A SCHOEN THEOREM FOR MINIMAL SURFACES IN $\mathbb{H}^2 \times \mathbb{R}$

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Abstract. In this paper we prove that a complete minimal surface immersed in $\mathbb{H}^2 \times \mathbb{R}$, with finite total curvature and two ends, each one asymptotic to a vertical geodesic plane, must be a horizontal catenoid. Moreover, we give a geometric description of minimal ends of finite total curvature in $\mathbb{H}^2 \times \mathbb{R}$. We also prove that a minimal complete end E with finite total curvature is properly immersed and that the Gaussian curvature of E is locally bounded in terms of the geodesic distance to its boundary.

1. Introduction

In the early eighties, R. Schoen [22] proved a beautiful theorem about minimal surfaces in Euclidean space. Namely, a complete and connected minimal surface immersed in $\mathbb{R}^3$ with two embedded ends of finite total curvature is a catenoid.
In his article, R. Schoen described the structure of finite total curvature ends minimally embedded in $\mathbb{R}^3$, relying on the results of A. Huber [10] and R. Osserman [16] about the Weierstrass representation of such ends.
At the beginning of this century, the discovery of a generalized Hopf differential by U. Abresch and H. Rosenberg [1] stimulated the study of minimal surfaces in three-dimensional homogeneous manifolds. Many new embedded and complete minimal surfaces have been found in $\mathbb{H}^2 \times \mathbb{R}$. In particular J. Pyo [17] and F. Morabito and M. Rodriguez [14] have constructed, independently, a family of minimal embedded annuli with finite total curvature. Each end of such annuli is asymptotic to a vertical geodesic plane. Such surface is called a *horizontal catenoid*, see Figure 1.

In this article we prove the following theorem.

Main Theorem. *A complete and connected minimal surface immersed in $\mathbb{H}^2 \times \mathbb{R}$ with nonzero finite total curvature and two ends, each one asymptotic to a vertical geodesic plane, is a horizontal catenoid.*

Following the same spirit of Schoen's work, we describe the full geometry of minimal ends of finite total curvature in $\mathbb{H}^2 \times \mathbb{R}$ and we give an interpretation of it in terms of closed polygonal curves (see Definition 2.4 and Proposition 2.4). The study of such ends was first developed by the first author and H. Rosenberg in [9].
We recall that in $\mathbb{R}^3$, there are only two kinds of embedded minimal ends with finite total curvature: such an end is necessarily asymptotic to a catenoid (*catenoidal end*) or to a

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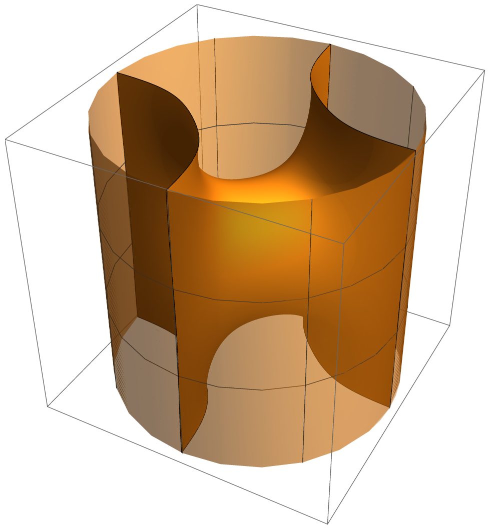

FIGURE 1. A horizontal catenoid in $\mathbb{H}^2 \times \mathbb{R}$ (*courtesy of the referee*)

plane (*planar end*). It is worthwhile to notice that in $\mathbb{H}^2 \times \mathbb{R}$ there are many more such ends. Namely, in the Poincaré disk model of the hyperbolic plane, consider the domain D with boundary the ideal polygon Γ with vertices the $2n$ points $e^{i\frac{\pi}{k}} \in \partial_\infty \mathbb{H}^2$, $k = 1, \dots, 2n$, $n \geqslant 2$. Then, P. Collin and H. Rosenberg have proved in [3, Theorem 1] a Jenkins-Serrin type result: there exists a minimal vertical graph over D taking the asymptotic values $+\infty$ and $-\infty$ alternatively on the sides of Γ. Those examples show that there exist infinitely many minimal embedded ends with finite total curvature in $\mathbb{H}^2 \times \mathbb{R}$.
We observe that each one of those examples is properly embedded, has finite total curvature and one end. If M is a properly embedded minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ with finite total curvature and two ends, it is not known if each end must be asymptotic to a vertical totally geodesic plane. For example, is it possible to connect two disjoint minimal vertical graphs as above, with a vertical neck of a catenoid?

The technical tools developped in order to prove the Main Theorem, allow us to prove two further results. A minimal complete end with finite total curvature is properly immersed (Theorem 2.2), and on such an end, say E, the Gaussian curvature is locally bounded in terms of the geodesic distance to the boundary of E (Theorem 2.3).

The paper is organized as follows.
In Section 2, we study the geometry of minimal ends of finite total curvature. The main geometric property is that horizontal sections of finite total curvature ends converge towards a horizontal geodesic. In Section 3, we prove the Main Theorem. In the Appendix, we study the geometry of curves with bounded curvature in the hyperbolic plane.

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2. Minimal ends with finite total curvature in $\mathbb{H}^2 \times \mathbb{R}$

In this section we give the geometrical structure of a finite total curvature end. We rely on the complex analysis involved in the theory of minimal surfaces [8], [9], [20] and on the theory of harmonic maps developed by Z. Han, L. Tan, A. Treiberg and T. Wan [7] and Y. N. Minsky in [13].

Let M be a Riemann surface and let $X = (F, h) : M \to \mathbb{H}^2 \times \mathbb{R}$ be a conformal and minimal immersion. The map $F : M \to \mathbb{H}^2$ is harmonic and h is a harmonic function on M. Let z be a local conformal coordinate on M and let $ds^2 = \sigma^2(u)\,|du|^2$ be the hyperbolic metric on $\mathbb{H}^2$ in the model of the unit disk. We set

$$Q(F) := (\sigma \circ F)^2 F_z \overline{F}_z dz^2 = \phi(z) dz^2,$$

then $Q(F)$ is a quadratic holomorphic differential globally defined on M, known as the *quadratic Hopf differential associated to F.*

Since we consider conformal immersion we have

$$\begin{cases} (\sigma \circ F)^2 |F_x|^2 + h_x^2 = (\sigma \circ F)^2 |F_y|^2 + h_y^2 \\ \\ (\sigma \circ F)^2 \langle F_x, F_y \rangle|^2 + h_x^2 = 0. \end{cases}$$

Therefore we have $(h_z)^2 (dz)^2 = -Q(F)$ (see [20, Proposition 1]). Then $Q(F)$ has two square roots globally defined on M. We denote by $\sqrt{\phi}\, dz$ the square root of $Q(F)$ so that

$$h = -2 \operatorname{Re} \int i \sqrt{\phi}\, dz = 2 \operatorname{Im} \int \sqrt{\phi}\, dz.$$

The metric induced on M by the immersion X is

$$ds^2 = (\sigma \circ F)^2 \Big(|F_z| + |F_{\overline{z}}| \Big)^2 |dz|^2.$$

From a result by A. Huber [10, Theorem 15], we deduce that a minimal end E of finite total curvature is parabolic, so that it can be parametrized by $U := \{z \in \mathbb{C} \mid |z| > 1\}$.

Let $X = (F, h) : U \to \mathbb{H}^2 \times \mathbb{R}$ be a conformal and complete parametrization of the end $E = X(U)$. As it is shown in [9], the conformal structure of the end is given by the following Theorem which relies the complex analysis involved in the theory of minimal surfaces [8], [9], [20], on the theory of harmonic maps developed by Z. Han, L. Tan, A. Treiberg and T. Wan [7] and by Y. N. Minsky in [13].

Theorem [9]**.** *Let $X := (F, h) : M \to \mathbb{H}^2 \times \mathbb{R}$ with finite total curvature. Then*

(1) *M is conformally $\bar{M} - \{p_1, ... p_n\}$ a Riemann surface punctured in a finite number of points.*
(2) *Q is holomorphic on M and extends meromorphically to each puncture.*
(3) *The third coordinate of the unit normal vector n_3 tends to zero uniformly at each puncture.*
(4) *The total curvature is a multiple of 2π, namely*

$$\int_M (-K dA) = 2\pi (2 - 2g - 2k - \sum_{i=1}^{n} m_i),$$

where m_i is defined in Definition 2.1 below.

This theorem contains informations on the geometrical structure of a finite total curvature end at infinity.

By the previous Theorem, $\phi(z)$ extends meromorphically to the puncture $z = \infty$. Thus we can write ϕ in the following form

$$\phi(z) = \big(\sum_{k\geqslant 1} \frac{a_{-k}}{z^k} + P(z)\big)^2, \tag{1}$$

where P is a polynomial function. If we choose $\sqrt{\phi} = \sum_{k\geqslant 1} \frac{a_{-k}}{z^k} + P(z)$, then

$$h = 2\,\mathrm{Im} \int \big(\sum_{k\geqslant 1} \frac{a_{-k}}{z^k} + P(z)\big) dz.$$

Definition 2.1. Let $m \geqslant 0$ be the degree of P. We will say that E *is an end of degree m with respect to the parametrization X.*

Since the height function is well defined on U, the real part of a_{-1} is zero. Let $\beta \in \mathbb{R}$ such that $a_{-1} = i\beta$.

Lemma 2.1. *The polynomial function P is not identically zero.*

Proof. Assume by contradiction that $P \equiv 0$. If $a_{-1} = 0$ we obtain that

$$\int_U |\phi(z)| dA < \infty,$$

and it is shown in [9] that the minimal end E would have finite area. From [5] (Theorem 3 and Remark 4) we deduce that for any $p \in E$ and for any real number $\mu < d_E(p, \partial E)$, we have $Area\big(B(p,\mu)\big) \geqslant \pi\mu^2$, where $B(p,\mu)$ is the geodesic disk in E centered at p, with radius μ. Considering a suitable diverging sequence of points (p_n) in E, we deduce that E has infinite area. This gives a contradiction.

Assume now that $a_{-1} \neq 0$. Since $a_{-1} = i\beta$, we obtain (up to an additive constant)

$$h = 2\,\mathrm{Im} \int \big(\sum_{k\geqslant 1} \frac{a_{-k}}{z^k}\big) dz = 2\beta \log|z| + o(1),$$

where $o(1)$ is a function depending of z and $o(1) \to 0$ when $|z| \to \infty$.
For $R > 1$, let $A_R = \{R \leqslant |z| \leqslant R^2\}$. Thus, $X(A_R)$ is a compact and minimal annulus immersed in $\mathbb{H}^2 \times \mathbb{R}$, whose boundary has two connected components. For R large enough, the vertical distance between those two boundary components is larger than 2π, while the family of the catenoids stays in a slab of height smaller than π [15, Proposition 5.1]. Therefore, we can compare $X(A_R)$ with the catenoids and obtain a contradiction by the maximum principle since the height of $X(A_R)$ is greater that 2π. This concludes the proof. q.e.d.

Let E be an end of degree m. Up to a change of variable, we can assume that the coefficient of the leading term of P is one. Then, for suitable complex number $a_0, \dots, a_{m-1}$, one has

$$P(z) = z^m + a_{m-1}z^{m-1} + \cdots + a_0 \quad \text{and} \quad \sqrt{\phi} = z^m(1 + o(1)). \tag{2}$$

For any $R > 1$, we set $U_R := \{z \in \mathbb{C} \mid |z| > R\}$, $S_R := \{z \in \mathbb{C} \mid |z| = R\} = \partial U_R$ and $E_R := X(U_R)$.

We set

$$W(z) := \int \sqrt{\phi(z)}\, dz = \int \big(\sum_{k\geqslant 1} \frac{a_{-k}}{z^k} + a_0 + \cdots + z^m\big)\, dz,$$

so that $h(z) = 2\,\mathrm{Im}\, W(z)$. If $\beta = 0$, the function W is well defined on U. If $\beta \neq 0$, the function W is only locally defined and has a real period equal to $-2\pi\beta$. We denote by $\theta \in \mathbb{R}$ a determination of the argument of $z \in U$, therefore

$$\frac{1}{2}h(z) = \mathrm{Im}\, W(z) = \beta \log|z| + \frac{|z|^{m+1}}{m+1}\big(\sin(m+1)\theta + o(1)\big) \tag{3}$$

and, locally

$$\mathrm{Re}\, W(z) = -\beta\theta + \frac{|z|^{m+1}}{m+1}\big(\cos(m+1)\theta + o(1)\big). \tag{4}$$

The image of W and the level sets of $\mathrm{Im}\, W$

Definition 2.2. (1) For any $R \geqslant 1$, a *semi-complete curve* in U_R is the image of a map $c : [0, +\infty[\to U_R$ such that $|c(t)| \xrightarrow[t\to\infty]{} +\infty$.
(2) Let $c : [0, +\infty[\to U_R$ be a semi-complete curve and let θ_0 be a real number. We say that the image of c has the ray $\{re^{i\theta_0},\ r > 0\}$ as *asymptotic direction*, if $\theta(t) \xrightarrow[t\to\infty]{} \theta_0$, where $\theta(t)$ is the determination of the argument of $c(t)$ in $[\theta_0 - \pi, \theta_0 + \pi[$.

From formula (3) above, by a continuity argument we deduce the following facts.

Lemma 2.2. (1) *There exists $R_0 > 1$ so that, for $k = 0, \ldots, 2m+1$ and for any $R \geqslant R_0$, the function $\mathrm{Im}\, W$ is strictly monotonous along the pairwise disjoint arcs*

$$A_k(R) := \big\{z \in S_R,\ \frac{k\pi}{m+1} - \frac{\pi}{10(m+1)} < \arg(z) < \frac{k\pi}{m+1} + \frac{\pi}{10(m+1)}\big\}.$$

(2) *For any fixed $C \in \mathbb{R}$ one has*

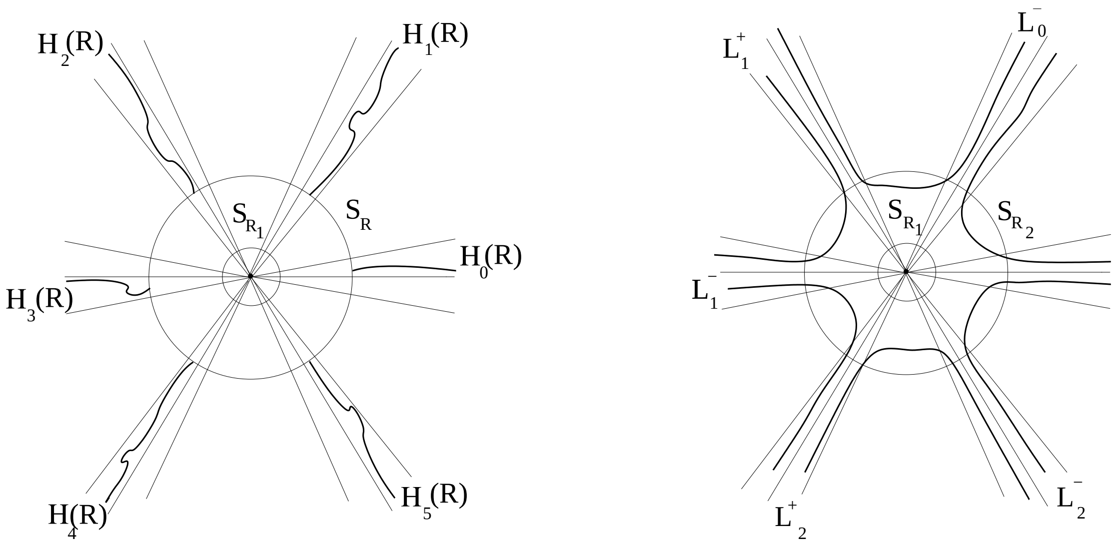
- *If (z_n) is a sequence of complex numbers such that $|z_n| \to \infty$ and $\mathrm{Im}\, W(z_n) \equiv C$, then $\sin\big((m+1)\arg z_n\big) \to 0$.*
- *There exists $r(C) > R_0$ such that, for any $R \geqslant r(C)$, there are exactly $2m+2$ points $Re^{i\theta_k}$, $k = 0, \ldots, 2m+1$, on the circle S_R verifying $\mathrm{Im}\, W(Re^{i\theta_k}) = C$ and $Re^{i\theta_k} \in A_k(R)$. Moreover, we have $\theta_k \xrightarrow[R\to\infty]{} \frac{k}{m+1}\pi$.*
- *For any $R \geqslant r(C)$, the set $U_R \cap \{\mathrm{Im}\, W(z) = C\}$ is composed of $2m+2$ semi-complete curves $H_k(C,R)$, $k = 0, \ldots, 2m+1$. Moreover $H_k(C,R)$ has the ray $\{re^{i\frac{k}{m+1}\pi},\ r > 0\}$ as asymptotic direction.*

Let $k = 0, \ldots, 2m+1$. We take $C = 0$ in Lemma 2.2 and define $H_k(R) := H_k(0,R)$, $R_1 := r(0) > R_0$. Moreover set $\alpha_k := \frac{k\pi}{m+1}$. Then, we deduce the following result.

Corollary 2.1. *For any $R \geqslant R_1$, the level set $U_R \cap \{\mathrm{Im}\, W(z) = 0\}$ is composed of $2m+2$ semi-complete curves $H_k(R)$, $k = 0, \ldots, 2m+1$, having the following properties (see Figure 2(a)).*

- *Each curve $H_k(R)$ has a unique boundary point, it belongs to the open arc $A_k(R)$.*
- *Each curve $H_k(R)$ has the ray $\{re^{i\frac{k\pi}{m+1}},\ r > 0\}$ as asymptotic direction.*
- *Each curve $H_k(R)$ is contained in the truncated sector $\Delta_k(R)$ defined as follows*

$$\Delta_k(R) := \big\{|z| > R \textit{ and } \alpha_k - \frac{\pi}{10(m+1)} < \arg(z) < \alpha_k + \frac{\pi}{10(m+1)}\big\}$$

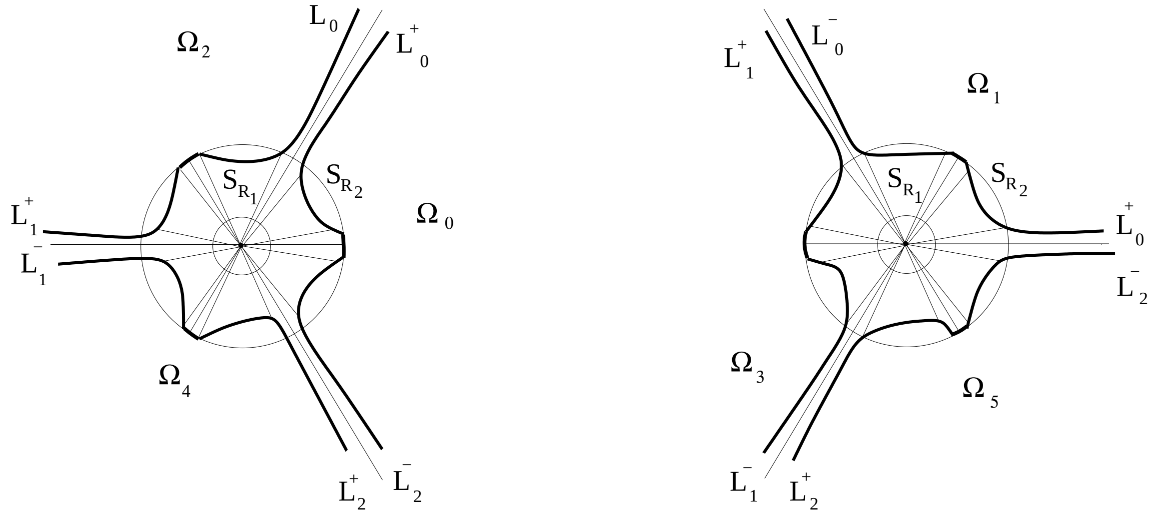


(a) The curves $H_k(R)$ (b) The curves L_j^+ and L_j^-

FIGURE 2. The curves $H_k(R)$, L_j^+ and L_j^- for $m = 2$

Let us state some consequences of the properties of the harmonic function $\operatorname{Im} W$.
Let $C_0 > 0$ be a real number such that $C_0 > \max\{|\operatorname{Im} W(z)|,\ z \in S_{R_1}\}$. Let R_2 be a real number satisfying $R_2 > r(C_0), r(-C_0), R_1$, where $r(C_0), r(-C_0)$ and $R_1 = r(0)$ are as in Lemma 2.2. Note that the set $U_{R_1} \cap \{\operatorname{Im} W(z) = C_0\}$, is composed of $m+1$ proper and complete curves without boundary $L_0^+, \dots, L_m^+$ (see Figure **??**).
For each $j = 0, \dots, m$, the level curve L_j^+ is contained in the domain of $\mathbb{C}$ which does not contain 0 and which is bounded by $H_{2j}(R_1)$, $H_{2j+1}(R_1)$ and an arc of S_{R_1} contained in the arc $\{z \in S_{R_1} \mid \alpha_{2j} - \frac{\pi}{10(m+1)} < \arg z < \alpha_{2j+1} + \frac{\pi}{10(m+1)}\}$.
In the same way, the set $U_{R_1} \cap \{\operatorname{Im} W(z) = -C_0\}$ is composed of $m+1$ proper and complete curves without boundary $L_0^-, \dots, L_m^-$. Each level curve L_j^- is contained in the domain of $\mathbb{C}$ which does not contain 0 and which is bounded by $H_{2j+1}(R_1)$, $H_{2j+2}(R_1)$ and an arc of S_{R_1} contained in the arc $\{z \in S_{R_1} \mid \alpha_{2j+1} - \frac{\pi}{10(m+1)} < \arg z < \alpha_{2j+2} + \frac{\pi}{10(m+1)}\}$, were we set $H_{2m+2}(R_1) := H_0(R_1)$.
For each level curve $L_j^\pm$, we denote by $\mathcal{L}_j^\pm$, the connected component of $\mathbb{C} \setminus L_j^\pm$ which does not contain the circle S_{R_1}.
For each $k = 0, \dots, 2m+1$ we define the open set Ω_k setting:

$$\Omega_k := \begin{cases} \mathcal{L}^-_{\frac{k}{2}-1} \cup \mathcal{L}^+_{\frac{k}{2}} \cup \Delta_k(R_2) & \text{if } k \text{ is even,} \\ \mathcal{L}^+_{\frac{k-1}{2}} \cup \mathcal{L}^-_{\frac{k-1}{2}} \cup \Delta_k(R_2) & \text{if } k \text{ is odd,} \end{cases} \tag{5}$$



(a) k even (b) k odd

FIGURE 3. The domains Ω_k for $m = 2$

where we set $\mathcal{L}_{-1}^{-} := \mathcal{L}_m^{-}$ (see Figure 3).
By construction, we have that each Ω_k is a simply connected domain and, setting $\mathcal{U} := \cup_{k=0}^{2m+1}\Omega_k$, we have $U_{R_2} \subset \mathcal{U} \subset U_{R_1}$. Since Ω_k is simply connected, we can define a continuous determination of the argument of z in Ω_k such that

$$\Omega_k \subset \Big\{|z| > R_1 \text{ and } \alpha_{k-1} - \frac{\pi}{10(m+1)} < \arg(z) < \alpha_{k+1} + \frac{\pi}{10(m+1)}\Big\},$$

(recall that $\alpha_{-1} := -\pi/(m+1)$ and $\alpha_{2m+2} := 2\pi$).

We summarize the above construction as follows.

Lemma 2.3. *Let C be a real number, then the following facts hold.*
(1) *If $C > C_0$ then*

- *the level set $\{\operatorname{Im} W(z) = C\} \cap \mathcal{U}$ is composed of $m+1$ proper and complete curves without boundary $L_0(C), \ldots, L_m(C)$ satisfying $L_j(C) \subset \mathcal{L}_j^{+}$ and, therefore, $L_j(C) \subset \Omega_{2j} \cap \Omega_{2j+1}$, $j = 0, \ldots, m$.*
- *the level set $\{\operatorname{Im} W(z) = -C\} \cap \mathcal{U}$ is composed of $m+1$ proper and complete curves without boundary $L_0(-C), \ldots, L_m(-C)$ satisfying $L_j(-C) \subset \mathcal{L}_j^{-}$ and, therefore, $L_j(-C) \subset \Omega_{2j+1} \cap \Omega_{2j+2}$, $j = 0, \ldots, m$, where $\Omega_{2m+2} := \Omega_0$.*

(2) *If $-C_0 \leqslant C \leqslant C_0$ then the level set $\{\operatorname{Im} W(z) = C\} \cap \mathcal{U}$ is composed of $2m+2$ proper curves $H_k(C) \subset \Omega_k$, $k = 0, \ldots, 2m+1$, satisfying the same properties as the level curves $H_k(R_2)$ in Corollary 2.1, with $R = R_2$.*

Proposition 2.1. *For $k = 0, \ldots, 2m+1$, the restriction of W to Ω_k is a well defined complex function, denoted by W_k. Furthermore, $W_k : \Omega_k \to \mathbb{C}$ is one-to-one and defines a conformal diffeomorphism from Ω_k onto a simply connected domain $\widetilde{\Omega}_k := W_k(\Omega_k)$ in the w complex plane.*

Proof. Since Ω_k is a simply connected domain which does not contain the origin, the function W is well defined on Ω_k.
Let $z_1, z_2 \in \Omega_k$ be such that $W_k(z_1) = W_k(z_2)$. We deduce from Lemma 2.3 that for any $C \in \mathbb{R}$, the level set $\{\operatorname{Im} W_k(z) = C\}$ has a unique connected component in Ω_k. Therefore, z_1 and z_2 belong to the same level curve $L \subset \Omega_k$. Since $W_k'(z) = \sqrt{\phi(z)}$ and ϕ does not vanish on $\mathcal{U}$, we deduce that the function $\operatorname{Re} W$ is strictly monotonous on L. We conclude that $z_1 = z_2$ as desired. q.e.d.

In the w-complex plane the domains $\widetilde{\Omega}_k$, $k = 0, \ldots, 2m+1$, defined in Proposition 2.1, have a nice structure, that will be crucial in the following.

Corollary 2.2. *Let k be an even number, $k = 2j$. Then, $\widetilde{\Omega}_k$ is the complementary of a horizontal half-strip. The non horizontal component of $\partial\widetilde{\Omega}_k$ is a compact arc that is the image by W_k of the boundary arc of Ω_k in $A_k(R_2)$ joining L_j^+ and L_{j-1}^-. Thus, $\operatorname{Im} W$ is strictly monotonous along such non horizontal component and $\operatorname{Re} w$ is bounded from above by a real number a_k for any $w \in \partial\widetilde{\Omega}_k$ (see Figure 4(a)).*
If k is an odd number, then $\widetilde{\Omega}_k$ has a similar description, except that on the half-strip the real part of w is now bounded from below, i.e. for some real number b_k we have $\operatorname{Re} w > b_k$ for any $w \in \partial\widetilde{\Omega}_k$ (see Figure 4(b)).

We get a proof of Corollary 2.2 by invoking Lemma 2.3.

(a) k even (b) k odd

FIGURE 4. The domains $\widetilde{\Omega}_k$

By the equalities in (2), we can take R_2 in (5) large enough so that

$$\frac{1}{2}|z|^m < \sqrt{|\phi(z)|} < 2|z|^m \tag{6}$$

when $|z| \geqslant R_2$. With this choice, we can prove the following result.

Lemma 2.4. *There is a real constant $c_1 > 0$ such that, for any z satisfying $|z| > 2R_2$, there exists $k \in \{0, \ldots, 2m+1\}$ such that*

$$z \in \Omega_k \quad \textit{and} \quad d_\phi(z, \partial\Omega_k) > c_1\,|z|,$$

where d_ϕ stands for the distance on Ω_k with respect to the ϕ-metric given by $|\phi(z)|\,|dz|^2$.

Proof. First assume that $m \geqslant 1$. Let $z \in \mathcal{U}$ such that $|z| \geqslant 2R_2$. We choose the determination of the argument of z in the interval $[0, 2\pi[$.

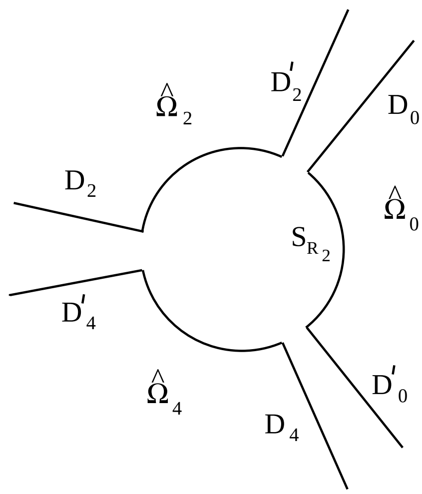


FIGURE 5. The domains $\widehat{\Omega}_k$ for $m = 2$ and k even

Recall that $\alpha_k = \frac{k\pi}{m+1}$ for $k = -1, \ldots, 2m+2$. There exists a unique $k \in \{0, \ldots, 2m+1\}$ such that either $(\alpha_k + \alpha_{k+1})/2 \leqslant \arg z < \alpha_{k+1}$ or $\alpha_k \leqslant \arg z < (\alpha_k + \alpha_{k+1})/2$. Without loss of generality, we can assume that the latter occurs. Therefore $z \in \Omega_k$.
For any $k = 0, \cdots, 2m+1$, we define the following rays:

$$D_k := \{\rho e^{i(\alpha_{k+1} - \pi/10(m+1))},\ \rho \geqslant R_2\},$$
$$D'_k := \{\rho e^{i(\alpha_{k-1} + \pi/10(m+1))},\ \rho \geqslant R_2\}.$$

By assumption, z belongs to the subdomain $\widehat{\Omega}_k$ of Ω_k bounded by D_k, D'_k and the arc $\Gamma(R_2)$ of S_{R_2} corresponding to $\alpha_{k-1} + \pi/10(m+1) \leqslant \theta \leqslant \alpha_{k+1} - \pi/10(m+1)$ (see Figure 5). Then, we have

$$d_\phi(z, \partial\Omega_k) \geqslant \min\left\{d_\phi(z, D_k),\ d_\phi(z, D'_k),\ d_\phi(z, \Gamma(R_2))\right\}.$$

Let $\gamma : [0,1] \to \Omega_k$ be any smooth arc satisfying $\gamma(0) = z$, $\gamma(1) \in D_k$ and $|\gamma(t)| > R_2$ for any $t \in [0,1]$. Denoting by $L_\phi(\gamma)$ the length of γ for the ϕ-metric and using (6), we have:

$$L_\phi(\gamma) = \int_0^1 \sqrt{|\phi(\gamma(t))|}\,|\gamma'(t)|\,dt \geqslant \frac{1}{2}\int_0^1 |\gamma(t)|^m |\gamma'(t)|\,dt \geqslant \frac{R_2^m}{2}\int_0^1 |\gamma'(t)|\,dt,$$

so that $L_\phi(\gamma) \geqslant (1/2)R_2^m L(\gamma)$, where $L(\gamma)$ is the Euclidean length of γ. Since
$4\pi/10(m+1) \leqslant \alpha_{k+1} - \pi/10(m+1) - \arg z \leqslant 9\pi/10(m+1) < \pi/2$,
we get

$$L(\gamma) > d(z, D_k) > \sin\left(\frac{4\pi}{10(m+1)}\right)|z|$$

where $d(z, D_k)$ stands for the Euclidean distance between z and D_k. From the last inequality, we deduce

$$d_\phi(z, D_k) \geqslant \frac{R_2^m}{2}\sin\left(\frac{4\pi}{10(m+1)}\right)|z|. \tag{7}$$

Let now $\gamma : [0,1] \to \Omega_k$ be any smooth arc satisfying $\gamma(0) = z$, $\gamma(1) \in D'_k$ and $|\gamma(t)| > R_2$ for any $t \in [0,1]$. In the same way, we can show that

$$L_\phi(\gamma) \geqslant \frac{R_2^m}{2} L(\gamma) \geqslant \frac{R_2^m}{2} d(z, D'_k).$$

Since $9\pi/10(m+1) \leqslant \arg z - \alpha_{k-1} - \pi/10(m+1) \leqslant 14\pi/10(m+1)$ we obtain

$$d_\phi(z, D'_k) \geqslant \min\Big\{ \sin\big(\frac{9\pi}{10(m+1)}\big), \sin\big(\frac{14\pi}{10(m+1)}\big)\Big\} \frac{R_2^m}{2} |z|. \tag{8}$$

Finally, let $\gamma : [0,1] \to \Omega_k$ be any smooth arc satisfying $\gamma(0) = z$, $|\gamma(1)| = R_2$ and $|\gamma(t)| > R_2$ for any $t \in [0,1[$. As before we have

$$L_\phi(\gamma) \geqslant \frac{R_2^m}{2} L(\gamma) \geqslant \frac{R_2^m}{2} \big(|z| - R_2\big).$$

Since $|z| > 2R_2$ we get

$$d_\phi(z, S_{R_2}) > \frac{R_2^m}{4} |z|. \tag{9}$$

Using estimates (7), (8) and (9), we are done in the case $m \geqslant 1$.

Now we consider the case $m = 0$. Then, there are only two domains: Ω_0, Ω_1, and we have $\alpha_0 = 0$, $\alpha_1 = \pi$ and $\alpha_2 = 2\pi$.
Let $z \in \mathcal{U}$ such that $|z| \geqslant 2R_2$. For some $k \in \{0,1\}$, we have either $(\alpha_k + \alpha_{k+1})/2 \leqslant \arg z < \alpha_{k+1}$ or $\alpha_k \leqslant \arg z < (\alpha_k + \alpha_{k+1})/2$. Without loss of generality, we can assume that the former occurs and that $k = 0$, that is: $\pi/2 \leqslant \arg z < \pi$ and, therefore, $z \in \Omega_1$.
We set

$$D := \{\rho e^{i\pi/10},\ \rho \geqslant R_2\}, \qquad D' := \{\rho e^{-i\pi/10},\ \rho \geqslant R_2\}.$$

We have $d(z, D) \geqslant |z|/2$ and $d(z, D') \geqslant d(z, D)$. Moreover, it can be shown in the same way as in the case $m \geqslant 1$, that $d_\phi(z, S_{R_2}) > |z|/4$. We obtain that $d_\phi(z, \partial\Omega_k) > |z|/4$, which concludes the proof. q.e.d.

Remark 2.1. For $k = 0, \ldots, 2m+1$, the map $W_k : \Omega_k \to \widetilde{\Omega}_k$ is a conformal diffeomorphism. Since $W'_k(z) = \sqrt{\phi(z)}$, W_k is an isometry when Ω_k is equipped with the ϕ-metric $|\phi(z)|\,|dz|^2$ and $\widetilde{\Omega}_k$ is equipped with the Euclidean metric $|dw|^2$.
We denote by $Z_k : \widetilde{\Omega}_k \to \Omega_k$ the inverse function of W_k.

The image of the level sets of $\operatorname{Im} W$ by the harmonic map F

Let $N := (n_1, n_2, n_3)$ be the unit normal vector field along the end E such that (X_x, X_y, N) has the positive orientation. We get from [20, Proposition 4] that $n_3 = \frac{|F_z| - |F_{\overline{z}}|}{|F_z| + |F_{\overline{z}}|}$. We define a function (possibly with poles) ω on U setting [8, Formula 14]

$$n_3 = \tanh \omega. \tag{10}$$

For $k = 0, \ldots, 2m+1$, we denote the restriction of ω to Ω_k by ω_k. The function $\widetilde{\omega}_k : \widetilde{\Omega}_k \to \mathbb{R}$ is defined by setting $\widetilde{\omega}_k(w) := (\omega_k \circ Z_k)(w)$ for any $w \in \widetilde{\Omega}_k$.
The induced metric ds^2 on U reads as

$$ds^2 = 4\cosh^2(\omega)\, |\phi|\, |dz|^2, \tag{11}$$

see [8, Equation 14].

Remark 2.2. Since ϕ has no zero on U, the function ω has no pole and the tangent plane of E is never horizontal. This means that the end E is transversal to any slice $\mathbb{H}^2 \times \{t\}$. Thus, the intersection of E with any slice is composed of analytic curves.

Let us denote by Δ_z (resp. Δ_w) the laplacian restricted to Ω_k (resp. $\widetilde{\Omega}_k$) for $k = 0, \dots, 2m+1$, with respect to the Euclidean metric $|dz|^2$ (resp. $|dw|^2$). Since $\Delta_z \omega_k = 2\sinh(2\omega_k)\,|\phi(z)|$ (see [8, Equation 13]), we deduce

$$\Delta_w \widetilde{\omega}_k = 2\sinh(2\widetilde{\omega}_k). \tag{12}$$

For any $w \in \widetilde{\Omega}_k$ we denote by $d_k(w)$ the Euclidean distance between w and the boundary of $\widetilde{\Omega}_k$.

The following estimate (13) can be found in [9] (see also [13, Lemma 3.3]).

Proposition 2.2. *There exists a constant $K_0 > 0$ such that for $k = 0, \dots, 2m+1$ and for any $w \in \widetilde{\Omega}_k$ with $d_k(w) > 1$, we have*

$$|\widetilde{\omega}_k(w)| \leqslant \frac{K_0}{\cosh d_k(w)} < 2K_0 e^{-d_k(w)}. \tag{13}$$

Consequently, the tangent planes to the end become vertical at infinity.

The last assertion is a consequence of the estimate (13), Lemma 2.4, and Remark 2.1.

We recall that the energy density of the harmonic function F with respect to the metric $|\phi(z)|\,|dz|^2$ on U and the hyperbolic metric on $\mathbb{H}^2$ is the real function defined on U by

$$e(z) := \frac{(\sigma \circ F)^2(z)}{|\phi(z)|}\big(|F_z|^2 + |F_{\overline{z}}|^2\big)$$

Then one has

$$e(z) = \frac{|F_z|}{|F_{\overline{z}}|} + \frac{|F_{\overline{z}}|}{|F_z|} = 2\cosh 2\omega.$$

where the first equality follows from the definition of ϕ and the second equality follows from the definition of ω. Observe that $e^{2\omega} = \frac{|F_z|}{|F_{\overline{z}}|}$.

For $k = 0, \dots, 2m+1$ we denote by $\widetilde{F}_k$ the harmonic map $\widetilde{F}_k := F \circ Z_k : \widetilde{\Omega}_k \to \mathbb{H}^2$.

Recall that the relation between the coordinate z in Ω and the coordinate w in $\widetilde{\Omega}$ is $w = W(z)$ and $\frac{dz}{dw} = 1/\sqrt{\phi \circ Z}$. The energy density $\widetilde{e}$ of $\widetilde{F}$ with respect to the Euclidean metric $|dw|^2$ on $\widetilde{\Omega}$ and the hyperbolic metric on $\mathbb{H}^2$ is defined on $\widetilde{\Omega}$ by

$$\widetilde{e}(w) := (\sigma \circ \widetilde{F})^2(w)\big(|\widetilde{F}_w|^2 + |\widetilde{F}_{\overline{w}}|^2\big)$$

As before, we have

$$\widetilde{e}(w) = \frac{|\widetilde{F}_w|}{|\widetilde{F}_{\overline{w}}|} + \frac{|\widetilde{F}_{\overline{w}}|}{|\widetilde{F}_w|} = 2\cosh 2\widetilde{\omega}.$$

Thus $\widetilde{e}(w) = e(z)$, if $z = Z(w)$.

Definition 2.3. Let $\kappa : U \to \mathbb{R}$ be defined as follows: for any $z_0 \in U$, $\kappa(z_0)$ is the *geodesic curvature* in $\mathbb{H}^2$ (with respect to the normal orientation induced by the unit normal vector field N on E) of the connected component of $F(\{\operatorname{Im} W = \operatorname{Im} W(z_0)\})$ passing through the point $F(z_0)$.
For any $k = 0, \dots, 2m+1$, let $\widetilde{\kappa} : \widetilde{\Omega}_k \to \mathbb{R}$ be defined by setting $\widetilde{\kappa}(w_0) = \kappa(z_0)$, where $w_0 = W(z_0)$

As a consequence of Remark 2.2, we have that the function κ is analytic

Lemma 2.5. *Fix a number $k \in \{0, \dots 2m+1\}$ and consider the simply connected domain Ω_k defined in* (5)*. Then, setting $w = u + iv$ on $\widetilde{\Omega}_k$, the pullback by the harmonic map $\widetilde{F}_k : \widetilde{\Omega}_k \to \mathbb{H}^2$ of the hyperbolic metric $\sigma^2(\xi)|d\xi|^2$ is given by*

$$\widetilde{F}_k^*\left(\sigma^2(\xi)|d\xi|^2\right) = 4\cosh^2 \widetilde{\omega}_k \, du^2 + 4\sinh^2 \widetilde{\omega}_k \, dv^2. \tag{14}$$

Moreover, for any horizontal coordinate curve $\widetilde{\gamma} := \{v = const\}$ in $\widetilde{\Omega}_k$, the absolute value of the geodesic curvature $\widetilde{\kappa}$ of the curve $\widetilde{F}_k(\widetilde{\gamma})$ in $\mathbb{H}^2$ is given by

$$|\widetilde{\kappa}(w)| = \frac{1}{2\cosh \widetilde{\omega}_k} \left| \frac{\partial \widetilde{\omega}_k}{\partial v} \right| (w) \tag{15}$$

for any $w \in \widetilde{\gamma}$.

Proof. A straightforward computation shows that

$$F_k^*\left(\sigma^2(\xi) d\xi d\overline{\xi}\right) = \phi(z) dz^2 + \overline{\phi}(z) d\overline{z}^2 + e(z)\, |\phi(z)| dz d\overline{z}.$$

Since $dw = \sqrt{\phi(z)}dz$ and $d\overline{w} = \sqrt{\overline{\phi}(z)}d\overline{z}$, in the coordinate $w = u + iv$, we have

$$\widetilde{F}_k^*\left(\sigma^2(\xi) d\xi d\overline{\xi}\right) = (\widetilde{e} + 2)du^2 + (\widetilde{e} - 2)dv^2 = 4\cosh^2 \widetilde{\omega}_k \, du^2 + 4\sinh^2 \widetilde{\omega}_k \, dv^2.$$

Then equality (14) is proved.
Now, let $w_0 \in \widetilde{\gamma}$ and assume $\widetilde{\omega}_k(w_0) \neq 0$. Then, by (14), the pullback by $\widetilde{F}_k$ of the hyperbolic metric is a regular metric in a neighborhood of w_0 in $\widetilde{\Omega}_k$. Consequently, the geodesic curvature of $\widetilde{F}_k(\widetilde{\gamma})$ at w_0 is given by

$$\begin{aligned}
\widetilde{\kappa}(w_0) &= -\frac{1}{2} \frac{1}{4\cosh^2 \widetilde{\omega}_k} \frac{1}{2|\sinh \widetilde{\omega}_k|} \frac{\partial}{\partial v}(4\cosh^2 \widetilde{\omega}_k)(w_0) \\
&= -\frac{1}{2} \frac{1}{\cosh \widetilde{\omega}_k} \frac{\sinh \widetilde{\omega}_k}{|\sinh \widetilde{\omega}_k|} \frac{\partial \widetilde{\omega}_k}{\partial v}(w_0),
\end{aligned}$$

(see [11, Formula (42.8)]). Therefore, the proof is finished in the case $\widetilde{\omega}_k(w_0) \neq 0$.

Assume now that $\widetilde{\omega}_k(w_0) = 0$. If $\widetilde{\omega}_k$ vanishes identically in a neighborhood of w_0, then the tangent plane of the minimal end E is always vertical in a open neighborhood of $X\big(Z_k(w_0)\big)$. This means that such a neighborhood is contained in a vertical cylinder in $\mathbb{H}^2 \times \mathbb{R}$. Since E is minimal, the vertical cylinder is a part of a vertical geodesic plane and, by analyticity, the whole end E is contained in the geodesic plane. Consequently the curve $\widetilde{F}_k(\widetilde{\gamma})$ is a part of a geodesic of $\mathbb{H}^2$ and formula (15) is trivially satisfied.
If $\widetilde{\omega}_k$ is not identically zero in a neighborhood of w_0, then there exists a sequence $(w_n)_{n \in \mathbb{N}^*}$ in $\widetilde{\Omega}_k$ converging to w_0 such that $\widetilde{\omega}_k(w_n) \neq 0$ for any $n > 0$. Since formula (15) holds at any point w_n and $|\kappa|$ is a continuous function, then (15) holds also at w_0. q.e.d.

The following Proposition is crucial in order to understand the geometry of the horizontal sections.

Proposition 2.3. *Let $z_0 \in U$ and let $\kappa(z_0)$ be the geodesic curvature of the level curve $F(\{\operatorname{Im} W(z) = \operatorname{Im} W(z_0)\}) \subset \mathbb{H}^2$. We set $R_3 = \max\{2R_2, 2/c_1\}$, where $c_1 > 0$ is the constant given by Lemma 2.4. Then, there exists a constant $c_2 > 0$ such that, for any $z_0 \in U_{R_3}$, we have*

$$|\kappa(z_0)| < c_2 e^{-c_1|z_0|}.$$

Proof. Let $z_0 \in U$ be any point such that $|z_0| > R_3$. It follows from Lemma 2.4 that there exists $k \in \{0, \ldots, 2m+1\}$ such that

$$z_0 \in \Omega_k \quad \text{and} \quad d_\phi(z_0, \partial\Omega_k) > c_1|z_0|.$$

Setting $w_0 := W(z_0) \in \widetilde{\Omega}_k$, we get $d_k(w_0) := d_k(w_0, \partial\widetilde{\Omega}_k) = d_\phi(z_0, \partial\Omega_k)$, where d_k denotes the Euclidean distance in $\widetilde{\Omega}_k$. Therefore, we obtain $d_k(w_0) > 2$, since $c_1|z_0| > c_1 R_3 > 2$.
Let $\widetilde{D}$ be the unit disk in the w-complex plane, centered at w_0, thus $\widetilde{D} \subset \widetilde{\Omega}_k$. For any $w \in \widetilde{D}$, we denote by $d(w)$ the Euclidean distance between w and $\partial\widetilde{D}$. Recall that the function $\widetilde{\omega}_k$ satisfies Equation (12) on $\widetilde{\Omega}_k$. We restrict $\widetilde{\omega}_k$ to $\widetilde{D}$ and we apply the interior *a-priori* gradient estimate for the Poisson Equation [6, Theorem 3.9], then

$$\sup_{\widetilde{D}} \big(d(w)|\nabla\widetilde{\omega}_k|\big) < K_1\big(\sup_{\widetilde{D}}|\widetilde{\omega}_k| + 2\sup_{\widetilde{D}} d^2(w)|\sinh 2\widetilde{\omega}_k|\big),$$

for some constant $K_1 > 0$, where ∇ means the Euclidean gradient.
Since $d(w_0) = 1$ and $d(w) \leqslant 1$ for any $w \in \widetilde{D}$, we get

$$|\nabla\widetilde{\omega}_k|(w_0) < K_1\big(\sup_{\widetilde{D}}|\widetilde{\omega}_k| + 2\sup_{\widetilde{D}}|\sinh 2\widetilde{\omega}_k|\big)$$

Moreover, since $d_k(w) \geqslant d_k(w_0) - 1$ for any $w \in \widetilde{D}$, we deduce from Proposition 2.2 that

$$|\widetilde{\omega}_k(w)| \leqslant \frac{K_0}{\cosh(d_k(w_0) - 1)}$$

for any $w \in \widetilde{D}$. Using the inequality $\cosh(t-1) > \frac{e^t}{10}$ for any $t \in \mathbb{R}$ we obtain

$$|\nabla\widetilde{\omega}_k|(w_0) < K_1\big(10K_0 e^{-d_k(w_0)} + 2\sinh(20K_0 e^{-d_k(w_0)})\big).$$

The function $x \mapsto \frac{\sinh x}{x}$ is strictly increasing for $x > 0$. As $d_k(w_0) > 2$, then we obtain that $\sinh(20K_0 e^{-d_k(w_0)}) < e^2 \sinh(20K_0 e^{-2})\, e^{-d_k(w_0)}$. This proves that there exists a constant $\delta > 0$ such that

$$|\nabla\widetilde{\omega}_k|(w_0) < \delta\, e^{-d_k(w_0)} \tag{16}$$

for any $w_0 \in \widetilde{\Omega}_k$ such that $d_k(w_0) > 2$. From formula (15) and from the previous computations, setting $c_2 := \delta/2$, we conclude that

$$|\widetilde{\kappa}(w_0)| < c_2 e^{-d_k(w_0)}.$$

As $\kappa(z_0) = \widetilde{\kappa}(w_0)$ and $d_k(w_0) := d_\phi(z_0, \partial\Omega_k) > c_1|z_0|$ by Lemma 2.4, this completes the proof. q.e.d.

In view of Lemma 2.3, let us sum up some notations previously established in the sequence of Corollary 2.1. For any $C \in \mathbb{R}$ and for $k = 0, \ldots, 2m+1$, $H_k(C, R) \subset \mathcal{U} \subset U$, denotes the semi-complete level curve of the function $\operatorname{Im} W$ whose asymptotic direction is $\{re^{i\alpha_k},\ r>0\}$, where $\alpha_k = k\pi/(m+1)$. That is

$$\operatorname{Im} W(z) = C \text{ for any } z \in H_k(C,R) \text{ and } \arg z \xrightarrow[|z|\to\infty]{} \alpha_k, \quad z \in H_k(C,R).$$

For any $C > C_0$, we denote by $L_j(C)$ (resp. $L_j(-C)$), $j = 0,\dots,m$, the proper and complete level curves given by $\{\operatorname{Im} W = C\}$ (resp. $\{\operatorname{Im} W = -C\}$). We have, for any R, $H_{2j}(C,R) \cup H_{2j+1}(C,R) \subset L_j(C)$ and $H_{2j+1}(-C,R) \cup H_{2j+2}(-C,R) \subset L_j(-C)$, $j = 0,\dots,m$, where $H_{2m+2}(-C,R) := H_0(-C,R)$.

The notion of *convergence in the C^1 topology* in the next Theorem is given in the statement of Definition 4.1.

Theorem 2.1. (1) *For any $C \in \mathbb{R}$, let $r(C)$ be defined as in Lemma 2.2. Then, for any $C \in \mathbb{R}$, for any $k \in \{0,\dots,2m+1\}$ and $R > r(C)$, the level curve $F(H_k(C,R)) \subset \mathbb{H}^2$ is a proper semi-complete curve which has no limit point in $\mathbb{H}^2$ and with a unique asymptotic point in $\partial_\infty \mathbb{H}^2$.*

(2) *For any $C_1, C_2 \in \mathbb{R}$ and for any $k \in \{0,\dots,2m+1\}$, the level curves $F(H_k(C_1,R))$, $F(H_k(C_2,R)) \subset \mathbb{H}^2$ are asymptotic. More precisely, for any $\varepsilon > 0$ there is a compact subset $K \subset \mathbb{H}^2$ such that for any C between C_1 and C_2 the level curve $F(H_k(C,R)) \setminus K$ remains in a ε-neighborhood of $F(H_k(C_1,R)) \setminus K$.*

Consequently, $F(H_k(C_1,R))$ and $F(H_k(C_2,R))$ have the same asymptotic point $\theta_k \in \partial_\infty \mathbb{H}^2$.

(3) *For $k = 0,\dots,2m+1$, the asymptotic points θ_k and θ_{k+1} are distinct, ($\theta_{2m+2} := \theta_0$).*

(4) *Let $j \in \{0,\dots,m\}$.*
- *When $C \to +\infty$, then the proper and complete level curves $F(L_j(C)) \subset \mathbb{H}^2$ converge for the C^1 topology to the geodesic in $\mathbb{H}^2$ with asymptotic boundary $\{\theta_{2j}, \theta_{2j+1}\}$.*
- *When $C \to +\infty$, then the proper and complete level curves $F(L_j(-C)) \subset \mathbb{H}^2$ converge for the C^1 topology to the geodesic in $\mathbb{H}^2$ with asymptotic boundary $\{\theta_{2j+1}, \theta_{2j+2}\}$, ($\theta_{2m+2} := \theta_0$).*

Proof. Assertion (1) is a straightforward consequence of the curvature estimates given in Proposition 2.3, together with Proposition 4.1.

Let us prove Assertion (2). Let $k \in \{0,\dots,2m+1\}$. By Lemma 2.3, for any $R > r(C_i)$, we have $H_k(C_i,R) \subset \Omega_k$, $i = 1,2$. From formula (4) we deduce that $\operatorname{Re} W(z) \xrightarrow[|z|\to\infty]{} +\infty$, $z \in H_k(C_i,R)$ (resp. $-\infty$) if k is an even (resp. odd) number, $i = 1,2$.
Assume now that k is even (the argument is analogous in the other case). Then, by the geometry of the sets $\widetilde{\Omega}_k = W(\Omega_k)$ (see Corollary 2.2), setting $\widetilde{p}_u := u + iC_1$ and $\widetilde{q}_u := u + iC_2$, for any real number $u > 0$ large enough, we have $\widetilde{p}_u \in W\big(H_k(C_1)\big) \subset \widetilde{\Omega}_k$ and $\widetilde{q}_u \in W\big(H_k(C_2)\big) \subset \widetilde{\Omega}_k$. Moreover, setting $\widetilde{\gamma}_u := \{(1-t)\widetilde{p}_u + t\widetilde{q}_u,\ 0 \leqslant t \leqslant 1\}$, we have $\widetilde{\gamma}_u \subset \widetilde{\Omega}_k$.
Let us set $p_u = Z_k(\widetilde{p}_u)$, $q_u = Z_k(\widetilde{q}_u)$ and $\gamma_u = Z_k(\widetilde{\gamma}_u)$, where $Z_k : \widetilde{\Omega}_k \to \Omega_k$ is the inverse function of W restricted to Ω_k as defined in Remark 2.1. Thus, we have:
- $\partial\gamma_u = \{p_u, q_u\}$, $p_u \in H_k(C_1,R)$ and $q_u \in H_k(C_2,R)$.
- $\operatorname{Re} W(z) = u$, for any $z \in \gamma_u$.

The distance between $F_k(p_u)$ and $F_k(q_u)$ in $\mathbb{H}^2$ is smaller than the length of $F_k(\gamma_u)$ which, by construction, is equal to the length of $\widetilde{F}_k(\widetilde{\gamma}_u)$.
We need to prove the following Claim.

Claim. *Let $\gamma \subset \gamma_u$ be an open arc along which the restriction of ω to Ω_k vanishes. Then, $F_k(\gamma) \subset \mathbb{H}^2$ is reduced to a single point.*

Indeed, let $c :]0,1[\to \gamma \subset \gamma_u$ be a smooth parametrization of γ. Since $\operatorname{Re} W\big(c(t)\big) \equiv u$ for $0 < t < 1$, setting $w = W(z)$ and differentiating with respect to t, we get $\frac{dw}{dz}\frac{dc}{dt} + \overline{\left(\frac{dw}{dz}\right)}\,\overline{\left(\frac{dc}{dt}\right)} = 0$. Moreover, as $\omega_k(c(t)) \equiv 0$ we have $|F_z| = |F_{\overline{z}}|$ along γ. Recall that $\frac{dw}{dz} = \sqrt{\phi}$ and $\phi = (\sigma \circ F)^2 F_z \overline{F}_z$. Combining those relations, we obtain that $F_z \frac{dc}{dt} + F_{\overline{z}}\overline{\left(\frac{dc}{dt}\right)} \equiv 0$. Then $\frac{d}{dt}(F \circ c)(t) \equiv 0$, which proves the Claim.

Since the function ω is real analytic, its restriction to the analytic arc γ_u vanishes identically or has a finite number of zeroes. Consequently, the length of $F_k(\gamma_u)$ in $\mathbb{H}^2$ is equal to the length of $\widetilde{\gamma}_u$ with respect to the pseudo-metric (14) on $\widetilde{\Omega}_k$, denoted by $L_k(\widetilde{\gamma}_u)$. Corollary 2.2 yields that for any $w \in \widetilde{\gamma}_u$ and for u large enough we have

$$d_k(w) := d_k(w, \partial\widetilde{\Omega}_k) \geqslant u - a_k,$$

where, as usual, d_k is the Euclidean metric on $\widetilde{\Omega}_k$. We have

$$d_{\mathbb{H}^2}\big(F_k(p_u), F_k(q_u)\big) = d_{\mathbb{H}^2}\big(\widetilde{F}_k(\widetilde{p}_u), \widetilde{F}_k(\widetilde{q}_u)\big) \leqslant L_{\mathbb{H}^2}\big(\widetilde{F}_k(\widetilde{\gamma}_u)\big) = L_k(\widetilde{\gamma}_u)$$

where $d_{\mathbb{H}^2}$ (resp. $L_{\mathbb{H}^2}$) is the distance (resp. the length) in the hyperbolic metric.
As $\operatorname{Re} W \equiv u$ along $\widetilde{\gamma}_u$, we obtain

$$\begin{aligned} L_k(\widetilde{\gamma}_u) &= 2|C_1 - C_2| \int_0^1 |\sinh \widetilde{\omega}_k\big(\widetilde{\gamma}_u(t)\big)|\, dt \\ &\leqslant 2\sinh\left(\frac{K_0}{\cosh(u - a_k)}\right)|C_1 - C_2|, \end{aligned}$$

where the inequality comes from formula (13). Hence, we have

$$d_{\mathbb{H}^2}\big(F_k(p_u), F_k(q_u)\big) \to 0 \quad \text{when} \quad u \to +\infty.$$

This completes the proof of Assertion (2).

Let us prove Assertion (3). Assume, for instance, that $\theta_0 = \theta_1$. Then, for any $C > C_0$, there exists a complete level curve $L_0(C) \subset \Omega_0$ such that $F(L_0(C)) \subset \mathbb{H}^2$ is a proper and complete curve with $\partial_\infty F(L_0(C)) = \{\theta_0\}$. We deduce from Proposition 2.3 and formula (3) that for C large enough, the absolute value of the geodesic curvature of $F(L_0(C)) \subset \mathbb{H}^2$ is smaller than $1/4$. Let $\Gamma \subset \mathbb{H}^2$ be any complete geodesic such that $\theta_0 \in \partial_\infty\Gamma$. Then, we obtain a contradiction with the maximum principle, comparing $F(L_0(C))$ with the family of complete curves γ_p, $p \in \Gamma$, orthogonal to Γ at p, with constant curvature $1/2$, and such that θ_0 belongs to the asymptotic boundary of the mean convex component of $\mathbb{H}^2 \setminus \gamma_p$. This completes the proof of Assertion (3).

Assertion (4) is a straightforward consequence of Assertion (3), Propositions 2.3 and 4.2.
q.e.d.

Remark 2.3. (1) We deduce from Theorem 2.1 that the asymptotic boundary of $F(U)$ is composed of exactly $2m + 2$ points, counting with multiplicity. In particular, if $m = 0$ then $\partial_\infty F(U)$ has exactly two distinct points.

(2) Observe that the $2m+2$ asymptotic points $\theta_0, \dots, \theta_{2m+1}$ of $F(U)$ need not to be distinct. They even not need to be well ordered as we can see in some examples found by J. Pyo and M. Rodriguez [18].
We can construct *artificial* examples for which the asymptotic points are not distinct: just consider the covering maps $\psi_n : U \to U$, $n \geqslant 2$, defined by $\psi_n(z) = z^n$, and the minimal ends $X_n := X \circ \psi_n : U \to \mathbb{H}^2 \times \mathbb{R}$.

We will give an alternative geometric interpretation of Theorem 2.1 in terms of polygonal curves. In order to to this, we need some definitions.
The asymptotic boundary $\partial_\infty(\mathbb{H}^2 \times \mathbb{R})$ is topologically equivalent to the following open cylinder joint with two closed disks:

$$\mathcal{C} = \{\mathbb{S}^1 \times (-1,1)\} \cup D(+1) \cup D(-1)$$

where $D(-1) = \{u \in \mathbb{C}; |u| \leqslant 1\} \times \{-1\}$ and $D(+1) = \{u \in \mathbb{C}; |u| \leqslant 1\} \times \{+1\}$. We identify $int(D(+1))$ and $int(D(-1))$ with the hyperbolic plane. Let $t : (-1,+1) \to \mathbb{R}$ be a homeomorphism. For any $y \in (-1,1)$, we identify $\mathbb{S}^1 \times \{y\}$ with the asymptotic boundary of $\mathbb{H}^2 \times \{t(y)\}$. The sets $int(D(+1))$ and $int(D(-1))$ represent the closure of vertical geodesics $\{p\} \times \mathbb{R}$, $p \in \mathbb{H}^2$.

Definition 2.4. We say that $\mathcal{P}$ is a *closed polygonal curve* if it is a closed curve contained in $\mathcal{C}$, that is union of a finite number of hyperbolic geodesics in $int(D(+1))$ and $int(D(+1))$, jointed by vertical segment in $\mathbb{S}^1 \times (-1,1)$, joint with their endpoints.

Notice that the closed polygonal curve $\mathcal{P}$ may happen to be not embedded and some of its sides may have multiplicity greater than one.
Now, we give the promised alternative interpretation of Theorem 2.1.

Proposition 2.4. *Let $X := (F,h) : U \to \mathbb{H}^2 \times \mathbb{R}$ be a properly immersed finite total curvature end $E = X(U)$, then $\partial_\infty E$ can be identified with a closed polygonal curve $\mathcal{P}$ in $\mathcal{C}$, where:*

- *A geodesic γ_{+1} of $\mathcal{P}$ contained in $D(+1)$ means that the end E contains a topological half plane which is asymptotic to $\gamma_{+1} \times \mathbb{R}_+$ when h tends to $+\infty$.*
- *A geodesic γ_{-1} of $\mathcal{P}$ contained in $D(-1)$ means that the end E contains a topological half plane which is asymptotic to $\gamma_{-1} \times \mathbb{R}_-$ when h tends to $-\infty$.*
- *A vertical segment $\{p\} \times (-1,+1)$ of $\mathcal{P}$ means that $p \times \mathbb{R}$ belongs to the asymptotic boundary of E.*

An interesting problem is to determine the correspondence between the space of closed polygonal curve $\mathcal{P}$ and the set of finite total curvature ends. We would like to understand the relation between the geometry of the end and the geometry of $\mathcal{P}$.

We remark that embedded ends can be only observed when $\mathcal{P}$ is an embedded polygonal curve. Properties of $\mathcal{P}$ can be derived from its projection $\pi(\mathcal{P})$ on a horizontal hyperbolic plane:

$$\pi : \mathcal{P} \to \mathbb{H}^2 \times \{0\}.$$

M. Rodriguez and J. Pyo has constructed an interesting example of a properly embedded minimal surface in $\mathbb{H}^2 \times \mathbb{R}$. The example is simply connected so that it has only one end. The polygonal curve $\mathcal{P}$ associated to the end is embedded with non embedded projection $\pi(\mathcal{P})$. The end has finite total curvature, contains a vertical geodesic $\{p\} \times \mathbb{R}$, and it is not a graph.

Let us state some results that have an independent interest in this theory.

Theorem 2.2. *Let be a complete minimal end with finite total curvature. Then, E is properly immersed.*

Proof. Let $m \in \mathbb{N}$ be the degree of the end E (see Definition 2.1). Let (p_n) be a sequence in U such that $|p_n| \to +\infty$. We want to show that $(X(p_n))$ is not a bounded sequence in $\mathbb{H}^2 \times \mathbb{R}$. Up to choose a subsequence, we can assume that there is $k \in \{0, \cdots, 2m+1\}$ such that, for any n, we have $p_n \in \Omega_k$. Recall that $h(z) = 2\,\mathrm{Im}\,W(z)$ for any z in U.
If $h(p_n) \to \infty$ we are done.
Assume that the sequence $(h(p_n))$ of real number is bounded. Thus, up to considering a subsequence, there exists a real number C_1 such that $h(p_n) \to C_1$. We set $S(C_1) := \{z \in \Omega_k \mid h(z) = C_1\}$. Thus, $S(C_1)$ is either a complete curve or a semi-complete curve. As in the proof of Theorem 2.1, we can construct a sequence (q_n) in Ω_k such that

$$\forall n \in \mathbb{N}, \quad q_n \in S(C_1) \quad \text{and} \quad d_{\mathbb{H}^2}\big(F(p_n), F(q_n)\big) \to 0.$$

Since $F(S(C_1))$ has no limit point in $\mathbb{H}^2$ and has an asymptotic point $p_\infty \in \partial_\infty \mathbb{H}^2$ (see (1) in Theorem 2.1), we get that $F(q_n) \to p_\infty$ and consequently $F(p_n) \to p_\infty$. Therefore, the sequence $X(p_n)$ is not bounded, which concludes the proof. q.e.d.

Theorem 2.3. *Let $X := (F, h) : U \to E \subset \mathbb{H}^2 \times \mathbb{R}$ be a minimal, complete end with finite total curvature. Then, there exists a constant c_3 such that, for any $p \in E$, we have*

$$|K_E(p)| \leqslant c_3\, e^{-d_E(p, \partial E)}, \tag{17}$$

where K_E denotes the intrinsic Gauss curvature and $d_E(\cdot, \partial E)$ stands for the intrinsic distance on E.

Proof. Let $m \in \mathbb{N}$ be the degree of the end E with respect to the parametrization X. We consider the open sets $\Omega_k \subset U$, $k = 0, \cdots, 2m+1$, as defined in (5) and the real number $R_3 > 1$ given in Proposition 2.3. In this proof we will use the notations previously established for the function $W = \mathrm{Im} \int \sqrt{\phi(z)}\, dz$, $\widetilde{\Omega}_k = W(\Omega_k)$, and $Z_k : \widetilde{\Omega}_k \longrightarrow \Omega_k$.
Let $p \in E$ and let $z \in U$ such that $X(z) = p$. Assume first that $|z| > R_3$. Therefore, by Lemma 2.4, there exists $k \in \{0, \ldots, 2m+1\}$ such that $z \in \Omega_k$ and $d_\phi(z, \partial\Omega_k) > c_1|z|$. We set $w = W(z)$, so that $w \in \widetilde{\Omega}_k$. We deduce from formula (11) that the metric $\widetilde{ds}^2$ induced on $\widetilde{\Omega}_k$ by the minimal immersion $\widetilde{X} := X \circ Z_k : \widetilde{\Omega}_k \to \mathbb{H}^2 \times \mathbb{R}$ is given by

$$\widetilde{ds}^2 = 4\cosh^2 \widetilde{\omega}_k(w)\, |dw|^2.$$

Therefore, we obtain

$$K_E(p) = K_{\widetilde{ds}^2}(w) = -\frac{\tanh \widetilde{\omega}_k}{4\cosh^2 \widetilde{\omega}_k}\, \Delta\widetilde{\omega}_k - \frac{1}{4\cosh^4 \widetilde{\omega}_k}\, |\nabla \widetilde{\omega}_k|^2.$$

It follows from the proof of Proposition 2.3 that $d_k(w) > 2$. Now, by a straightforward computation, using formulas (12), (13) and (16) (where $\delta = 2c_2$), we obtain

$$|K_{\widetilde{ds}^2}(w)| < c_3 e^{-2d_k(w)},$$

for some constant $c_3 > 0$ which does not depend on w. Observe that

$$d_E(p, \partial E) = d_E(\widetilde{X}(w), \partial E) \geqslant d_E(\widetilde{X}(w), \widetilde{X}(\partial\widetilde{\Omega}_k)) = d_{\widetilde{ds}^2}(w, \partial\widetilde{\Omega}_k).$$

From the comparison of the metric $d_{\widetilde{ds}^2}$ with the Euclidean metric, we infer

$$d_{\widetilde{ds}^2}(w, \partial\widetilde{\Omega}_k) \geqslant 2d_k(w).$$

Formula (17) follows by the previous inequalities for $|z| > R_3$, i.e. outside a compact subset of the end E. Finally, it suffices to observe that the continuous function $p \mapsto |K_E(p)|\, e^{d_E(p,\partial E)}$ is bounded on any compact subset of E. q.e.d.

Remark 2.4. A straightforward consequence of Theorem 2.3 is the following: for any complete and connected minimal surface $\Sigma \subset \mathbb{H}^2 \times \mathbb{R}$ with finite total curvature, and for any $p_0 \in \Sigma$, there exists a constant $c_4 = c_4(p_0, \Sigma)$ such that for any $p \in \Sigma$ we have

$$|K_\Sigma(p)| \leqslant c_4\, e^{-d_\Sigma(p,p_0)}.$$

Lemma 2.6. *Let $X := (F, h) : U \to E \subset \mathbb{H}^2 \times \mathbb{R}$ be a complete minimal end with finite total curvature. Let $m \in \mathbb{N}$, be the degree of the end E.*
For any $k \in \{0, \dots, 2m-1\}$, there exists a compact subset $K \subset \mathbb{C}$ such that the restricted minimal and conformal immersion $X = (F, h) : \Omega_k \setminus K \to \mathbb{H}^2 \times \mathbb{R}$ is an embedding.

Proof. To simplify the notations we give the proof for $k = 1$.
Recall that the immersion X is proper, Theorem 2.2, and that $h = 2\,\mathrm{Im}\, W$. Therefore we deduce from Lemma 2.3 and Proposition 2.3 that there exists a real number $C_1 > C_0$ with the property that for any $C \geqslant C_1$, the level set $\{\mathrm{Im}\, W(z) = C\}$ (resp. $\{\mathrm{Im}\, W(z) = -C\}$) in Ω_1 consists of a complete curve $L(C)$ (resp. $L(-C)$) such that the geodesic curvature of $F(L(C))$ (resp. $F(L(-C))$) in $\mathbb{H}^2$ is smaller than $1/4$ in absolute value.
Consequently for any C satisfying $C \geqslant C_1$, we get that $F(L(C))$ and $F(L(-C))$ are complete and embedded curves in $\mathbb{H}^2$. We deduce from Theorem 2.1 that there exist $\theta_0, \theta_1, \theta_2 \in \partial_\infty \mathbb{H}^2$, with $\theta_0 \neq \theta_1$, $\theta_1 \neq \theta_2$, and such that

$$\partial_\infty F(L(C)) = \{\theta_0, \theta_1\} \qquad \text{and} \qquad \partial_\infty F(L(-C)) = \{\theta_1, \theta_2\}.$$

Considering the height function h, we deduce that the restriction of X to the nonconnected subset of Ω_1 bounded by $L(C_1) \cup L(-C_1)$ is an embedding.
Let $\varepsilon > 0$. We deduce from (the proof of) Theorem 2.1 - (2), that there exist $a^+ \in L(C_1)$, $a^- \in L(-C_1)$ and a compact arc $\gamma_1 \subset \Omega_1$, joining a^+ and a^- and verifying

- $\mathrm{Re}\, W$ is constant along γ_1.
- $\mathrm{Im}\, W$ is strictly monotonous along γ_1.
- Denoting by $L_1(C_1)$ (resp. $L_1(-C_1)$ the component of $L(C_1) \setminus \{a^+\}$ (resp. $L(-C_1) \setminus \{a^-\}$) with asymptotic direction the ray $\{re^{i\frac{\pi}{m+1}}\}$, then $F(L_1(C_1))$ remains in a ε-neighborhood of $F(L_1(-C_1))$ in $\mathbb{H}^2$ (and also $F(L_1(-C_1))$ remains in a ε-neighborhood of $F(L_1(C_1))$). Therefore we have $\partial_\infty F(L_1(C_1)) = \partial_\infty F(L_1(-C_1)) = \theta_1$ (say).
- For any $C \in [-C_1, C_1]$, denoting by $H_1(C) \subset \Omega_1$ the semi-complete level curve $\{\mathrm{Im}\, W(z) = C\} \cap \Omega_1$ issue from γ_1 and with asymptotic direction the ray $\{re^{i\frac{\pi}{m+1}}\}$, then $F(H_1(C)) \subset \mathbb{H}^2$ remains in a ε-neighborhood of both $F(L_1(C_1))$ and $F(L_1(-C_1))$.

We deduce that the restriction of X to the connected component of Ω_1 bounded by γ_1 and a part of $L(C_1) \cup L(-C_1)$ and containing $L_1(C_1) \cup L_1(-C_1)$ is an embedding.

Recall that $L_0^+ := \{\mathrm{Im}\, W(z) = C_0\} \cap \Omega_1$ (resp. $L_0^- := \{\mathrm{Im}\, W(z) = -C_0\} \cap \Omega_1$) is a complete curve and that $\partial_\infty F(L_0^+) = \{\theta_0, \theta_1\}$ and $\partial_\infty F(L_0^-) = \{\theta_1, \theta_2\}$.

We deduce again from (the proof of) Theorem 2.1 - (2), that there exist $b_0^- \in L_0^-$, $b_1^- \in L(-C_1)$, a compact arc $\gamma_2 \subset \overline{\Omega}_1$ joining b_0^- and b_1^- verifying

- $\mathrm{Re}\, W$ is constant along γ_2.
- $\mathrm{Im}\, W$ is strictly monotonous along γ_2.

- Denoting by $(L_0^-)_2$ (resp. $L_2(-C_1)$ the component of $L_0^- \setminus \{b_0^-\}$ (resp. $L(-C_1) \setminus \{b_1^-\}$) with asymptotic direction the ray $\{re^{i\frac{2\pi}{m+1}}\}$, then $F((L_0^-)_2)$ remains in a ε-neighborhood of $F(L_2(-C_1))$ in $\mathbb{H}^2$ (and also $F(L_2(-C_1))$ remains in a ε-neighborhood of $F((L_0^-)_2))$.
- For any $C \in [-C_1, -C_0]$, denoting by $H_2(C) \subset \Omega_1$ the semi-complete level curve $\{\mathrm{Im}\, W(z) = C\} \cap \Omega_1$ issue from γ_2 and with asymptotic direction the ray $\{re^{i\frac{2\pi}{m+1}}\}$, then $F(H_2(C)) \subset \mathbb{H}^2$ remains in a ε-neighborhood of both $F((L_0^-)_2)$ and $F(L_2(-C_1))$.

Since $\partial_\infty F(L_2(-C_1)) = \theta_2$ and $\partial_\infty F(L_1(-C_1)) = \theta_1 \neq \theta_2$, we may assume that the points a^+, a^-, b_0^+ and b_0^- above are chosen so that the curves $F(L_1(-C_1))$ and $F(L_2(-C_1))$ are far away from each other.
Consequently, for any $C \in [-C_1, -C_0]$, we have $F(H_2(C)) \cap F(H_1(C)) = \emptyset$. We deduce that the restriction of X to the subset of Ω_1 bounded by $(L_0^-)_2$, γ_2, a compact part of $L(-C_1)$, γ_1 and a part of $L(C_1)$, and containing $L_2(-C_1)$ (and $L_1(-C_1)$) is an embedding.

In the same way, there exist $d_0^+ \in L_0^+$, $d_1^+ \in L(C_1)$, and a compact arc $\gamma_0 \subset \overline{\Omega}_1$ joining d_0^+ and d_1^+, such that the restriction of X to the non bounded connected subset V_1 of Ω_1 with boundary $\gamma_0 \cup \gamma_1 \cup \gamma_2$ a part of $L_0^+ \cup L_0^-$ and a compact part of $L(C_1) \cup L(-C_1)$, is an embedding. By construction, $\overline{\Omega}_1 \setminus V_1$ is a compact part of $\overline{\Omega}_1$. q.e.d.

3. Complete minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with finite total curvature

The aim of this section is to prove the Main Theorem stated in the Introduction. The proof makes essential use of the geometric properties of the horizontal sections of a finite total curvature end, that were established in Section 2.
In the following, $\mathbb{H}^2 \times \{0\}$ will be identified with $\mathbb{H}^2$.

Definition 3.1. Let $X := (F, h) : U \to E \subset \mathbb{H}^2 \times \mathbb{R}$ be a conformal and complete minimal annular end, where $U := \{|z| > 1\}$, and let $\gamma \subset \mathbb{H}^2$ be a geodesic.
We say that the end $X(U) \subset \mathbb{H}^2 \times \mathbb{R}$ is *asymptotic* to the vertical geodesic plane $\gamma \times \mathbb{R}$ if, for any real number C with $|C|$ large enough, $E \cap \{t = C\}$ is a complete curve of $\mathbb{H}^2 \times \{C\}$ and if, for any $\varepsilon > 0$, there exists a compact subset $K \subset U$ such that the distance between any point of $X(U \setminus K)$ and $\gamma \times \mathbb{R}$ is smaller than ε.

Lemma 3.1. *Let $X := (F, h) : U \to \mathbb{H}^2 \times \mathbb{R}$ be a conformal and complete minimal annular end asymptotic to a vertical geodesic plane. Let $m \in \mathbb{N}$ be the degree of the end $E := X(U)$ with respect to the parametrization X (see Definition 2.1).*
Then E is embedded (up to a compact part). Furthermore, up to a compact part, there exists a covering map $\pi : U \to U$ with degree $m + 1$, and a conformal minimal immersion $Y : U \to \mathbb{H}^2 \times \mathbb{R}$ such that:

- *$X = Y \circ \pi$,*
- *Y is an embedding,*
- *the degree of the end E with respect to the parametrisation Y is 0.*

Therefore, up to choose a new parametrization, we can assume that such an end has degree zero

Proof. We consider the open sets $\Omega_k \subset U$, $k = 0, \cdots, 2m+1$, as defined in (5). With the aid of Lemma 2.6, up to remove a compact part of U, we may assume that the restriction of X to each Ω_k is an embedding.

On one hand, we know that there exists $C_1 > 0$ such that for any $C > C_1$ the level set $\{h(z) = C\}$ is composed of $(m+1)$ complete curves $L_0(C), \dots, L_{m+1}(C)$ with $L_j(C) \subset \Omega_{2j}$, $j = 0, \dots, m$, see Lemma 2.3.

On the other hand, since the end E is asymptotic to a vertical geodesic plane, there exists $C_2 > 0$ such that for any $C > C_2$ the intersection $E \cap \{t = C\}$ is composed of a complete curve.

Consequently, for any $C > C_1 + C_2$ we have that

$$X\big(L_0(C)\big) = X\big(L_1(C)\big) = \cdots = X\big(L_{m+1}(C)\big).$$

By making vary C in $]C_1 + C_2, +\infty[$, we obtain that $X(\Omega_0), X(\Omega_2), \dots, X(\Omega_{2m})$ agree on an open set. We deduce with the analytic continuation principle that, up to a compact part, we have $X(\Omega_0) = X(\Omega_2) = \cdots = X(\Omega_{2m})$

For analogous reasons, since we have also $L_j(C) \subset \Omega_{2j+1}$, $j = 0, \dots, m$, we obtain that $X(\Omega_1) = X(\Omega_3) = \cdots = X(\Omega_{2m+1})$ up to a compact part.
Thus, up to remove a compact part of U and E, we can assume that $X : U \to E$ is a covering map with degree $m + 1$.
For any $z_1, z_2 \in U$ we set $z_1 \sim z_2$ if $X(z_1) = X(z_2)$. Then the canonical projection $\pi : U \to U/\!\sim$, is a covering map with degree $m + 1$. For any $p \in U/\!\sim$, we set $Y(p) = X(z)$ for any $z \in U$ verifying $\pi(z) = p$, by construction $Y(p)$ does not depend on the choice of such a z.
Observe that $U/\sim$ is homeomorphic to an annulus. Since Y is a conformal and minimal immersion with finite total curvature, we deduce that $U/\sim$ is conformally equivalent to U. Thus we may assume that $U/\!\sim = U$ and $Y : U \to E \subset \mathbb{H}^2 \times \mathbb{R}$ is a complete and minimal immersion with finite total curvature. We deduce from Lemma 2.6 that Y is an embedding.
By construction, for any $C > 0$ large enough, $Y^{-1}(\{t = C\})$ is composed of a unique and complete curve, namely $\pi(L_k(C))$, for any $k \in \{0, \dots, m+1\}$. Consequently, if $n \in \mathbb{N}$ denotes the degree of the end E with respect to the parametrization Y, since $Y^{-1}(\{t = C\})$ is composed of $n+1$ complete and disjoint curves, we deduce that $n = 0$. Thus the degree of the end E with respect to the parametrization Y is zero, this completes the proof. q.e.d.

Definition 3.2. Let $\gamma \subset \mathbb{H}^2$ be a geodesic. We say that a nonempty set $S \subset \mathbb{H}^2 \times \mathbb{R}$ is a *horizontal graph with respect to the geodesic* γ, if for any equidistant line $\widetilde{\gamma}$ of γ and for any $t \in \mathbb{R}$, the curve $\widetilde{\gamma} \times \{t\}$ intersects S at most at one point.

Remark 3.1. We notice that a different notion of horizontal graph appears in [19], in order to treat different kinds of problems about minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$.

Proposition 3.1. *Let γ_1 and γ_2 be two distintc geodesics in $\mathbb{H}^2$ with a common asymptotic point. Then, there is no complete, connected, immersed minimal surface with finite total curvature and two ends, one being asymptotic to $\gamma_1 \times \mathbb{R}$ and the other asymptotic to $\gamma_2 \times \mathbb{R}$.*

Proof. (see Figure 6). We set $\partial_\infty \gamma_1 = \{a_\infty, p_\infty\}$ and $\partial_\infty \gamma_2 = \{b_\infty, p_\infty\}$, so that p_∞ is the common asymptotic point of γ_1 and γ_2. We denote by $\gamma_\perp$ the geodesic such that γ_1 is the reflection of γ_2 across $\gamma_\perp$, we then have $p_\infty \in \partial_\infty \gamma_\perp$. We denote by γ_0 the geodesic such that $\partial_\infty \gamma_0 = \{a_\infty, b_\infty\}$. Observe that γ_0 meets $\gamma_\perp$ orthogonally at some point $p_0 \in \gamma_0 \cap \gamma_\perp$. For any $s > 0$ we denote by p_s the point in the half geodesic $[p_0, p_\infty[\subset \gamma_\perp$ such that $d_{\mathbb{H}^2}(p_0, p_s) = s$. For any $s > 0$ let γ_s be the geodesic orthogonal to $\gamma_\perp$ at p_s. We set $P_s := \gamma_s \times \mathbb{R}$.
Assume by contradiction that there exists a complete and connected minimal surface Σ with finite total curvature and two ends, one asymptotic to $\gamma_1 \times \mathbb{R}$ and the other asymptotic to

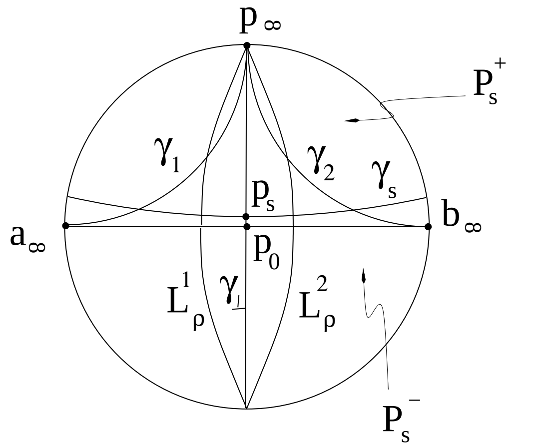


FIGURE 6

$\gamma_2 \times \mathbb{R}$. By a result from A. Huber [10, Theorems 13 and 15], such a surface is parametrized by a Riemann surface M conformally equivalent to a compact Riemann surface $\overline{M}$ punctured at two points z_1, z_2, $M \simeq \overline{M} \setminus \{z_1, z_2\}$. We denote by $X = (F, h) : M \to \Sigma \subset \mathbb{H}^2 \times \mathbb{R}$ the minimal and conformal immersion. Thus, $F : M \to \mathbb{H}^2$ is a harmonic map and $h : M \to \mathbb{R}$ is a harmonic function.

We want to show that Σ is a horizontal graph with respect to $\gamma_\perp$, afterwards we will derive a contradiction to conclude that such a surface does not exist.

For any $s > 0$, we denote by P_s^- the component of $(\mathbb{H}^2 \times \mathbb{R}) \setminus P_s$ containing $\{p_0\} \times \mathbb{R}$ and we denote by P_s^+ the other component. Thus $\{p_\infty\} \times \mathbb{R} \subset \partial_\infty P_s^+$. For any $s > 0$ we set $\Sigma_s^- := \Sigma \cap P_s^-$, $\Sigma_s^+ := \Sigma \cap P_s^+$ and we denote by Σ_s^{-*} the reflection of Σ_s^- across the vertical geodesic plane P_s.
For any $\rho > 0$, we denote by L_ρ^1 (resp. L_ρ^2) the equidistant line of $\gamma_\perp$ with distance ρ, intersecting γ_1 (resp. γ_2). We denote by $\mathcal{C}_\rho$ the domain of $\mathbb{H}^2 \times \mathbb{R}$ bounded by $(L_\rho^1 \cup L_\rho^2) \times \mathbb{R}$ and we set $\mathcal{Q}_\rho := (\mathbb{H}^2 \times \mathbb{R}) \setminus \mathcal{C}_\rho$. Thus, $\{a_\infty\} \times \mathbb{R}, \{b_\infty\} \times \mathbb{R} \subset \partial_\infty \mathcal{Q}_\rho$ for any $\rho > 0$. Since each end of Σ is asymptotic to one of the vertical planes $\gamma_i \times \mathbb{R}$, $i = 1, 2$, for any $s > 0$ there exists $\rho_s > 0$, large enough, such that $\Sigma \cap \mathcal{Q}_{\rho_s} \subset P_s^-$.
Following Lemma 3.1 it can be assumed that each end has degree zero with respect to a suitable parametrization. Therefore, we deduce from Lemma 2.3 that for $s > 0$ small enough, for each end E and for any real number t, the level set $E \cap (\mathbb{H}^2 \times \{t\}) \cap P_s^-$ has only one non bounded component. As a consequence of Propositions 2.3 and 4.3, we have that if $s > 0$ is small enough, then for any $t \in \mathbb{R}$, the level set $\Sigma_s^- \cap (\mathbb{H}^2 \times \{t\})$ is a horizontal graph with respect to the geodesic $\gamma_\perp$. Consequently, there exists $s_1 > 0$ such that $\Sigma_{s_1}^-$ is a horizontal graph with respect to $\gamma_\perp$ and $\Sigma_s^{-*} \cap \Sigma_s^+ = \emptyset$ for any $0 < s < s_1$.

We set

$$I := \{\sigma \geqslant 0 \mid \Sigma_s^{-*} \cap \Sigma_s^+ = \emptyset, \text{ for any } 0 < s \leqslant \sigma\}.$$

In order to ensure that Σ is a horizontal graph, we must show that $I = [0, +\infty[$.
The set I is nonempty because $[0, s_1] \subset I$.
We set $s_2 = \sup I$. If $s_2 = +\infty$ we are done. Assume that $s_2 \neq +\infty$. By a continuity argument we have $\Sigma_{s_2}^{-*} \cap \Sigma_{s_2}^+ = \emptyset$, so that $s_2 \in I$.

Recall that one end of Σ is asymptotic to $\gamma_1 \times \mathbb{R}$ and the other end is asymptotic to $\gamma_2 \times \mathbb{R}$. Moreover, from Lemma 2.3, formula (3) and Proposition 2.3, it follows that, for any $\varepsilon > 0$, there exists $t_0 > 0$ such that for any $t > t_0$, the intersection $\Sigma \cap (\mathbb{H}^2 \times \{t\})$ is composed of two complete curves, c_1^t and c_2^t verifying

$$\partial_\infty c_1^t = \{a_\infty, p_\infty\},\ \partial_\infty c_2^t = \{b_\infty, p_\infty\} \quad \text{and} \quad \sup_{c_i^t} |\kappa(q)| < \varepsilon,\ i = 1, 2,$$

where κ denotes the geodesic curvature. From Proposition 4.2 we deduce that that c_i^t is C^1-close to γ_i, $i = 1, 2$, if ε is small enough. Analogously $\Sigma \cap (\mathbb{H}^2 \times \{-t\})$ is composed of two complete curves, c_1^{-t} and c_2^{-t}, C^1-close to γ_1 and γ_2 respectively.

Claim 1. *There exist $t_0 > 0$ and $\eta_1 > 0$ such that $\Sigma^-_{s_2+\eta_1} \cap (\mathbb{H}^2 \times \{|t| > t_0\})$ is a horizontal graph with respect to the geodesic $\gamma_\perp$ and $\big(\Sigma^{-*}_{s_2+\eta_1} \cap \Sigma^+_{s_2+\eta_1}\big) \cap (\mathbb{H}^2 \times \{|t| > t_0\}) = \emptyset$.*

We set $p_1^t = c_1^t \cap P_{s_2}$. Observe that the (nonoriented) angle between c_1^t and the equidistant line to $\gamma_\perp$ passing through p_1^t is close to the angle between the same equidistant line and γ_1 at the common point. Hence, there is $\alpha \in (0, \pi/2)$ such that, if t_0 is large enough, this angle is larger than α for any $t > t_0$, and the same is true for the analogous angles defined for $P_{s_2} \cap c_2^t$, $P_{s_2} \cap c_1^{-t}$ and $P_{s_2} \cap c_2^{-t}$. Therefore, there exists $\eta_1 > 0$ such that $\Sigma^-_{s_2+\eta_1} \cap (\mathbb{H}^2 \times \{|t| > t_0\})$ is a horizontal graph with respect to $\gamma_\perp$ and $\big(\Sigma^{-*}_{s_2+\eta_1} \cap \Sigma^+_{s_2+\eta_1}\big) \cap (\mathbb{H}^2 \times \{|t| > t_0\}) = \emptyset$. Then the claim is proved.

Claim 2. *There exists $\eta_2 > 0$ such that $\Sigma^-_{s_2+\eta_2} \cap (\mathbb{H}^2 \times \{|t| \leqslant t_0\})$ is a horizontal graph with respect to the geodesic $\gamma_\perp$ and $\big(\Sigma^{-*}_{s_2+\eta_2} \cap \Sigma^+_{s_2+\eta_2}\big) \cap (\mathbb{H}^2 \times \{|t| \leqslant t_0\}) = \emptyset$.*

Observe first that, at any point of $\Sigma \cap P_{s_2}$, the equidistant line to $\gamma_\perp$ passing through this point is not tangent to Σ. Indeed, suppose that at some point $p \in \Sigma \cap P_{s_2}$ the equidistant line to $\gamma_\perp$ passing through p is tangent to Σ. Thus Σ is orthogonal to P_{s_2} at p and, therefore, $\Sigma^{-*}_{s_2}$ and $\Sigma^+_{s_2}$ are tangent at the point p of their common boundary. Since $\Sigma^{-*}_{s_2} \cap \Sigma^+_{s_2} = \emptyset$, the boundary maximum principle would imply that $\Sigma^{-*}_{s_2} = \Sigma^+_{s_2}$. This gives a contradiction, since the asymptotic boundary of Σ is not symmetric with respect to any vertical geodesic plane P_s.

Therefore, since $(\Sigma \cap P_{s_2}) \cap \{|t| \leqslant t_0\}$ is compact, there is $\beta \in (0, \pi/2)$ such that the nonoriented angle between the equidistant lines to $\gamma_\perp$ and Σ at any point of $(\Sigma \cap P_{s_2}) \cap \{|t| \leqslant t_0\}$ is larger than β. By a compactness argument again, there is $\eta_2 > 0$ such that $\Sigma^-_{s_2+\eta_2} \cap (\mathbb{H}^2 \times \{|t| \leqslant t_0\})$ is a horizontal graph and $\big(\Sigma^{-*}_{s_2+\eta_2} \cap \Sigma^+_{s_2+\eta_2}\big) \cap (\mathbb{H}^2 \times \{|t| \leqslant t_0\}) = \emptyset$. This proves the claim.

We set $\eta = \min\{\eta_1, \eta_2\}$. From Claims 1 and 2, we get that $s_2 + \eta \in I$. This gives a contradiction with the maximality of s_2. Therefore, $I = [0, +\infty[$ and Σ is a horizontal graph with respect to $\gamma_\perp$.

Now we can conclude the proof.

Let $Q(F)$ be the quadratic Hopf differential associated to F. We know that $Q(F)$ is holomorphic on M and has a pole at the ends $z_1, z_2 \in \overline{M}$. Let us denote by $m_1, m_2 \in \mathbb{N}$ the degrees of the ends of Σ with respect to the parametrization X. Therefore, one end is a pole of order $2m_1 + 4$ of $Q(F)$ and the other end is a pole of order $2m_2 + 4$ of $Q(F)$. According to the Riemann relation for $Q(F)$, we have that

$$\mathrm{Pole}(Q(F)) - \mathrm{Zero}(Q(F)) = 2\chi(\overline{M}),$$

thus $\mathrm{Zero}(Q(F)) = \mathrm{Pole}(Q(F)) - 2\chi(\overline{M}) = 2(m_1 + m_2) + 8 - 2\chi(\overline{M}) \geqslant 4$. Consequently, there exists $z_0 \in M$ which is a zero of $Q(F)$. Since $Q(F) = \phi(z)dz^2$, we deduce from (11) that z_0 is a pole of ω and then, the tangent plane of Σ at $X(z_0)$ is horizontal (see formula (10)).
Let $s' > 0$ such that $X(z_0) \in P_{s'}$. We get a contradiction by the boundary maximum principle since, on one hand $\Sigma_{s'}^{-*} \cap \Sigma_{s'}^{+} = \emptyset$ (since Σ is a horizontal graph with respect to $\gamma_\perp$) but on the other hand $\Sigma_{s'}^{-*}$ and $\Sigma_{s'}^{+}$ are tangent at their common boundary point $X(z_0)$. q.e.d.

Proposition 3.2. *Let γ_1 and γ_2 be two distinct geodesics in $\mathbb{H}^2$ intersecting at some point. Let $\Sigma \subset \mathbb{H}^2 \times \mathbb{R}$ be a complete immersed minimal surface with finite total curvature and two ends, one being asymptotic to $\gamma_1 \times \mathbb{R}$ and the other asymptotic to $\gamma_2 \times \mathbb{R}$.
Then we have $\Sigma = (\gamma_1 \times \mathbb{R}) \cup (\gamma_2 \times \mathbb{R})$ and, consequently, Σ has zero total curvature.*

Proof. We set $\{w\} := \gamma_1 \cap \gamma_2$. We denote by α and β the two geodesics passing through w such that the reflection of γ_1 across α is γ_2 and the reflection of γ_1 across β is γ_2. Thus, α intersects β orthogonally at w (see Figure 7).

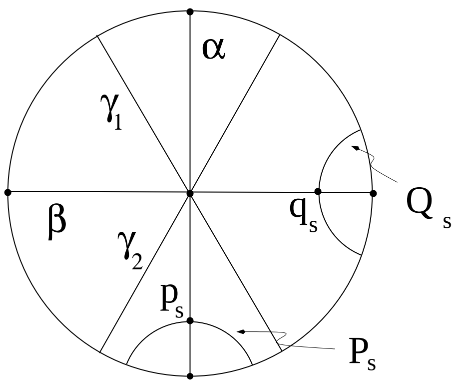


FIGURE 7. $w = \gamma_1 \cap \gamma_2 = 0$.

We choose an orientation on α and β. For any $s \in \mathbb{R}$, we denote by p_s (resp. q_s) the point of α (resp. β) whose signed distance to w is s, observe that $q_0 = p_0 = w$. Furthermore, for any $s \in \mathbb{R}$, we denote by P_s (resp. Q_s) the vertical geodesic plane passing through p_s (resp. q_s) and orthogonal to the geodesic α (resp. β), note that $P_0 = \beta \times \mathbb{R}$ and $Q_0 = \alpha \times \mathbb{R}$.
We set $P_0^+ := \cup_{s>0} P_s$, $P_0^- := \cup_{s>0} P_s$, $Q_0^+ := \cup_{s>0} Q_s$ and $Q_0^- := \cup_{s>0} Q_s$.
Assume that there exists a complete minimal surface Σ with finite total curvature and two ends, one being asymptotic to $\gamma_1 \times \mathbb{R}$ and the other being asymptotic to $\gamma_2 \times \mathbb{R}$.
Using the Alexandrov reflection principle with respect to the vertical planes P_s, $s \in \mathbb{R}$, we can show, as in the proof of Proposition 3.1, that Σ is symmetric with respect to P_0, and that $\Sigma \cap \overline{P_0^+}$ is a horizontal graph with respect to the geodesic α and so is $\Sigma \cap \overline{P_0^-}$. In the same way, we can show that Σ is symmetric with respect to Q_0, and that $\Sigma \cap \overline{Q_0^+}$ and $\Sigma \cap \overline{Q_0^-}$ are both horizontal graphs with respect to the geodesic β.
We deduce that Σ is transversal to both P_0 and Q_0. Therefore the intersections $\Sigma \cap P_0$ and $\Sigma \cap Q_0$ are analytic sets.
Now we proceed as in the proof of [22, Theorem 3, Case 1].
We set $L := P_0 \cap Q_0 = \{w\} \times \mathbb{R}$. Since $\Sigma \cap \overline{P_0^+}$ and $\Sigma \cap \overline{P_0^-}$ are horizontal graphs with respect to α, the self intersection set S of Σ is contained in P_0. By the same argument, we

have $S \subset Q_0$, so that $S \subset L$. Since an end of Σ is asymptotic to $\gamma_1 \times \mathbb{R}$ and the other end is asymptotic to $\gamma_2 \times \mathbb{R}$, we have $S \neq \emptyset$. By the analyticity of the sets $\Sigma \cap P_0$ and $\Sigma \cap Q_0$, we get that $S = L$. Moreover, since $\Sigma \cap \overline{P_0^+}$ and $\Sigma \cap \overline{P_0^-}$ are horizontal graphs with respect to α, we deduce that $\Sigma \cap Q_0 = L$. Analogously, $\Sigma \cap P_0 = L$. Therefore, $\Sigma \setminus L$ consists of four connected components $\Sigma_i,\ i = 1, \cdots, 4$ with:

$$\Sigma_1 \subset P_0^+ \cap Q_0^+, \quad \Sigma_2 \subset P_0^+ \cap Q_0^-, \quad \Sigma_3 \subset P_0^- \cap Q_0^- \quad \text{and} \quad \Sigma_4 \subset P_0^- \cap Q_0^+.$$

Denoting by σ the rotation about the vertical geodesic L with angle π, the reflection principle shows that $\sigma(\Sigma_1) = \Sigma_3$, so that $\Sigma' := \Sigma_1 \cup \Sigma_3 \cup L$ is a smooth and complete minimal surface embedded in $\mathbb{H}^2 \times \mathbb{R}$. Up to a change of numbering, we can assume that Σ' is asymptotic to $\gamma_1 \times \mathbb{R}$. Therefore it can be shown using the Alexandrov reflection principle that $\Sigma' = \gamma_1 \times \mathbb{R}$. In the same way, it can be shown that $\Sigma_2 \cup \Sigma_4 \cup L = \gamma_2 \times \mathbb{R}$. Therefore, we get that $\Sigma = (\gamma_1 \times \mathbb{R}) \cup (\gamma_2 \times \mathbb{R})$, which concludes the proof. q.e.d.

Now we can restate the Main Theorem, announced in the Introduction, in the following way.

Theorem 3.1. *Let Σ be a complete, connected minimal surface immersed in $\mathbb{H}^2 \times \mathbb{R}$ with finite nonzero total curvature and two ends. Assume that each end is asymptotic to a vertical geodesic plane $\gamma_i \times \mathbb{R}$, where each γ_i, $i = 1, 2$, is a geodesic.*
Then, we have $\gamma_1 \cap \gamma_2 = \emptyset$, $\partial_\infty \gamma_1 \cap \partial_\infty \gamma_2 = \emptyset$. Furthermore, Σ is a properly embedded annulus and is a horizontal catenoid.

In order to prove Theorem 3.1 we fix some notations and prove some lemmas.

Notations. Let $\gamma_1, \gamma_2 \subset \mathbb{H}^2$ be two geodesics satisfying

$$\gamma_1 \cap \gamma_2 = \emptyset \quad \text{and} \quad \partial_\infty \gamma_1 \cap \partial_\infty \gamma_2 = \emptyset. \tag{18}$$

We denote by $\gamma_0 \subset \mathbb{H}^2$ the geodesic orthogonal to both γ_1 and γ_2. We set $p_1 = \gamma_1 \cap \gamma_0$ and $p_2 = \gamma_2 \cap \gamma_0$. We call p_0 the middle point of the segment of γ_0 between p_1 and p_2. We denote by $\Gamma \subset \mathbb{H}^2$ the geodesic passing through p_0 and orthogonal to γ_0 (see Figure 8).

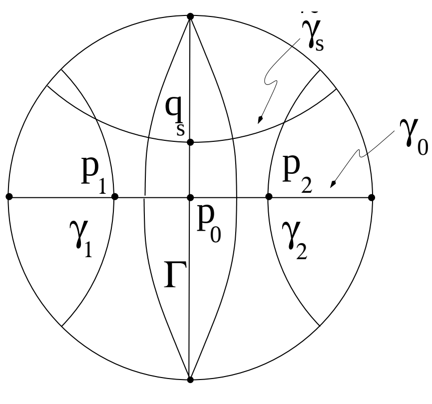


FIGURE 8

In the following lemmas the surface Σ satisfies the hypothesis of Theorem 3.1.

Lemma 3.2. *Suppose that γ_1 and γ_2 are geodesics satisfying the properties in* (18). *Then, the surface Σ is symmetric with respect to the vertical geodesic plane $\gamma_0 \times \mathbb{R}$ and the closure of each component of $\Sigma \setminus (\gamma_0 \times \mathbb{R})$ is a horizontal graph with respect to Γ.*

Proof. We choose an orientation on Γ. For any $s \in \mathbb{R}$, we denote by q_s the unique point in Γ whose signed distance to p_0 is s, thus $q_0 = p_0$. For any $s \in \mathbb{R}$, we denote by $\widetilde{\gamma}_s \subset \mathbb{H}^2$ the geodesic orthogonal to Γ and passing through q_s. Observe that $\widetilde{\gamma}_0 = \gamma_0$.
For any $s \in \mathbb{R}$, we set $Q_s := \widetilde{\gamma}_s \times \mathbb{R}$. Moreover, for any $s \neq 0$, we denote by Q_s^+ the component of $(\mathbb{H}^2 \times \mathbb{R}) \setminus Q_s$ containing $\{p_0\} \times \mathbb{R}$ and by Q_s^- the other component. For any $s \neq 0$ we set $\Sigma_s^- := \Sigma \cap Q_s^-$, $\Sigma_s^+ := \Sigma \cap Q_s^+$ and we denote by Σ_s^{-*} the reflection of Σ_s^- across the vertical geodesic plane Q_s.
As in the proof of Proposition 3.1 (Claim 1 and Claim 2), we can show that for any $s \neq 0$, Σ_s^- is a horizontal graph with respect to Γ and that $\Sigma_s^{-*} \cap \Sigma_s^+ = \emptyset$.
Then, passing to the limit for $s \to 0$ from both sides, we conclude that Σ is symmetric with respect to $\gamma_0 \times \mathbb{R}$ and that each component of $\Sigma \setminus (\gamma_0 \times \mathbb{R})$ is a horizontal graph with respect to Γ. q.e.d.

Remark 3.2. It follows from the proof of Lemma 3.2 that the tangent plane at any point of $\Sigma \setminus (\gamma_0 \times \mathbb{R})$ is never horizontal.

Lemma 3.3. *Suppose that γ_1 and γ_2 are geodesics satisfying the properties in* (18). *Then, the surface Σ is symmetric with respect to the vertical geodesic plane $\Gamma \times \mathbb{R}$ and the closure of each component of $\Sigma \setminus (\Gamma \times \mathbb{R})$ is a horizontal graph with respect to γ_0. Furthermore Σ is embedded.*

Proof. Let $d > 0$ be the distance between p_0 and p_1, thus we have $d = d_{\mathbb{H}^2}(p_0, p_1) = d_{\mathbb{H}^2}(p_0, p_2)$. For any $s \in [0, d]$ we denote by $\widetilde{p}_s \in \gamma_0$ the unique point between p_0 and p_1, whose distance to p_0 is s. Thus $\widetilde{p}_0 = p_0$ and $\widetilde{p}_d = p_1$.

We denote by $\Gamma_s \subset \mathbb{H}^2$ the geodesic orthogonal to γ_0 and passing through $\widetilde{p}_s$, thus $\Gamma_d = \gamma_1$. We set $P_s := \Gamma_s \times \mathbb{R}$. For any $s \in [0, d[$ we denote by P_s^- the connected component of $(\mathbb{H}^2 \times \mathbb{R}) \setminus P_s$ containing $\{p_1\} \times \mathbb{R}$ and by P_s^+ the other component. We set $\Sigma_s^- := \Sigma \cap P_s^-$ and $\Sigma_s^+ := \Sigma \cap P_s^+$. Furthermore, Σ_s^{-*} denotes the reflection of Σ_s^- across P_s. We give to the geodesic γ_0 the orientation going from p_1 to p_2.
We will say that $\Sigma_s^{-*} \leqslant \Sigma_s^+$ if Σ_s^{-*} remains under Σ_s^+ with respect to the orientation of γ_0.
As in the proof of Proposition 3.1, it can be shown that there exists $\varepsilon > 0$ such that for any $s \in [d - \varepsilon, d[$, Σ_s^- is a horizontal graph with respect to γ_0. Therefore, for any $s \in [d - \varepsilon/2, d[$, we have $\Sigma_s^{-*} \leqslant \Sigma_s^+$.
We set

$$I = \big\{ s \in [0, d] \mid \Sigma_r^{-*} \leqslant \Sigma_r^+ \text{ for any } r \in]d - s, d[\big\}.$$

We have $I \neq \emptyset$, since $\varepsilon/2 \in I$. We set $s_0 := \sup I$, we want to prove that $s_0 = d$.

Assume that $s_0 \neq d$. By continuity we get that $\Sigma_{s_0}^{-*} \leqslant \Sigma_{s_0}^+$. On the other hand we have $\Sigma_{s_0}^{-*} \neq \Sigma_{s_0}^+$, since the asymptotic boundaries of those two parts are not equal. Observe that $\partial \Sigma_{s_0}^{-*} = \Sigma \cap P_{s_0}$ is compact and the boundary maximum principle shows that Σ is never orthogonal to P_{s_0} along their intersection. We deduce that there exists $\varepsilon_1 > 0$ such that $\Sigma_{d-s_0-\varepsilon_1}^-$ is a horizontal graph with respect to γ_0 and $\Sigma_r^{-*} \leqslant \Sigma_r^+$ for any $r \in]d - s_0 - \varepsilon_1, d[$, which gives a contradiction with the maximality of s_0. We deduce that $s_0 = d$ and then $\Sigma_0^{-*} \leqslant \Sigma_0^+$.

Using the same arguments coming from the other side, that is from p_2 to p_0, we can show that $\Sigma_0^{+*} \geqslant \Sigma_0^-$. We conclude that $\Sigma_0^{+*} = \Sigma_0^-$, that is Σ is symmetric with respect to $P_0 = \Gamma \times \mathbb{R}$, as desired.
The proof that Σ is embedded can be established in the same way as in [22, Theorem 2]. q.e.d.

Remark 3.3. In [12, Proposition 2.4], F. Martin, R. Mazzeo and M. Rodriguez have given an independent proof of Lemmas 3.2 and 3.3.

Lemma 3.4. *Suppose that γ_1 and γ_2 are geodesic satisfying the properties in* (18). *Then, the surface Σ is symmetric with respect to some slice $\mathbb{H}^2 \times \{t_0\}$ and the closure each component of $\Sigma \setminus (\mathbb{H}^2 \times \{t_0\})$ is a vertical graph.*

Proof. From Lemma 3.3 we know that: Σ is embedded, each component of $\Sigma \setminus (\Gamma \times \mathbb{R})$ is a horizontal graph with respect to γ_0 and that Σ is symmetric with respect to the geodesic vertical plane $\Gamma \times \mathbb{R}$.
We first deduce that Σ is transversal to $\Gamma \times \mathbb{R}$, then that Σ is actually orthogonal to $\Gamma \times \mathbb{R}$. Therefore the intersection $\mathcal{C} := \Sigma \cap (\Gamma \times \mathbb{R})$ is composed of a finite number of Jordan curves. Since each component of $\Sigma \setminus (\gamma_0 \times \mathbb{R})$ is a horizontal graph with respect to Γ (see Lemma 3.2), we get that the interiors of the Jordan curves of $\mathcal{C}$ are pairwise disjoint.

For $i = 1, 2$, we call Σ_i the component of $\Sigma \setminus (\Gamma \times \mathbb{R})$ which is asymptotic to the vertical plane $\gamma_i \times \mathbb{R}$. For any $t \in \mathbb{R}$, we set $\Pi_t := \mathbb{H}^2 \times \{t\}$. Let $t_1 \in \mathbb{R}$ be such that $\partial\Sigma_1 \cap \Pi_t = \emptyset$ for any $t > t_1$ and $\partial\Sigma_1 \cap \Pi_{t_1} \neq \emptyset$. Such a t_1 exists since $\partial\Sigma_1 = \mathcal{C}$ is compact.
For any $t \in \mathbb{R}$ we set: $\Pi_t^+ := \mathbb{H}^2 \times \{s \mid s > t\}$, $\Pi_t^- := \mathbb{H}^2 \times \{s \mid s < t\}$, $\Sigma_{1,t}^+ = \Sigma_1 \cap \Pi_t^+$, $\Sigma_{1,t}^- = \Sigma_1 \cap \Pi_t^-$ and we denote by $\Sigma_{1,t}^{+*}$ the reflection of $\Sigma_{1,t}^+$ across Π_t. Moreover, $\Sigma_{1,t}^{+*} \geqslant \Sigma_{1,t}^-$ means that $\Sigma_{1,t}^{+*}$ stays above $\Sigma_{1,t}^-$.

Claim 1. *For any $t \geqslant t_1$ we have $\Sigma_{1,t}^{+*} \geqslant \Sigma_{1,t}^-$. Consequently, Σ_{1,t_1}^+ is a vertical graph.*

Indeed, for $t > t_1$ we know from Lemma 3.3 that the intersection $\Sigma_1 \cap \Pi_t$ is a complete curve, that is a horizontal graph with respect to γ_0 and whose asymptotic boundary is $\partial_\infty\gamma_1 \times \{t\}$. For any $s \in \mathbb{R}$ we denote by T_s the horizontal translation along γ_0 of signed length s in the direction going from p_2 to p_1.
Suppose that $\Sigma_{1,t}^{+*}$ does not remain above $\Sigma_{1,t}^-$. Then, for $\varepsilon > 0$ small enough, the translated $T_\varepsilon(\Sigma_{1,t}^{+*})$ does not remain above $\Sigma_{1,t}^-$. Observe that $\partial_\infty T_\varepsilon(\Sigma_{1,t}^{+*}) \cap \partial_\infty\Sigma_{1,t}^- = \emptyset$. Moreover, $\partial T_s(\Sigma_{1,t}^{+*}) \cap \overline{\Sigma_{1,t}^-} = \emptyset$, for any $s > 0$, since Σ_1 is a horizontal graph with respect to γ_0. We deduce that, for $\varepsilon > 0$ small enough, the part of $\Sigma_{1,t}^-$ which remains above $T_\varepsilon(\Sigma_{1,t}^{+*})$ has compact closure. Therefore, there exists $s_1 > 0$ such that $T_s(\Sigma_{1,t}^{+*}) \cap \Sigma_{1,t}^- = \emptyset$ for any $s > s_1$ and $T_{s_1}(\Sigma_{1,t}^{+*}) \cap \Sigma_{1,t}^- \neq \emptyset$. This means that $T_{s_1}(\Sigma_{1,t}^{+*})$ and $\Sigma_{1,t}^-$ are tangent at some point and one surface remains in one side of the other, which gives a contradiction with the maximum principle and proves the claim.

Claim 2. *For any $\varepsilon > 0$ small enough, we have $\Sigma_{1,t_1-\varepsilon}^{+*} \geqslant \Sigma_{1,t_1-\varepsilon}^-$.*

For any $s \in \mathbb{R}$, let T_s be the horizontal translation defined as in Claim 1. Since the closure of Σ_1 is a horizontal graph with respect to γ_0, we have $T_s(\mathcal{C}) \cap \Sigma_1 = \emptyset$ for any $s > 0$.

Furthermore, since the whole surface Σ is symmetric with respect to $\Gamma \times \mathbb{R}$ we have that $T_s(\mathcal{C}) \cap \Sigma_1 = \emptyset$ for any $s \neq 0$. Let $\mathcal{D} \subset \Gamma \times \mathbb{R}$ be the bounded subset with boundary $\mathcal{C}$. Since Σ is connected, we have $T_s(\mathcal{D}) \cap \Sigma = \emptyset$ for any $s \neq 0$.
Let $\varepsilon > 0$ such that $\mathcal{C}^{+*}_{t_1-\varepsilon} \geqslant \mathcal{C}^{-}_{t_1-\varepsilon}$, where $\mathcal{C}^{+,*}_{t_1-\varepsilon}$ etc. are obviously defined. Then, using an argument analogous to that of Claim 1, considering translations T_s, it can be shown that $\Sigma^{+*}_{1,t_1-\varepsilon} \geqslant \Sigma^{-}_{1,t_1-\varepsilon}$. This proves Claim 2.

Now we can conclude the proof.
Since $\mathcal{C}$ is composed of a finite number of Jordan curves, there exists a component, say C, and a real number $t_0 < t_1$ satisfying $\Sigma^{+*}_{1,t} \geqslant \Sigma^{-}_{1,t}$ for any $t > t_0$, such that at least one of the following properties occurs:

(1) $C^{+*}_{t_0} \geqslant C^{-}_{t_0}$ and $C^{+*}_{t_0}$ is tangent to $C^{-}_{t_0}$ at some interior point.
(2) $C^{+*}_{t_0} \geqslant C^{-}_{t_0}$ and $C^{+*}_{t_0}$ and $C^{-}_{t_0}$ are tangent along their common boundary.

Recall that Σ is orthogonal to $\Gamma \times \mathbb{R}$ along $\mathcal{C}$, since $\mathcal{C} = \Sigma \cap (\Gamma \times \mathbb{R})$ and Σ is symmetric with respect to $\Gamma \times \mathbb{R}$.
Thus, in the first case, applying the boundary maximum principle to the surfaces Σ^{+*}_{1,t_0} and Σ^{-}_{1,t_0}, we conclude that $\Sigma^{+*}_{1,t_0} = \Sigma^{-}_{1,t_0}$ and then $\Sigma^{+*}_{t_0} = \Sigma^{-}_{t_0}$.
In the second case we apply the boundary maximum principle to the surfaces $\Sigma^{+*}_{t_0}$ and $\Sigma^{-}_{t_0}$ in order to infer $\Sigma^{+*}_{t_0} = \Sigma^{-}_{t_0}$.
Consequently, the surface Σ is symmetric with respect to the horizontal plane $\Pi_{t_0} = \mathbb{H}^2 \times \{t_0\}$, as desired. q.e.d.

Remark 3.4. It follows from the proof of Lemma 3.4 that the tangent plane at any point of $\Sigma \setminus (\mathbb{H}^2 \times \{t_0\})$ is never vertical.

Proof of Theorem 3.1. The maximum principle shows that $\gamma_1 \neq \gamma_2$, since Σ is not a vertical plane. We know from Proposition 3.1 that $\partial_\infty \gamma_1 \cap \partial_\infty \gamma_2 = \emptyset$ and from Proposition 3.2 that $\gamma_1 \cap \gamma_2 = \emptyset$. Thus, the geodesics γ_1 and γ_2 satisfy the properties (18). Therefore we deduce from Lemma 3.3 that Σ is embedded.
Furthermore, we deduce from Lemmas 3.2, 3.3 and 3.4 that Σ is symmetric with respect to the vertical planes $\gamma_0 \times \mathbb{R}$ and $\Gamma \times \mathbb{R}$ and also with respect to the slice $\Pi_0 := \mathbb{H}^2 \times \{0\}$ (up to a vertical translation).
We call S_0 the reflection across the slice Π_0, S_Γ the reflection across the vertical plane $\Gamma \times \mathbb{R}$ and S_{γ_0} the reflection across the vertical plane $\gamma_0 \times \mathbb{R}$.

For any real number $s \neq 0$, we denote by Γ_s the equidistant line to Γ with distance equal to $|s|$, which intersects γ_0 between p_0 and p_1 (resp. p_0 and p_2) if $s > 0$ (resp. $s < 0$). We set $\Gamma_0 = \Gamma$. For any $s \in \mathbb{R}$, we set $P_s := \Gamma_s \times \mathbb{R}$.

We define $\Sigma^+ := \Sigma \cap (\mathbb{H}^2 \times]0, +\infty[)$. Since Σ^+ is a vertical graph, the tangent plane is never vertical along Σ^+. Consequently Σ^+ intersects any P_s transversally. Since each component of $\Sigma^+ \setminus (\gamma_0 \times \mathbb{R})$ is a horizontal graph with respect to Γ and since Σ is symmetric with respect to Π_0 and $\gamma_0 \times \mathbb{R}$, we deduce that for any $s \in \mathbb{R}$ the intersection $\Sigma \cap P_s$ consists of a Jordan curve. Therefore, Σ is homeomorphic to an annulus. Since Σ has finite total curvature, we get that Σ is conformally parametrized by $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$.

Let $X : \mathbb{C}^* \to \Sigma \subset \mathbb{H}^2 \times \mathbb{R}$ be a conformal parametrization of Σ. Since Σ is embedded we may assume that X is an embedding. We deduce from Lemma 3.1 that each end of Σ has degree zero.

The symmetry S_Γ corresponds to a anticonformal diffeomorphism $s_\Gamma : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ satisfying $s_\Gamma(0) = \infty$ and $s_\Gamma(\infty) = 0$. Since the set of fixed points of S_Γ in Σ is a Jordan curve, the set of fixed points of s_Γ is a circle c_Γ. Up to a conformal change of coordinates, we can assume that $c_\Gamma \subset \mathbb{C}$ is the unit circle centered at the origin. Thus, we get $s_\Gamma(z) = 1/\overline{z}$ for any $z \in \mathbb{C}^*$.
Then, we denote by $s_{\gamma_0} : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ the anticonformal diffeomorphism corresponding to S_{γ_0}. The set of fixed points of s_{γ_0} in $\mathbb{C}^*$ is a straight line L_γ passing to and punctured at the origin. Up to a rotation we can assume that $L_\gamma = \{\operatorname{Re} z = 0\}$. Thus, we have $s_{\gamma_0}(z) = -\overline{z}$ for any $z \in \mathbb{C}$.
At last, let us call $s_0 : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ the anticonformal diffeomorphism corresponding to S_0. The set of fixed points of s_0 in $\mathbb{C}^*$ is a straight line L passing and punctured at the origin. Since we have $(S_0 \circ S_{\gamma_0}) \circ (S_0 \circ S_{\gamma_0}) = Id$ on Σ, we must have $(s_0 \circ s_{\gamma_0}) \circ (s_0 \circ s_{\gamma_0}) = Id$ on $\mathbb{C}^*$. Thus, L must be orthogonal to L_γ and we get $L = \{\operatorname{Im} z = 0\}$. Then, we have $s_0(z) = \overline{z}$ for any $z \in \mathbb{C}$.
We call P_0^+ the component of $(\mathbb{H}^2 \times \mathbb{R}) \setminus (\Gamma \times \mathbb{R})$ containing $\gamma_1 \times \mathbb{R}$ and we set $\Sigma^{++} := \Sigma^+ \cap P_0^+$. Finally, we call Σ_0 any of the two components of $\Sigma^{++} \setminus (\gamma_0 \times \mathbb{R})$. Thus, we recover the whole surface Σ by applying the symmetries S_0, S_Γ and S_{γ_0} to the closure of Σ_0.
We can assume that Σ_0 is parametrized by the subset

$$U_0 := \{z \in \mathbb{C} \mid |z| > 1,\ \operatorname{Re} z < 0,\ \operatorname{Im} z > 0\}.$$

Since Σ_0 is simply connected, we can consider its conjugate Σ_0^* which is a minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ uniquely defined up to an ambient isometry.
From now on, for any object x relative to Σ_0 we denote by x^* the corresponding object relative to the conjugate surface Σ_0^*. Thus, $X^* := (F^*, h^*) : U_0 \to \Sigma_0^* \subset \mathbb{H}^2 \times \mathbb{R}$ is a conformal parametrization of Σ_0^*.
Observe that the boundary of Σ_0 is composed of three arcs:

(1) A semi-complete curve $b_1 \subset (\gamma_0 \times \mathbb{R})$ with boundary point q.
(2) A compact arc $b_2 \subset (\Gamma \times \mathbb{R})$ with boundary q and $\widetilde{q}$.
(3) A semi-complete curve $b_3 \subset \Pi_0$ with boundary point $\widetilde{q}$.

In order to visualize the following discussion we consider the model of the unit disk for $\mathbb{H}^2$ (see Figure 9).
Up to an isometry we can assume that $q^* = 0 \in \mathbb{H}^2$.
Since b_1 is contained in a vertical plane, its conjugate b_1^* must be a horizontal half-geodesic issue from 0 (see [4, end of Section 4.1]). We can assume that $\partial_\infty b_1^* = \{i\}$.
Moreover, since b_2 is contained in a vertical plane, its conjugate b_2^* must be a compact geodesic arc orthogonal to b_1^* with endpoints $\widetilde{q}^*$ and 0. Finally, since b_3 is contained in a horizontal plane, its conjugate b_3^* must be a vertical half-geodesic issue from $\widetilde{q}^*$. We can assume that $b_3^* = \widetilde{q}^* \times \{t \geqslant 0\}$.
We denote by $C \subset \mathbb{H}^2$ the geodesic passing through $\widetilde{q}^*$, having i in its asymptotic boundary and we denote by C_0 the half-geodesic of C issue from $\widetilde{q}^*$ verifying $\partial_\infty C_0 = \{i\}$ We call $\mathcal{D}$ the domain of $\mathbb{H}^2$ bounded by b_1^*, b_2^* and C_0 such that $\partial_\infty \mathcal{D} = \{i\}$.
In order to prove that Σ is a horizontal catenoid, it is enough to prove that Σ_0^* is a vertical graph over $\mathcal{D}$, with infinite data on C_0 and zero data on the two sides b_1^* and b_2^* (see [17]).

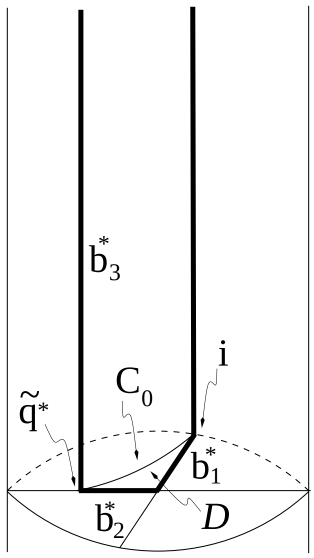


FIGURE 9

We call σ the reflection in $\mathbb{H}^2 \times \mathbb{R}$ across the vertical geodesic containing b_3^*.
Since Σ_0^* is parametrized by U_0, the interior of $\overline{\Sigma}_0^* \cup \sigma(\Sigma_0^*)$ is parametrized by

$$\widetilde{U}_0 := \big\{ z \in \mathbb{C} \mid |z| > 1,\ \operatorname{Re} z < 0 \big\}.$$

We recall that $\phi^* = -\phi$, $h^* = -2\operatorname{Re} W$, since $h = 2\operatorname{Im} W$ and h^* is the harmonic conjugate of h. Thus $W^* = -iW$. We may suppose that $h^* = 0$ on $b_1^* \cup b_2^*$. We are able to study the behavior of F^* and h^* on $\widetilde{U}_0$ in the same way as we did for F and h in Section 2. Recall that $m = 0$ for the end of Σ parametrized by $\{|z| > 1\}$.

Since the interior of $\overline{\Sigma}_0 \cup S_0(\Sigma_0)$ is a horizontal graph with respect to Γ, we get that the tangent plane along it is never horizontal. Thus we get $\phi \neq 0$ on $\widetilde{U}_0$, and since $(h_z)^2 = -\phi$, we get also that $h_z \neq 0$ on $\widetilde{U}_0$. Therefore h is strictly monotonous along any level curve of h^*. Also, since $h \equiv 0$ on $L^- := \{\operatorname{Im} z = 0\} \cap \widetilde{U}_0$, we get that h^* is strictly monotonous and unbounded along L^-.
We deduce that for any $\mu > 0$, the level set $\{h^* = \mu\}$ is composed of a unique complete curve $L_\mu \subset \widetilde{U}_0$: a part of L_μ has the ray $\{ri,\ r > 0\}$ as asymptotic direction and the other part has the ray $\{ri,\ r < 0\}$ as asymptotic direction. Furthermore L_μ intersects L^- at a unique point. We deduce also that for any $\mu < 0$, the level set $\{h^* = \mu\}$ is empty.

Since h is strictly monotonous along any curve L_μ, we deduce that W^* is one-to-one on $\widetilde{U}_0$. The results established in Theorem 2.1 yields that the curves $F^*(L_\mu)$ in $\mathbb{H}^2$ converge to the geodesic C when $\mu \to +\infty$. We deduce that $\partial_\infty \Sigma_0^+ \subset \{i\} \times \mathbb{R}$.

Let us call B_i the geodesic containing b_i^*, $i = 1, 2$.

Claim 1. *We have* $\Sigma_0^* \subset \mathcal{D} \times \mathbb{R}$.

By construction, B_1 and B_2 meet orthogonally at the origin and B_2 is the geodesic with asymptotic points $(-1,0)$, $(1,0)$. For $s>0$, we call $Q_s \subset \mathbb{H}^2\times\mathbb{R}$ the vertical plane orthogonal to B_2 at $(s,0)$ and we call Q_s^- the component of $(\mathbb{H}^2\times\mathbb{R})\setminus Q_s$ which does not contain $\mathcal{D}\times\mathbb{R}$. Recall that $\partial_\infty\Sigma_0^* \subset \{i\}\times\mathbb{R}$. Moreover, if (z_n) is a sequence in U_0 such that $h^*(z_n)\to+\infty$, then we have $d_{\mathbb{H}^2}\big(F^*(z_n),C\big)\to 0$. Consequently, for any $s>0$, the intersection $\Sigma_0^*\cap Q_s^-$ is either empty or have compact closure. Assume the latter is true. In this case, we find s such that $\Sigma_0^*\cap Q_s^-=\emptyset$. Then, we start to decrease s. By the maximum principle, we can decrease s till 0 and obtain that $\Sigma_0^*\cap Q_s^-=\emptyset$ for any $s>0$. Therefore, Σ_0^* remains in the component of $(\mathbb{H}^2\times\mathbb{R})\setminus(B_1\times\mathbb{R})$ containing $\mathcal{D}\times\mathbb{R}$.
By the same reasoning as above we can prove that Σ_0^* remains in the component of $(\mathbb{H}^2\times\mathbb{R})\setminus(B_2\times\mathbb{R})$ containing $\mathcal{D}\times\mathbb{R}$ and also Σ_0^* remains in the component of $(\mathbb{H}^2\times\mathbb{R})\setminus(C\times\mathbb{R})$ containing $\mathcal{D}\times\mathbb{R}$. We conclude that $\Sigma_0^*\subset\mathcal{D}\times\mathbb{R}$.

Claim 2. *We have $F^*(U_0)\subset\mathcal{D}$. Furthermore, the map $F^*: U_0\to\mathcal{D}$ is proper.*

Let $Pr:\mathbb{H}^2\times\mathbb{R}\to\mathbb{H}^2$ be the projection on the first component. Since $F^*=Pr\circ X^*$ we deduce from Claim 1 that $F^*(U_0)\subset\mathcal{D}$.
We must prove that for any compact set $K\subset\mathcal{D}$, $(F^*)^{-1}(K)$ is a compact subset of U_0. In order to prove it, it is enough to show that for any sequence (z_n) in $(F^*)^{-1}(K)$, there is a subsequence of (z_n) converging in $(F^*)^{-1}(K)$.
Since K is far from the geodesic C, the height function is bounded on K. Therefore, there exists a constant $\mu>0$ such that $(F^*)^{-1}(K)$ remains in the subset of U_0 bounded by the level set L_μ and the half-axis $\{iy \mid y>0\}$. Let (z_n) be a sequence in $(F^*)^{-1}(K)$. Suppose that (z_n) is not bounded. Therefore, there exists a subsequence $(z_{\varphi(n)})$ of (z_n) such that $|z_{\varphi(n)}|\to+\infty$. As in the proof of Theorem 2.1 (Assertion 2), we can show that there exists a sequence (iy_n) such that $y_n\to+\infty$ and $d_{\mathbb{H}^2}\big(F^*(z_{\varphi(n)}),F^*(iy_n)\big)\to 0$. But this is absurd since we have, by construction, $F^*(z_{\varphi(n)})\in K$ and $F^*(iy_n)\to i\in\partial_\infty\mathbb{H}^2$.
Thus (z_n) is a bounded sequence of U_0. Therefore, since $U_0\subset\mathbb{C}$, we can extract a subsequence $(z_{\psi(n)})$ converging to some point $z\in\overline{U}_0$. We want to show that $z\in(F^*)^{-1}(K)$.
Observe that F^* maps the boundary of U_0 onto the boundary of $\mathcal{D}$, and moreover $d_{\mathbb{H}^2}\big(K,\partial\mathcal{D}\big)>0$, since K is a compact subset of $\mathcal{D}$. Therefore, we deduce that $z\in U_0$. Since F^* is continuous and K is compact, we obtain $F^*(z)\in K$, from which we get $z\in(F^*)^{-1}(K)$. Therefore $F^*:U_0\to\mathcal{D}$ is a proper map, as desired.

Claim 3. *We have $F^*(U_0)=\mathcal{D}$ and Σ_0^* is a vertical graph over $\mathcal{D}$.*

We know from Claim 2 that $F^*(U_0)\subset\mathcal{D}$. Therefore, it suffices to prove that $F^*(U_0)$ is a closed and open subset of $\mathcal{D}$.
It is known that $n_3^*=\pm n_3$ (see [8, Remark 9]). On the other hand, since Σ_0 is a vertical graph, we have $n_3\neq 0$ along Σ_0. We deduce that the tangent plane is never vertical along Σ_0^* and that the map $F^*:U_0\to\mathcal{D}\subset\mathbb{H}^2$ is a local smooth diffeomorphism. Then, F^* is an open map. As U_0 is open, we get that $F^*(U_0)$ is an open subset of $\mathcal{D}$.
Now we prove that $F^*(U_0)$ is also a closed subset of $\mathcal{D}$.
Let (q_n) be a sequence in $F^*(U_0)$ converging to some point $q\in\mathcal{D}$. We want to prove that $q\in F^*(U_0)$.
We set $K:=\{q\}\cup\{q_n,\ n\in\mathbb{N}\}$, then K is a compact subset of $\mathcal{D}$. From Claim 2 we deduce that $(F^*)^{-1}(K)$ is a compact subset of U_0. For any $n\in\mathbb{N}$ there exists $z_n\in(F^*)^{-1}(K)$ such that $F^*(z_n)=q_n$. Therefore, we can extract a subsequence $(z_{\varphi(n)})$ which converges to

some $z \in (F^*)^{-1}(K) \subset U_0$. Since F^* is continuous we obtain $F^*(z_{\varphi(n)}) \to F^*(z)$, that is $q_{\varphi(n)} \to F^*(z)$. We deduce that $q = F^*(z)$ and then $q \in F^*(U_0)$, therefore $F^*(U_0)$ is a closed subset of $\mathcal{D}$. Consequently we get $F^*(U_0) = \mathcal{D}$.
Hence, the map $F^* : U_0 \to \mathcal{D}$ is a local smooth diffeomorphism. Moreover F^* is proper and surjective. We deduce that it is a covering map. Since $\mathcal{D}$ is connected and simply connected and U_0 is connected, we deduce that F^* is a global diffeomorphism from U_0 onto $\mathcal{D}$, that is Σ_0^* is a vertical graph over $\mathcal{D}$, as desired.

Since $h*$ is strictly monotonous and non bounded along L^-, we obtain that Σ_0^* is a vertical graph over $\mathcal{D}$ with infinite data on C_0 and zero data on b_1^* and b_2^*, this concludes the proof. q.e.d.

4. Appendix. Basic geometry in $\mathbb{H}^2$

In this section, we establish some background material about C^2-curves in $\mathbb{H}^2$ whose absolute value of geodesic curvature is strictly smaller than one. We observe that the condition on the curvature implies that such a curve is embedded, see for example [21, Proposition 2.6.32].

Proposition 4.1. *Let $c : [0, +\infty[\to \mathbb{H}^2$ be a regular C^2-curve with infinite length. Let $\kappa(t)$ be the geodesic curvature of c at the point $c(t)$. Assume that $|\kappa(c(t))| < k < 1$, for any $t \geqslant 0$, and that c is one-to-one.*
Then, the curve $C := c([0, +\infty[)$ has no limit point in $\mathbb{H}^2$, and the asymptotic boundary of C consists of only one point $\{p_\infty\} = \partial_\infty C$.

Proof. If $k = 0$ then C is a part of a geodesic and the assertions are obvious. Therefore we assume that $0 < k < 1$.

Claim 1. *C has no limit point in $\mathbb{H}^2$.*
Indeed, assume by contradiction that there exists $p \in \mathbb{H}^2$ and a sequence of positive numbers (t_n) such that $t_n \to +\infty$ and $p_n := c(t_n) \to p$ when $n \to \infty$.
Assume first that there exists a point $q \in C$, $q = c(t_0)$ for some $t_0 > 0$, such that C is orthogonal at q to the geodesic passing through q and p. Let $H_q \subset \mathbb{H}^2$ be the horocycle through q, tangent to the curve C, such that p belongs to the convex component of $\mathbb{H}^2 \setminus H_q$. Recall that $|\kappa(c(t))| < 1$ and the absolute value of the curvature of the horocycles is 1. Thus, the maximum principle for curves, see [21, Theorem 2.6.27], ensures that $C_0 := c([t_0, +\infty[)$ belongs to the non convex component of $\mathbb{H}^2 \setminus H_q$ and then, p cannot not be in the closure of C.
Hence we infer that the function $t \mapsto d_{\mathbb{H}^2}(c(t), p)$ is strictly decreasing.
For $t > 0$ we denote by $\alpha(t) \in [0, \pi]$, the nonoriented angle at $c(t)$ between the tangent vector $c'(t)$ and the geodesic segment $[c(t), p]$. Since the function $t \mapsto d_{\mathbb{H}^2}(c(t), p)$ is strictly decreasing we have $\alpha(t) \in [0, \pi/2[$ for any $t > 0$.
Actually, we have $\alpha(t) \to 0$ as $t \to +\infty$. Indeed, assume by contradiction that there exist a sequence (t_n), and a real number $\alpha_0 \in]0, \pi/2[$, such that $t_n \to +\infty$ and $\alpha(t_n) \to \alpha_0$. For any $n \in \mathbb{N}$, we denote by γ_n the geodesic of $\mathbb{H}^2$ through $c(t_n)$ tangent to C. Let us denote by $H_n \subset \mathbb{H}^2$, $n \in \mathbb{N}$, the horocycle through $c(t_n)$ tangent to the curve C and contained in the same component of $\mathbb{H}^2 \setminus \gamma_n$ as p. Therefore, for n large enough, the point $c(t_n)$ is very close to p and the angle $\alpha(t)$ is very close to α_0. This would imply, for n large enough, that p belongs to the convex component of $\mathbb{H}^2 \setminus H_n$ and this would give again a contradiction with the maximum principle for curves. Therefore we get that $\alpha(t) \to 0$ as $t \to +\infty$.

To conclude the argument, we choose for $\mathbb{H}^2$ the model of the unit disk equipped with the metric $g_{\mathbb{D}} = \lambda^2(z)\,|dz|^2$, where $\lambda(z) = 2/(1-|z|^2)$. We can assume that $p = 0$ and that C is parametrized by arclength.
In polar coordinates we have $c(t) = (r(t)\cos\theta(t), r(t)\sin\theta(t))$ where $r(t) = |c(t)| > 0$ and $\theta(t) \in \mathbb{R}$. We set $\partial_r := (\cos\theta, \sin\theta)$. Since $\alpha(t) \to 0$, we have

$$\frac{\langle c'(t)\,;\,\partial_r\rangle_{\mathbb{D}}}{|\partial_r|_{\mathbb{D}}} \to -1,$$

where the scalar product and the norm are considered with respect to the metric $g_{\mathbb{D}}$. From which we get that $\lambda(c(t))r'(t) \to -1$. Using that $c(t) \to p = 0$ for $t \to \infty$, we obtain $\lambda(c(t)) \to 2$ for $t \to \infty$, therefore $r'(t) \to -1/2$ and then $r(t) \to -\infty$ as $t \to +\infty$. This is a contradiction and this concludes the proof of the claim.

Since C has infinite length and has no limit point in $\mathbb{H}^2$, we deduce that its asymptotic boundary is not empty. Let $p_\infty \in \partial_\infty C$ be an asymptotic point of C.

Claim 2. *p_∞ is the unique asymptotic point of C.*

Let $\Gamma \subset \mathbb{H}^2$ be a complete curve with constant curvature k such that $p_\infty \notin \partial_\infty\Gamma$ and p_∞ belongs to the asymptotic boundary of the convex component of $\mathbb{H}^2 \setminus \Gamma$.
Let $\gamma \subset \mathbb{H}^2$ be a geodesic intersecting Γ such that $p_\infty \in \partial_\infty\gamma$. For any $s \in \mathbb{R}$, let Γ_s be the translated copy of Γ along γ at distance $|s|$, towards p_∞ if $s > 0$, and in the opposite direction otherwise. We denote by Γ_s^+ the convex component of $\mathbb{H}^2 \setminus \Gamma_s$. Thus, we have $\Gamma_0 = \Gamma$ and $p_\infty \in \partial_\infty\Gamma_s^+$ for any $s \geqslant 0$. Observe that $\cap_{s\geqslant 0}\partial_\infty\Gamma_s^+ = \{p_\infty\}$.
If we assume that for any $s > 0$, there exists $t_s > 0$ such that $c([t_s, +\infty[) \subset \overline{\Gamma_s^+}$, then we deduce that p_∞ is the unique asymptotic point of C, as desired. Therefore, we are left with the proof of the assumption above.

Suppose by contradiction that there exists $r > 0$ such that for any $t > 0$ the curve $c([t, +\infty[)$ is not entirely contained in $\overline{\Gamma_r^+}$. Therefore, there is an arc $C_1 \subset C$ such that $\partial C_1 \subset \Gamma_r$ and $C_1 \cap \Gamma_r^+ = \emptyset$, that is C_1 stays outside Γ_r^+. Note that $\overline{C}_1$ is a compact arc with boundary on Γ_r. Considering the curves Γ_s, for s going from r to $-\infty$, we get a real number $\rho < r$ such that $C_1 \subset \overline{\Gamma_\rho^+}$ and C_1 and Γ_ρ are tangent at some interior point of C_1. This gives a contradiction by the maximum principle, keeping in mind the hypothesis about the curvature of C and Γ_ρ and the fact that C_1 belongs to the closure of the convex component of $\mathbb{H}^2 \setminus \Gamma_\rho$. This concludes the proof . q.e.d.

Definition 4.1. (1) Let (p_n) be a sequence in $\mathbb{H}^2$ converging to some point $p \in \mathbb{H}^2$. Let $v \in T_p\mathbb{H}^2$ and $v_n \in T_{p_n}\mathbb{H}^2$ be non zero vectors. Assuming that $p_n \neq p$, we denote by c_n the geodesic passing through p and p_n, and by T_n the translation along c_n such that $T_n(p) = p_n$. If $p_n = p$, we set $T_n = Id$. Let $\alpha_n \in [0, \pi]$ be the non-oriented angle between v_n and $T_n(v)$.

We say that the sequence (v_n) *converges* to v (denoted shortly by $v_n \to v$) if $\alpha_n \to 0$ and $|v_n|_{\mathbb{H}^2} \to |v|_{\mathbb{H}^2}$.

(2) Let γ be a geodesic in $\mathbb{H}^2$ and (γ_n) be a sequence of complete and regular C^1-curves in $\mathbb{H}^2$. We say that the sequence (γ_n) *converges to γ in the C^1 topology* if:
   (a) For any $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that for any $n > n_0$ the curve γ_n stays in the region of $\mathbb{H}^2$ bounded by the two equidistant lines of γ with distance ε from γ.

(b) Let $p \in \gamma$, and let (p_n), $p_n \in \gamma_n$, be any sequence converging to p. Let $v_n \in T_{p_n}\gamma_n$ be a unit tangent vector of γ_n at p_n. If the sequence v_n converges to a unit vector $v \in T_p\mathbb{H}^2$, then $v \in T_p\gamma$, that is v is tangent to γ at p.

Proposition 4.2. *Let $\gamma \subset \mathbb{H}^2$ be a geodesic . Let (γ_n) be a sequence of complete and regular C^2-curves such that:*

- *$\partial_\infty \gamma_n = \partial_\infty \gamma$ for any $n \in \mathbb{N}$.*
- *$\sup_{q\in\gamma_n} \{|\kappa_{\gamma_n}(q)|\} \xrightarrow[n\to\infty]{} 0$, where $\kappa_{\gamma_n}(q)$ is the geodesic curvature of γ_n at point $q \in \gamma_n$.*

Then, the sequence (γ_n) converges to γ in the C^1 topology.

Proof. Let us prove (2a) of Definition 4.1.
Let $\varepsilon > 0$ and let $n_0 \in \mathbb{N}$ such that for any $n > n_0$ we have

$$\tanh\varepsilon > \sup_{q\in\gamma_n} \{|\kappa_{\gamma_n}(q)|\}.$$

Denote by L^1_ε and L^2_ε the two equidistant lines of γ with distance ε from γ. It suffices to show that, for any $n > n_0$, the curve γ_n belongs to the convex component of both $\mathbb{H}^2 \setminus L^1_\varepsilon$ and $\mathbb{H}^2 \setminus L^2_\varepsilon$. We prove that fact for $L_\varepsilon := L^1_\varepsilon$. The proof for L^2_ε is analogous.

Let γ' be any geodesic of $\mathbb{H}^2$ different from γ, intersecting γ. Let $p'_\infty \in \partial_\infty\mathbb{H}^2$ be the asymptotic point of γ' which stays in the asymptotic boundary of the non convex component of $\mathbb{H}^2 \setminus L_\varepsilon$. Consider the translations along γ', towards p'_∞. Assume by contradiction that for any n_0 there exists $n > n_0$ such that γ_n does not belong to the convex component of $\mathbb{H}^2 \setminus L_\varepsilon$. There exists a translated copy L'_ε of L_ε such that:

- L'_ε intersects the curve γ_n at one point q_n.
- L'_ε and γ_n are tangent at q_n.
- γ_n belongs to the closure of the convex component of $\mathbb{H}^2 \setminus L'_\varepsilon$.

Since the geodesic curvature of L'_ε is $\tanh\varepsilon$ (with respect to the normal direction pointing towards the convex component of $\mathbb{H}^2 \setminus L'_\varepsilon$) and since $\tanh\varepsilon > \sup_{q\in\gamma_n} \{|\kappa_{\gamma_n}(q)|\}$, we obtain a contradiction with the maximum principle. This completes the proof of (2a).

Now, we prove (2b) of Definition 4.1.
By contradiction, assume that the unit vector $v \in T_p\mathbb{H}^2$ is not tangent to γ.
Let $\varepsilon > 0$ and let $L \subset \mathbb{H}^2$ be one of the two complete curves passing through p, tangent to v whose absolute value of the geodesic curvature is $\tanh\varepsilon$. Since v is not tangent to γ, if ε is small enough, then the curve L separates the two points of the asymptotic boundary of γ, say p_∞ and q_∞.
As in the Definition 4.1, we denote by c_n the geodesic passing through p and p_n. Let T_n be the hyperbolic translation along c_n such that $T_n(p) = p_n$. Let R_n be the rotation in $\mathbb{H}^2$ around p_n such that $R_n(T_n(v)) = v_n$. Therefore, $L_n := (R_n \circ T_n)(L)$ is a complete curve through p_n, tangent to γ_n at p_n, with constant (absolute value) curvature equal to $\tanh\varepsilon$. If n is large enough then the curve L_n separates p_∞ and q_∞, since this is true for L.
Observe that, if n is large enough, we have $\sup_{q\in\gamma_n} \{|\kappa_{\gamma_n}(q)|\} < \tanh\varepsilon$. Consequently, using the maximum principle for curves in the same way as before, we can show that γ_n entirely belongs to the closure of one of the two components of $\mathbb{H}^2 \setminus L_n$. But this gives a contradiction with the assumption that $\partial_\infty\gamma_n = \partial_\infty\gamma = \{p_\infty, q_\infty\}$. We conclude that v is tangent to γ, as desired. q.e.d.

Remark 4.1. We can extend Definition 4.1 to any dimensional hyperbolic space $\mathbb{H}^n$, $n \geqslant 2$. Moreover, we can prove in the same way as in Proposition 4.2, that if $\Pi \subset \mathbb{H}^n$ is a geodesic hyperplane and if (Π_n) is a sequence of complete and regular C^2-hypersurfaces of $\mathbb{H}^n$ such that $\partial_\infty \Pi_n = \partial_\infty \Pi$ for any n and $\sup_{q \in \Pi_n}\{|H_n(q)|\} \to 0$, where $H_n(q)$ denotes the mean curvature of Π_n at q, then the sequence (Π_n) converges C^1 to Π.

Proposition 4.3. *Let $\gamma_1 \subset \mathbb{H}^2$ be a geodesic and let $p_\infty \in \partial_\infty \mathbb{H}^2$ such that $p_\infty \notin \partial_\infty \gamma_1$. For any $\rho > 0$, let L_ρ be the equidistant line to γ_1 whose distance to γ_1 is ρ, such that p_∞ belongs to the asymptotic boundary of the non convex component of $\mathbb{H}^2 \setminus L_\rho$.*
Let $0 < k < 1$ and let $c : [0, \infty[\to \mathbb{H}^2$ be a regular C^2-curve such that $\partial_\infty c([0, \infty[) = \{p_\infty\}$ and such that $|\kappa(c(t))| < k$ for any $t \geqslant 0$. Set $\rho_0 = \max\{d_{\mathbb{H}^2}\big(c(0), \gamma_1\big), \tanh^{-1}(k)\}$.
Then, for any $\rho > \rho_0$, one has the following facts.

1. *The equidistant line L_ρ cuts the curve $c([0, \infty[)$ at a unique point. Therefore, there exists $t_0 > 0$ such that the curve $c([t_0, \infty[)$ is a horizontal graph with respect to γ_1.*
2. *The equidistant line L_ρ is transversal to $c([0, \infty[)$.*

Proof. Let $\rho > \rho_0$ and let $C = c\big([0, \infty[\big)$. Since $\partial_\infty C = p_\infty$, the equidistant line L_ρ must intersect the curve C at least at one point. Assume by contradiction that L_ρ cut C in at least two points. By construction, $c(0)$ belongs to the convex component of $\mathbb{H}^2 \setminus L_\rho$. Let $p_1 \in L_\rho \cap C$ be the first intersection point from $c(0)$. The boundary maximum principle for curves shows that the curves L_ρ and C are not tangent at p_1. Let $p_2 \in L_\rho \cap C$ be the first intersection point after p_1. Thus, the whole arc of C between p_1 and p_2 belongs to the non convex component of $\mathbb{H}^2 \setminus L_\rho$. Now we obtain a contradiction with the maximum principle in the following way.
Let $p'_\infty \in \partial_\infty \mathbb{H}^2$ be a point in the asymptotic boundary of the convex component of $\mathbb{H}^2 \setminus L_\rho$. Let $\gamma \subset \mathbb{H}^2$ be the geodesic such that $\partial_\infty \gamma = \{p_\infty, p'_\infty\}$. Considering the hyperbolic translations along γ towards p_∞, we obtain a translated copy L'_ρ of L_ρ such that:

- L'_ρ intersects the arc of C between p_1 and p_2 at one point q.
- L'_ρ and the arc of C between p_1 and p_2 are tangent at q.
- The arc of C between p_1 and p_2 belongs to the closure of the convex component of $\mathbb{H}^2 \setminus L'_\rho$.

Since the geodesic curvature of L'_ρ is $\tanh \rho$ (with respect to the normal direction pointing towards the convex component of $\mathbb{H}^2 \setminus L'_\rho$) and since $\tanh \rho > k > \sup_{q \in C}\{|\kappa(C)|\}$, we obtain a contradiction with the maximum principle. So Assertion (1) is proved.

Now we prove Assertion (2).
Suppose, by contradiction, that for some $\rho > \rho_0$, the equidistant line L_ρ is tangent to the curve C at some point p_1. Recall that the curvature of L_ρ is strictly greater, in absolute value, than the curvature of C. We deduce from the maximum principle that an open arc of C, containing p_1, remains in the non convex component of $\mathbb{H}^2 \setminus L_\rho$.
We set $C_1 = c\big(]\rho_0, \infty[\big)$. Then, the first part of the proof shows that the curve $C_1 \setminus \{p_1\}$ remains in the non convex component of $\mathbb{H}^2 \setminus L_\rho$. Therefore, for $\varepsilon > 0$ small enough, the equidistant line $L_{\rho+\varepsilon}$ intersects the curve C at least at two different points near p_1, giving a contradiction with assertion (1). q.e.d.

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