodesics is stable, as well. This follows from the classical criterion of stability for minimal surfaces: Let M be an oriented connected minimal surface immersed into $\mathbb{H}^2 \times \mathbb{R}$. If there exists a positive smooth function u on a bounded domain Ω of M satisfying $\mathcal{L}u = 0$, where $\mathcal{L}$ is the stability operator [1, Section 2.2], then Ω is stable, see [7] or [3, Lemma 1.36].

Notice first that there are many minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ whose finite asymptotic boundary is the union of regular curves, see, for instance, M. Rodriguez and F. Martin [18] and the authors [26], [27]. However, there are "local obstructions" to a curve be the asymptotic boundary of a minimal surface in $\mathbb{H}^2 \times \mathbb{R}$, see [28, Theorem 2.1]. Also, B. Coskunuzer gave a necessary and sufficient condition on a finite collection of Jordan curves in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ to be the asymptotic boundary of a complete area minimizing surface in $\mathbb{H}^2 \times \mathbb{R}$, [5, Theorem 2.13]. Afterward, B. Kloeckner and R. Mazzeo generalized this result for a finite collection of Jordan curves in $\partial_\infty(\mathbb{H}^2 \times \mathbb{R})$, [16, Proposition 4.4].

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2. Asymptotic boundary

Definition 2.1 (Convergence to an asymptotic boundary point of $\mathbb{H}^2$)**.** Let $y_0 \in \mathbb{H}^2$ be a fixed point of $\mathbb{H}^2$ and let $x_\infty \in \partial_\infty \mathbb{H}^2$. We denote by $[y_0, x_\infty) \subset \mathbb{H}^2$ the geodesic ray issuing from y_0 and with asymptotic boundary x_∞. For any $\rho > 0$ we denote by $\gamma_\rho \subset \mathbb{H}^2$ the geodesic intersecting the ray $[y_0, x_\infty)$ orthogonally at point y_ρ such that $d_{\mathbb{H}^2}(y_0, y_\rho) = \rho$. Let γ_ρ^+ be the component of $\mathbb{H}^2 \backslash \gamma_\rho$ which contains x_∞ in its asymptotic boundary: $x_\infty \in \partial_\infty \gamma_\rho^+$.
Let (x_n) be a sequence of points of $\mathbb{H}^2$. We say that (x_n) *converges to* x_∞, denoted by $x_n \to x_\infty$, if for any $\rho > 0$ there exists $n_\rho \in \mathbb{N}$ such that $x_n \in \gamma_\rho^+$ for any $n \geqslant n_\rho$.

We observe that if we choose the Poincaré disc model of $\mathbb{H}^2$, then $x_n \to x_\infty$ if and only if the sequence (x_n) converges to x_∞ in Euclidean sense.
Also, let us consider the Poincaré half-plane model of $\mathbb{H}^2$, then in this model $\partial_\infty \mathbb{H}^2 = \mathbb{R} \cup \{\infty\}$. We have:

- If $x_\infty \in \mathbb{R}$ then $x_n \to x_\infty$ if and only if the sequence (x_n) converges to x_∞ in Euclidean sense.
- If $x_\infty = \infty$ then $x_n \to x_\infty$ if and only if $|x_n| \to +\infty$.

Definition 2.2 (Asymptotic boundary)**.**

(1) We define the *asymptotic boundary* of $\mathbb{H}^2 \times \mathbb{R}$ setting:

$$\partial_\infty(\mathbb{H}^2 \times \mathbb{R}) := \big(\partial_\infty \mathbb{H}^2 \times \mathbb{R}\big) \cup \big(\mathbb{H}^2 \times \{-\infty, +\infty\}\big) \cup \big(\partial_\infty \mathbb{H}^2 \times \{-\infty, +\infty\}\big).$$

This decomposition means that for a divergent sequence (p_n) of $\mathbb{H}^2 \times \mathbb{R}$ there are three possibilities for *converging to infinity* (up to extracting a subsequence). That is, setting $p_n = (x_n, t_n) \in \mathbb{H}^2 \times \mathbb{R}$, we have the following cases:

- $x_n \to x_\infty \in \partial_\infty \mathbb{H}^2$ (see Definition 2.1) and $t_n \to t_0 \in \mathbb{R}$. We say that $p_\infty := (x_\infty, t_0) \in \partial_\infty \mathbb{H}^2 \times \mathbb{R}$ is an asymptotic point *at finite height.*
- $x_n \to x_0 \in \mathbb{H}^2$ and $t_n \to \pm\infty$. That is (p_n) converges to $p_\infty := (x_0, \pm\infty) \in \mathbb{H}^2 \times \{-\infty, +\infty\}$.
- $x_n \to x_\infty \in \partial_\infty \mathbb{H}^2$ and $t_n \to \pm\infty$. That is (p_n) converges to $p_\infty := (x_\infty, \pm\infty) \in \partial_\infty \mathbb{H}^2 \times \{-\infty, +\infty\}$.

(2) Let $\Omega \subset \mathbb{H}^2 \times \mathbb{R}$ be a nonempty subset. We say that a point $p_\infty \in \partial_\infty(\mathbb{H}^2 \times \mathbb{R})$ is an *asymptotic point* of Ω if there is a sequence (p_n) of Ω converging to p_∞.

The set of asymptotic points of Ω, called the *asymptotic boundary of* Ω, is denoted by $\partial_\infty \Omega$.

(3) Let $\Omega \subset \mathbb{H}^2 \times \mathbb{R}$ be a nonempty subset. The set of asymptotic points at finite height is called the *finite asymptotic boundary* and is denoted by $\partial_\infty^f \Omega$.

The complement $\partial_\infty \Omega \setminus \partial_\infty^f \Omega$ is called the *non finite asymptotic boundary of* Ω.

We say that the finite asymptotic boundary $\partial_\infty^f \Omega$ has *bounded vertical height* if

$$\exists t_1 > 0,\, \partial_\infty^f \Omega \subset \{(e^{i\theta}, t) \in \partial_\infty \mathbb{H}^2 \times \mathbb{R},\ |t| < t_1\}$$

The boundary of a surface is defined as usual.

3. Geometric Lemmas

The following result describes, in particular, the local behavior at infinity of a minimal surface M in $\mathbb{H}^2 \times \mathbb{R}$, whose finite asymptotic boundary is an arc in $\partial_\infty^f M \setminus \partial_\infty(\partial M)$, which is not contained in a vertical line.

Lemma 3.1. *Let M be a connected immersed minimal surface in $\mathbb{H}^2 \times \mathbb{R}$. Assume that:*

(1) *The finite asymptotic boundary of M is composed of an arc α properly embedded in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$.*

(2) *There exists an open and simple arc $\alpha_0 \subset \alpha$ in $\partial_\infty^f M \setminus \partial_\infty(\partial M)$ which is not contained in a vertical line.*

Let $p_\infty := (x_\infty, t_0) \in \partial_\infty \mathbb{H}^2 \times \mathbb{R}$, be any point of α_0 which does not belong to a vertical segment contained in α_0.
Then, for any $\varepsilon > 0$, there exist a vertical plane P_ε and a component P_ε^+ of $(\mathbb{H}^2 \times \mathbb{R}) \setminus P_\varepsilon$, such that, setting $S_\varepsilon := M \cap P_\varepsilon^+$, we have

1. *$S_\varepsilon \subset \mathbb{H}^2 \times (t_0 - \varepsilon, t_0 + \varepsilon)$ and $\partial S_\varepsilon \subset P_\varepsilon$.*
2. *The asymptotic boundary of S_ε is a subarc α_ε of α_0 which is not contained in a vertical line: $\partial_\infty S_\varepsilon = \alpha_\varepsilon \subset \alpha_0$. Furthermore $p_\infty \in \alpha_\varepsilon$ and $p_\infty \notin \partial_\infty P_\varepsilon$.*
3. *$\pi(\alpha_\varepsilon) = \pi(\partial_\infty P_\varepsilon^+)$, where $\pi : \partial_\infty \mathbb{H}^2 \times \mathbb{R} \to \partial_\infty \mathbb{H}^2$ is the first projection.*
4. *Assume that M is not contained in the slice $\mathbb{H}^2 \times \{t_0\}$. Then for any $\varepsilon > 0$ there exists $\varepsilon_0 < \varepsilon$ such that for any $\varepsilon' < \varepsilon_0$, $P_{\varepsilon'}^+$ is strictly contained in P_ε^+. Hence $S_{\varepsilon'}$ is strictly contained in S_ε. Furthermore $\bigcap_{\varepsilon > 0} P_\varepsilon^+ = \emptyset$ and $\partial_\infty S_\varepsilon = \partial_\infty^f S_\varepsilon$.*

Proof. In the following we identify $\mathbb{H}^2 \times \{0\}$ with $\mathbb{H}^2$.
Let $p_\infty = (x_\infty, t_0) \in \alpha_0$ be a point as in the statement.
Let $y_0 \in \mathbb{H}^2$ be a fixed point. We denote by $\gamma_0^+ \subset \mathbb{H}^2$, the geodesic ray issuing from y_0 and with asymptotic boundary x_∞.
For any $\rho > 0$ we denote by $\Pi_\rho \subset \mathbb{H}^2 \times \mathbb{R}$ the geodesic vertical plane intersecting the ray γ_0^+ orthogonally at point y_ρ such that $d_{\mathbb{H}^2}(y_0, y_\rho) = \rho$. Let Π_ρ^+ be the component of $(\mathbb{H}^2 \times \mathbb{R}) \setminus \Pi_\rho$ which contains x_∞ in its asymptotic boundary, thus we have $x_\infty \in \partial_\infty \Pi_\rho^+$ and $x_\infty \notin \partial_\infty \Pi_\rho$.
For any $\rho > 0$ we denote by $\beta_\rho \subset \alpha_0$ the connected component of $\alpha_0 \cap \partial_\infty \Pi_\rho^+$ containing p_∞: $p_\infty \in \beta_\rho \subset \alpha_0 \cap \partial_\infty \Pi_\rho^+$.
Let $\pi : \partial_\infty \mathbb{H}^2 \times \mathbb{R} \to \partial_\infty \mathbb{H}^2$ be the first projection. Recalling that α_0 is properly embedded, it follows from [28, Theorem 2.1] that there exists $\rho_0 > 0$ such that for any $\rho \geqslant \rho_0$ then $\pi(\beta_\rho)$ is an arc and

for any $z_\infty \in \pi(\beta_\rho)$ its inverse image by π in β_ρ is either a single point or a vertical segment. (1)

We can also assume that ρ_0 is large enough so that $M \cap \Pi_{\rho_0} \neq \emptyset$.
Denoting by $p_\rho^+, p_\rho^- \in \partial_\infty \Pi_\rho$ the two endpoints of β_ρ, we have $\pi(p_\rho^+) \neq \pi(p_\rho^-)$. Therefore we get $\pi(\beta_\rho) = \pi(\partial_\infty \Pi_\rho^+)$ if $\rho \geqslant \rho_0$.
Observe that, by a continuity argument, for any $\varepsilon > 0$ there is $\rho_\varepsilon > \rho_0$ such that for any $\rho > \rho_\varepsilon$ we have $\beta_\rho \subset \partial_\infty \mathbb{H}^2 \times [t_0 - \varepsilon, t_0 + \varepsilon]$.
Furthermore, since the finite asymptotic boundary of M is a properly embedded arc, if $\varepsilon > 0$ is small enough we have

$$\beta_\rho = \partial_\infty^f M \cap \partial_\infty \Pi_\rho^+ \cap \big(\partial_\infty \mathbb{H}^2 \times [t_0 - \varepsilon, t_0 + \varepsilon]\big), \tag{2}$$

for any $\rho \geqslant \rho_\varepsilon$.
For any $\rho > \rho_0$, we denote by M_ρ the union of the connected components M' of $M \cap \Pi_\rho^+$ such that its finite asymptotic boundary meets β_ρ, that is $\partial_\infty^f M' \cap \beta_\rho \neq \emptyset$. Therefore we have

- $M_\rho \subset \Pi_\rho^+$,
- $\partial_\infty^f M_\rho = \beta_\rho$
- $M_{\rho_2} \subset M_{\rho_1}$ if $\rho_0 < \rho_1 < \rho_2$.

We claim that there exists $\rho'(\varepsilon) > \rho_\varepsilon$ such that

$$M_{\rho'(\varepsilon)} \subset \mathbb{H}^2 \times (t_0 - \varepsilon, t_0 + \varepsilon).$$

Indeed, otherwise there would exist a strictly increasing sequence (ρ_n) such that

- $\rho_n > \rho_\varepsilon$ for any n and $\rho_n \to +\infty$,
- for any n, M_{ρ_n} intersects $\big(\mathbb{H}^2 \times \{t_0 + \varepsilon\}\big) \cap \Pi_{\rho_n}^+$, or $\big(\mathbb{H}^2 \times \{t_0 - \varepsilon\}\big) \cap \Pi_{\rho_n}^+$, at some point $(y_n, t_0 \pm \varepsilon)$.

Observe that by construction we have $y_n \to x_\infty$. Letting n going to $+\infty$ we obtain that the asymptotic point $(x_\infty, t_0 \pm \varepsilon)$ belongs to the finite asymptotic boundary of M, which gives a contradiction with (1) and (2), with $\rho = \rho_\varepsilon$, and the assumption that p_∞ does not belong to a vertical segment contained in α_0.
Now we set

$$P_\varepsilon := \Pi_{\rho'(\varepsilon)},\ S_\varepsilon := M_{\rho'(\varepsilon)},\ \text{and}\ \alpha_\varepsilon := \beta_{\rho'(\varepsilon)},$$

We have just seen that

- $S_\varepsilon \subset \mathbb{H}^2 \times (t_0 - \varepsilon, t_0 + \varepsilon)$,
- $\partial_\infty^f S_\varepsilon = \alpha_\varepsilon$,
- $\pi(\alpha_\varepsilon) = \pi(P_\varepsilon^+)$,

therefore we have $\partial_\infty S_\varepsilon = \alpha_\varepsilon$. Since p_∞ does not belong to the asymptotic boundary of ∂M, we can choose ρ_ε so large that for any $\rho > \rho_\varepsilon$ we have

$$\partial M \cap \Pi_\rho^+ \cap \Big(\mathbb{H}^2 \times (t_0 - \varepsilon, t_0 + \varepsilon)\Big) = \emptyset.$$

Therefore we get that $\partial S_\varepsilon \cap P_\varepsilon^+ = \emptyset$ and then $\partial S_\varepsilon \subset P_\varepsilon$. This proves Assertions 1-3.

Now we suppose that M is not contained in $\mathbb{H}^2 \times \{t_0\}$.
Let $\varepsilon > 0$ be such that Assertion (4) does not hold. Then there exists a strictly decreasing positive sequence (ε_n) such that $\varepsilon_n \to 0$ and $P_{\varepsilon_n}^+$ is not contained in P_ε^+. Recall that, for any n, we have $P_{\varepsilon_n}^+ = \Pi_{\rho_n}^+$ for some $\rho_n > 0$. Terefore, if $P_{\varepsilon_n}^+$ is not contained in P_ε^+ we obtain that P_ε^+ is contained in $P_{\varepsilon_n}^+$, for any n. Thus $S_\varepsilon \subset S_{\varepsilon_n}$, and consequently $S_\varepsilon \subset \mathbb{H}^2 \times [t_0 - \varepsilon_n, t_0 + \varepsilon_n]$ for any n. Letting n going to $+\infty$ we get $S_\varepsilon \subset \mathbb{H}^2 \times \{t_0\}$. By the analytic continuation property, we get that $M \subset \mathbb{H}^2 \times \{t_0\}$, which leads to a contradiction.
Notice that the same argument shows that, if M is not contained in $\mathbb{H}^2 \times \{t_0\}$, P_ε^+ goes to infinity as ε goes to zero, that is $\rho'(\varepsilon) \to +\infty$ if $\varepsilon \to 0$. Therefore we get that $\bigcap_{\varepsilon > 0} P_\varepsilon^+ = \emptyset$. This accomplishes the proof of the Lemma. $\square$

In our context, it is natural to expect that the area of a minimal surface M in $\mathbb{H}^2 \times \mathbb{R}$ is infinite. More precisely, we derive the following result.

Lemma 3.2. *Let M be a minimal surface immersed into $\mathbb{H}^2 \times \mathbb{R}$. Assume that the finite asymptotic boundary of the boundary of M is not equal to the finite asymptotic boundary of the boundary of M, that is $\partial_\infty^f(\partial M) \neq \partial_\infty^f M$. Then M has infinite area.*

Proof. By assumption there exists a finite asymptotic point p_∞ of M which is not an asymptotic point of the boundary of M: $p \in \partial_\infty^f M \setminus \partial_\infty^f(\partial M)$.
Let (p_n) be a sequence of points of M which converges to p_∞, see Definition 2.2. Let $\delta > 0$ be a fixed real number. Then, since p_∞ is not an asymptotic boundary point of ∂M, there exists $n_0 \in \mathbb{N}$ such that $d_M(p_n, \partial M) > 2\delta$ for any $n \geqslant n_0$, where d_M means the intrinsic distance on M.
For each n, let $D(p_n, \delta) \subset M$ be the geodesic disc on M centered at p_n and with radius δ. Then for any $n \geqslant n_0$ we have $D(p_n, \delta) \cap \partial M = \emptyset$.
From the one hand, up to extracting a subsequence, we can assume that there exists $n_1 \in \mathbb{N}$, $n_1 > n_0$, such that $D(p_n, \delta) \cap D(p_m, \delta) = \emptyset$ for any $m, n \geqslant n_1$.
In an other hand, a result of K. Frensel [8, Theorem 3 and Remark 4] states that there exists a fixed real number $\alpha > 0$ such that $\text{Area}\,(D(p_n, \delta)) > \alpha$ for any $n \geqslant n_1$. We conclude that M has infinite area. □

Remark 3.1. The assumption on the asymptotic boundary in Lemma 3.2 is crucial as we can see from the following examples.

(1) A geodesic triangle in $\mathbb{H}^2$ with one, or more, vertices in the asymptotic boundary of $\mathbb{H}^2$ has finite area [29, Lemme 2.5.23 and Théorème. 2.5.24]. We observe that the asymptotic boundary of the triangle is equal to the asymptotic boundary of its boundary.

(2) We construct a domain in $\mathbb{H}^2$ with finite area, or equivalently finite total curvature, and whose asymptotic boundary is the whole $\partial_\infty \mathbb{H}^2$.

Consider the Poincaré disc model of $\mathbb{H}^2$. For any $\rho > 0$ we denote by $C_\rho \subset \mathbb{H}^2$ the circle centered at 0 with radius ρ.

Let (ρ_n) be a strictly increasing sequence of positive real numbers such that $\rho_n \to +\infty$. Now we consider another sequence of positive real numbers (ρ'_n), $\rho'_n > \rho_n$, such that, calling A_n the open annulus bounded by the circles C_{ρ_n} and $C_{\rho'_n}$, we have:

- the closed annuli A_n are mutually disjoint,
- $\text{Area}\,(A_n) < \frac{1}{n^2}$ for any $n \in \mathbb{N}^*$.

Let $y_0 \in \mathbb{H}^2$, $y_0 \neq 0$, be any point on the imaginary axis such that its hyperbolic distance to 0 is lesser that $\rho_1/2$. We call T the open geodesic triangle with vertices $0, y_0$ and 1, observe that this last vertex is the unique vertex of T belonging to the asymptotic boundary of $\mathbb{H}^2$.

Then we set $U := T \cup \bigcup_{n \geqslant 1} A_n$. By construction U is a domain of $\mathbb{H}^2$ satisfying:

- $\text{Area}\,(U)$ is finite, (since Area (T) is finite [29, Lemme 2.5.23]).
- $\partial_\infty U = \partial_\infty \mathbb{H}^2$ and also $\partial_\infty(\partial U) = \partial_\infty \mathbb{H}^2$. In particular the asymptotic boundary of U is equal to the asymptotic boundary of its boundary.

Observe that the domain U is infinitely connected. We can modify slightly U in order to obtain a simply connected domain. For that we consider a fixed point $x_0 \in \mathbb{H}^2$, $x_0 \neq 0$, on the real axis such that its hyperbolic distance to 0 is lesser that $\rho_1/2$. We call Σ^+ (resp. Σ^-) the closed geodesic triangle in $\mathbb{H}^2$ with vertices $0, x_0$ and i (resp. $0, x_0$ and $-i$). We set

$$\Omega := T \cup \big(\bigcup_{k\geqslant 1} A_{2k} \setminus \Sigma^+\big) \cup \big(\bigcup_{k\geqslant 1} A_{2k-1} \setminus \Sigma^-\big).$$

We have by construction $\Omega \subset U$, therefore Ω has finite area. Furthermore:

- Ω is a simply connected domain,
- $\partial_\infty \Omega = \partial_\infty \mathbb{H}^2$ and also $\partial_\infty(\partial\Omega) = \partial_\infty \mathbb{H}^2$. In particular the asymptotic boundary of Ω is equal to the asymptotic boundary of its boundary.

(3) Now we construct many non planar examples of minimal surfaces M with finite area and finite total curvature, and whose asymptotic boundary is the same as the asymptotic boundary of its boundary, more precisely $\partial_\infty^f(\partial M) = \partial_\infty^f M$.

We consider a Jordan curve $\Gamma \subset \partial_\infty \mathbb{H}^2 \times \mathbb{R}$ which is the vertical graph of a C^1 function over $\partial_\infty \mathbb{H}^2$. Then we resolve the Dirichlet problem for the minimal equation over $\mathbb{H}^2$, see [20, Theorem 4] or [28, Remark 4-(2)]. We obtain a complete minimal surface S which is a vertical graph over $\mathbb{H}^2$ and such that $\partial_\infty S = \partial_\infty^f S = \Gamma$.

For any part $O \subset \mathbb{H}^2$ we denote by $\widetilde{O} \subset S$ the corresponding part of S which is a graph over O. Now we proceed as in the example (2) before, we use the same notations.

We choose the positive sequences (ρ_n) and (ρ'_n) such that:

- the closed annuli $\overline{A}_n$ are mutually disjoint,
- Area $(\widetilde{A}_n) < \frac{1}{n^2}$.

Consider the triangle $T \subset \mathbb{H}^2$ (with vertices $0, y_0$ and 1). Let n_3 be the third component of the Gauss map of S. In the Main Theorem 1.1 it is proved that for any $p_\infty \in \partial_\infty \mathbb{H}^2$ we have $|n_3(p)| \to 1$ as $p \to p_\infty$, $p \in \mathbb{H}^2$. Furthermore a computation shows that

$$\text{Area}(\widetilde{T}) = \int_T \frac{1}{|n_3|}\, dA$$

where dA is the area element of $\mathbb{H}^2$. Since T has finite area and since $n_3(p) \to 1$ when $p \to 1$, we get that Area $(\widetilde{T})$ is finite. Finally we set

$$M := \widetilde{T} \cup \big(\bigcup_{k\geqslant 1} \widetilde{A}_{2k} \setminus \widetilde{\Sigma}^+\big) \cup \big(\bigcup_{k\geqslant 1} \widetilde{A}_{2k-1} \setminus \widetilde{\Sigma}^-\big).$$

By construction we have:

- $M \subset \mathbb{H}^2 \times \mathbb{R}$ is a connected and simply connected minimal surface,
- Area (M) is finite,
- $\partial_\infty M = \Gamma = \partial_\infty(\partial M)$.

Since S is a vertical graph, S is stable. We deduce from [25, Main Theorem] that the second fundamental form of S is bounded and thus S has bounded extrinsic curvature. Since the sectional curvatures of $\mathbb{H}^2 \times \mathbb{R}$ are bounded, S has also bounded Gaussian curvature (see also Formula (3) below).

Finally, since M has finite area and bounded Gaussian curvature, M has finite total curvature as desired.

4. Proof of the Main Theorem

Proof of the Main Theorem. Observe that, taking into account the proof of Lemma 3.2, if M is contained in a slice $\mathbb{H}^2 \times \{t_0\}$ then there is nothing to prove. Thus from now we assume that M is not contained in a slice.
We recall the Gauss equation of the immersion (see [14, Lemma 4]):

$$K = K_{\text{ext}} - n_3^2, \tag{3}$$

where K is the Gaussian curvature and K_{ext} is the extrinsic curvature.
Since M is a minimal surface, we have $K \leqslant -n_3^2$.

By assumption, the finite asymptotic boundary of M is an arc α, and there exists a simple arc $\alpha_0 \subset \partial_\infty^f M \setminus \partial_\infty(\partial M)$, $\alpha_0 \subset \alpha$, which is not contained in a vertical line. Let $p_\infty := (x_\infty, t_0) \in \alpha_0$ as in the statement.
For any $\varepsilon > 0$ we consider the minimal surface $S_\varepsilon \subset M$ given by Lemma 3.1.

Claim *For any real number $c \in (0,1)$, there exists $\varepsilon > 0$ such that $|n_3(p)| > c$ for any $p \in S_\varepsilon$. Consequently,*

$$|n_3(p)| \to 1, \text{ if } p \to p_\infty, \ p \in M. \tag{4}$$

Let us assume momentarily that the Claim holds.
Using the Claim above and the Gauss equation (3), we have

$$\int_M K\, dA \leqslant \int_{S_\varepsilon} K\, dA \leqslant -c^2 \text{Area}(S_\varepsilon).$$

By combining with Lemma 3.2, we deduce therefore that M has infinite total curvature, as desired. Thus it remains to prove the Claim.

Proof of the Claim.
Assume, by contradiction, that the Claim does not hold. Then, there exists a fixed number $c \in (0,1)$ such that for any $n \in \mathbb{N}^*$ there is a point $p_n \in S_{1/n}$ satisfying

$$|n_3(p_n)| \leqslant c \tag{5}$$

It follows from Lemma 3.1-(4) (or from its proof), that $p_n \to p_\infty$ as $n \to \infty$, see Definition 2.2.

Let $n_0 \in \mathbb{N}^*$ be a positive integer. We have $p_n \in S_{\frac{1}{n_0}}$ for any integer $n \geqslant n_0$ large enough. Therefore, up to extracting a subsequence, we can assume that for any $n \in \mathbb{N}^*$, $n > n_0$, we have $d_M(p_n, \partial S_{\frac{1}{n_0}}) > 1$, where d_M is the intrinsic distance on M.

From now on we consider the Poincaré disc model of $\mathbb{H}^2$. Letting $p_n := (x_n, t_n) \in \mathbb{H}^2 \times \mathbb{R}$, for any $n > n_0$ we denote by T_n the hyperbolic translation on $\mathbb{H}^2$ along the geodesic passing through x_n and 0, such that $T_n(x_n) = 0$. We also denote by T_n the horizontal translation of $\mathbb{H}^2 \times \mathbb{R}$ induced by this isometry of $\mathbb{H}^2$.

Now we proceed as in the proof of [31, Theorem 2.5].

Observe that for any $n > n_0$ the translated surface $T_n\left(S_{\frac{1}{n_0}}\right)$ is stable and oriented. We deduce from [25, Main Theorem] that, far away from the boundary, we have uniform a priori upper estimates of the norm of the second fundamental form of $T_n\left(S_{\frac{1}{n_0}}\right)$.

We consider $\mathbb{H}^2 \times \mathbb{R}$ as an open set of Euclidean space $\mathbb{R}^3$. We deduce from [25, Proposition 2.3] and from [31, Proposition A.1], that there exists a real number $\delta > 0$, which does not depend on n nor on n_0, such that for any $n > n_0$, a part Σ_n of $T_n\left(S_{\frac{1}{n_0}}\right)$ is the Euclidean graph of a function u defined on the disc centered at point $T_n(p_n)$ with Euclidean radius δ in the tangent plane of Σ_n at $T_n(p_n)$. Furthermore, the norm of the Euclidean gradient of the function u is bounded above by 1.

As a matter of fact, from the discussion after the proof of [3, Lemma 2.4], we get the following.

Fact: *for any $r \in (0,1)$ there exists $\delta(r) \in (0, \delta)$ such that the norm of the gradient of the function u is bounded above by r on the disc of Euclidean radius $\delta(r)$* (6)

Observe that we can use [3, Lemma 2.4] since we have a priori estimates for the norm of the Euclidean second fundamental form. Those estimates follow from [31, Proposition A.1].

Observe that, since $d_M\left(T_n(p_n), \partial T_n\left(S_{\frac{1}{n_0}}\right)\right) > 1$ for any $n > n_0 > 0$, the constant δ can be chosen so that $\Sigma_n \cap \partial T_n\left(S_{\frac{1}{n_0}}\right) = \emptyset$.

Let ν_n be the unitary normal along $T_n\left(S_{\frac{1}{n_0}}\right)$ in the Euclidean metric. We denote by $\nu_{n,3}$ the vertical component of ν_n. Recall that $|n_3(p_n)| \leqslant c$ for any $n > n_0 > 0$. Comparing the product metric of $\mathbb{H}^2 \times \mathbb{R}$ with the Euclidean metric, it can be shown that there exists $c' \in (0,1)$, which does not depend on n nor on n_0, such that $|\nu_{n,3}(T(p_n))| < c'$ for any $n > n_0 > 0$, (see the formula of the unit normal vector field of a vertical graph in the proof of [30, Proposition 3.2]).

This implies that the tangent planes of Σ_n at points $T(p_n)$ have Euclidean slope bounded from below uniformly (with respect to $n > n_0$).

By Lemma 3.1, $S_{\frac{1}{n_0}} \subset \mathbb{H}^2 \times (t_0 - \frac{1}{n_0}, t_0 + \frac{1}{n_0})$, thus the same occurs for any Σ_n. We infer therefore a contradiction with the fact (6) above since then, for n_0 large enough and $n > n_0$, the surface Σ_n would intersect $\mathbb{H}^2 \times \{t_0 \pm \frac{1}{n_0}\}$. This concludes the proof. $\square$

Remark 4.1. As a matter of fact, in Theorem 1.1 the stability assumption is only used to ensure a priori estimates for the second fundamental form of M. We think that stability is a hypothesis simpler to handle than bounded second fundamental form since, for example, any vertical or horizontal minimal graph is stable.

Remark 4.2. Given a bounded function g on $\partial_\infty \mathbb{H}^2 \times \{0\}$, continuous except perhaps at a finite set of points S, there exists a minimal entire extension u of g [28, Corollary 4.1, Remark 4-(2)]. We remark that the problem of Dirichlet at infinity (g is continuous) was solved by B. Nelli and H. Rosenberg [20], [21].
The Main Theorem 1.1 ensures that these entire graphs have infinite total curvature. However, the fact that all non trivial ($g \not\equiv \text{cst}$) such entire graphs have infinite total curvature, follows directly from Huber theorem [15, Theorem 15], see also [32, Théorème 2. 4. 10].
In fact, on the one hand, a complete simply connected minimal surface immersed into $\mathbb{H}^2 \times \mathbb{R}$ of finite total curvature is conformally equivalent to $\mathbb{C}$. On the other hand, it is well-known that the height function of a minimal surface M conformally immersed into $\mathbb{H}^2 \times \mathbb{R}$ is a harmonic function on M, see for instance [27, Proposition 7]. As there is no non constant bounded harmonic function over $\mathbb{C}$, the finite total curvature assumption leads to a contradiction.

Corollary 4.1. *Let M be a minimal graph in $\mathbb{H}^2 \times \mathbb{R}$ such that its finite asymptotic boundary is a graph over an arc of $\partial_\infty \mathbb{H}^2 \times \{0\}$ and is different from the asymptotic boundary of ∂M. Then M has infinite total curvature.*
Furthermore, for any interior point p_∞ of $\partial_\infty^f M$ such that $p_\infty \notin \partial_\infty^f(\partial M)$, we have $|n_3(p)| \to 1$ if $p \to p_\infty$, $p \in M$.

Corollary 4.2. *Let M be an oriented stable minimal surface immersed into $\mathbb{H}^2 \times \mathbb{R}$ with compact boundary (e.g. a minimal graph with compact boundary), whose asymptotic boundary is a (continuous) graph over the whole $\partial_\infty \mathbb{H}^2 \times \{0\}$. Then M has infinite total curvature.*
Furthermore, if $p_\infty \in \partial_\infty M$, we have $|n_3(p)| \to 1$ if $p \to p_\infty$, $p \in M$.

Remark 4.3. By Corollary 4.2, any minimal graph with compact boundary whose asymptotic boundary is a graph over $\partial_\infty \mathbb{H}^2 \times \{0\}$ has infinite (intrinsic) total curvature. We refer to [28, Theorem 5.1] for an existence result of such graphs, when the boundary is a Jordan curve $C \subset \mathbb{H}^2 \times \{0\}$ satisfying an "exterior circle of radius ρ condition". So, all such graphs have infinite total curvature.
In the particular case of the end of a catenoid, the result of Corollary 4.2 follows from an explicit computation carry out in [1, Proof of the Proposition 3.3].

5. Some examples of infinite total curvature minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and their asymptotic boundary

Next we exhibit complete and non-complete minimal surfaces M generated by vertical graphs, pointing out some geometric properties. All the examples have infinite total curvature. For this purpose we choose the Poincaré disc model of hyperbolic plane.

Example 5.1 (Main Example)**.** In this example we construct a minimal graph such that in a domain whose asymptotic boundary is a vertical segment the total curvature is finite, but any neighborhood of another part of the asymptotic boundary has infinite total curvature by the Main Theorem 1.1.
Let $\theta_0 \in (0, \pi/2)$ be a fixed number. Let $\gamma \subset \mathbb{H}^2$ be the geodesic with asymptotic boundary $\{1, e^{i\theta_0}\}$.
Let $U \subset \mathbb{H}^2$ be the domain whose boundary is the union of the geodesic rays $[0, 1)$ and $[0, i)$ with the geodesic γ, whose asymptotic boundary is the asymptotic arc $\gamma_\infty := \{e^{i\theta},\ \theta_0 \leqslant \theta \leqslant \pi/2\}$ of $\partial_\infty \mathbb{H}^2$ with the point 1.
Let $A > \pi$ be a real number to be chosen later. We consider the Dirichlet problem (P) on U with boundary data

- 0 on the geodesic rays $[0, 1)$ and $[0, i)$,
- $-A$ on the asymptotic arc γ_∞,
- $+\infty$ on the geodesic γ.

Using [28, Theorem 4.1] and the minimal surface M_1 described in [28, proposition 2.1 (2)], as in [28, Example 4.1] or as in [30, Theorem 5.1 (n=2)], we can solve the Dirichlet problem (P) above and find a solution $g : U \to \mathbb{R}$ whose finite asymptotic boundary of the graph, M, is

$$\Big(\{i\} \times [-A, 0]\Big) \cup \Big(\gamma_\infty \times \{-A\}\Big) \cup \Big(\{e^{i\theta_0}\} \times [-A, +\infty)\Big) \cup \Big(\{1\} \times [0, +\infty)\Big)$$

and the non finite asymptotic boundary of M is

$$\Big(\gamma \times \{+\infty\}\Big) \cup \Big(\{1, e^{i\theta_0}\} \times \{+\infty\}\Big).$$

We claim that the following phenomena hold.

Claim 1. Let p_∞, q_∞ be points in γ_∞ such that $p_\infty, q_\infty \neq e^{i\theta_0}, i$. Let $\alpha \subset \mathbb{H}^2$ be the geodesic whose asymptotic boundary is $\{p_\infty, q_\infty\}$. We call $U_1 \subset \mathbb{H}^2$ the component of $\mathbb{H}^2 \setminus \alpha$ whose asymptotic boundary is the subarc $[p_\infty, q_\infty]$ of γ_∞. We have $U_1 \subset U$. Let $S_1 \subset M$ be the graph of g restrited to U_1.
Then it follows from Corollary 4.1 that S_1 has infinite total curvature.
Furthermore, we have $|n_3(q)| \to 1$ if $q \to \gamma_\infty \setminus \{e^{i\theta_0}, i\}$, $q \in U_1$.

Claim 2. Let $S_2 \subset M$ be a domain such that its asymptotic boundary is a compact arc of $\Big(\{1\} \times [0, +\infty)\Big)$. Then it can be showed that S_2 has finite total curvature.

The proof of Claim (2) is rather long so we divide it in the statement and the proof of the following facts.

(1) *Let $V \subset U$ be a subdomain such that its asymptotic boundary is constituted of zero, one or two points of $\partial_\infty U$. If g is constant along the boundary of V, then g is constant on V, which leads to a contradiction.*

To prove this fact we use the maximum principle and the family of complete minimal surfaces M_d, $d > 1$, described in [28, Proposition 2.1-(1)] and in the proof of [22, Theorem 3.2].

(2) *The function g has no critical points on U.*

Indeed, if g would have a critical point $p \in U$, with $g(p) = c > -A$, then in a neighborhhod of p, the level set $g^{-1}(\{c\})$ is constituted of at least four analytic arcs issuing from p. Observe that any level set of g cannot have end points in U. Observe also that the asymptotic boundary of the level set $g^{-1}(\{c\})$ is included in $\{1, e^{i\theta_0}, i\}$. Therefore, continuing any of the analytic arcs issuing from p, we obtain a domain V as in item (1), which leads to a contradiction.

(3) *For any real number $c \in (-A, +\infty)$, the level curve $g^{-1}(\{c\})$ is constituted of an unique simple divergent curve in U and its asymptotic boundary is contained in $\{1, e^{i\theta_0}, i\}$.*

To proof this assertion, we first study the different possible cases of the level set $g^{-1}(\{0\})$. Then for each one of those cases we apply the items (1) and (2).

(4) *Using the reflection principle along the two geodesic rays starting from the origine and whose asymptotic boundary are $\{1\}$ and $\{i\}$ respectively, we obtain a complete minimal surface $\widetilde{M} \subset \mathbb{H}^2 \times \mathbb{R}$ which is a graph. Hence $\widetilde{M}$ is stable and from* [25, Main Theorem] *we obtain global upper estimates for the norm of the second fundamental form of M. Observe that those upper estimates do not depend on A.*

We denote by $(0,1)$ the open geodesic ray starting at 0 whose asymptotic boundary is 1. We denote by R the reflection in $\mathbb{H}^2 \times \mathbb{R}$ with respect to the geodesic ray $(0,1)$ and we set $M^* := M \cup (0,1) \cup R(M)$. Then M^* is a minimal surface which is a graph over the domain $U_1 := U \cup (0,1) \cup R(U)$ of $\mathbb{D}$.

(5) *For any $\rho > 0$ we set $\mathcal{Z}_\rho = \{\xi \in U_1,\ d_{\mathbb{H}^2}(\xi, \gamma) < \rho\}$. Then, for any $c \in (0,1)$ there exists $\rho_c > 0$ such that $|n_3(\xi)| < c$, for any $\xi \in \mathcal{Z}_{\rho_c}$. Furthermore the number ρ_c does not depend on A.*

Indeed, if the assertion is not true, there would exist a sequence (p_n) in U_1 such that

- $d_{\mathbb{H}^2}(p_n, \gamma) \to 0$,
- $|n_3(p_n)| \geqslant c$ for any n.

Let $\xi_0 \in \gamma$ be any fixed point and set $D_1 := \{\xi \in U,\ d_{\mathbb{H}^2}(\xi, \xi_0) < 1\}$.

Observe that for any n large enough we can use a translation T_n along the geodesic γ to take p_n to a point $T_n(p_n)$ in the domain D_1. By construction we have that $d_{\mathbb{H}^2}(T_n(p_n),\gamma)\to 0$. Using the global upper estimates for the norm of the second fundamental form of M, we can proceed as in the proof of the Claim in Theorem 1.1 to reach a contradiction. Since those upper estimates do not depend on A, we obtain also that the number ρ_c does not depend on A.

(6) *Let $c\in(0,1)$ be a fixed number and let $\rho_c>0$ be the positive real number given in item (5). We call $\alpha\in U$ the geodesic whose asymptotic boundary is $\{i,e^{i\theta_0}\}$. We denote by M_{d_A}, $d_A>1$, the surface of the family M_d, described in* [28, Proposition 2.1-(1)] *and in the proof of* [22, Theorem 3.2]*, such that*
- *the height of M_{d_A} is A,*
- *M_{d_A} is symmetric with respect to the slice $\mathbb{H}^2\times\{0\}$,*
- *for any $t\in(-\frac{A}{2},\frac{A}{2})$ the intersection $M_{d_A}\cap\big(\mathbb{H}^2\times\{t\}\big)$ is an equidistant curve of the geodesic α.*

Then, we have $M\cap M_{d_A}=\emptyset$. Consequently, we have $M\cap(M_{d_A}+(0,0,t))=\emptyset$ for any $t\geqslant 0$.

Observe that, using the notations of [28, Proposition 2.1-(1)] we have $A=2H(d_A)$. Moreover the asymptotic boundary of M_{d_A} is

$$\left(\{i,e^{i\theta_0}\}\times\Big[-\frac{A}{2},\frac{A}{2}\Big]\right)\cup\left(\gamma_\infty\times\Big\{-\frac{A}{2},\frac{A}{2}\Big\}\right).$$

where $\gamma_\infty:=\{e^{i\theta},\ \theta_0\leqslant\theta\leqslant\pi/2\}\subset\partial_\infty\mathbb{H}^2$.

Let $\theta_1\in(\theta_0,\pi/2)$ be a fixed number and let $\beta\subset\mathbb{H}^2$ be the geodesic whose asymptotic boundary is $\{-e^{i\theta_1},e^{i\theta_1}\}$.

To prove the first assertion we consider the hyperbolic translation along the geodesic β and proceed as in [22, Theorem 3.2]. The second assertion is a consequence of the first one.

(7) *Let $\delta_0\subset\mathbb{H}^2$ be the geodesic ray issuing from 0 and with asymptotic boundary $\{e^{i\theta_0}\}$. For any $r>0$ we denote by Q_r the vertical geodesic plane intersecting orthogonally δ_0 at distance r from 0. Let $Q_r^+\subset\mathbb{H}^2\times\mathbb{R}$ be the component of $(\mathbb{H}^2\times\mathbb{R})\setminus Q_r$ containing $e^{i\theta_0}$ in its asymptotic boundary.*

Fix $c\in(0,1)$ and let $\rho_c>0$ as in (5).

Then, if A is large enough, there exists $r>0$ so that

$$\big(M\cap Q_r^+\big)\cap\{t\geqslant 0\}\subset\mathcal{Z}_{\rho_c}\times[0,+\infty).\tag{7}$$

The proof of the assertion is based upon the following observation.

Since $\rho_c>0$ does not depend on A, observe that for $A>0$ large enough we have $M_{d_A}\cap\big(\mathcal{Z}_{\rho_c}\times[0,+\infty)\big)\neq\emptyset$. For such a number A, using the last affirmation of item (6) certainly we can find a number $r>0$ large enough satisfying (7).

Before going on with the proof of Claim 2, we need to recall some facts derived from [12], [14] and [31].

Let $X : \mathbb{D} \to \mathbb{H}^2 \times \mathbb{R}$ be a conformal parametrization of M. We set as in [31] $X = (F, h)$, thus $F : (\mathbb{D}, g_{\text{euc}}) \to \mathbb{H}^2$ is a harmonic map and $h : \mathbb{D} \to \mathbb{R}$ is a harmonic function, where g_{euc} is the Euclidean metric.

Since g has no critical point on U, we have $|n_3| \neq 1$ along M. Therefore we can define a real function ω on $\mathbb{D}$ by the relation: $\tanh \omega = n_3$.

We consider also the function ϕ on $\mathbb{D}$ defined by $\phi := (\sigma \circ F) F_z \overline{F}_z$, where σ is the conformal factor of the hyperbolic metric of $\mathbb{H}^2$. Since F is a harmonic map, ϕ is a holomorphic function.

The metric induced on $\mathbb{D}$ by the immersion X is

$$ds^2 = 4 \cosh^2(\omega) \, |\phi| \, |dz|^2.$$

Moreover we have $\phi(z) = -(h_z(z))^2$, [27, Proposition 1]. Now we define a holomorphic function W on $\mathbb{D}$ setting

$$W(z) = \int \sqrt{\phi(z)} \, dz,$$

where the square root of ϕ is chosen so that

$$h = 2 \operatorname{Im} W(z) \tag{8}$$

(8) *The function W is a univalent map, hence W is a holomorphic diffeomorphism between $\mathbb{D}$ and the open subset $\widetilde{\Omega} := W(\mathbb{D})$ of $\mathbb{C}$.*

It follows from item (3) that for any $c \in (-A, +\infty)$, the level curve $h^{-1}(\{c\})$ is constituted of a unique simple divergent curve in $\mathbb{D}$. We deduce from item (2) that h has no critical point. Consequently the conjugate function ${}^*h$ is strictly monotonous along any level curve of h. Combining this with Formula (8) we conclude that W is an univalent map.

Now we define the function $\widetilde{\omega}$ on $\widetilde{\Omega}$ setting

$$\widetilde{\omega} := \omega \circ W^{-1}. \tag{9}$$

We know from [12, Formula (12)] that the function $\widetilde{\omega}$ satisfies

$$\Delta \widetilde{\omega} = 2 \sinh 2\widetilde{\omega}, \tag{10}$$

where Δ is the Laplacian for the Euclidean metric.

We consider also the new conformal parametrization $\widetilde{X} : \widetilde{\Omega} \to \mathbb{H}^2 \times \mathbb{R}$ of M given by $\widetilde{X} = X \circ W^{-1}$. Denoting by w the coordinate on $\widetilde{\Omega}$, the induced metric on $\widetilde{\Omega}$ reads as

$$d\widetilde{s}^2 = 4 \cosh^2 \widetilde{\omega}(w) \, |dw|^2, \tag{11}$$

We define also the function $W_0 : U \to \widetilde{\Omega} \subset \mathbb{C}$ setting $W_0 := W \circ (\Pi \circ X)^{-1}$, where $\Pi : \mathbb{H}^2 \times \mathbb{R} \to \mathbb{H}^2$ is the first projection. As a matter of fact, W_0 is nothing but the function W read on U, in particular W_0 is an open map. Observe also that by means of the reflections with respect to the geodesic rays $(0, 1)$ and $(0, i)$ issuing from 0 with asymptotic boundary $\{1\}$ and $\{i\}$ respectively, the map W_0 can be extended to a larger open set $\widehat{U} \subset \mathbb{D}$ containing U and the open geodesic rays, and this extended map is still an open map.

Observe also that $g = 2\,\mathrm{Im}\, W_0$.

Since $g = h \circ (\Pi \circ X)^{-1}$, we define the "conjugate function" ${}^*g$ setting ${}^*g := {}^*h \circ (\Pi \circ X)^{-1}$. Since g has no critical point on U, we observe that ${}^*g$ is strictly monotonous on any level curve of g. From the relation (8) we get that ${}^*g = -2\,\mathrm{Re}\, W_0$.

(9) *The level curve $L_0 := g^{-1}(\{0\})$ cannot be a simple curve with asymptotic boundary the set $\{1, e^{i\theta_0}\}$. Consequently, the level curve L_0 must have one of the following behaviors.*
- $\partial_\infty L_0 = \{e^{i\theta_0}, i\}$,
- $\partial_\infty L_0 = \{e^{i\theta_0}\}$ *and L_0 has an end point on* $[0,1) \cup (0,i)$.

Let us assume, by contradiction, that $\partial_\infty L_0 = \{1, e^{i\theta_0}\}$. Let $p_0 \in L_0$ be a fixed point. We denote by L_0^+ and L_0^- the components of $L_0 \setminus \{p\}$ with asymptotic boundary 1 and $e^{i\theta_0}$ respectively.

Let $c \in (0,1)$, $\rho_c > 0$ fixed as in (5) and (6) so that $|n_3(\xi)| < c$ for any $\xi \in \mathcal{Z}_{\rho_c}$.

We deduce from item (7) that, up to extracting a compact part, we can assume that $L_0^+ \subset \mathcal{Z}_{\rho_c}$ and $L_0^- \subset \mathcal{Z}_{\rho_c}$.

Since the graph of L_0^+ has infinite length in M, then (11) implies that the curve $W_0(L_0^+) \subset \widetilde{\Omega}$ must have infinite length as well, for the metric $\widetilde{ds}^2$.

As $|n_3| < c$ on $\mathcal{Z}_{\rho_c}$, we get that the function $\widetilde{\omega}$ is bounded on $\mathcal{Z}_{\rho_c}$. Thus, the curve $W_0(L_0^+) \subset \widetilde{\Omega} \subset \mathbb{C}$ must have infinite Euclidean length. We deduce that

$$\mathrm{Re}\, W_0(\xi) \to \pm\infty, \quad \text{if } \xi \to 1,\ \xi \in L_0.$$

Without loss of generality, we can assume that $\mathrm{Re}\, W_0(\xi) \to +\infty$ when $\xi \to 1$, $\xi \in L_0$.

In the same way, and using the fact that $\mathrm{Re}\, W_0$ is strictly monotonous on any level curve of g, we get that

$$\mathrm{Re}\, W_0(\xi) \to -\infty, \quad \text{if } \xi \to e^{i\theta_0},\ \xi \in L_0.$$

Consequently the image of the level curve L_0, $W_0(L_0)$, is the whole real axis in $\mathbb{C}$, that is $W_0(L_0) = \mathbb{R} \subset \mathbb{C}$. We get a contradiction as follows.

Note that g can be extended across the geodesic ray $(0,1)$ by means of the reflection principle. Note also that the critical points of the extended map, if any, are isolated.

Let $p_1 \in (0,i)$ be a fixed point in the geodesic ray which is not a critical point of g. Thus W_0 is a local diffeomorphism near p_1. Let $\mathcal{O}_1$ be an open neighborhood of p_1 such that W_0 is one-to-one on $\mathcal{O}_1$. Since $W_0(L_0) = \mathbb{R} \subset \mathbb{C}$, there exists $p_2 \in L_0$ such that $W_0(p_1) = W_0(p_2)$. Now let $\mathcal{O}_2 \subset U$ be any open neighborhood of p_2. As W_0 is an open map, we get that $W_0(\mathcal{O}_1) \cap W_0(\mathcal{O}_2)$ is an open set containing $W_0(p_2)$, which gives a contradiction with the fact W_0 is one-to-one on U.

(10) *Let* $U^+ := \{\xi \in U,\ g(\xi) > 0\}$ *and let* $\Omega^+ := \{w \in \mathbb{C}, \operatorname{Im} w > 0\}$. *Then* $W_0(U^+) = \Omega^+ \subset \widetilde{\Omega}$.

Indeed, for any $c > 0$ we set $L_c := g^{-1}(\{c\})$. Thus $W_0(L_c)$ is contained in the horizontal line of Ω^+ at height $2c$, that is $W_0(L_c) \subset \{w \in \mathbb{C}, \operatorname{Im} w = 2c\}$, since $g = 2 \operatorname{Im} W_0$.

With the same arguments used for item (9), we can prove that $W_0(L_c)$ is the whole line $\{w \in \mathbb{C}, \operatorname{Im} w = 2c\}$. We conclude that $W_0(U^+) = \Omega^+ \subset \widetilde{\Omega}$.

(11) *Let* $p_0 \in (0,1)$ *be any fixed point of the geodesic ray* $(0,1)$, *such that* $u_0 := W_0(p_0) > 0$. *We consider the subset* $\Omega_1 := \{w \in \Omega^+,\ \operatorname{Re} w > u_0\}$. *Let* $S_2 \subset M$ *such that* $S_2 = \widetilde{X}(\Omega_1)$, *(see the discussion after the item (8)). Then,* S_2 *has finite total curvature.*

Since $\widetilde{X} : (\widetilde{\Omega}, \widetilde{ds^2}) \to \mathbb{H}^2 \times \mathbb{R}$ is an isometric immersion, it is equivalent to prove that Ω_1 has finite total curvature with respect to the metric $\widetilde{ds^2}$.

For any $C > 0$, we consider the square $R(C) \subset \Omega_1$ with horizontal sides $H_0(C)$ and $H_1(C)$ and vertical sides $V_0(C)$ and $V_1(C)$ defined by

- $H_0(C)$ is the horizontal segment with end points u_0 and $u_0 + C$,
- $H_1(C)$ is the horizontal segment with end points $u_0 + iC$ and $u_0 + C + iC$,
- $V_0(C)$ is the vertical segment with end points u_0 and $u_0 + iC$,
- $V_1(C)$ is the vertical segment with end points $u_0 + C$ and $u_0 + C + iC$,

The Gauss-Bonnet theorem applied to the square $R(C)$ gives

$$\int_{R(C)} K\, dA = -\int_{\partial R(C)} k_g\, ds, \tag{12}$$

where dA is the area element of $(\widetilde{\Omega}, \widetilde{ds^2})$, K is the Gaussian curvature, k_g is the geodesic curvature along $\partial R(C)$ parametrized by the arc length s. Hence it suffices to show that the right hand-side integral in (12) is bounded if $C \to +\infty$.

We have

$$\int_{\partial R(C)} k_g\, ds = \int_{H_0(C)} k_g\, ds + \int_{H_1(C)} k_g\, ds + \int_{V_0(C)} k_g\, ds + \int_{V_1(C)} k_g\, ds.$$

Since the geodesic ray $(0,1)$ is a geodesic of M, we get that $H_0(C)$ is a geodesic of S for any $C > 0$.Thus

$$\int_{H_0(C)} k_g\, ds = 0$$

for any $C > 0$. We are going to prove that the integral on $V_0(C)$ is bounded when $C \to +\infty$.

We choose the following parametrization of $V_0(C)$,

$$\gamma(t) = u_0 + i\, tC, \quad t \in [0,1].$$

Let $w = u + iv$ be the coordinates on $\widetilde{\Omega}$. We deduce from the expression of the metric $d\widetilde{s}^2$ (see Formula (11)) and from [17, Formula (42.8)], that the geodesic curvature of the curve γ is given by

$$k_g(\gamma(t)) = \pm \frac{\sinh \widetilde{\omega}}{2\cosh^2 \widetilde{\omega}} \frac{\partial \widetilde{\omega}}{\partial u}\big(\gamma(t)\big). \tag{13}$$

Let $\Theta \subset \mathbb{C}$ be any domain on which the function $\widetilde{\omega}$ is defined and satisfies the equation (10). For any $w \in \Theta$ we denote by $d(w, \partial\Theta)$ the Euclidean distance between w and $\partial\Theta$. It is shown in the proof of [12, Proposition 2.3] that there exists a positive constant δ such that for any $w \in \Theta$ with $d(w, \partial\Theta) > 2$ we have

$$|\nabla \widetilde{\omega}|(w) < \delta e^{-d(w,\partial\Theta)},$$

where ∇ means the Euclidean gradient.

Hence, choosing $\Theta = \Omega^+$ we get

$$|\nabla \widetilde{\omega}|(w) < \delta e^{-\operatorname{Im} w} \tag{14}$$

for any $w \in \Omega^+$ such that $\operatorname{Im} w > 2$. For any $C > 3$ we have

$$\int_{V_0(C)} k_g \, ds = 2\int_0^1 k_g(\gamma(t)) \cosh \widetilde{\omega}(\gamma(t))\, |\gamma'(t)|\, dt =$$

$$2\int_0^{3/C} k_g(\gamma(t)) \cosh \widetilde{\omega}(\gamma(t))\, |\gamma'(t)|\, dt + 2\int_{3/C}^1 k_g(\gamma(t)) \cosh \widetilde{\omega}(\gamma(t))\, |\gamma'(t)|\, dt.$$

From the one hand, since $V_0(C)$ is a smooth curve, there exists a constant number $a > 0$ such that

$$2\int_0^{3/C} |k_g(\gamma(t))| \cosh \widetilde{\omega}(\gamma(t))\, |\gamma'(t)|\, dt < a,$$

for any $C > 4$. On the other hand, from the formulae (13) and (14) we get for any $t > 3/C$

$$2|k_g(\gamma(t))| \cosh \widetilde{\omega}(\gamma(t)) \leqslant |\nabla \widetilde{\omega}|(\gamma(t)) < \delta e^{-tC}.$$

Therefore

$$\begin{aligned}
2\int_{3/C}^1 |k_g(\gamma(t))| \cosh \widetilde{\omega}(\gamma(t))\, |\gamma'(t)|\, dt &\leqslant 2\delta C \int_{3/C}^1 e^{-tC}\, dt \\
&\leqslant 2\delta C\, \frac{(e^{-3} - e^{-C})}{C} \\
&\leqslant 2\delta(e^{-3} - e^{-C}) \\
&\leqslant 2\delta e^{-3}.
\end{aligned}$$

This proves that the integral $\int_{V_0(C)} k_g \, ds$ is bounded when $C \to +\infty$.

Choosing again $\Theta = \Omega^+$, we can prove in the same way that $\int_{H_1(C)} k_g \, ds \to 0$ when $C \to +\infty$.

Finally, recall that the minimal surface M can be extended across the geodesic ray $(0,1)$ by means of the reflection principle. Therefore the function $\widetilde{\omega}$ can be extended to the domain $\Theta := \{z \in \mathbb{C},\ \mathrm{Re}(z) > u_0\}$. Consequently we can prove that $\int_{V_1(C)} k_g\, ds \to 0$ when $C \to +\infty$.

We conclude that S_2 has finite total curvature and Claim 2 is proved.

The following peculiar examples are obtained by applying the reflection principle on suitable minimal graphs.

Example 5.2. *M is non-complete, properly embedded and its asymptotic boundary is the union of a discrete set in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$, with*

- *either the whole $(\mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2) \times \{-\infty, +\infty\}$,*
- *or a finite subset of $\partial_\infty \mathbb{H}^2 \times \{-\infty, +\infty\}$ and $\bigcup_{i=1}^n \gamma_i \times \{-\infty, +\infty\}$, where $\gamma_i \subset \mathbb{H}^2$ is a complete geodesic, $i = 1, \dots, n$.*

To obtain such a surface we consider, for $\theta \in (0, \pi/2)$ the geodesic triangle T with vertices 0, 1 and $e^{i\theta}$. Let $c > 0$ be a positive real number. Let $\gamma \subset \mathbb{H}^2$ be the geodesic with asymptotic boundary the points 1 and $e^{i\theta}$.
Let $f : \gamma \to \mathbb{R}$ be a continuous and one to one function, such that $f(\xi) \to 0$ if $\xi \to e^{i\theta}$ and $f(\xi) \to c$ if $\xi \to 1$.
We consider the Dirichlet problem for the minimal surface equation on $\mathrm{int}(T)$ taking the boundary data

$$
\begin{aligned}
&- c \ \text{ on the geodesic ray } (0,1),\\
&- 0 \ \text{ on the geodesic ray } (0, e^{i\theta}),\\
&- f \ \text{ on } \gamma.
\end{aligned}
$$

We deduce from [28, Theorem 4.1] that there exists a solution u to this problem. Thus the graph S of u is a minimal surface whose boundary contains the geodesic rays $(0,1) \times \{c\}$ and $(0, e^{i\theta}) \times \{0\}$, the vertical segment $\{(0,t) \in \mathbb{H}^2 \times \mathbb{R},\ 0 \leqslant t \leqslant c\}$ and the graph of f on γ. Now we perform the reflections of S with respect to the geodesic rays $(0,1)$, $(0, e^{i\theta})$, the vertical geodesic $\{0\} \times \mathbb{R}$ and the new geodesic rays appearing in this process.
In this way we get a non complete and properly embedded minimal surface M invariant by a discrete group of screw-motions. The finite asymptotic boundary is a discrete set of $\mathbb{H}^2 \times \mathbb{R}$
To describe the non finite asymptotic boundary of M we consider two cases.

- If the angle θ/π is irrational then the non finite asymptotic boundary is the whole $(\mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2) \times \{-\infty, +\infty\}$.
- If the angle θ/π is rational then the non finite asymptotic boundary is composed of a finite subset $\{\pm\xi_1, \dots, \pm\xi_n\} \times \{-\infty, +\infty\}$ of $\partial_\infty \mathbb{H}^2 \times \{-\infty, +\infty\}$ and $\bigcup_{i=1}^n \gamma_i \times \{-\infty, +\infty\}$, where $\gamma_i \subset \mathbb{H}^2$ is the complete geodesic with asymptotic boundary $\{-\xi_i, \xi_i\}$, $i = 1, \dots, n$.

Since S is a vertical graph we deduce from [14, Corollary 5] that $K < 0$ on S, where K is the intrinsic Gaussian curvature of S. Therefore S has non zero total curvature. Observe that M is composed of an infinite number of isometric copies of S. We deduce that M has infinite total curvature.

Example 5.3. *M is complete, properly embedded and its finite asymptotic boundary consists of the union of two helix type curves. The rest of the asymptotic boundary consists*

- *either of the whole $(\mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2) \times \{-\infty, +\infty\}$,*
- *or of a finite subset of $\partial_\infty \mathbb{H}^2 \times \{-\infty, +\infty\}$ and $\bigcup_{i=1}^n \gamma_i \times \{-\infty, +\infty\}$, where $\gamma_i \subset \mathbb{H}^2$ is a complete geodesic, $i = 1, \dots, n$.*

Let $\theta \in (0, \pi/2)$ be a fixed number. We denote by $\Gamma_\theta \subset \partial_\infty \mathbb{H}^2$ the closed arc of $\partial_\infty \mathbb{H}^2$ bounded by 1 and $e^{i\theta}$ which does not contain i.
Let $D_\theta \subset \mathbb{H}^2$ be the domain bounded by the geodesic rays $(0, 1)$ and $(0, e^{i\theta})$ and whose asymptotic boundary is Γ_θ.
Let $f : \Gamma_\theta \to \mathbb{R}$ be a continuous and one to one function, such that $f(e^{i\theta}) = 0$ and and $f(1) = c$, where $c > 0$ is a positive number.
We consider the Dirichlet problem for the minimal surface equation on D_θ taking the boundary data

$$
\begin{aligned}
&- c \ \text{ on the geodesic ray } (0, 1),\\
&- 0 \ \text{ on the geodesic ray } (0, e^{i\theta}),\\
&- f \ \text{ on } \Gamma_\theta.
\end{aligned}
$$

We deduce from [28, Theorem 4.1] that there exists a solution u to this problem. Thus the graph S of u is a minimal surface whose boundary contains the geodesic rays $(0, 1)$ and $(0, e^{i\theta})$ and the vertical segment $\{(0, t) \in \mathbb{H}^2 \times \mathbb{R},\ 0 \leqslant t \leqslant c\}$. Since a vertical graph is stable, we deduce from Theorem 1.1 that S has infinite total curvature.
Now we perform the reflections of S with respect to the vertical geodesic $\{0\} \times \mathbb{R}$ and with respect to the geodesic rays $(0, 1)$, $(0, e^{i\theta})$ and the new geodesic rays appearing in this process.
In this way we get a complete and properly embedded minimal surface M invariant by a discrete group of screw-motions. The finite asymptotic boundary is composed of to “helix type” curves.
To describe the non finite asymptotic boundary of M we consider two cases.

- If the angle θ/π is irrational then the non finite asymptotic boundary is the whole $(\mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2) \times \{-\infty, +\infty\}$.
- If the angle θ/π is rational then the non finite asymptotic boundary is composed of a finite subset $\{\pm\xi_1, \dots, \pm\xi_n\} \times \{-\infty, +\infty\}$ of $\partial_\infty \mathbb{H}^2 \times \{-\infty, +\infty\}$ and $\bigcup_{i=1}^n \gamma_i \times \{-\infty, +\infty\}$, where $\gamma_i \subset \mathbb{H}^2$ is the complete geodesic with asymptotic boundary $\{-\xi_i, \xi_i\}$, $i = 1, \dots, n$.

Moreover M has infinite total curvature since $S \subset M$.

Example 5.4. *M is non properly immersed and its asymptotic boundary is an annulus $\partial_\infty \mathbb{H}^2 \times [-a, a]$, where $a > 0$.*

In order to construct such an example, we proceed as in Example 5.3 above, setting $c = 0$, $f(1) = a$, $f(e^{i\theta}) = 0$ and θ/π is irrational. Observe that this surface is complete far away from the origin. Furthermore this surface has infinite total curvature for the same reason as in Example 5.3.

Example 5.5. *The asymptotic boundary of M is either $\partial_\infty(\mathbb{H}^2 \times \mathbb{R}) \setminus (D \times \{-\infty, +\infty\})$, where D is an open geodesic disc of $\mathbb{H}^2$ or the whole asymptotic boundary of $\mathbb{H}^2 \times \mathbb{R}$.*

We proceed as in Example 5.2 above, now $c \geqslant 0$ is a nonnegative constant and $f \equiv +\infty$ on the geodesic γ.
It can be shown, using [28, Theorem 4.1] that this Dirichlet problem has a solution.
We choose θ such that θ/π is irrational.
After performing all reflections, we get a minimal surface M. To describe the surface M we consider two cases.

- $c = 0$. In this case M is complete far away from the origin and is non properly immersed. Its asymptotic boundary is $\partial_\infty(\mathbb{H}^2 \times \mathbb{R}) \setminus (D \times \{-\infty, +\infty\})$, where $D \subset \mathbb{H}^2$ is the open geodesic disc centered at 0 and such that $D \cap \gamma = \emptyset$ and $\overline{D} \cap \gamma \neq \emptyset$.
- $c > 0$. In this case M is complete, properly immersed, and its asymptotic boundary is the whole asymptotic boundary of $\mathbb{H}^2 \times \mathbb{R}$.

It can be shown as in Example 5.2 that M has infinite total curvature.

Example 5.6. *M is complete and dense in $\mathbb{H}^2 \times \mathbb{R}$. Therefore its asymptotic boundary is the whole asymptotic boundary of $\mathbb{H}^2 \times \mathbb{R}$.*

We proceed as in Example 5.2 above with the following modifications.

- $f \equiv +\infty$ on the geodesic γ.
- On the geodesic ray $(0, e^{i\theta})$ we consider the constant boundary data 0.
- On the geodesic ray $(0, 1)$ we consider the boundary data g given by

$$g = \begin{cases} \pi \text{ on } (0, 1/3) \\ 1 \text{ on } (1/3, 2/3) \\ 0 \text{ on } (2/3, 1) \end{cases}$$

We choose θ such that θ/π is irrational.
It can be shown, using [28, Theorem 4.1] that this Dirichlet problem has a solution.
The complete minimal surface M obtained by doing all reflections is dense in $\mathbb{H}^2 \times \mathbb{R}$.
Moreover it can be shown as in Example 5.2 that M has infinite total curvature.

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Departamento de Matemática
Pontifícia Universidade Católica do Rio de Janeiro
Rio de Janeiro
22453-900 RJ
Brazil
E-mail address: rsaearp@gmail.com

Institut de Mathématiques de Jussieu - Paris Rive Gauche
Université Paris Diderot - Paris 7
Equipe Géométrie et Dynamique, UMR 7586
Bâtiment Sophie Germain
Case 7012
75205 Paris Cedex 13
France
E-mail address: eric.toubiana@imj-prg.fr