SOME APPLICATIONS OF MAXIMUM PRINCIPLE TO HYPERSURFACES IN EUCLIDEAN AND HYPERBOLIC SPACE

RICARDO SA EARP
Departamento de Matemática
Pontifícia Universidade Católica
Rua Marquês de São Vicente, 225
22453-900 Gávea Rio de Janeiro
Brasil

ERIC TOUBIANA
Département de Mathématiques
Université Paris VII
2, Place Jussieu
75251 Paris - Cedex 05
France

ABSTRACT. Throughout this paper we apply maximum principle to prove several results in both euclidean and hyperbolic space. We first solve the exterior Dirichlet problem for the classical minimal surface equation over a domain $\mathbb{R}^2 - \Omega$, where Ω satisfies the interior circle of radius a condition and $\partial\Omega$ is the union of C^0 simple, closed and disjoint curves on the plane with pairwise disjoint interiors. Furthermore, we shall construct by a quite geometrical and simple argument, a minimal solution u with logarithm growth b, for any $b \in [o, a]$. Concerning minimal vertical graphs in $\mathbb{H}^{n+1}$ over a bounded annular domain we prove an existence and uniqueness theorem; moreover, we prove a Phragmen Lindelöf type result for graphs of mean curvature 1. Finally, we prove Alexandrov type results for compact and noncompact constant mean curvature hypersurfaces with (connected and nonconnected) boundary in hyperbolic space.

s

1. Introduction

In this paper we shall intent to add some insight on the amazing *interlink* between PDE and Differential Geometry in view to apply maximum principle to several situations of the hypersurface theory. A pioneer of this field of research is R. Finn by establishing some properties of classical minimal and constant mean curvature equation. These properties derive from a geometrical approach of certain quasi-linear elliptic equations (See [7]). A remarkable union between Analysis and Geometry is the celebrated Alexandrov Reflection Principle (See [2]). This method have been intensively used to investigate symmetry arising from the boundary of constant mean curvature hypersurfaces in space form (See [1]). The analytical counterpart is based on maximum principle for second order elliptic equations (See [23]). We digress here momentarily to say the following: It is worth mention that in the early

The first author was partially supported by CNPq of Brasil, and the second author by FAPERJ, Brasil.

1992 AMS Subject Classification: 53A10, 53C42,35J25

Seventies Serrin wrote a very elegant paper showing the converse, i.e, Alexandrov geometrical method applied to Analysis also hold (See [35]). In fact he have used Alexandrov techniques to prove symmetry theorems for overdetermined boundary value problems in euclidean space. For similar results in hyperbolic space the reader is referred to the derivations in [19].

In order to achieve the results, we will be concerned with certain *knowhow* in this matter taking account as much classical as forthcoming papers. Moreover, we shall discuss some intriguing open questions issuing from ours developments.

We now will start to report the statements proved throughout the paper given simultaneouly some explanation of the techniques, together with a few of selected enlightening and immediate applications.

In section *2* we shall focalize the classical Dirichlet problem for the minimal surface equation in euclidean three space with the aim to prove a new existence result using maximum principle in a quite geometrical and simple way. The geometry of an minimal embedded *end* M in euclidean space with finite total curvature is well-known (See [34]). We remark that up to a isometry of ambient space such an end M (with vertical normal limit) is a graph of smooth function $z = u(x, y)$ over an exterior domain $\mathbb{R}^2 - \Omega$. Precisely M converges geometrically to an catenoid, i.e, u have an a asymptotic behavior

$$u(x, y) = a \log R + b + o(1)$$

where $R = \sqrt{x^2 + y^2}$ and b is a constant.

It is said that u have a *logarithmic growth* a. With regard to minimal graphs in $\mathbb{R}^3$ there are some basic well-known facts: There is only one solution of the Dirichlet problem $z = u(x, y)$ on a *compact* domain $\mathcal{D}$ with smooth boundary data on $\partial\mathcal{D}$. Indeed, there are at least two ways to see this. The classical one is just an application of Hopf's maximum principle to the difference of two solutions of the mean curvature equation (See [23]). The other is a suitable application of the divergence theorem (See [7], [22]). Another standard application of Hopf's maximum principle is that the maximum of the gradient is reached at a boundary point of $\mathcal{D}$. This fact is easily derived by differentiating the mean curvature equation, taking account that the gaussian curvature of M is non-positive. Of course, we may conclude from above that to pursue the study of an embedded end M converging to an embedded catenoid , it suffices to suppose M is given by a function u over some exterior domain with positive growth $a > 0$, assuming a zero value on $\Gamma = \partial\Omega$ (where Ω is a smooth domain on $\mathbb{R}^2$). A beautiful application of maximum principle to minimal surfaces is inferred in [10].

In section *2* we shall investigate the existence of a solution to the following classical exterior Dirichlet problem :

$$\operatorname{div}\left(\frac{\nabla u}{W_u}\right) = 0 \quad \text{on } \mathbb{R}^2 - \Omega \tag{1.1}$$

$$u \quad = 0 \quad \text{on } \partial\Omega \tag{1.2}$$

where $W_u = \sqrt{1 + |\nabla u|^2}$.

The above equation satisfies the well known geometrical maximum principle related to the comparison of two surfaces tangent at a point p one in a side of the other in a neighborhood of p.(See [34]). We shall say that the domain Ω satisfy an *interior circle of radius a condition* if there is a number $a > 0$, such that at any point $p \in \partial\Omega$ there exists a ball $\mathrm{B_a}$ of radius a , $\mathrm{B}_a \subset \Omega$ with $p \in \partial \mathrm{B}_a$ The main existence result produced in section *2* may be described as follows:

We shall solve the exterior Dirichlet problem given by equations (1.1) and (1.2) when Ω satisfies the *interior circle of radius a condition* and $\partial\Omega$ is the union of C° simple, closed and disjoint curves in $\mathbb{R}^2$ with pairwise disjoint interiors . Precisely, we shall construct a minimal solution u with logarithm growth b, for any $b \in [0, a]$. This existence theorem generalize a result of [25]] (we remark that in [17] there is a more general result for C^2 domains). See also [16].

We will deal with an appropriate notion of *vertical graph* in hyperbolic space. We observe that in hyperbolic space there are several possible definitions of graph of a function to the treatement of constant mean hypersurfaces. For other notion of graph ("horizontal graphs"), related results comparing some common and distinct properties, see the paper published in this Volume entitled "On two mean curvature equations in hyperbolic space" [28]. The reader is referred also to [4], [5], or [20]. In sections *3* and *5* we use the following: Take the half space model $\{(x_1, x_2, \ldots, x_{n+1}); x_{n+1} > 0\}$ for $\mathbb{H}^{n+1}$ in such way that the *asymptotic boundary* $\partial_\infty \mathbb{H}^{n+1}$ is represented by $\{x_{n+1} = 0\}$. Let $\mathcal{D}$ be a domain in $\partial_\infty \mathbb{H}^{n+1}$ (considered as a submanifold of $\mathbb{R}^{n+1}$) whose boundary is Γ. Given a function $u : \bar{\mathcal{D}} \to \mathbb{R}$, the graph of u is defined as the set

$$G(u) = \{x_1, \ldots, x_n, u(x_1, \ldots, x_n); (x_1, \ldots, x_n) \in \bar{\mathcal{D}}\}.$$

Considering the above coordinate system, with respect to the unit upper normal, one easily infer the following mean curvature H equation :

$$\operatorname{div}\left(\frac{\nabla u}{W_u}\right) = \frac{n}{u}\left(\mathrm{H} - \frac{1}{W_u}\right) \quad \text{on } \mathcal{D} \tag{1.3}$$

The above equation satisfies the geometrical maximum principle produzing a well known comparison between two surfaces (and respectively mean curvatures) tangent at a point p one in a side of the other in a neighborhood of p. Notice that as a consequence of this is that *if M is a graph of a solution u over a domain $\mathcal{D}$ of* (1.3) *continuous up to the boundary with $u \geqslant 1$ on $\partial\mathcal{D}$, and* H *satisfying* $\mathrm{H}^2 \leqslant 1$, *then the restriction of u to $\mathcal{D}$ is contained inside the region* $\{x_{n+1} > 1\}$.
Indeed, otherwise one may move the 1-parameter family of horospheres

$\mathcal{H}_t := \{z = t\,, t > 0\}$, $\mathcal{H} = \mathcal{H}_1$, coming from the asymptotic boundary towards M as $t \uparrow 1$ to reach a first interior contact point. It is evident that euclidean vertical translations applied to the family $\mathcal{H}_t$ yield the same movement obtained by homothetically expanding, which give rise to hyperbolic translations. Whence, one get a contradiction by maximum principle, since the mean curvature vector $\vec{H}$ of each horosphere $\mathcal{H}_t$ is pointing to the positive x_{n+1} direction. Clearly to carry out the details of comparison principle applied to this geometrical situation it suffices to restrict the focus to curves in hyperbolic plane, because horospheres are totally umbilic. Notice that the proof of maximum principle for curves in $\mathbb{H}^2$ follows from elementary arguments (See [31] or [32]).

We remark that the *principle* behind the previous argument with the aid of the *flux formula* has been used to give a characterization of totally umbilic caps with mean curvature $0 \leqslant \mathrm{H} \leqslant 1$ (See [5]). We now provide the statement that can be inferred by the maximum principle – the hypersurface is *trapped* into a region with the aid of horospheres – as follows : *Let M be a compact hypersurface in $\mathbb{H}^{n+1}$ with mean curvature not greater than 1. Then M is contained in the region obtained by the intersection of the regions determined by the horospheres that involve ∂M inside its interior.*

On the other hand, in opposition to the minimal graphs in euclidean space, we assert that in general there are more than one solution of the related Dirichlet problem $z = u(x, y)$ on a *compact* domain $\mathcal{D}$ with smooth boundary data on $\partial\mathcal{D}$. This fact may be easily checked doing an inspection of embedded catenoids cousin which are graphs (in the sense above) over the entire asymptotic plane (See [21]). We have seen that the study of constant mean hypersurfaces leads naturally to the search of properties of graphs over unbounded domains. Standard examples of constant mean curvature such graphs are given by hyperbolic cylinders (euclidean cones) with vertical geodesic axis ($\mathrm{H} > 1$), equidistant hypersurfaces with mean curvature $\mathrm{H}^2 < 1$ (tilted euclidean hyperplanes meeting transversely $\partial_\infty H^{n+1}$), horospheres (horizontal euclidean hyperplanes) and catenoid cousin ($\mathrm{H} = 1$). We remark that, P. Collin was able to prove Nitsche conjecture having first obtained (during his doctoral thesis) mastership of minimal graphs in euclidean space.

In section *3* we shall prove a Phragmen Lindelöf type theorem. The underlying idea was conceived by Collin and Krust in [6] (See also [26]). At present there are many unknown aspects of this interesting problem to be explored which are associated to the general problem concerning existence of constant mean curvature 1 graphs over unbounded domains. We now wish to set up a natural problem considering the particular case that M lies above the horosphere $\mathcal{H} = \{z = 1\}$ and ∂M is contained inside $\mathcal{H}$:

Is there a graph of constant mean curvature 1 over an exterior domain for some structure condition on the domain ?

Nevertheless, when $\mathrm{H} = 0$ we shall infer in section *5* some existence results that implies solution of the minimal equation (1.3) over a bounded annular domain $\mathcal{D}$ taking constant boundary value data, provided there is a strict control over the size of the domain and magnitude of the boundary value data. In fact, requirements of this type are somewhat necessary by virtue of the following reasons: *We should emphasize that if Ω is bounded then the exterior Dirichlet problem over $\mathbb{R}^2 - \Omega$, for constant $\mathrm{H} \in [0, 1]$ (see eq (1.3)) and continuous boundary data, just arise when* $\mathrm{H} = 1$. Indeed: Given $\mathrm{H} \in [0, 1)$ there is a *barrier* family $\mathcal{C}_t$ of embedded rotational H-surfaces with same axis, which are contained in a region $\mathcal{R}$ of $\mathbb{H}^{n+1}$ between two close parallels planes orthogonal to the axis (see [9]). If $\mathrm{H} \in [0, 1)$ as ∂M is compact one may place the region $\mathcal{R}$ in such a way that ∂M is outside $\mathcal{R}$, i.e , $\mathcal{R} \cap \partial M = \emptyset$. Now moving the family $\mathcal{C}_t$ throughout $\mathcal{R}$ coming from the infinity towards M, one get a first tangent point for some t. This yields a contradiction by applying maximum principle, since both M and $\mathcal{C}_t$ have the same mean curvature (in absolute value) with the mean curvature vector $\vec{H}$ of $\mathcal{C}_t$ pointing upward. Besides, one can also easily see that if C_2 and C_1 are two circles on $\partial_\infty \mathbb{H}^3$ of radius r_2 and r_1, respectively, with $r_2 \gg r_1$ then there is no immersed minimal surface in $\mathbb{H}^3$ with asymptotic boundary $C_2 \cup C_1$. The argument is the same as before using a family of *minimal catenoids* in a suitable way to get a contradiction by the maximum principle. However, we shall prove in section *5* uniqueness of our minimal graphs in the set of minimal graphs with the same boundary. We remark that regularity properties of the minimal equation with zero boundary value data have been established in [18]. We remark that to infer our result we shall use techniques taken from ([8]), based on the maximum principle, global gradient estimates, i.e Schauder's theory and Ladyzhenskaya–Ural'tseva global Hölder estimates. Finally, it is evident that there are many open questions which may be asked, related to the general Dirichlet problem for equation (1.3) (See [28]) .

We shall consider in section *4* a special case of the following general question setted up by several authors (see for instance [4], [29]) : *Under which further condition a constant mean curvature hypersurface M inherits the symmetries of its boundary ?*

Some additional condition seems to be necessary, since N. Kapouleas ([15]) have stated the existence of non spherical surfaces of genus greater than two, immersed in $\mathbb{R}^3$, with constant mean curvature whose boundary is a plane circle. Searching for Alexandrov type results when M is non compact we infer the following: Suppose that M is a properly embedded constant mean curvature hypersurface in $\mathbb{H}^{n+1}$. If ∂M is a $n-1$- sphere lying in some horosphere $\mathcal{H}$, if M is contained inside the region bounded by $\mathcal{H}$ then M is rotationally symmetric. In particular, M is a genus zero hypersurface. Clearly, when $\mathrm{H}=1$ then M is part of an embedded catenoid cousin. It should be mentioned that the deduction of the previous theorem shows a more general result: Under an analogous assumption *if the boundary of M is merely strictly convex then M inherits the plane of symmetry of the boundary.* In [4], it is stated that when the boundary consists of two circles invariant by a group of rotations of $\mathbb{H}^{n+1}$, then (strong) stability implies the hypersurface is also invariant by rotations. There J. L. Barbosa and the first author asked the question if the assumption of stability may be exchanged by embeddedness. When the boundary of M is non connected we obtain the following : Assume M is mean curvature 1 hypersurface in $\mathbb{H}^{n+1}$. We prove that if $\partial M = S_1 \cup S_2$, where S_1 and S_2 are $n-1$- dimensional spheres of same radius apart enough and ∂M is invariant by rotations then M is a piece of an embedded catenoid cousin. It should be emphasize that those Alexandrov type results may be extended, with slight modifications, to elliptic Weingarten hypersurfaces (see [12]). Namely, in the case of surfaces ($n=2$), we have shown this is true in [29] for a general class of $f-$ surfaces, see also [24]. There is in [3] a characterization of f–surfaces with boundary a round circle announced in [27].

2. On the classical exterior Dirichlet problem

First, we wish to sketch the scheme of the proof relative to the result mentioned at the introduction that we shall prove in this section :

To get the desired minimal graph we shall use in theorem 1 below the hypothesis Ω satisfies the uniform interior circle condition in order to work with the family of catenoids as *barriers*. Using this we are able to establish apriori estimates (by applying maximum principle) to a Dirichlet problem for the minimal equation (see (1.1)) over a C° exterior domain. To see how to accomplish the proof we remark that our construction allow us to make use of Perron process followed by a compactness argument.

2.1 Theorem. *Let Ω be a closed domain in the xy plane satisfying an interior circle of radius a condition. Let $b \in [0,a]$. If $\partial\Omega$ is the union of simple closed disjoint curves in $\mathbb{R}^2$ with pairwise disjoint interior regions $\Omega_1, \ldots, \Omega_k$, then there is a minimal solution $z = u(x,y)$ of* (1.1) *and* (1.2) *(taking zero boundary value) defined in the exterior domain $\mathbb{R}^2 - \Omega$ which is continuous up to the boundary . Furthermore, u has logarithm growth b.*

PROOF. We commence setting the main construction : We first consider a half catenoid $\mathcal{C}_b$ of logarithm growth $0 < b \leqslant a$, which boundary is a circle C_b of radius b, such that up to a horizontal displacement we have $\partial\mathcal{C}_b \subset \Omega_i$ and $\partial\mathcal{C}_b \cap \partial\Omega_i = \emptyset$, $i = 1, \ldots, k$. Clearly, for any $0 < b < a$, such a catenoid exists for ours assumptions. (think $\mathcal{C}_b$ contained in the upper half space of $\mathbb{R}^3$) . Let $\mathcal{C}_b^\epsilon$ be a fixed slight vertical downward displacement of $\mathcal{C}_b$ restricted to the upper halfspace. We note that $\partial\mathcal{C}_b^\epsilon$ is a circle of radius $r > b$, but r is chosen near b so that, $\partial\mathcal{C}_b^\epsilon \subset \Omega_j$, $j \in \{1, \ldots, k\}$. . Let $\mathcal{C}_b^\Omega$ be a fixed part of a catenoid, which is the restriction to the upper halfspace of a big downward translation of $\mathcal{C}_b$ in such a way that $\partial\mathcal{C}_b^\Omega$ is a big circle involving Ω. It is worth observing now that we can do horizontal displacements of $\mathcal{C}_b$ to obtain a family of "supersolutions " to be used as a barrier, with the purpose to control up to the boundary the convergence in Perron process. We also note that $\mathcal{C}_b^\epsilon$ can be made the support of the sequence of solutions over an arbitrarily big annular domain that converges to the desired solution. Moreover, $\mathcal{C}_b^\Omega$ can give rises to the barrier to ensure the desired logarithmic growth b. In order to infer the theorem, one may put all this geometrical knowledge together, combining with well–known theoretical arguments.

We next shall proceed to give the details of this deduction. Let Γ_n be a large circle of radius R_n so that Γ_n involves Ω and $\Gamma_n \gg n$. We will denote the restriction to Γ_n of the function u which graph is the catenoid $\mathcal{C}_b^\epsilon$ by the same symbol, thus we will commit an abuse of notation setting $u = \mathcal{C}_b^\epsilon$ over Γ_n. Let $\mathcal{A}_n$ be the annular bounded domain with boundary equal to $\partial\Omega \cup \Gamma_n$. So we now consider the following auxiliary Dirichlet problem:

$$\operatorname{div}\left(\frac{\nabla u}{W_u}\right) = 0 \quad \text{on } \mathcal{A}_n \tag{2.1}$$

$$u \quad = 0 \quad \text{on } \partial\Omega \tag{2.2}$$

$$u \quad = \mathcal{C}_b^\epsilon \quad \text{on } \Gamma_n \tag{2.3}$$

We now point out that it is well–known that Perron process hold for the minimal surface equation (See [JS1,2]). We also note that the xy plane and $\mathcal{C}_b$ are natural subsolution and supersolution to the above auxiliary Dirichlet problem, respectively. Indeed, to see that any solution u of equations (2.1), (2.2) and (2.3) is bellow $\mathcal{C}_b$, just place $\mathcal{C}_b$ above the graph M of u, by means of vertical translations and then descending $\mathcal{C}_b$ to the former position, we see that it do not touch M, by taking account of maximum principle, for $\mathcal{C}_b$ is above $\mathcal{C}_b^\epsilon$. Clearly, the horizontal xy plane is a subsolution, since any solution of (2.1) that reaches a minimum is constant. Now to get a solution to the auxiliary Dirichlet problem (2.1) taking the prescribed boundary value data (2.2) and (2.3) it suffices to find a barrier at each point of the boundary (see [14]). But then we have to look over the inner boundary $\partial\Omega$, since Scherk's surfaces, as usual, provide a natural barrier at a convex arc of the outer boundary circle (See [13]). We now use the fact that both catenoids $\mathcal{C}_b$ and $\mathcal{C}_b^\epsilon$ have same logarithmic growth b to infer that if Γ_n is chosen sufficiently large we can be sure that any horizontal displacement of $\mathcal{C}_b$ throughout $\bar{\Omega}$ does not fit the restriction of $\mathcal{C}_b^\epsilon$ to Γ_n. Whence, by putting this together with the crucial assumption that $\partial\Omega$ satisfies the interior circle of radius a condition, where $a > b$, it follows that the family of horizontal displacement of $\mathcal{C}_b$ provides the required barrier at any point of $\partial\Omega$. Then we obtain a solution u_n over $\mathcal{A}_n$ of the Dirichlet problem given by equations (2.1), (2.2) and (2.3) as desired. To accomplish the proof of the theorem we have to infer that there is a subsequence above $\mathcal{C}_b^\Omega$ that converges to the wanted minimal solution. But this can be inferred as follows : We notice that it is well–known that the derivatives of a solution u_n of the minimal surface equation at a point $p \in \Omega$ depends only on the distance to the boundary and apriori height estimates (See [8] or [36]). We now are able to make use of Arzela's theorem combining with standard compactness argument to get a minimal solution u over the exterior domain $\mathbb{R}^2 - \Omega$. To carry out the remaining part of the proof, we notice that our construction of the sequence u_n and the barrier obtained at any point of $\partial\Omega$ imply u takes zero boundary data. It also arises from the construction and maximum principle that the family of graphs given by the solutions u_n are above the catenoid $\mathcal{C}_b^\Omega$, hence we infer that u have logarithm growth b. Actually, we have conclude that for any b, satisfying $0 < b < a$, we get a solution over the exterior domain $\mathbb{R}^2 - \Omega$ taking zero boundary value data, with u having logarithm growth b. Therefore, applying again an analogous compactness argument as before, taking account that the catenoid $\mathcal{C}_a$ provides a necessary barrier, we can obtain a solution to our problem with logarithm growth a, as desired.

3. A Phragmen Lindelöf type theorem

We have discussed previously the notion of vertical graphs in hyperbolic space. We shall need some further definitions. Given $r > 0$, let $\mathcal{D}_r := \{(x, y) \in \mathcal{D};\, x^2 + y^2 < r\}$ and let $C_r := \{(x, y) \in \mathcal{D};\, x^2 + y^2 = r\}$. Denote $\mathrm{M}(r) := \sup\{v(x, y)\ , (x, y) \in C_r\}$. Under the preceding conventions we infer the following:

3.1 Theorem. *Let $M \subset \mathbb{H}^3$ be a vertical graph of a function u over an unbounded domain $\mathcal{D}$ with $u = 1$ on $\partial\mathcal{D}$ and $u \not\equiv 1$ in $\mathcal{D}$. If $H^2 \leqslant 1$ and $u \geqslant \varepsilon > 0$ then $M - \partial M$ is entirely above the horosphere $\mathcal{H} := \{z = 1\}$. Furthermore, if $H = 1$ setting $v := u - 1$ $(v \geqslant 0)$, given $r_0 > 0$, $\beta \in (0, 1]$ there is a positive constant c^2 such that if $R > r_0$ then the following estimate hold*

$$\mathrm{M}^2(R) \geqslant \beta^2 \left(\frac{c^2}{32\pi^2} \log^2 \frac{R}{r_0} + \frac{b_\beta(R)}{8\pi} \frac{\mathrm{M}^{1-\beta}(\sqrt{R\, r_0})}{\beta} \log \frac{R}{r_0} \right)$$

$$\textit{where}\ \ b_\beta(R)=\int\limits_{\mathcal{D}_{\sqrt{Rr_0}}}\frac{2v^\beta}{v+1}\big(1-\frac{1}{W_v}\big)$$

PROOF. We will commence the proof of the first assertion as follows: We will argue by absurd. Suppose that part of M lies bellow the horosphere $\mathcal{H}$. Clearly there are no compact components by maximum principle doing exactly the same argument as we have showed in the introduction. So we may assume for simplicity M is asymptotic to the horosphere $\mathcal{H}_\epsilon$ given by $\{z=\varepsilon\}$. Recall that ours assumptions imply M stay into the region bounded by $\mathcal{H}_\epsilon$, that is M is above $\mathcal{H}_\epsilon$. Now notice that up to an isometry of ambient space an embedded end $\mathcal{A}$ of an immersed catenoid cousin having vertical normal limits is a graph of smooth function $z=u(x,y)$ satisfying (1.3) over an exterior domain $\mathbb{R}^2-\ \mathrm{D}_R$, where D_R is an open disk of radius R. Furthermore, one may derive that the principal term of such an end $\mathcal{A}$ have the following asymptotic behavior:

$$\mathrm{c\ R}^\alpha \quad \text{as} \quad \mathrm{R}^2=\mathrm{x}^2+\mathrm{y}^2\longrightarrow\infty,$$

where $c>0,\ \alpha<0$ (See [30]). Now the desired claim follows by working with an apropriate 1-parameter family $\mathcal{A}_t$, $0<t\leqslant t_0$ of such ends with the aid of maximum principle exactly as in the *half-space theorem* of Hofmann and Meeks. We shall give the details of this construction as follows : Each $\mathcal{A}_t$ will be a graph of a function u over a exterior domain $\mathcal{D}_t$, assuming on $\partial\mathcal{D}_t$ for all t a suitable constant value $0<\varepsilon+\delta<1$, where $\delta>0$ is sufficiently small so that the boundary of each end do not touch M and the boundary of each end lies in some fixed horizontal horosphere given by $\{z=\varepsilon+\delta\}$. That is, $\mathcal{A}_t$ is chosen to obtain $\partial\mathcal{A}_t\cap M=\emptyset$. Moreover , we shall choose the initial end $\mathcal{A}_{t_0}$ carefully to guarantee $\mathcal{A}_{t_0}\cap M=\emptyset$.
We may describe precisely the geometrical configuration in the following way: $\mathcal{D}_t$ is the complement of an open disk D_t of radius R(t), $t\in(0,t_0]$ $\mathrm{R}(t)\longrightarrow 0$ as $t\to 0$. We shall choose all such disks concentric and satisfying if $t_1<t_2$ then $\mathrm{D}_{t_1}\subsetneq\mathrm{D}_{t_2}$. Moreover, it follows that the family end $\mathcal{A}_t$ varies continuously on t and have growth $\alpha(t)$ with $\alpha(t)\longrightarrow 0$ when $t\to 0$ (See [21] for explicit parametrizations of catenoids cousin and related figures drawed by the *Mathematica* software. See also [30] for further details with respect to the mentioned asymptotic behavior). Finally one may do the Hoffman -Meeks procedure (see [10]) starting from $\mathcal{A}_{t_0}$ moving the family $\mathcal{A}_t$ towards the horosphere $\{z=\varepsilon+\delta\}$, by making $t\downarrow 0$, until a member the family fit M at a first tangent interior point. This lead to a contradiction since the mean curvature vector of each member of the family $\mathcal{A}_t$ is pointing through the upper halfspace of $\mathbb{R}^3$, concluding the proof of the first assertion as desired.

We now will procced the proof of the second assertion:

Consider the formula

$$\mathrm{div}\big(v^\beta\frac{\nabla v}{W_v}\big)=\beta v^{\beta-1}\frac{\parallel\nabla v\parallel^2}{W_v}+v^\beta\,\mathrm{div}\frac{\nabla v}{W_v}$$

By assumption v vanishes along $\mathcal{D}$, hence from divergence theorem it follows that

$$\int_{\mathcal{D}_r} v^\beta \mathrm{div} \frac{\nabla v}{W_v} + \int_{\mathcal{D}_r} \beta v^{\beta-1} \frac{\| \nabla v \|^2}{W_v} = \int_{C_r} v^\beta \frac{\nabla v}{W_v} \,.\, \nu \tag{3.1}$$

where . denotes the standard inner product, ν is the unit exterior conormal vector along $\partial\mathcal{D}_r$.

Let $\eta(r) := \displaystyle\int_{C_r} \frac{\| \nabla v \|}{W_v}$.

Let

$$b_\beta := \int_{\mathcal{D}_{R_0}} \frac{v^\beta}{v+1}\big(1 - \frac{1}{W_v}\big) \gg 0$$

and

$$\mu_\beta := \int_{\mathcal{D}_{R_0}} \beta v^{\beta-1} \frac{\| \nabla v \|^2}{W_v}.$$

Thus, by substituting the above quantities into equation (3.1), taking account the mean curvature 1 equation (equation (1.3)) we have

$$b_\beta + \mu_\beta + \int_{\mathcal{D}_r - \mathcal{D}_{R_0}} \beta v^{\beta-1} \frac{\| \nabla v \|^2}{W_v} \leqslant \mathrm{M}^\beta(r)\eta(r).$$

But then invoking comparison principle (with horospheres) we get

$$v \leqslant \mathrm{M}(r)$$

over $\mathcal{D}_r$.

Notice that the last inequality hold for the mean curvature vector $\vec{H}$ is pointing upward. Clearly if $\beta \in (0,1)$ $v \mapsto v^{\beta-1}$ is a non increasing function.
Hence if $R_1 > R_0$ we infer

$$\begin{aligned}\forall r \in [R_0, R_1], \qquad \int_{\mathcal{D}_r - \mathcal{D}_{R_0}} \frac{\| \nabla v \|^2}{W_v} &\leqslant \frac{\mathrm{M}(r)\ \eta(r)}{\beta} - \frac{(b_\beta + \mu_\beta)}{\beta}\ \mathrm{M}^{1-\beta}(r) \\ &\leqslant \frac{\mathrm{M}(R_1)\ \eta(r)}{\beta} - \frac{(b_\beta + \mu_\beta)}{\beta}\ \mathrm{M}^{1-\beta}(R_0)\end{aligned}$$

Schwarz inequality yields

$$\forall r \in [R_0, R_1], \qquad \int_{\mathcal{D}_r - \mathcal{D}_{R_0}} \frac{\| \nabla v \|^2}{W_v} \geqslant \int_{R_0}^r \frac{\eta^2(\rho)}{2\pi\rho}\, d\rho \tag{3.2}$$

Henceforth, for r varying in the interval between R_0 and R_1 we infer the following crucial inequality :

$$\big(\frac{b_\beta+\mu_\beta}{\beta}\big)\ \mathrm{M}^{1-\beta}(R_0)+\int_{R_0}^{r}\frac{\eta^2(\rho)}{2\pi\rho}\,d\rho\leqslant\frac{\mathrm{M}(R_1)}{\beta}\ \eta(r) \tag{3.3}$$

Setting $A_1=\dfrac{\mathrm{M}(R_1)}{\beta}$ and $\mu_1=\big(\dfrac{b_\beta+\mu_\beta}{\beta}\big)\ \mathrm{M}^{1-\beta}(R_0)$ we are able to reproduce the argument of Collin and Krust in [6] with the aid of the first assertion to get the desired result. For the sake of completeness we will fulfill the details:

Consider ξ the solution of the differential equation in the interval $J=[R_0,R_0\ e^{4\pi A_1^2/\mu_1})$, given by

$$\frac{2A_1}{\mu_1}-\frac{1}{\xi}=\frac{1}{2\pi A_1}\ \int_{R_0}^{r}\frac{1}{\rho}\,d\rho$$

that is, ξ satisfies

$$\xi^{'}=\frac{\xi^2}{2\pi A_1 r}$$

with initial condition $\xi(R_0)=\frac{\mu_1}{2A_1}$.

Clearly,

$$\xi\longrightarrow\infty \text{ as } r\to R_0\ e^{4\pi A_1^2/\mu_1}.$$

In the interval $J\cap[R_0,R_1]$ one may verify that

$$\xi(r)<\eta(r).$$

Indeed, by taking account equation (3.3) it is straightforward to show that this set is open and closed with $\xi(R_0)<\eta(R_0)$.

We obtain therefore

$$R_1<R_0 e^{\dfrac{4\pi A_1^2}{\mu_1}}.$$

Hence

$$A_1=\frac{\mathrm{M}(R_1)}{\beta}\geqslant\left(\frac{(b_\beta+\mu_\beta)}{4\pi\beta}\ \mathrm{M}^{1-\beta}(R_0)\ \log\frac{R_1}{R_0}\right)^{1/2} \tag{3.4}$$

Now using divergence theorem again we have

$$\begin{aligned}\eta(r)&=\int_{C_r}\frac{\|\nabla v\|}{W_v}\geqslant|\int_{C_r}\frac{\nabla v}{W_v}\ .\ \nu|\\&=|\int_{\partial\mathcal{D}_r}\frac{\nabla v}{W_v}\ .\ \nu-\int_{\partial\mathcal{D}_r-C_r}\frac{\nabla v}{W_v}\ .\ \nu|\\&=\int_{\mathcal{D}_r}\mathrm{div}\frac{\nabla v}{W_v}\ -\ \int_{\partial\mathcal{D}_r-C_r}\frac{\nabla v}{W_v}\ .\ \nu\end{aligned}$$

Thereby $\eta(r)$ is positive. Indeed, this can be inferred by combining together the last equality, with the equation (1.3) (when H=1) and the first assertion proved before. Furthermore,

$$\eta(r) \geqslant c > 0$$

where

$$c := -\int_{\partial\mathcal{D}\cap\mathcal{D}_{r_0}} \frac{\nabla v}{W_v} \, . \, \nu > 0$$

Then making $R_1 = \dfrac{R_0^2}{r_0}$ by standard computations and equation (3.2) it follows that

$$\mu_\beta = \int_{\mathcal{D}_{R_0}} \beta v^{\beta-1} \, \frac{\| \nabla v \|^2}{W_v} \tag{3.5}$$

$$\geqslant \int_{\mathcal{D}_{R_0}} \beta \, \mathrm{M}^{\beta-1}(R_0) \, \frac{\| \nabla v \|^2}{W_v}$$

$$\geqslant \beta \, \mathrm{M}^{\beta-1}(R_0) \, \frac{c^2}{2\pi} \, \log \frac{R_0}{r_0}$$

Finally the proof of second assertion follows by putting together equations (3.4) and (3.5). This conclude the proof of the theorem as desired.

Of course it will be more striking to get an asymptotic type result as above for the difference of two solutions of the constant mean 1 equation. This is an open challenging problem.

4. Symmetry of constant mean 1 hypersurfaces

Notice that a part of Riemann minimal surface lies inside a half-space in euclidean space with boundary a round circle. On the other hand, the symmetry of hyperbolic space yields the following result in the context of constant mean curvature or minimal hypersurfaces (non necessarily compact):

4.1 Theorem. *Let M be a properly embedded hypersurface in $\mathbb{H}^{n+1}$ with constant mean curvature. Assume the boundary of M is a $n-1$ sphere S lying in a horosphere $\mathcal{H}$. Then if M is entirely contained inside the region bounded by $\mathcal{H}$, it follows that M is rotationally symmetric.*

PROOF. We first suppose H $\neq 0$. The case M is compact is trivial, since horizontal reflections are either hyperbolic and euclidean isometries, so that the proof follows as in euclidean space (see, for instance [27]). Hence, we suppose M is non compact. Now we note that the geometrical situation and maximum principle imply an interior orientation of M, given by the mean curvature vector $\vec{H}$. It is now easy to infer that M lies in one side of the unique hyperplane $\mathcal{P}$ that contains S. Let α be the unique geodesic line cutting orthogonally the hyperplane $\mathcal{P}$ at the center O of S. Let P be an arbitrarly hyperplane passing through α. Without loss of generality we may assume P is a vertical hyperplane (in euclidean sense), so that $\mathcal{H}$ is a horizontal euclidean hyperplane. Notice that to proceed the proof of the statement it suffices to show M is symmetric on P. Let γ be the geodesic line in $\mathcal{P}$ which is orthogonal to P at O. Consider P_t , $t \in \mathbb{R}$ the 1-parameter family of totally geodesic hyperplanes with $P = P_0$ generated by moving P along γ by means of hyperbolic translations. The crucial observation is that

$$P_t \cap M \qquad \textit{is compact}$$

for any $t \neq 0$. That is, any translation of P meet M at a compact part. So we do not have to take care about a point of contact at infinity during the Alexandrov process. Furthermore, each hyperplane P_t is orthogonal to $\mathcal{P}$, so symmetry on P_t leaves $\mathcal{P}$ invariant. Clearly, if P_t fit S, then reflection on P_t carries a cap of S on its symmetrical about P_t in $\mathcal{P}$. Putting those facts together one may easily apply Alexandrov reflection principle to achieve the wanted result. For sake of clarity we will give more details as follows : Let $\mathcal{A}$ be a connect component of $\mathbb{H}^{n+1} - P$. We now start to run Alexandrov process moving the family P_t towards M, coming from the infinity . Accordingly to the above observation it follows that the symmetry of the part of $M - P_t \cap M$, lying in the component of $\mathcal{H}^{n+1} - P_t$ not containing P, about P_t do not touch M untill P_t reaches P (i.e, t=0). On the other hand the symmetry and invariance of S, by Alexandrov movement, imply that the boundary do not interferes in the process. Working in the same way along the the other component of $\mathbb{H}^{n+1} - P$, we get M have symmetry about P, as desired. It is evident that when H = 0 we can apply the same argument , since we do not need anymore to use orientation nor embeddness to achieve the desired result.

We now recall that there is a constant d such that any 1-parameter family $\mathcal{C}_t$ of rotation minimal hypersurfaces in $\mathbb{H}^{n+1}$ with same axis and same plane of symmetry lies between two parallels planes a distance d apart , where d is the maximum value of the function $x(y_0)$, given by:

$$x(y_0) = \int_{y_0}^{+\infty} \frac{(\sinh y_0)^{n-2} \cosh y_0}{\cosh y} \Big(\frac{1}{(\sinh y)^{2n-4} (coshy)^2 - (\sinh y_0)^{2n-4} (\cosh y_0)^2} \Big)^{1/2} dy$$

This formula follows from the first integral of the second order differential equation satisfied by the generating curve [9] or [11]. Clearly, the above function is bounded and numerical computations shows that $d < 0.6$.

From now on, we will denote by d the positive real number given by the previous estimate. The proof of the following theorem is done in [29]. We shall writedown here the details for the sake of completeness and reader' s convenience.

4.2 Theorem. *Let M be constant mean curvature hypersurface in $\mathbb{H}^{n+1}$.Assume $\partial M = S_1 \cup S_2$, where S_1 and S_2 are $n-1$- dimensional spheres of same radius. Suppose that ∂M is invariant by rotations. Then there is a positive real number d, such that if $\mid H \mid = 1$ and $dist(S_1, S_2) \geq 2d$, M is a piece of a n – dimensional embedded catenoid cousin.*

PROOF. Up to a motion of ambient space we may suppose $\partial M = S_1 \cup S_2$ is obtained by intersecting a cylinder C (which is a euclidean cone) with two horospheres (which are euclidean parallel hyperplanes) given by $\mathcal{H}_1 = \{z = t_1\}$ and $\mathcal{H}_2 = \{z = t_2\}$, respectively. Assume $t_1 < t_2$. That is, $S_1 = \mathcal{H}_1 \cap C$ and $S_2 = \mathcal{H}_2 \cap C$. We shall prove first the following claim .

Claim 1

Under the above conventions if $M \subset \overline{int(C)}$ and $M \cap C = \partial M$, M then has a plane of symmetry. Moreover if $dist(S_1, S_2) \geq 2d$ then M lies in the region $\mathcal{R}$ between the horospheres $\mathcal{H}_1$ and $\mathcal{H}_2$, i.e, M is contained in the slab bounded by the parallel euclidean planes containing S_1 and S_2.

Proof of Claim 1

We denote by P the hyperplane of symmetry of $S_1 \cup S_2$, which exists by assumption. Let γ be the axis of C, that is, γ is the geodesic vertical line passing through the centers of S_1 and S_2. We will prove P is a hyperplane of symmetry of M as follows: We start by moving P along γ by doing hyperbolic translations. This movement give rises to a 1-parameter family $\{P_t\}$ of geodesic hyperplanes starting from $P = P_0$ and cutting ortogonally γ. We may choose a parameter t such that $t =$ oriented distance between P_t and P. We claim that ours assumptions allow us to apply Alexandrov reflection, by the means of the family $\{P_t\}$, to conclude M inherits the symmetry of ∂M: Actually, to explain how to do this, we will denote by ${M_t}^*$ the reflection on P_t for $t \neq 0$ of the components of $M - P_t$ lying in the connected component of $\mathbb{H}^3 - P_t$ not containing P. We begin the standard procedure by moving P until P_t is disjoint of M, *i.e*, $P_t \cap M = \emptyset$. Then moving back P_t towards P, doing Alexandrov reflection during this movement, we find a first point of contact between M and ${M_t}^*$. Since C is invariant by reflection on P_t, then if by absurd $P_t \neq P$ this first point of contact cannot occurs at a boundary point of ∂M, because $\partial M \subset C$ and $M \cap C = \partial M$. So, if $t \neq 0$ we get a tangent point of contact, both M and ${M_t}^*$ have the correct orientation at this point. We arrive to a contradiction for the maximum principle yields $P_t, t \neq 0$, is a hyperplane of symmetry of M. Then $P_t = P$, that is P is a plane of symmetry of M, as required.

To prove the second assertion we will proceed as follows: Let P_d and P_{-d} be the geodesic planes passing through S_1 and S_2. We will denote by $\mathfrak{R}$ the region of $\mathbb{H}^3$ bounded by P_d and P_{-d}. We will denote by ${P_t}^+$ the connected component of $\mathbb{H}^3 - P_t$ not containing P. Under those conventions, notice that $M \cup (P_d^+ \cap C) \cup (P_{-d}^+ \cap C)$ is the boundary of a region V in $\mathbb{H}^3$; V is contained in $\overline{int(C)}$, ∂V is not smooth over ∂M. We point out either the mean curvature vector $\vec{H}$ of M is pointing into the interior of V or else $\vec{H}$ is pointing towards the exterior of V. We claim $\vec{H}$ is an inward pointing normal vector. Indeed, let α be a geodesic line cutting orthogonally γ at a point p equidistant from S_1 and S_2, *i.e*, $dist(p, S_1) = dist(p, S_2)$. Let $\{\zeta_t\}$ be the 1-parameter family of minimal hypersurfaces revolution with axis γ and generating curve c_t such that α is the symmetry line of ζ_t and $t = dist(\zeta_t, \gamma)$. Note that as $dist(S_1, S_2) \geq 2d$ the family $\{\zeta_t\}$ is inside $\mathcal{R}$. Now we move ζ_t coming from the infinity towards M making $t \to 0$ to reach a first interior point of contact. By the comparison principle $\vec{H}$ is pointing into V, as required. Notice that if $M \cap ext(\mathfrak{R}) \neq \emptyset$, one derives a contradiction by the same argument as before, making use of the family $\{P_t\}$ coming from the infinity towards $\partial \mathfrak{R}$. To accomplish the proof of Claim 1 consider the 1 - parameter family of horospheres $\mathcal{H}_t = \{z = t\}$ for $t > t_2$. Recall that we may move this family along γ by making use of hyperbolic translations. Now, doing $t \downarrow t_2$ coming from the infinity towards M it follows from maximum principle that M is entirely contained inside the region $\mathcal{R}$.

Claim 2

Let M be an embedded constant mean curvature hypersurface in $\mathbb{H}^{n+1}$ which boundary is the union of two $n-1$ spheres of not necessarily same radius, lying inside two parallel horizontal euclidean hyperplanes. If M is contained in the slab bounded by the hyperplanes, and if the centers of the $n-1$ spheres are lying in some common fixed vertical line, then M is rotationally symmetric.

Proof of Claim 2

The proof of Claim 1 is now standard, since the boundary is symmetric and do not interferes in the Alexandrov performance. We will outline the proof as follows: Fix any direction $v = (v_1, v_2, \ldots, 0)$ and consider the family of hyper planes P_t orthogonal to v. Clearly, by moving this family P_t (varying t), by means of euclidean horizontal translations (which are hyperbolic isometries), with the aid of successive reflections on P_t during this movement, one may reach a geodesic plane of symmetry P_{t_0} passing through γ. Clearly the procedure is justified by the assumption ∂M is invariant by rotations around γ.

To achieve the proof of the theorem we will proceed as follows : Let $\{C_t\}$, $t \geq 0$ be the 1-parameter family of cylinder of same axis γ with $C_0 = C$ and $t = dist(C_t, \gamma)$. We assert that $M \subset \overline{int(C)}$ and $M \cap C = \partial M$. Indeed, if this is not the case then as $t \longrightarrow +\infty$ one may find a "last" cylinder C_{t_1} such that $C_{t_1} \cap M \neq \emptyset$ and $C_t \cap M = \emptyset$, for $t > t_1$. If $M \cap (\mathbb{H}^3 - \overline{int(C)}) \neq \emptyset$ then $t_1 > 0$, hence we get a contradiction with the standard comparison principle (for the mean curvature of M is not greater than 1). For the same reasons, $M \cap C = \partial M$. It follows now from above that M is an embedded constant mean curvature 1 hypersurface of revolution whose generating curve attaints a local minimum. The classification of rotational constant mean curvature 1 hypersurfaces ensures our desired result (See [9]). This concludes the proof of the theorem.

5. On the constant mean curvature equation in hyperbolic space

We first are mainly concerned with the deduction of apriori hight and gradient estimates of C^2 solutions u of the following family of Dirichlet problems in $\mathbb{H}^{n+1}$ (cf. equation (1.3)). We shall assume that the vertical graph M of u satisfies $H^2 \leqslant 1$.

$$\operatorname{div}\left(\frac{\nabla u}{W_u}\right) = \frac{n\mathrm{B}(t)}{u}\left(\mathrm{H} - \frac{1}{W_u}\right) \quad \text{on} \quad \mathcal{D} \tag{5.1}$$

$$u = 1 \quad \text{on} \quad \Gamma_1 \tag{5.2}$$

$$u = c \quad \text{on} \quad \Gamma_2 \tag{5.3}$$

where $\mathcal{D}$ is a C^2 bounded annular domain with $\partial\mathcal{D} = \Gamma_1 \cup \Gamma_2$. Furthermore, we require that B is a smooth function and c is a constant satisfying

$$0 \leqslant \mathrm{B} \leqslant 1 \tag{5.4}$$

and

$$c \geqslant 1 \tag{5.5}$$

We note specially that as $0 \leqslant \mathrm{H} \leqslant 1$ it follows that the family of minimal hyperbolic catenoids provide geometric barriers to the above Dirichlet problem : Indeed, suppose $M_t \subset \mathbb{H}^{n+1}$ is the graph of a solution u of the above Dirichlet problem. Let H(t) be the mean curvature of M_t with respect the upward (or exterior) unit normal N. Then a simple calculation shows that

$$\mathrm{H}(t) = \frac{1 - \mathrm{B}}{W_u} + \mathrm{HB}.$$

Thus

$$0 \leqslant \mathrm{H}(t) \leqslant 1 \tag{5.6}$$

so that the mean curvature vector $\vec{H}_t$ of M_t is an exterior pointing normal. Now it suffices to move the family of minimal catenoids coming from the infinity towards M_t, as we have exhaustively explained before, in order to ensure the assertion. We next will infer the estimates that we shall need later.

5.1 Lemma. *Let $u \in C^2(\bar{\mathcal{D}})$ satisfy equations (5.1) and (5.4) in the bounded domain $\mathcal{D}$. Suppose that the boundary data condition $u \geqslant 1$ over $\partial\mathcal{D}$ is fulfilled. Then we have the estimate*

(1) $1 \leqslant u \leqslant \mathrm{C}$, where C is a constant that depends only on the boundary value data, i.e, the restriction of u to $\partial\mathcal{D}$.

(2) Suppose that H = 0. Then, if $\mid \nabla u \mid$ is apriori bounded in $\partial\mathcal{D}$ it follows that $\mid \nabla u \mid$ is apriori bounded in $\mathcal{D}$. Furthermore, $\sup_{\mathcal{D}} \mid \nabla u \mid = \sup_{\partial\mathcal{D}} \mid \nabla u \mid$.

PROOF. We observe first that (1) is inferred from equation (5.6) by dealing with a family of horospheres as barriers, by analogy with the form that hyperplanes provide barriers to minimal euclidean hypersurfaces, as is suggested by the *principle* stated in the introduction. To this derivation we make use of the same action drawned in the introduction. Actually, one may find a fixed horosphere $\mathcal{H}_0$ which asymptotic boundary is the origin of $\mathbb{R}^{n+1}$ depending only on the boundary value data such that any solution of (5.1) satisfying (5.4) is contained in the interior of the region bounded by $\mathcal{H}_0$. The “optimum cap ” is obtained by the so–called principle cited above. We now observe that when H = 0, (5.1) is a nice quasilinear second order elliptic equation, since the left side is the euclidean mean curvature of the graph and the right side has a good sign. Hence, differentiating equation (5.1) with respect to the variable x_j , $j \in \{1, \ldots, n\}$ we infer that the derivative $w = u_j$ satisfies a strictly elliptic second order linear equation with bounded coeficients and $c \leqslant 0$

$$a_{ij} D_{ij} w + b_j D_j w + cw = 0$$

since $c \leqslant 0$ it follows that the maximum of w is attained at the boundary of $\mathcal{D}$. This implies (2), by applying Hopf's maximum principle (See [23]).

It is worth pointing out that solvability of the Dirichlet problem associated to equation (5.1) is a new stimulating problem. However, we now turn out attention to the derivation of certain special existence result. Let Ω be a smooth strictly convex bounded domain in $\partial_\infty \mathbb{H}^{n+1}$, namely, all the principal curvatures of $\Gamma_1 := \partial\Omega$ are greater than zero. Let $\Gamma_2 := \mathrm{S}_r$ be a $n-1$- dimensional sphere of radius r. Suppose that S_r lies in the interior of Ω. Let $\mathcal{D}$ be the annular domain bounded by $\Gamma_1 \cup \Gamma_2$, i.e, $\mathcal{D} := \Omega - \mathrm{D}_r$, where D_r is the n-dimensional disk of radius r with boundary S_r. We recall that as a theoretical consequence of the argument inferred just before the above lemma, we must make restrictions on the size of the Dirichlet data to expect find solutions of the related Dirichlet problem.

First, we shall need to fix further notations. Let $\mathrm{S}^{\mathrm{a}}_{\mathrm{r}}$ denote the $n-1$ - sphere in the horosphere $\{x_{n+1} = a\}$ given by $\{(x_1, x_2, \ldots, x_n, a); (x_1, x_2, \ldots, x_n) \in \mathrm{S}_r, a > 0\}$. Let P be the totally geodesic hyperplane that contains $\mathrm{S}^{\mathrm{c}}_{\mathrm{r}}$ (Recall this is the graph of u restricted to S_r, i.e, $u = c$ on S_{r}). Let $\mathcal{H}$ be the horosphere given by $\{x_{n+1} = 1\}$. Denote $\Omega^1 := \{(x_1, x_2, \ldots, x_n, 1) \in \mathbb{H}^{n+1}; (x_1, x_2, \ldots, x_n) \in \Omega\}$. Finally, denote $\mathrm{D}^{\mathrm{c}}_{\mathrm{r}}$ the disk in the horosphere $\mathcal{H}^c := \{x_{n+1} = c\}$ which boundary is $\mathrm{S}^{\mathrm{c}}_{\mathrm{r}}$.

Let us now impose the following structure conditions:

Suppose that $P \cap \mathcal{H}$ involves Ω^1 in its interior. Suppose also that c is chosen close enough to 1 so that there is a minimal euclidean catenoid C such that C intersects each horosphere $\mathcal{H}$ and $\mathcal{H}^c$ along $n-1$ -dimensional spheres S_1 and S_2 with $\mathrm{S}_2 = \mathrm{S}^{\mathrm{c}}_{\mathrm{r}}$, $\mathrm{S}_1 \subset \Omega^1$ and $\mathrm{D}_1 \supset \mathrm{S}^1_{\mathrm{r}}$, where D_1 is the disk with boundary S_1. We suppose further that this part of the catenoid C connecting S_1 and S_2 is a smooth graph up to the boundary

(5.7)

We set $\partial M := \partial\Omega^1 \cup \mathrm{S}^{\mathrm{c}}_{\mathrm{r}}$.

Under the previous conventions we have the following :

5.2 Theorem. *Suppose that the structure conditions (5.7) are fulfilled. Then, there is a minimal ($\mathrm{H} = 0$) solution u over $\mathcal{D}$ of equation (1.3), taking boundary value data given by equations (5.2), (5.3) and (5.5). Furthermore, the graph of u is also a radial graph about a point $\mathrm{O} \in \partial_\infty \mathbb{H}^{n+1}$ and is the unique compact minimal graph in $\mathbb{H}^{n+1}$ with boundary equal to ∂M.*

PROOF. We first observe that the right side of equation (5.1) is not well defined for z = u= 0. However, we shall prove that C^1 apriori estimates up to the boundary of an arbitrary solution u of equations (5.1), (5.2) and (5.3) is sufficient to guarantee the desired result. This is inferred, with the aid of classical elliptic quasilinear theory, by the following assertion : Any solution of an suitable extended auxiliary quasilinear operator with C^α coefficents is, in fact, a solution u of the referred Dirichlet problem. We commence to derive this assertion analysing an auxiliary Dirichlet problem that we will call *family t*:

$$\mathrm{div}\left(\frac{\nabla w}{W_w}\right) = n\mathrm{B}(t)h(w)\left(\mathrm{H} - \frac{1}{W_w}\right) \quad \text{on} \qquad \mathcal{D} \tag{5.8}$$

$$w \qquad = 0 \qquad\qquad \text{on} \qquad \Gamma_1 \tag{5.9}$$

$$w \qquad = t(c-1) \qquad\qquad \text{on} \qquad \Gamma_2 \tag{5.10}$$

where $h = h(z)$ is a C^α function defined for all z, given by

$$h(z) = \begin{cases} \frac{1}{z+1} & \text{if } z \geqslant 0; \\ 2z+1 & \text{if } \frac{-1}{2} \leqslant z \leqslant 0; \\ 0 & \text{if } z \leqslant \frac{-1}{2}; \end{cases}$$

Let w be a solution of equations (5.8), (5.9) and (5.10). We point out that if $\mathbf{B}(\mathrm{t}) \equiv 1$ and $w \geqslant 0$ then $u = w + 1$ satisfies the hyperbolic mean equation $\mathbf{H}$ (see (1.3)) taking boundary value data given by equations (5.2) and (5.3). Now to infer the last assertion it suffices to show that $w \geqslant 0$. But with the observation that $\mathbf{H} \in [0,1]$ we assert that this is a consequence of maximum principle applied twice. Namely, we first note that if $w < \frac{-1}{2}$ then w satisfies the minimal euclidean hypersurface equation, so that classical maximum principle implies this is not possible; so that we have $w \geqslant \frac{-1}{2}$. On the other hand, if $w < 0$ then the hyperbolic mean curvature $\mathbf{H}(t)$ of the graph in hyperbolic space given by $u = w + 1$ varies between 0 and 1. Thus, using a variant of a reasoning repeatedly written before, working with the family of horospheres, we infer $w \geqslant 0$, as desired. Let we denote $M(t)$ the graph in hyperbolic space of the solution $u := w + 1$ to the family t problem.
From now on, we shall focus *minimal vertical graphs*, that is $\mathbf{H} \equiv 0$: We now need to make some basic observations concerning our family t of Dirichlet problems in the context of quasilinear elliptic theory. First, notice that the assumptions in the statement imply there is a (unique) smooth solution of the euclidean minimal hypersurface equation (analytic up to the boundary) on $\bar{\mathcal{D}}$ taking the required boundary data given by equations (5.9) and (5.10). This is consequence of the fact that the structure conditions imply we have a barrier at any point of the boundary, so the classical minimal theory in euclidean space guarantee our assertion. Second, we observe that the coefficients of the equation (5.8), written in the form $a_{ij}\left(\nabla w\right) D_{ij} w + b\left(w, \nabla w; t\right) = 0$, verifies $a_{ij} \in C^1(\mathbb{R}^n)$ and for fixed t we have $b \in C^0(\mathbb{R} \times \mathbb{R}^n)$. Now putting those facts together, it follows from the fundamental global estimates of Ladyzhenskaya and Ural'tseva (see [8]) that we can get global $C^{1,\beta}$, $\beta \in (0,1]$ apriori estimates of $C^2(\bar{\mathcal{D}})$ solutions of our problem, *if* we are able to obtain global C^1 apriori estimates of $C^2(\bar{\mathcal{D}})$ solutions of the family t.

Thus we shall derive now height and gradient estimates up to the boundary: We observe that Lemma 5.1 yields C^0 apriori bounds. To obtain C^1 apriori bounds according again to the same Lemma we have to prove apriori gradients estimates at the boundary. We can infer this in the following way :

Setting $\mathbf{B}(1) = 1$, we note that gradient estimates for t=1 restricted to the component S_r is a consequence of the structure conditions using a barrier argument: Indeed, we can move downward totally geodesic planes to impose strict control on the gradient along S_r

from above, and we can lift upward minimal euclidean catenoids to bring the gradient along S_r under control from bellow. Now in order to obtain control over the family t *from below* we use the same procedure working with minimal catenoids, as before. On the other hand, to get boundary gradients estimates from above we will proceed as follows: Let $\mathbf{E}(t)$ be the unique equidistant hypersurface, given by a function u_{E}, passing through $S := P \cap \mathcal{H}$ and the sphere $S_r^{t(c-1)+1}$ (this is the graph of u restricted to Γ_2). We claim that by choosing appropriately $\mathbf{B}(t)$ we may use this family $\mathbf{E}(t)$ of equidistant hypersurface as the required barriers. Indeed, we choose $\mathbf{B}(t) < \dfrac{2t}{c+1-t^2(c-1)}$, for $0 < t < 1$. Making this choise to prove our claim it suffices to show that $\mathbf{E}(t)$ (which is an euclidean spherical cap) involves $M(t)$(in the euclidean sense). But in contrary, it is not difficult to see that

$$W_u = W_{u_{\mathrm{E}}} \qquad \text{implies} \qquad \mathbf{H}(t) > H_{\mathrm{E}}(t)$$

where $\mathbf{H}(t)$ and $H_{\mathrm{E}}(t)$ are the mean curvature of $M(t)$ and $\mathbf{E}(t)$, respectively. Let E_t^+ be the connected component of $\mathbb{H}^{n+1} - \mathbf{E}(t)$ not containing Ω^1 (exterior open region bounded by $\mathbf{E}(t)$, in the hyperbolic sense). We get

$$E_t^+ \cap M(t) = \emptyset,$$

otherwise by doing hyperbolic translations on $\mathbf{E}(t)$ (i.e, euclidean homotheties from the origin) we find a last point of contact of a copy of $\mathbf{E}(t)$ and $M(t)$. But this is an absurd, by taking account the preceding inequality and comparison principle.

It remains to show that we can get the gradient at the boundary of Ω under control. With the aid of the fact proved before, that is $w \geqslant 0$(or $u \geqslant 1$), it is easy to see that one side is well controlled. On the other hand, using the convexity assumption, it is easy to infer that the outer side is also controlled by moving the unique hyperplane P that contains S_r^c along hyperbolic translations and euclidean horizontal displacements in a suitable way. Namely, we have to continue the movement until *any* point of the boundary of Ω^1 is fitted by a hyperplane P_s of the family, taking care that $M(t)$ stay in one side of P_s with P_s touching $M(t)$ just on $\partial\Omega^1$. We set $\mathbf{B}(0) = 0$. At the level $t = 0$, we have $z = 1$ as the unique solution of the problem. We have concluded the derivation of the needed apriori height and gradients estimates *independent of t*. We now remark that a_{ij} and $b \in C^{\alpha}\left(\bar{\mathcal{D}} \times \mathbb{R} \times \mathbb{R}^n\right)$, for each $t \in [0,1]$. Moreover, as mappings from $[0,1]$ into $C^{\alpha}\left(\bar{\mathcal{D}} \times \mathbb{R} \times \mathbb{R}^n\right)$ the functions a_{ij} and b are continuous. Now we may applied Schauder's theory and classical existence theorem for quasilinear strictly elliptic equation to accomplish the proof of the existence result in the first part of the statement [8].

To prove the second statement in the theorem we will proceed as follows : Let O be the asymptotic point of the vertical geodesic line γ passing through the center of S_r^c (the other asymptotic point is ∞). We assert that M is a radial graph about O. Indeed, it is inferred by maximum principle that M lies inside the unbounded region $\mathcal{R}$ in $\mathbb{H}^{n+1}$ which boundary is the union of the euclidean " cones" formed for all rayons issuing from O that joint O to each component of ∂M. Hence, we do have a permit to apply reflections on hyperplanes ortogonal to γ using minimality to infer the assertion. We now get uniqueness by standard argument, as is done to minimal surfaces in euclidean space, since we have proved M is a radial graph and homotheties about O are hyperbolic translations. This concludes the proof of the theorem, as desired.

Finally, we wish to mention a work on general existence and uniqueness theorems for minimal vertical graphs in hyperbolic space (See [33]): We solve, in hyperbolic space the Dirichlet problem for the (vertical) minimal equation over a convex domain taking arbitrary continuous non-negative boundary data. We prove some existence and uniqueness of some unbounded vertical graphs. In particular, we derive existence of complete minimal graphs invariant by either a discrete subgroup of an parabolic, elliptic or hyperbolic isometry.

References

1. A. D. Alexandrov, Uniqueness theorems for surfaces in the large I, *Vestnik Leningrad Univ. Math.* **13** (1958), 5–8.
2. A. D. Alexandrov, Uniqueness theorems for surfaces in the large V, *English Translation AMS Transl.* **21** (1982), 412–416.
3. F. Braga Brito and R. Sa Earp, Special Weingarten surfaces with boundary a round circle, *Ann. Fac. Sci. Toulouse 2* **VI** (1997), 243–255.
4. J. L.Barbosa and Sa Earp, R., New results on prescribed mean curvature hypersurfaces in space forms 1, *An. Acad. Bras. Ci. 1* **67** (1995), 1–5.
5. J. L. Barbosa and R. Sa Earp, Prescribed mean curvature hypersurfaces in $H^{n+1}(-1)$ with convex planar boundary I, to appear in Geom. Dedicata.
6. P. Collin and R. Krust, Le Problème de Dirichlet pour l'équation des surfaces minimales sur un domaine non borné, *Bull. Soc. Math. Fr.* **119** (1991), 443–462.
7. R. Finn, Remarks relevant to minimal surfaces and to surfaces of prescribed mean curvature, *J. Analyse Math.* **14** (1965), 139–160.
8. D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order.* Springer-Verlag, 1983.
9. J. M. Gomes, *Sobre hipersuperfícies com curvatura média constante no espaço hiperbólico.* Doctoral thesis, IMPA, 1985.
10. D. Hoffman and W. Meeks, The strong halfspace theorem for minimal surfaces, *Inven. Math.* **101** (1990), 373–377.
11. W. Hsiang, On generalization of theorems of A. D. Alexandrov and C. Delaunay on hypersurfaces of constant mean curvature, *Duke Math. J.* **49** (1982), 485–496.
12. H. Hopf, *Differential geometry in the large.* Lecture Notes in Mathematics, vol. 1000, Springer-Verlag, 1983.
13. H. Jenkins and J. Serrin, Variational problem of minimal surface type II. Boundary value problems for the minimal surface equation, *Arc. Rat. Mech. Anal. 4* **21** (1966), 21-342.

14. H. Jenkins and J. Serrin, The Dirichlet problem for the minimal surface equation in higher dimensions, *J. Reine Angew. Math.* **229** (1968), 170–187.
15. N. Kapouleas, Compact constant mean curvature surfaces in Euclidean three–space, *J. Differential Geometry* **33** (1991), 683–715.
16. R. Krust, Remarques sur le problème de Plateau, *Duke math. J.* **59** (1989), 161–173.
17. N. Kutev and F. Tomi, Existence and nonexistence in the exterior Dirichlet problem for the minimal surface equation in the plane, preprint.
18. F. H. Lin, On the Dirichlet problem for minimal graphs in hyperbolic space, *Inven. Math.* **96** (1989), 593–612.
19. R. Molzon, Symmetry and overdetermined boundary value problems, *Forum Math.* **3** (1991), 143–156.
20. B. Nelli and R. Sa Earp, Some properties of hypersurfaces of prescribed mean curvature in H^{n+1}, *Bull. Soc. Sci. Math.* **120** (1996), 537–553.
21. J. Ordonez, *Superfícies helicoidais com curvatura constante no espaço de formas tridimensional.* Doctoral thesis, PUC-Rio, 1995.
22. R. Osserman, *A survey of minimal surfaces.* Dover publ.,inc , New York, 1986.
23. M. H. Protter and H. F. Weinberger, *Maximum principles in differential equations.* Elglewood Cliffs., New Jersey Prentice-Hall, 1967.
24. Th. M. Rassias and R. Sa Earp, Some problems of analysis and geometry, To appear in the Volume *Complex in several variables*, Hadronic Press Inc., Florida, USA, 1998.
25. J. Ripoll and F. Tomi, Some existence theorems for minimal graphs over non convex planar domains, preprint.
26. J. Ripoll and F. Tomi, Maximum principles for minimal surfaces in R^3 having noncompact boundary and a uniqueness theorem for the helicoid, *Manusc. math.* **87** (1995), 417–434.
27. R. Sa Earp, Recent developments on the structure of compact surfaces with planar boundary, In: The Problem of Plateau (edited by Th. M. Rassias),pp. 245–257, World Scientific, 1992.
28. R. Sa Earp, On two mean curvature equations in hyperbolic space , Published in this Volume.
29. R. Sa Earp and E. Toubiana, Symmetry of properly embedded special Weingarten surfaces in H^3, to appear.
30. R. Sa Earp and E. Toubiana, Remarks on the geometry of surfaces with mean curvature 1 in hyperbolic space, to appear.
31. R. Sa Earp and E. Toubiana, *Cours de geometrie hyperbolique et de surfaces de Riemann.* Monografias do dept. Mat., PUC-Rio, 1996.
32. R. Sa Earp and E. Toubiana, *Introduction à la géométrie hyperbolique et aux surfaces de Riemann.* Diderot Editeur, Paris, 1997.
33. R. Sa Earp and E. Toubiana, Existence and uniqueness of minimal graphs in hyperbolic space, preprint.
34. R. Schoen, Uniqueness, symmetry, and embeddedness of minimal surfaces, *J. Differential Geometry* **18** (1983), 791–809.
35. J. Serrin, A symmetry problem in potential theory, *Arch. Rat. Mech. Anal.* **43** (1971), 304–318.
36. J. Serrin, A priori estimates for solutions of the minimal surface equation, *Arch. Rat. Mech. Anal.* **14** (1963), 376–383.