ON THE STRUCTURE OF CERTAIN WEINGARTEN SURFACES WITH BOUNDARY A CIRCLE

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INTRODUCTION

We study in this paper a certain class of surfaces M in R^3 satisfying a Weingarten relation of the form

$$H = f(H^2 - K) \tag{1}$$

where H is the mean curvature, K is the Gaussian curvature and f is a real smooth function defined on a interval $[-\epsilon, \infty), \epsilon > 0$.

Furthermore, we require that f satisfies the inequality

$$4t(f'(t))^2 < 1 \tag{2}$$

We call such a function f, elliptic, when it satisfies (2). The reason for this denomination is that equation (1) and inequality (2) give rise to a fully nonlinear elliptic equation. We call M a special surface when M satisfies $H = f(H^2 - K)$ for f elliptic. They have been studied by Hopf [8], Hartman and Wintner [7], Chern [5] and Bryant [3]. Here, we extend some results for constant mean curvature surfaces obtained in [2] and [6],when M is topologically a disk. Precisely we prove the following theorems:

Theorem 1:*Let M be a disk type special surface immersed in R^3. Assume ∂M is a circle S^1 of radius 1. Suppose f is analytic with $f(0) > 0$. Then*

  a) $f(0) \leq 1$
  b) *If $f(0) = 1, M$ is a halfsphere*

Theorem 2: *Let M be a disk type special surface embedded in R^3. Assume ∂M is a circle S^1 of radius 1 contained in the horizontal plane $\mathcal{H} = \{z = 0\}$. Suppose $f > 0$, $f(0) > 0$ and M cuts transversely $\mathcal{H}$ along ∂M. Then M is a spherical cap.*

We remark that the ellipticity condition (2) on M allow us to apply maximum principle (for special surfaces), and Alexandrov reflection principle techniques as it was applied in [6] and [10],for constant mean curvature surfaces (see Hopf's book [8] for further details). Futhermore, we notice that R.Bryant constructed a global quadratic form Q on a surface M satisfying (1) such that the zeros of Q are the umbilical points of M (see [3]). These facts emphasize the analogy between special surfaces and constant mean curvature surfaces. Now we state and prove the *maximum principle for special Weingarten surfaces in R^3* satisfying (1) and (2) in the form we shall need: If M_1, M_2 are tangent at p, M, on one side of M_2 near p, both M_1, M_2 satisfying (1) and (2) with respect to the same normal N at p then $M_1 = M_2$ near p. By a standard argument $M_1 = M_2$ everywhere.

INTERIOR MAXIMUM PRINCIPLE

Suppose M_1, M_2 are C^2 surfaces in R^3, which are given as graphs of C^2 functions $u, v : \Omega \subset R^2 \to R$.

Suppose the tangent planes of both M_1, M_2 agree at a point (x, y, z) ; $i.e$ $T_{(x,y,z)}M_1 = T_{(x,y,z)}M_2$ for $z = u(x, y) = v(x, y), (x, y) \in \Omega$.

Let $H(N_1)$ and $H(N_2)$ be the mean curvature functions of u and v with respect to unit normals N_1 and N_2 that agree at (x, y, z). Let K_i be the Gaussian curvature of $M_i, i = 1, 2$.

Suppose M_i satisfy

$$H(N_i) = f(H_i^2 - K_i), i = 1, 2$$

for f satisfying (2).

If $u \leq v$ near (x, y) then $M_1 = M_2$ near $(x, y, z), i.e, u = v$ in a neighbourhood of (x, y).

BOUNDARY MAXIMUM PRINCIPLE

Suppose M_1, M_2 as in the statement of the interior maximum principle with C^2 boundaries B_1, B_2 given by restrictions of u and v to part of the boundary $\partial\Omega$.

Suppose $T_{(x,y,z)}M_1 = T_{(x,y,z)}M_2$ and $T_{(x,y,z)}B_1 = T_{(x,y,z)}B_2$ for $z = u(x, y) = v(x, y)$, with (x, y, z) in the interior of both B_1 and B_2.

Suppose M_1, M_2 satisfy (1) and (2) with respect the same normal N at (x, y, z).

If $u \leq v$ near (x, y) then $M_1 = M_2$ near $(x, y, z), i.e, u = v$ in a neighbourhood of (x, y).

PROOF OF THE INTERIOR AND BOUNDARY MAXIMUM PRINCIPLE

Clearly, by applying a rigid motion of R^3 which do not change the geometry of the statements, we may suppose the tangent planes of both M_1, M_2 at (x, y, z) are the horizontal xy plane $P = \{z = 0\}$, and the unit normals N_1, N_2 at (x, y, z) are equal to $N = (0, 0, 1)$.

First, we fix some notations. We denote

$$p_1 = \frac{\partial u}{\partial x}, q_1 = \frac{\partial u}{\partial y}, p_2 = \frac{\partial v}{\partial x}, q_2 = \frac{\partial v}{\partial y}$$

$$r_1 = \frac{\partial^2 u}{\partial x^2}, \tau_1 = \frac{\partial^2 u}{\partial y^2}, s_1 = \frac{\partial^2 u}{\partial x \partial y}$$

$$r_2 = \frac{\partial^2 v}{\partial x^2}, \tau_2 = \frac{\partial^2 v}{\partial y^2}, s_2 = \frac{\partial^2 v}{\partial x \partial y}$$

With this convention the normals N_1 and N_2 are given by

$$N_i = \frac{1}{(1+p_i^2+q_i^2)^{\frac{1}{2}}}(-p_i, -q_i, 1), \quad i = 1,2.$$

The mean curvature H_i and the Gaussian curvature K_i are given by

$$2H_i = \frac{1}{(1+p_i^2+q_i^2)^{\frac{3}{2}}}\left((1+p_i^2)\tau_i - 2p_iq_is_i + (1+q_i^2)r_i\right)$$

$$K_i = \frac{1}{(1+p_i^2+q_i^2)^2}(r_i\tau_i - s_i^2)$$

for $i = 1,2$.

We may write equation (1) for M_1 and M_2 in the following way

$$F(p_i, q_i, r_i, s_i, \tau_i) = H_i - f(H_i^2 - K_i) = 0 \tag{3}$$

for $i = 1,2$, where F is a C^1 function in the p, q, r, s, τ variables.
We fix $(x,y) \in \Omega$ and we define for $t \in [0,1]$

$$\alpha(t) = F(tp_1 + (1-t)p_2, tq_1 + (1-t)q_2, tr_1 + (1-t)r_2, ts_1 + 1(1-t)s_2, t\tau_1 + (1-t)\tau_2) \tag{4}$$

Let $w = u - v$.
By applying the mean value theorem, using equation (3) and differentiating equation (4) we are led to the linearized operator on Ω defined by

$$Lw := \frac{\partial F}{\partial r}(\xi)\frac{\partial^2 w}{\partial x^2} + \frac{\partial F}{\partial s}(\xi)\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial F}{\partial \tau}(\xi)\frac{\partial^2 w}{\partial y^2} + \frac{\partial F}{\partial p}(\xi)\frac{\partial w}{\partial x} + \frac{\partial F}{\partial q}(\xi)\frac{\partial w}{\partial y} = 0 \tag{5}$$

where

$$\xi = (p, q, r, s, \tau),$$
$$p = cp_1 + (1-c)p_2, q = cq_1 + (1-c)q_2$$
$$r = cr_1 + (1-c)r_2, s = cs_1 + (1-c)s_2, \tau = c\tau_1 + (1-c)\tau_2$$

for $0 < c(x,y) < 1$.
Notice that the principal part of L is given by the symmetric matrix

$$A = A(p, q, r, s, \tau) = \begin{bmatrix} \dfrac{\partial F}{\partial r} & \dfrac{1}{2}\dfrac{\partial F}{\partial s} \\ \dfrac{1}{2}\dfrac{\partial F}{\partial s} & \dfrac{\partial F}{\partial \tau} \end{bmatrix}$$

Computations show that if $p = q = 0$, then trace $A = 1$ and det $A = \frac{1}{4}(1 - 4t(f'(t))^2)$, where

$$t = \left[\frac{(1 + p^2)\tau - 2pqs + (1 + q^2)r}{2(1 + p^2 + q^2)^{\frac{3}{2}}} \right]^2 - \frac{1}{(1 + p^2 + q^2)^2}(r\tau - s^2) \qquad (6)$$

Now, consider in formula (6)

$$p = cp_1 + (1 - c)p_2, q = cq_1 + (1 - c)q_2$$
$$r = cr_1 + (1 - c)r_2, s = cs_1 + (1 - c)s_2, \tau = c\tau_1 + (1 - c)\tau_2,$$

where $p_i, q_i, r_i, s_i, \tau_i$ are varying in a neighbourhood of (x, y) and c is varying in the interval $[0,1]$. We see easily that the non negative quantity $t = t(p, q, r, s, \tau)$ is bounded from above. Hence $1 - 4t(f'(t))^2 \geq \mu > 0$ in this neighhourhood (c is varying between 0 and 1), for some positive real number μ. As $p_i = q_i = 0$ at $(x, y), i = 1, 2$, by continuity we have that in a neighbourhood V of (x, y) the matriz $A(\xi)$ is positive definite. Furthermore, there is a positive real number λ_0 such that

$$\frac{\partial F}{\partial r}(\xi)\eta_1^2 + \frac{\partial F}{\partial s}(\xi)\eta_1\eta_2 + \frac{\partial F}{\partial \tau}(\xi)\eta_2^2 \geq \lambda_0(\eta_1^2 + \eta_1^2)$$

for any (x, y) in V and any real numbers η_1, η_2. Consequently, L is a linear second order uniformly elliptic operator with bounded coefficients in a neighbourhood of (x, y). The same conclusion hold if (x, y) is a boundary point as in the hypothesis of the boundary maximum principle statement.

Finally we have in a neighbourhood of (x, y)

$$Lw = 0$$
$$w \leq 0 \quad , \quad w(x, y) = 0$$

If (x, y) is a interior point then $w = u - v = 0$ in a neighbourhood of (x, y), by applying the interior maximum principle of Hopf.

If (x, y) is a boundary point lying in the interior of a C^2 portion contained in Ω, then w attaint again a local maximum at (x,y) with $\frac{\partial w}{\partial \nu}(x, y) = 0$, where ν is the exterior unit normal to Ω at (x, y). This implies by using the boundary maximum principle of Hopf that $w = 0$ in a neighbourhood of (x, y), as desired. We conclude the proof of the maximum principal for special Weingarten surfaces in R^3.

We remark that the maximum principle above led to *Alexandrov theorem for special Weingarten surfaces*. That is, a closed embedded special Weingarten surface M given by equation (1) with respect to a unit global normal N, for f elliptic, is a sphere. Hence, $f(0) \neq 0$ and M is a sphere o f radius $R = \frac{1}{|f(0)|}$.

PROOF OF THEOREM 1

We consider M an immersed smooth special surface in R^3 and N an unit normal vector field. We denote by $<,>$ the inner product in R^3 and by $\bigtriangledown$ the standard covariant derivative in R^3. The mean curvature vector H of M at p is given by $H(p) = \left(\frac{\lambda_1(p)+\lambda_2(p)}{2}\right)N(p)$ where $\lambda_1(p)$, $\lambda_2(p)$ are the principal curvatures of M at p (respecting to N).

Let us prove assertion a):

Suppose first that there is an umbilical boundary point $p \in \partial M$. Denote by v a unit tangent field along $\partial M = S^1$. Then,

$$f(0) = H(p) = \langle \bigtriangledown_v v, \ N \rangle_p =\leq 1 \tag{1}$$

Suppose now there are no umbilical points on the boundary. Notice that the set U of umbilical points of M is finite. Otherwise M is a spherical cap and $f(0) \leq 1$. This follows from the proof of theorem 3.2, pg. 142 of H.Hopf's book (see [8]), and from the fact that M is compact.

Let $\lambda_1, \lambda_2 : M - U \to R$ be the principal curvature functions with $\lambda_1 < \lambda_2$ on $M - U$. Let us prove first that ellipticity condition yields

$$\lambda_2 > f(0) \tag{2}$$

on $M - U$
Indeed,

$$\lambda_2 = H + \sqrt{H^2 - K} = f(H^2 - K) + \sqrt{H^2 - K}$$

and the ellipticity condition

$$4t(f'(t))^2 < 1$$

assures

$$g(t) = f(t) + \sqrt{t}$$

is a monotonic increasing function for $t \geq 0$.

Denote by $\mathcal{F}_2$ the principal line distribution on $M - U$ associated to the principal curvature λ_2. Clearly, there is a point $p \in \partial M$ where $\mathcal{F}_2$ is tangent to ∂M at p, i.e., $T_p \partial M = \mathcal{F}_2(p)$. If not we would obtain a line foliation of M transverse to ∂M and finite number (possible none) of singularities of negative indices (see [8]), this is impossible since M has disk topological type. Choose then $p \in \partial M$ such that $T_p \partial M = \mathcal{F}_2(p)$

Clearly

$$\lambda_2(p) = \langle \bigtriangledown_v v, N \rangle_p \leq 1 \tag{3}$$

by inequalities (1), (2), (3)

$$f(0) \leq 1.$$

This proves assertion a).

To prove assertion b) notice first that there is an extension for M beyond ∂M satisfying $\mathrm{H} = f(\mathrm{H}^2 - K), f$ elliptic and analytic. This is so, because of the boundary regularity for the underlying analytic elliptic partial differential equation (see [4], [11]). If $f(0) = 1$ we will show that there are infinitely many umbilical points in ∂M. The resulting non-discreteness of U will so imply M is totally umbilical (see [8]).

Suppose by absurd ∂M had finitely many umbilical points. Observe that the foliation $\mathcal{F}_2$ defined on $M - U$ is transverse to $\partial M - U$. To prove this, suppose $p \in \partial M - U$ is such that $\mathcal{F}_2(\mathrm{p})$ is tangent to $\partial M - U$. By equations (2),(3),we derive a contradiction because $f(0) < \lambda_2(p) \leq 1$.

Suppose now, there are no umbilical points on the boundary ∂M. This means (by what we have just proved) that $\mathcal{F}_2$ is transverse to ∂M. In this case $\mathcal{F}_2$ may be seen as a foliation of M with finite number of singularities with negative index (see [8]). This is a contradiction since by our hypothesis M is a topological disk.

For the case where ∂M has a non zero finite number of umbilical points, consider a umbilical point $p \in \partial M$, and let $\tilde{M}$ to be an extension of M beyond the boundary ∂M.

FACT:p is a singularity of $\mathcal{F}_2$ with negative index and finite number of separatrices, all of them smooth at p. Moreover, there is at least on separatrix going from p to the interior of M. In other words there is at least one separatrix such that, its interior tangent vector at p, say u, satisfies $\langle u, \eta \rangle > 0$, where η is the interior co-normal of M at p. This is a consequence of a straightforward computation using Bryant holomorphic quadratic form (see [3]) that, in a neighbourhood of p, the foliation is diffeomorphically equivalent to the standard foliation

$$Im z^n (dz)^2 = 0$$

on the complex z-plane.

Observe now that the foliation $\mathcal{F}_2$ on $M - U$ is topologically equivalent to a foliation with finite number of singularities on M. Some of them are interior singularities on M. Others are in the boundary ∂M. Those which are in the boundary have separatices (at least one) coming transversally to ∂M (see figures [1]). In order to see this situation is topologically impossible, we just recall M is a topological disk and use double construction to obtain a foliation of a topological sphere S^2 with finite number of singularities, all of them with negative index.

This concludes prove of Theorem 1.

Figure 1

PROOF OF THEOREM 2

Suppose without loss of generality that M is locally contained in the upper halfspace $\mathcal{H}^+ = \{z \geq 0\}$ in a neighbourhood of ∂M. We also identify ∂M with the unit circle S^1 centered at the origin of $\mathcal{H}$.

We first show that boundary roundness determines the behavior of the mean curvature vector H along the boundary(in fact , only convexity of ∂M is required). Precisely we state:

CLAIM 1: Let $p \in \partial M$. Then $\langle H(p), p \rangle < 0$

PROOF OF CLAIM 1:

Suppose first that there is a umbilical point $p \in \partial M$. Take a unit vector field v tangent to ∂M. Then umbilicity yields

$$H(N) = \langle \bigtriangledown_v v, N \rangle_p$$

If $N = \frac{H}{|H|}$ then the mean curvature H is positive and $\langle \bigtriangledown_v v, N \rangle = |H| > 0$. So $\langle -p, H \rangle > 0$, as desired, for $\bigtriangledown_v v = -p$ is the acceleration vector of S^1.

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For the case where there is no umbilical points on ∂M we recall that the foliation $\mathcal{F}_2$ parallel to the line field associated to the bigger principal curvature λ_2 defined over $M - U$ has to be tangent to $\partial M = S^1$ in some point p. Let $p \in \partial M$ be such that $\mathcal{F}_2(\mathrm{p})$ is tangent to ∂M. Clearly

$$\lambda_2(p) = \left\langle \nabla_v v, \frac{H}{|H|} \right\rangle_p > 0$$

Notice that Claim 1 means the following: the orthogonal projection of the mean curvature vector H on $\mathcal{H}$ points into the interior of the planar domain D contained in $\mathcal{H}$ bounded by ∂M. We will denote D by $\mathrm{int}\partial M$.

We now define $M_1 \subset M$ to be the connected component of $M \cap \mathcal{H}^+$ which contains ∂M.

CLAIM 2: $M_1 \cap \mathcal{H} \subset \mathrm{int}\partial \mathcal{M}$

This follows from Claim 1 and from Alexander reflection Principle techniques used exactly in the same way it was used in the proof of Theorem 1 pg. 337 of [6].

Let us denote $C_{f(0)}$ the vertical cylinder on $\mathcal{H}$ over the circle $S_{f(0)}$ of radius $\frac{1}{f(0)}$ centered at the origin.

CLAIM 3: There is a point $p \in \partial M$ such that

$$\langle N, -p \rangle_p \geq f(0)$$

for $N = \frac{H}{|H|}$.

This means there is a point $p \in \partial M$ where the surface M has bigger (or equal) inclination respect to xy plane than the small spherical cap of radius $\frac{1}{f(0)}$ bounding ∂M.

PROOF OF CLAIM 3 :

Let $p \in \partial M$ be a point of ∂M where $\mathcal{F}_2(\mathrm{p})$ is tangent to ∂M at p (see proof of Claim 1). Then, at this point p we have

$$\langle -p, N \rangle_p = \langle \nabla_v v, N \rangle_p = \lambda_2(p) \geq f(0)$$

CLAIM 4: If $M \cap ext\, C_{f(0)} = \phi$ then M is a spherical cap.

Where $ext\, C_{f(0)}$ is the exterior of the cylinder $C_{f(0)}$ (i.e. it is the connected region of $R^3 - C_{f(0)}$ not containing the origin of $\mathcal{H}$).

PROOF OF CLAIM 4:

The proof follows by using Claim 3 and the maximum principle (for special surfaces), comparing M_1 with a half sphere of radius $\frac{1}{f(0)}$ (see, for instance [1]).

CLAIM 5 : If $M_1 \cap \mathrm{int}\partial M = \phi$ then M is a spherical cap.

PROOF OF CLAIM 5:

First notice, if $M_1 \cap \text{int}\partial M = \phi$ then, by Claim 2 it follows $M_1 \cap \mathcal{H} = \partial \mathcal{M}$ and M is globally contained in $\mathcal{H}^+$. Now, using Alexandrov Reflection Principle for planes normal to $\mathcal{H}$, we conclude M is rotationally symmetric (see, for instance [10]). Therefore, the round boundary is every where parallel to one of the principal curvature directions for M. Now because M is a topological closed disk, we conclude, by the same index reasons as before, that M is totally umbilical. This shows that M is a spherical cap (of radius $\frac{1}{f(0)}$).

We finish the proof of Theorem 2 supposing, by contradiction, that $M_1 \cap (ExtC_f(0)) \neq \phi$ and $M_1 \cap \text{int}\partial M \neq \phi$.

At this point we may suppose M to be globally transverse to $\mathcal{H}$ without loss of generality. Therefore $M \cap \mathcal{H}$ is a finite collection of closed simple curves of $\mathcal{H}$.

Notice first that under the contradiction hypothesis there should be a curve $\gamma \in M \cap \mathcal{H} - \partial \mathcal{M}$ which is homotopically non trivial curve in $\mathcal{H} - \partial \mathcal{M}$. This follows directely from the extended Graph Lemma for special surfaces (see lemma 3 pg 12, Remark pg 14 and final Remarks in [2]).

Let $\gamma_L \in M \cap \mathcal{H}$ be the outermost homotopically non trivial curve in $\mathcal{H} - \partial \mathcal{M}$. Observe that γ_L bounds a topological disk $D_L \subset M$. Moreover, D_L is locally contained in the upper half-space $\mathcal{H}^+$ along its boundary γ_L. In fact, if the disk D_L were locally contained in the lower halfspace $\mathcal{H}^-$ we would have a connected component, say C, of $M - (M \cap \text{int}\partial M)$ such that $C \cap \mathcal{H}$ contains at least two distint closed curves both of them homotopically non trivial in $\mathcal{H} - \partial \mathcal{M}$. This is a consequence of the fact that M_1 is locally contained in $\mathcal{H}^+$ along its boundary together with the hypothesis that the mean curvature vector H never vanishes and the maximum principle. This would lead to a contradiction by applying Alexander Reflection Principle by vertical planes as in [6].

Notice that $D_L \cap \mathcal{H}$ is the union of γ_L with null homotopic closed curves on $\mathcal{H} - \gamma_L$, and as consequence of the Graph Lemma proved on [2] (see Lemma 3 pgs 12, 13, 14 and Remark pg 14) each curve on $D_L \cap \mathcal{H} - \gamma_L$ other than γ_L bounds a graph over its Jordan interior. We denote the Jordan interior of γ_L in $\mathcal{H}$ by $\text{int}\gamma_L$. Now a standard orientation argument yields (since $H \neq 0 \text{ on } M$):

$$D_L \cap (int\gamma_L) = \phi$$

So $D_L \cup int\gamma_L$ is embedded (non smooth over γ_L) compact surface without boundary. Moreover M_1 is clearly contained in the closed compact solid S determined by $D_L \cup int\gamma_L = \partial S$ (see figure 2).

Figure 2

Let $M_1(\theta), 0 \leq \theta \leq 2\pi$. be the 1-parameter family of surfaces obtained by rotating $M_1 = M_1(0)$ around an axis z normal to $\mathcal{H}$ and passing by the center of the round circle S_1 bounding M. Clearly $M_1(\theta) \cap D_L = \phi$, for every $\theta \in [0, 2\pi]$. Otherwise there would be a first parameter $\theta_0 > 0$ such that $M_1(\theta_0)$ would be tangent to $D_L - \gamma_L$, and contained inside S, contradicting the maximum principle for special surfaces.

Now, let $p \in M_1$ be a point of maximum distance of M_1 to the z-axis, contained in the interior of the solid S. the radius of this circle C_1 is bigger than $\frac{1}{f(0)}$ because of the hypothesis of contradiction. Also $D_L \cap D_1 = \phi$, where D_1 is the horizontal disk bounding C_1. This is again a consequence of mean curvature orientation and maximum principle.

We now finish the contradiction argument by comparing D_L with a sphere of radius $\frac{1}{f(0)}$, which we can actually introduce through the barrier disk D_1. This proves Theorem 2.

ACKNOWLEDGMENT

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