MINIMAL GRAPHS IN $\mathbb{H}^n \times \mathbb{R}$ AND $\mathbb{R}^{n+1}$

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En l'honneur de Pierre Bérard and Sylvestre Gallot

Abstract.
We construct geometric barriers for minimal graphs in $\mathbb{H}^n \times \mathbb{R}$.
We prove the existence and uniqueness of a solution of the vertical minimal equation in the interior of a convex polyhedron in $\mathbb{H}^n$ extending continuously to the interior of each face, taking infinite boundary data on one face and zero boundary value data on the other faces.
In $\mathbb{H}^n \times \mathbb{R}$, we solve the Dirichlet problem for the vertical minimal equation in a C^0 convex domain $\Omega \subset \mathbb{H}^n$ taking arbitrarily continuous finite boundary and asymptotic boundary data.
We prove the existence of another Scherk type hypersurface, given by the solution of the vertical minimal equation in the interior of certain admissible polyhedron taking alternatively infinite values $+\infty$ and $-\infty$ on adjacent faces of this polyhedron.
We establish analogous results for minimal graphs when the ambient is the Euclidean space $\mathbb{R}^{n+1}$.

Key words: Dirichlet problem, minimal equation, vertical graph, Perron process, barrier, convex domain, asymptotic boundary, translation hypersurface, Scherk hypersurface.

1. Introduction

In Euclidean space, H. Jenkins and J. Serrin [9] showed that in a bounded C^2 domain D the Dirichlet problem for the minimal equation in D is solved for C^2 boundary data if and only if the boundary is mean convex. The theorem also holds in the case that the boundary data is C^0 (but the domain is still C^2) by an approximation argument [6, Theorem 16.8]. On the other hand, the authors solved the Dirichlet

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problem in $\mathbb{H}^3$ for the vertical minimal surface equation over a C^0 convex domain Ω in $\partial_\infty \mathbb{H}^3$, taking any prescribed continuous boundary data on $\partial\Omega$ [14]. There are also in this context the general results proved by M. Anderson [1] and [2].

In this paper we study the vertical minimal equation equation in $\mathbb{H}^n \times \mathbb{R}$ (Definition 3.1) in the same spirit of our previous work when $n = 2$ [15]. In that paper the authors have given a full description of the minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ invariant by translations (cf [13]). Afterwards, inspired on this construction, P. Bérard and the first author [3] have given the minimal translation hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$ and they showed that the geometric behavior is similar to the two dimensional case. There is also a one parameter family of such hypersurfaces, denoted again by M_d, $d > 0$. For instance, M_1 is a vertical graph over an open

half-space of $\mathbb{H}^n$ bounded by a geodesic hyperplane Π, taking infinite boundary value data on Π and zero asymptotic boundary value data. We show that the hypersurface M_1 provides a barrier to the Dirichlet problem at any point of the asymptotic boundary of Ω. Moreover, we prove that the hypersurfaces M_d $(d < 1)$ give a barrier to the Dirichlet problem at any strictly convex point of the finite boundary of Ω.

We prove the existence and the uniqueness of rotational Scherk hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$ and we prove that these hypersurfaces give a barrier to the Dirichlet problem at any convex point.

Given an admissible convex polyhedron (Definition 5.2), we prove the existence and uniqueness of a solution of the vertical minimal equation in $\mathrm{int}(\mathcal{P})$ extending continuously to the interior of each face, taking infinite boundary value on one face and zero boundary value data on the other faces. We call these minimal hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$ by *first Scherk type (minimal) hypersurface*. The hypersurface M_1 above plays a crucial role in the construction.

Using the rotational Scherk hypersurfaces as barriers, we solve the Dirichlet problem for the minimal vertical equation in a bounded C^0 convex domain $\Omega \subset \mathbb{H}^n$ taking arbitrarily continuous boundary data. Furthermore, using the hypersurface M_1 as well, we are able to solve the Dirichlet problem for the minimal vertical equation in a C^0 convex domain $\Omega \subset \mathbb{H}^n$ taking arbitrarily continuous data along the finite and asymptotic boundary.

We prove the existence of another Scherk type hypersurface, that we call *Scherk second type hypersurfaces*, given by the solution of the vertical minimal equation in the interior of a certain polyhedron taking alternatively infinite values $+\infty$ and $-\infty$ on adjacent faces of this polyhedron. Those polyhedra may be chosen convex or non convex.

We establish also that the above results, except the statements involving the asymptotic boundary, hold for minimal graphs in $\mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$.

Given a non convex admissible domain $\Omega \subset \mathbb{H}^n$ and given certain geometric conditions on the asymptotic boundary data $\Gamma_\infty \subset \partial_\infty \mathbb{H}^n \times \mathbb{R}$, we prove the existence of a minimal graph in $\mathbb{H}^n \times \mathbb{R}$ whose finite boundary is $\partial\Omega$ and whose asymptotic boundary data is Γ_∞.

A further interesting open problem is to prove a "Jenkins-Serrin" type results in $\mathbb{H}^n \times \mathbb{R}$. When $n = 2$ this task was carried out, for instance, by B. Nelli and H. Rosenberg [11] or by L. Mazet, M. M. Rodriguez and H. Rosenberg [10]. Recently, A. Coutant [5], under the supervision of F. Pacard, has obtained Scherk type hypersurfaces in $\mathbb{R}^{n+1}$ using a different approach.

The knowledge of the n-dimensional hyperbolic geometry is usefull in this paper. The reader is referred to [16].

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2. MINIMAL HYPERSURFACES INVARIANT BY HYPERBOLIC TRANSLATIONS IN $\mathbb{H}^n \times \mathbb{R}$

We recall shortly the geometric description of the family M_d of translation hypersurfaces. First consider a fixed geodesic hyperplane Π of $\mathbb{H}^n$. Let $O \in \Pi$ be any fixed point and let $\gamma \subset \mathbb{H}^n$ be the complete geodesic through O orthogonal to Π.

For any $d > 0$, the hypersurface M_d is generated by a curve in the vertical geodesic two-plane $\gamma \times \mathbb{R}$. The orbit of a point of the generating curve at level t is the equidistant hypersurface of Π in $\mathbb{H}^n \times \{t\}$ passing through this point.

As we said in the introduction, for $d = 1$, the hypersurface M_1 is a complete non entire vertical graph over a half-space of $\mathbb{H}^n \times \{0\}$ bounded by Π, taking infinite value data on Π and zero asymptotic boundary value data.

For any $d < 1$, the hypersurface M_d is an entire vertical graph. For $d > 1$, M_d is a bi-graph over the exterior of an equidistant hypersurface in $\mathbb{H}^n = \mathbb{H}^n \times \{0\}$.

The generating curve of M_d is given by the following explicit form:

$$t = \lambda(\rho) = \int_a^\rho \frac{d}{\sqrt{\cosh^{2n-2} u - d^2}} du, \qquad (a \geqslant 0) \tag{1}$$

where ρ denotes the signed distance on γ with respect to the point O. More precisely: if $d > 1$ then $a > 0$ satisfies $\cosh^{n-1}(a) = d$ and $\rho \geqslant a$, if $d = 1$ then $\rho \geqslant a > 0$ and if $d < 1$ then $a = 0$ and $\rho \in \mathbb{R}$. Observe that if $d < 1$ then λ is an odd function of $\rho \in \mathbb{R}$.

It can be proved in the same way as in Proposition 2.1 of [15] that for any $\rho > 0$ we have

$$\lambda(\rho) \to +\infty, \quad \text{if} \quad d \to 1 \quad (d \neq 1). \qquad (M_d\text{- Property}) \tag{2}$$

3. Vertical minimal equation in $\mathbb{H}^n \times \mathbb{R}$

Definition 3.1 (***Vertical graph***). *Let $\Omega \subset M$ be a domain in a n-dimensional Riemannian manifold M and let $u : \Omega \to \mathbb{R}$ be a C^2 function on Ω. A* vertical graph *in the product space $M \times \mathbb{R}$ is a set $G = \{(x, u(x)) \mid x \in \Omega\}$. We call u the height function.*

Let X be a vector field tangent to M. We denote by $\nabla_M u$ and by $\operatorname{div}_M X$ the gradient of u and the divergence of X, respectively. We define $W_M u := \sqrt{1 + \|\nabla_M u\|_M^2}$.

The following proposition is straightforward but we will write it in a suitable form to establish the *reflection principle* we need.

Proposition 3.1 (***Mean curvature equation in*** $M \times \mathbb{R}$). *Assume that the domain $\Omega \subset M$ in coordinates $(x_1, \ldots, x_n)$ is endowed by a conformal metric $\lambda^2(x_1, \ldots, x_n)\,(dx_1^2 + \cdots + dx_n^2)$. Let H be the mean curvature of a vertical graph G. Then the height function $u(x_1, \ldots, x_n)$ satisfies the following equation*

$$\begin{aligned} nH &= \operatorname{div}_M \left(\frac{\nabla_M u}{W_M u} \right) := \mathcal{M}(u) \\ &= \sum_{i=1}^{n} \frac{n \lambda_{x_i} u_{x_i}}{\lambda^3 \sqrt{1 + \lambda^{-2} \|\nabla u\|_{\mathbb{R}^n}^2}} + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\frac{\lambda^{-2}\, u_{x_i}}{\sqrt{1 + \lambda^{-2} \|\nabla u\|_{\mathbb{R}^n}^2}} \right) \end{aligned} \tag{3}$$

(Mean curvature equation)

Proof. Consider in the conformal coordinates $(x_1, \ldots, x_n)$ the frame field $X_k = \frac{\partial}{\partial x_k}, k = 1, \ldots, n$. Then the upper unit normal field N is given by

$$N = \frac{-\lambda^{-2} \sum\limits_{i=1}^{n} u_{x_i} \frac{\partial}{\partial x_i} + \frac{\partial}{\partial t}}{\sqrt{1 + \|\nabla u\|_M^2}} = -\frac{\nabla_M u}{W_M u} + \frac{1}{W_M u} \frac{\partial}{\partial t}.$$

We call $N^h := -\dfrac{\nabla_M u}{W_M u}$ the horizontal component of N (lifting of a vector field tangent to M). Now using the properties of the Riemannian connection, we infer that the divergence of N in the ambient space

$\mathbb{M}\times\mathbb{R}$ is given by $\operatorname{div}_{M\times\mathbb{R}} N = \operatorname{div}_M N^h$. On the other hand we have, $\operatorname{div}_{M\times\mathbb{R}} N = -nH$, hence we obtain the first equation in the statement of the proposition. Finally, the second equation follows from a simple derivation. □

From Proposition 3.1, we deduce the *minimal vertical equation* or simply *minimal equation* in $\mathbb{H}^n\times\mathbb{R}$ ($\mathcal{M}(u)=0$). We observe that this equation was obtained in a more general setting by Y.-L. Ou [12, Proposition 3.1].

Corollary 3.1 (***Minimal equation in*** $\mathbb{H}^n\times\mathbb{R}$). *Let us consider the upper half-space model of hyperbolic space:* $\mathbb{H}^n=\{(x_1,\dots,x_n)\in\mathbb{R}^n \mid x_n>0\}$. *If* $H=0$, *then the height function* $u(x_1,\dots,x_n)$ *of a vertical minimal graph* G *satisfies the following equation*

(4)

$$\mathcal{M}(u) := \operatorname{div}_{\mathbb{R}^n}\left(\frac{\nabla_{\mathbb{R}^n} u}{\sqrt{1+x_n^2(u_{x_1}^2+\cdots+u_{x_n}^2)}}\right) + \frac{(2-n)u_{x_n}}{x_n\sqrt{(1+x_n^2(u_{x_1}^2+\cdots+u_{x_n}^2)}} = 0,$$

or equivalently

$$\sum_{i=1}^n \Big(1+x_n^2(u_{x_1}^2+\cdots+\widehat{u_{x_i}^2}+\cdots+u_{x_n}^2)\Big)u_{x_ix_i} + \frac{(2-n)\big(1+x_n^2(u_{x_1}^2+\cdots+u_{x_n}^2)\big)u_{x_n}}{x_n} - 2x_n^2\sum_{i<k}u_{x_i}u_{x_k}u_{x_ix_k} - x_nu_{x_n}\left(u_{x_1}^2+\cdots+u_{x_n}^2\right) = 0 \qquad \text{(Minimal equation)}$$

For example the hypersurfaces M_d, $d\in(0,1)$, are entire vertical graphs whose the height function satisfies Equation (4). Other examples are provided by the half part of the hypersurfaces M_d, $d>1$, and the half part of the n-dimensional catenoid, [3] and [15].

Now we state the classical maximum principle and uniqueness for the equation (4).

Remark 3.1 (***Classical maximum principle***). Let $\Omega\subset\mathbb{H}^n$ be a bounded domain and let $g_1,g_2 : \partial\Omega\to\mathbb{R}$ be continuous functions satisfying $g_1\leqslant g_2$. Let $u_i:\overline{\Omega}\to\mathbb{R}$ be a continuous extension of g_i on $\overline{\Omega}$ satisfying the minimal equation (4) on Ω, $i=1,2$, then we have $u_1\leqslant u_2$ on Ω. Consequently, setting $g_1=g_2$, there is at most one continuous extension of g_1 on $\overline{\Omega}$ satisfying the minimal surface equation (4) on Ω.

We will need also a maximum principle involving the asymptotic boundary.

Let $\Omega \subset \mathbb{H}^n$ be an unbounded domain and let $g_1, g_2 : \partial\Omega \cup \partial_\infty \Omega \to \mathbb{R}$ be bounded functions satisfying $g_1 \leqslant g_2$. Assume that g_1 and g_2 are continuous on $\partial\Omega$. Let $u_i : \Omega \cup \partial\Omega \to \mathbb{R}$ be a continuous extension of g_i satisfying the minimal equation (4) on Ω, $i = 1, 2$, such that for any $p \in \partial_\infty \Omega$ we have

$$\limsup_{q \to p} u_1(q) \leqslant g_1(p) \leqslant g_2(p) \leqslant \liminf_{q \to p} u_2(q),$$

then we have $u_1 \leqslant u_2$ on Ω.

We observe that this maximum principle holds assuming the weaker assumptions $\mathcal{M}(u_1) \geqslant 0$ and $\mathcal{M}(u_2) \leqslant 0$ in Ω (instead of $\mathcal{M}(u_1) = \mathcal{M}(u_2) = 0$).

We shall need in the sequel the following important result of J. Spruck.

Remark 3.2 (***Spruck's result on graphs in*** $\mathbb{H}^n \times \mathbb{R}$). We remark that among other pioneering and general results on H-graphs in $M \times \mathbb{R}$, J. Spruck obtained interior a priori gradient estimates depending on a priori height estimates and the distance to the boundary, [17, Theorem 1.1]. Combining this with classical elliptic theory one obtains a *compactness principle*: any bounded sequence (u_n) of solutions of Equation (4) on a domain $\Omega \subset \mathbb{H}^n$ admits a subsequence that converges uniformly on any compact subset of Ω to a solution u of Equation (4) on Ω.

Lemma 3.1 (***Reflection principle for minimal graphs in*** $\mathbb{H}^n \times \mathbb{R}$). *Let $\Omega \subset \mathbb{H}^n$ be a domain whose boundary contains an open set V_Π of a geodesic hyperplane Π of $\mathbb{H}^n$. Assume that Ω is contained in one side of Π and that $\partial\Omega \cap \Pi = \overline{V_\Pi}$.*

Let I be the reflection in $\mathbb{H}^n$ with respect to Π and let $u : \Omega \to \mathbb{R}$ be a solution of the minimal equation (4) that is continuous up to V_Π and taking zero boundary value data on V_Π. Then u can be analytically extended across V_Π to a function $\widetilde{u} : \Omega \cup V_\Pi \cup I(\Omega) \to \mathbb{R}$ satisfying the minimal equation (4), setting $\widetilde{u} = u(p)$, if $p \in \Omega \cup V_\Pi$ and $\widetilde{u} = -u(I(p))$, if $p \in I(\Omega)$.

Proof. Without loss of generality, we will consider the upper half-space model for $\mathbb{H}^n$. Let $u : \Omega \subset \mathbb{H}^n \to \mathbb{R}$ be a C^2 solution of the minimal equation (4).

We first note that the proof of the assertion does not depend on the choice of the geodesic hyperplane Π. Therefore, by applying an ambient horizontal isometry to the minimal graph G, if necessary, we may assume that, without loss of generality, $\Pi = \{(x_1, x_2 \dots, x_n) \in \mathbb{H}^n \mid x_1 = 0\}$ and we assume that $\Omega \subset \Pi^+ := \{(x_1, x_2 \dots, x_n) \in \mathbb{H}^n \mid x_1 > 0\}$.

Notice that setting $w(x_1, x_2, \dots, x_n) := -u(-x_1, x_2, \dots, x_n)$ for any $(x_1, \dots, x_n) \in I(\Omega)$, then it is simple to verify, on account of (4), that w also satisfies the minimal equation on $I(\Omega)$. Now let p be an interior point of V_Π and let $B_r(p) \subset \mathbb{H}^n$ be a small ball around p of radius r entirely contained in $\Omega \cup V_\Pi \cup I(\Omega)$. Let $\partial B_r^+(p) := \partial B_r(p) \cap \Pi^+$ and let $f : \partial B_r^+(p) \to \mathbb{R}$ be the restriction of u to $\partial B_r^+(p)$. We now extend continuously f to the whole sphere $\partial B_r(p)$ of radius r by odd extension. For simplicity we still denote this extension by f. We call v the minimal extension of f on $B_r(p)$ given by Spruck [17, Theorem 1.5], and also by the proof of Theorem 4.1-(1). Notice that the maximum principle ensures that v is the unique solution of the minimal equation in $B_r(p)$ taking the continuous boundary value data f at $\partial B_r(p)$. Therefore we have $v(-x_1, x_2, \dots, x_n) = -v(x_1, \dots, x_n)$ for any $(x_1, \dots, x_n) \in B_r(p)$ and thus $v(0, x_2, \dots, x_n) = 0$ for any $(0, x_2, \dots, x_n) \in V_\Pi$.

The maximum principle again guarantees that v coincides with u on $\Omega \cap B_r(p)$, hence the existence of the minimal extension of f ensures the desired analytic extension of u to $B_r(p)$. This completes the proof. □

4. Perron process for the minimal equation in $\mathbb{H}^n \times \mathbb{R}$

The notions of subsolution, supersolution and barrier for equation (4) are the same as in the two dimensional case, which is treated with details by the authors in [14] and [15].

Definition 4.1 (***Problem*** (P)). In the product space $\mathbb{H}^n \times \mathbb{R}$, we consider the ball model for the hyperbolic plane $\mathbb{H}^n$. Let $\Omega \subset \mathbb{H}^n$, be a domain.

Let $g : \partial\Omega \cup \partial_\infty\Omega \to \mathbb{R}$ be a bounded function. We consider the Dirichlet problem, say problem (P), for the vertical minimal hypersurface equation (4) taking at any point of $\partial\Omega \cup \partial_\infty\Omega$ prescribed boundary (finite and asymptotic) value data g. More precisely,

$$(P)\begin{cases} u \in C^2(\Omega) \text{ and } \mathcal{M}(u) = 0 \text{ in } \Omega, \\ \text{for any } p \in \partial\Omega \cup \partial_\infty\Omega \text{ where } g \text{ is continuous, } u \text{ extends} \\ \text{ continuously at } p \text{ setting } u(p) = g(p). \end{cases}$$

Now, let $u : \Omega \cup \partial\Omega \to \mathbb{R}$ be a continuous function.

Let $U \subset \Omega$ be a closed round ball in $\mathbb{H}^n$. We then define the continuous function $M_U(u)$ on $\Omega \cup \partial\Omega$ by:

$$M_U(u)(x) \begin{cases} u(x) & \text{if } x \in \Omega \cup \partial\Omega \setminus U \\ \tilde{u}(x) & \text{if } x \in U \end{cases} \tag{5}$$

where $\tilde{u}$ is the minimal extension of $u_{|\partial U}$ on $\overline{U}$ given by Spruck [17, Theorem 1.5] and also by the proof of Theorem 4.1-(1).

We say that u is a *subsolution* (resp. *supersolution*) of (P) if:

i) For any closed round ball $U \subset \Omega$ we have $u \leqslant M_U(u)$ (resp. $u \geqslant M_U(u)$).
ii) $u_{|\partial\Omega} \leqslant g$ (resp. $u_{|\partial\Omega} \geqslant g$).
iii) We have $\limsup_{q \to p} u(q) \leqslant g(p)$ (resp. $\liminf_{q\to p} u(q) \geqslant g(p)$) for any $p \in \partial_\infty \Omega$.

Remark 4.1. We now give some classical facts about subsolutions and supersolutions (cf. [4], [14],[15]).

(1) It is easily seen that if u is C^2 on Ω, the condition i) above is equivalent to $\mathcal{M}(u) \geqslant 0$ for subsolution or $\mathcal{M}(u) \leqslant 0$ for supersolution.
(2) As usual if u and v are two subsolutions (resp. supersolutions) of (P) then $\sup(u, v)$ (resp. $\inf(u, v)$) again is a subsolution (resp. supersolution).
(3) Also if u is a subsolution (resp. supersolution) and $U \subset \Omega$ is a closed round ball then $M_U(u)$ is again a subsolution (resp. supersolution).
(4) Let ϕ (resp. u) be a supersolution (resp. a subsolution) of problem (P), then we have $u \leqslant \phi$ on Ω. Moreover, for any closed round ball $U \subset \Omega$ we have $u \leqslant M_U(u) \leqslant M_U(\phi) \leqslant \phi$.

Definition 4.2 (***Barriers***). We consider the Dirichlet problem (P), see Definition 4.1. Let $p \in \partial\Omega \cup \partial_\infty\Omega$ be a boundary point where g is continuous.

(1) • Assume first that $p \in \partial\Omega$. Suppose that for any $M > 0$ and for any $k \in \mathbb{N}$ there is an open neighborhood $\mathcal{N}_k$ of p in $\mathbb{H}^n$ and a function ω_k^+ (resp. ω_k^-) in $C^2(\mathcal{N}_k \cap \Omega) \cap C^0(\overline{\mathcal{N}_k \cap \Omega})$ such that
   i) $\omega_k^+(x)_{|\partial\Omega \cap \overline{\mathcal{N}_k}} \geqslant g(x)$ and $\omega_k^+(x)_{|\partial\mathcal{N}_k \cap \Omega} \geqslant M$ (resp. $\omega_k^-(x)_{|\partial\Omega\cap\overline{\mathcal{N}_k}} \leqslant g(x)$ and $\omega_k^-(x)_{|\partial\mathcal{N}_k\cap\Omega} \leqslant -M$).
   ii) $\mathcal{M}(\omega_k^+) \leqslant 0$ (resp. $\mathcal{M}(\omega_k^-) \geqslant 0$) in $\mathcal{N}_k \cap \Omega$.
   iii) $\lim_{k\to+\infty} \omega_k^+(p) = g(p)$ (resp. $\lim_{k\to+\infty} \omega_k^-(p) = g(p)$).

   • If $p \in \partial_\infty\Omega$, then we choose for $\mathcal{N}_k$ an open set of $\mathbb{H}^n$ containing a half-space with p in its asymptotic boundary. We recall that a half-space is a connected component of $\mathbb{H}^n \setminus \Pi$ for any geodesic hyperplane Π. Then the functions ω_k^+ and ω_k^- are in $C^2(\mathcal{N}_k \cap \Omega) \cap C^0(\overline{\mathcal{N}_k \cap \Omega})$and satisfy:

i) $\omega_k^+(x)_{|\partial\Omega\cap\overline{\mathcal{N}_k}} \geqslant g(x)$ and $\omega_k^+(x)_{|\partial\mathcal{N}_k\cap\Omega} \geqslant M$
(resp. $\omega_k^-(x)_{|\partial\Omega\cap\overline{\mathcal{N}_k}} \leqslant g(x)$ and $\omega_k^-(x)_{|\partial\mathcal{N}_k\cap\Omega} \leqslant -M$).

ii) For any $x \in \partial_\infty(\Omega \cap \mathcal{N}_k)$ we have $\liminf_{y\to x} \omega_k^+(y) \geqslant g(x)$ (for $y \in \mathcal{N}_k \cap \Omega$) (resp. $\limsup_{y\to x} \omega_k^-(y) \geqslant g(x)$).

iii) $\mathcal{M}(\omega_k^+) \leqslant 0$ (resp. $\mathcal{M}(\omega_k^-) \geqslant 0$) in $\mathcal{N}_k \cap \Omega$.

iv) $\lim_{k\to+\infty}\big(\liminf_{q\to p} \omega_k^+(q)\big) = g(p)$ and
$\lim_{k\to+\infty}\big(\limsup_{q\to p} \omega_k^-(q)\big) = g(p)$.

(2) Suppose that $p \in \partial\Omega$ and that there exists a supersolution ϕ (resp. a subsolution η) in $C^2(\Omega) \cap C^0(\overline{\Omega})$ such that $\phi(p) = g(p)$ (resp. $\eta(p) = g(p)$).

In both cases (1) or (2) we say that p admits an *upper barrier* ($\omega_k^+, k \in \mathbb{N}$ or ϕ) (resp. *lower barrier* $\omega_k^-, k \in \mathbb{N}$ or η) for the problem (P). If p admits an upper and a lower barrier we say more shortly that p admits a *barrier*.

Definition 4.3 (C^0 ***convex domains***).

(1) *We say that a C^0 domain Ω is convex at $p \in \partial\Omega$, if a neighborhood of p in $\overline{\Omega}$ lies in one side of some geodesic hyperplane of $\mathbb{H}^n$ passing through p.*

(2) *We say that a C^0 domain Ω is strictly convex at $p \in \partial\Omega$ if a neighborhood $U_p \subset \overline{\Omega}$ of p in $\overline{\Omega}$ lies in one side of some geodesic hyperplane Π of $\mathbb{H}^n$ passing through p and if $U_p \cap \Pi = \{p\}$.*

We are then able to state the following result.

Theorem 4.1 (***Perron process***). *Let $\Omega \subset \mathbb{H}^n$ be a domain and let $g : \partial\Omega \cup \partial_\infty\Omega \to \mathbb{R}$ be a bounded function. Let ϕ be a bounded supersolution of the Dirichlet problem (P), for example the constant function $\phi \equiv \sup g$.*

Set $\mathcal{S}_\phi = \{\varphi,$ subsolution of $(P),\ \varphi \leqslant\ \phi\}$. We define for each $x \in \Omega$

$$u(x) = \sup_{\varphi\in\mathcal{S}_\phi} \varphi(x).$$

(Observe that $\mathcal{S}_\phi \neq \emptyset$ since the constant function $\varphi \equiv \inf g$ belongs to $\mathcal{S}_\phi$.)

We have the following:

(1) *The function u is C^2 on Ω and satisfies the vertical minimal equation (4).*

(2) *Let $p \in \partial_\infty \Omega$ be an asymptotic boundary point where g is continuous. Then p admits a* barrier *and therefore u extends continuously at p setting $u(p) = g(p)$; that is, if (q_m) is a sequence in $\mathbb{H}^n$ such that $q_m \to p$, then $u(q_m) \to g(p)$. In particular, if g is continuous on $\partial_\infty \Omega$ then the asymptotic boundary of the graph of u is the restriction of the graph of g to $\partial_\infty \Omega$.*

(3) *Let $p \in \partial \Omega$ be a finite boundary point where g is continuous. Suppose that p admits a* barrier. *Then the solution u extends continuously at p setting $u(p) = g(p)$.*

(4) *If $\partial \Omega$ is C^0 strictly convex at p then u extends continuously at p setting $u(p) = g(p)$.*

Proof. The proof of (1) follows as in [14, Theorem 3.4]. We will give now some details. To obtain the solution u we need a *compactness principle* and we also need that for any $y \in \Omega$ there exists a round closed ball $B \subset \Omega$ such that $y \in \text{int}(B)$ and such that the Dirichlet problem (P) can be solved on B for any continuous boundary data on ∂B.

The compactness principle was shown by Spruck, see [17]. The resolution of the Dirichlet problem on B may also be encountered in [17], nevertheless we give some details for an alternative proof. Working in the half space model of $\mathbb{H}^n$, B can be seen as an Euclidean ball centered at y of radius $R > 0$. Assume first that h is a $C^{2,\alpha}$ function on ∂B. Observe that the eigenvalues of the symmetric matrix of the coefficients of $u_{x_i x_j}$ in Equation (4) are 1 and $(W_M u)^2 = 1 + x_n^2(u_{x_1}^2 + \cdots + u_{x_n}^2)$, the last with multiplicity $n-1$. Therefore, if R is small enough, then the equation (4) satisfies the structure conditions (14.33) in [6, Chapter 14]. Thus Corollary 14.5 in [6] shows that there exist a priori boundary gradient estimates. Then the classical elliptic theory provides a $C^{2,\alpha}$ solution of (P), see for example [6, Chapter 11]. Finally, for continuous boundary data h on ∂B, we use an approximation argument.

Let us proceed the proof of the assertion (2). Let $p \in \partial_\infty \Omega$, we want to show that the minimal hypersurface M_1 provides an upper and a lower barrier at p. Let $k \in \mathbb{N}^*$, since g is continuous at p, there exists a neighborhood U of p in $\mathbb{H}^n \cup \partial_\infty \mathbb{H}^n$ such that for any $q \in \big(\partial\Omega \cup \partial_\infty \Omega\big) \cap U$ we have $g(p) - 1/2k < g(q) < g(p) + 1/2k$.

Let Π be a geodesic hyperplane such that $\Pi \subset U$ and such that the connected component of $\mathbb{H}^n \setminus \Pi$ lying entirely in U contains p in its asymptotic boundary. We choose an equidistant hypersurface Π_k of Π in the same connected component of $\mathbb{H}^n \setminus \Pi$. We denote by $\mathcal{N}_k$

the connected component of $\mathbb{H}^n \setminus \Pi_k$ containing p in its asymptotic boundary.

We can choose Π_k such that there exist two copies M_1^+ and M_1^- of M_1 satisfying:

- M_1^+ takes the asymptotic boundary value data $g(p) + 1/2k$ on $\partial_\infty \mathcal{N}_k$, the value data $+\infty$ on Π and a finite value data $A > \max\big(g(p) + 1/2k, \sup_\Omega \phi\big)$ on Π_k.
- M_1^- takes the asymptotic boundary value data $g(p) - 1/2k$ on $\partial_\infty \mathcal{N}_k$, the value data $-\infty$ on Π and a finite value data $B < \inf g$ on Π_k.

Let us denote by ω_k^+ (resp. ω_k^-) the function on $\mathcal{N}_k \cap \Omega$ whose graph is the copy M_1^+ (resp. M_1^-) of M_1. We extend ω_k^- on $\overline{\Omega}$ setting $\omega_k^-(q) = B$ for any $q \in \overline{\Omega} \setminus \mathcal{N}_k$, keeping the same notation.

Claim 1. $\omega_k^- \in \mathcal{S}_\phi$, that is ω_k^- is a subsolution such that $\omega_k^- \leqslant \phi$.
Claim 2. For any subsolution $\varphi \in \mathcal{S}_\phi$ we have $\varphi_{|\mathcal{N}_k \cap \Omega} \leqslant \omega_k^+$.

We assume momentarily that the two claims hold. We then have for any $q \in \mathcal{N}_k \cap \Omega$: $\omega_k^-(q) \leqslant u(q)$ (since $\omega_k^- \in \mathcal{S}_\phi$ and by the very definition of u) and $\varphi(q) \leqslant \omega_k^+(q)$ for any subsolution $\varphi \in \mathcal{S}_\phi$. We deduce that

$$\omega_k^-(q) \leqslant u(q) \leqslant \omega_k^+(q)$$

for any $q \in \mathcal{N}_k \cap \Omega$ and for any $k \in \mathbb{N}^*$. The rest of the argument is straightforward but we will provide the details for the readers convenience.

We thus have for any $q \in \mathcal{N}_k \cap \Omega$:

$$\omega_k^-(q) - \Big(g(p) - \frac{1}{2k}\Big) - \frac{1}{2k} \leqslant u(q) - g(p) \leqslant \omega_k^+(q) - \Big(g(p) + \frac{1}{2k}\Big) + \frac{1}{2k}.$$

Let (q_m) be a sequence in Ω such that $q_m \to p$. By construction, for m big enough we have $q_m \in \mathcal{N}_k \cap \Omega$ and

$$|\omega_k^+(q_m) - \Big(g(p) + \frac{1}{2k}\Big)| \leqslant \frac{1}{2k}, \quad |\omega_k^-(q_m) - \Big(g(p) - \frac{1}{2k}\Big)| \leqslant \frac{1}{2k}.$$

We then have $|u(q_m) - g(p)| \leqslant 1/k$ for m big enough, hence $u(q_m) \to g(p)$. We conclude therefore that u extends continuously at p setting $u(p) = g(p)$.

Let us prove Claim 1. By construction, ω_k^- is continuous on $\overline{\Omega}$ and satisfies $\omega_k^-{}_{|\partial\Omega} \leqslant g$ and $\limsup_{y \to p} \omega_k^-(y) \leqslant g(p)$ $(y \in \overline{\Omega})$ for any $p \in \partial_\infty \Omega$. It is straightforward to show that for any closed round ball $U \subset \Omega$ we have $M_U(\omega_k^-) \geqslant \omega_k^-$, see (5) in Definition 4.1. Hence ω_k^- is a subsolution of our Dirichlet problem (P). Observe that we have $\omega_k^- \leqslant \phi$, see Remark 4.1-(4), thus $\omega_k^- \in \mathcal{S}_\phi$ as desired.

The proof of Claim 2 can be accomplished in the same way as the proof of Claim 1, but we give another proof as follows. Let $\varphi \in \mathcal{S}_\phi$. Assume by contradiction that $\sup_{|\mathcal{N}_k \cap \Omega}(\varphi - \omega_k^+) > 0$. Since φ and ω_k^+ are bounded on $\mathcal{N}_k \cap \Omega$ we have $\sup_{|\mathcal{N}_k \cap \Omega}(\varphi - \omega_k^+) < +\infty$. Let (q_m) be a sequence in $\mathcal{N}_k \cap \Omega$ such that $(\varphi - \omega_k^+)(q_m) \to \sup_{|\mathcal{N}_k \cap \Omega}(\varphi - \omega_k^+)$. Let $q \in \overline{\mathcal{N}_k \cap \Omega} \cup \partial_\infty(\mathcal{N}_k \cap \Omega)$ be any limit point of this sequence. Since

$$\varphi \leqslant \phi < A = \omega_k^+$$

on Π_k and

$$\varphi \leqslant g < g(p) + 1/2k \leqslant \omega_k^+$$

on $\partial\Omega \cap \mathcal{N}_k$, we must have

$$q \in \Omega \cap \mathcal{N}_k \quad \text{or} \quad q \in \partial_\infty \mathcal{N}_k.$$

The first possibility is discarded by the maximum principle. The second possibility is also discarded since $\omega_k^+ \geqslant g(p) + 1/2k$ on $\mathcal{N}_k$ and $\varphi(q_m) < g(p) + 1/2k$ if $q_m \in \mathcal{N}_k \cap \Omega$ is close enough of $\partial\Omega \cup \partial_\infty\Omega$.

We conclude that ω_k^+ (resp. ω_k^-) is an upper (resp. a lower) barrier at any asymptotic point of Ω in the sense of Definition 4.2-(1).

We remark that the proof of the assertion (3) is analogous to the proof of the assertion (2), see also [14, Theorem 3.4].

Finally, the proof of the assertion (4) is a consequence of the following.

Claim. The family M_d, $d \in (0, 1)$, provides a barrier at any boundary point where Ω is strictly convex and g is continuous.

We proceed the proof of the claim as follows. We choose the ball model for $\mathbb{H}^n$ and we may assume that $p = 0$. As p is a strictly convex point, there is a geodesic hyperplane $\Pi \subset \mathbb{H}^n$ such that, locally, we have:

$\Pi \cap \partial\Omega = \{0\}$ and, locally, Ω lies in one side, say Π^+, of Π.

Let $M > 0$ and $k \in \mathbb{N}^*$. We now construct a upper barrier at 0. Let $E(\rho)$ be the equidistant hypersurface to Π at distance ρ lying in Π^+. Let $E^+(\rho)$ be the connected component of $\mathbb{H}^n \setminus E(\rho)$ that contains 0. We call $\mathcal{N}$ the connected component of $E^+(\rho) \cap \Omega$ such that $0 \in \overline{\mathcal{N}}$. Consider the hypersurfaces M_d, $d < 1$, given by equation (1). We choose $\rho > 0$ such that $g(q) \leqslant g(0) + 1/k$ on $\overline{\mathcal{N}} \cap \partial\Omega$.

Using the M_d-Property (2), we may choose d near 1, $0 < d < 1$, such that $\lambda(\rho) > M - (g(0) - 1/k)$. We set w_k^+ to be the function on $\overline{\mathcal{N}}$ whose the graph is (a piece of) the vertical translated copy of M_d by $g(0) + 1/k$.

Clearly, the functions w_k^+ are continuous up to the boundary of $\mathcal{N}$ and give a upper barrier at p in the sense of Definition 4.2-(1). In the same way we can construct a lower barrier at p. This completes the proof of the theorem. □

5. Scherk type minimal hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$

Definition 5.1 (***Special rotational domain***). Let $\gamma, L \subset \mathbb{H}^n$ be two complete geodesic lines with L orthogonal to γ at some point $B \in \gamma \cap L$. Using the half-space model for $\mathbb{H}^n$, we can assume that γ is the vertical geodesic such that $\partial_\infty \gamma = \{0, \infty\}$. We call $P \subset \mathbb{H}^n$ the geodesic two-plane containing L and γ. We choose $A_0 \in (0, B) \subset \gamma$ and $A_1 \in L \setminus \gamma$ and we denote by $\alpha \subset P$ the euclidean segment joining A_0 and A_1. Therefore the hypersurface Σ generated by rotating α with respect to γ has the following properties.

(1) $\operatorname{int}(\Sigma)$ is smooth except at point A_0.
(2) Σ is strictly convex in hyperbolic meaning and convex in euclidean meaning.
(3) $\operatorname{int}(\Sigma) \setminus \{A_0\}$ is transversal to the Killing field generated by the translations along γ.

Consequently Σ lies in the mean convex side of the domain of $\mathbb{H}^n$ whose boundary is the hyperbolic cylinder with axis γ and passing through A_1. Let us call $\Pi \subset \mathbb{H}^n$ the geodesic hyperplane orthogonal to γ and passing through B. Observe that the boundary of Σ is a $n-2$ dimensional geodesic sphere of Π centered at B.

We denote by $U_\Sigma \subset \Pi$ the open geodesic ball centered at B whose boundary is the boundary of Σ. We call $\mathcal{D}_\Sigma \subset \mathbb{H}^n$ the closed domain whose boundary is $U_\Sigma \cup \Sigma$. Observe that $\partial \mathcal{D}_\Sigma$ is strictly convex at any point of Σ and convex at any point of U_Σ. Such a domain will be called a *special rotational domain.*

Proposition 5.1. *Let $\mathcal{D}_\Sigma \subset \mathbb{H}^n$ be a special rotational domain. For any number $t \in \mathbb{R}$, there is a unique solution v_t of the vertical minimal equation in* $\operatorname{int}(\mathcal{D}_\Sigma)$ *which extends continuously to* $\operatorname{int}(\Sigma) \cup U_\Sigma$*, taking prescribed zero boundary value data on the interior of Σ and prescribed boundary value data t on U_Σ.*

More precisely, for any $t \in \mathbb{R}$, the following Dirichlet problem (P_t) admits a unique solution v_t.

$$
(P_t)\begin{cases}
\mathcal{M}(u) = 0 \text{ in } \operatorname{int}(\mathcal{D}_\Sigma),\\
u = 0 \text{ on } \operatorname{int}(\Sigma),\\
u = t \text{ on } U_\Sigma,\\
u \in C^2\left(\operatorname{int}(\mathcal{D}_\Sigma)\right) \cap C^0\left(\mathcal{D}_\Sigma \setminus \partial\Sigma\right).
\end{cases}
$$

Furthermore, the solutions v_t are strictly increasing with respect to t and satisfy $0 < v_t < t$ *on* $\text{int}(\mathcal{D}_\Sigma)$.

Proof. Before beginning the proof of the existence part of the statement, we would like to remark that, as the ambient space has dimension n (arbitrary), we cannot use classical Plateau type arguments to obtain a regular minimal hypersurface in $\mathbb{H}^n \times \mathbb{R}$ whose the boundary is $\big(\Sigma \times \{0\}\big) \cup \big(U_\Sigma \times \{t\}\big) \cup \big(\partial\Sigma \times [0,t]\big)$.

We are not able to apply directly *Perron process* (Theorem 4.1) to solve this Dirichlet problem. For this reason, in order to prove the existence part of our statement, we need to consider an auxiliary Dirichlet problem, as follows.

We can assume that $t > 0$. For $k \in \mathbb{N}^*$ we set

$$V_k := \{p \in \Sigma \mid \text{dist}\,(p, \Pi) \leqslant \frac{1}{k}\},$$

where we recall that $\Pi \subset \mathbb{H}^n$ is the geodesic hyperplane containing U_Σ and where *dist* means the distance in $\mathbb{H}^n$.

We choose a translated copy M_{d_k} of the hypersurface M_d, see section 2, with $d_k < 1$, given by a function $\lambda_k(\rho)$ satisfying $\lambda_k(0) = t$ and $\lambda_k(1/k) \leqslant -1$. Since λ_k is an odd function for $d_k \in (0,1)$, the M_d-Property (2) insures that such a M_{d_k} exists for $d_k < 1$ close enough to 1. Then we choose a continuous function $f_k : V_k \longrightarrow [0,t]$ such that

(1) $f_k = t$ on $\partial\Sigma = V_k \cap \Pi$.
(2) $f_k = 0$ on $\partial V_k \cap \text{int}(\Sigma)$.
(3) The graph of f_k stands above the hypersurface M_{d_k}, that is $f_k \geqslant \lambda_k$ on V_k.

Now we define a function $g_k : \partial\mathcal{D}_\Sigma \longrightarrow [0,t]$ setting:

$$g_k(p) = \begin{cases} 0 & \text{if}\, p \in \Sigma \setminus V_k, \\ f_k & \text{if}\, p \in V_k, \\ t & \text{if } p \in U_\Sigma. \end{cases}$$

Note that g_k is a continuous function on $\partial\mathcal{D}_\Sigma$. Then we consider an auxiliary Dirichlet problem $(\widehat{P}_k)$ as follows:

$$(\widehat{P}_k) \begin{cases} \mathcal{M}(u) = 0 \text{ in } \text{int}(\mathcal{D}_\Sigma), \\ u = g_k \text{ on } \partial\mathcal{D}_\Sigma, \\ u \in C^2\left(\text{int}(\mathcal{D}_\Sigma)\right) \cap C^0\left(\mathcal{D}_\Sigma\right). \end{cases}$$

Observe that the hypersurface M_{d_k} provides a lower barrier at any point of U_Σ and that at such a point the constant function $\omega^+ \equiv t$ is an upper barrier in the sense of Definition 4.2-(2). Furthermore, $\partial\mathcal{D}_\Sigma$ is C^0 strictly convex at any other point, that is at any point of

Σ. Therefore the hypersurfaces M_d, $d < 1$, provide a barrier at these points, see the proof of Theorem 4.1-(4). Thus, any point of $\partial\mathcal{D}_\Sigma$ has a barrier. Applying *Perron Process* (Theorem 4.1), considering the set of subsolutions to problem $(\widehat{P}_k)$ below the constant supersolution identically equal to t, we find a solution w_k of the Dirichlet problem $(\widehat{P}_k)$. Observe that the zero function is a subsolution of $(\widehat{P}_k)$. Therefore we have $0 \leqslant w_k \leqslant t$ for any $k > 0$.

Using the *reflection principle* with respect to Π (Lemma 3.1), it follows that each point of U_Σ can be considered as an interior point of the domain of a function, denoted again by w_k, satisfying the minimal equation, bounded below by 0 and bounded above by $2t$. Observe that this estimate is independent of $k > 0$.

Consequently, using the *compactness principle*, we can find a subsequence that converges to a function $v_t \in C^2(\mathrm{int}(\mathcal{D}_\Sigma)) \cap C^0(\mathrm{int}(\mathcal{D}_\Sigma) \cup U_\Sigma)$ satisfying the minimal equation $\mathcal{M}(v_t) = 0$ and such that $v_t(p) = t$ at any $p \in U_\Sigma$. Since any point of $\mathrm{int}(\Sigma)$ has a barrier the function v_t extends continuously there, setting $v_t(p) = 0$ at any $p \in \mathrm{int}(\Sigma)$. We have therefore proved the existence of a solution v_t of the Dirichlet problem (P_t). Observe that by construction we have $0 < v_t < t$ on $\mathrm{int}(\mathcal{D}_\Sigma)$.

Let us prove now uniqueness of the solution of (P_t). Let u and v be two solutions of the Dirichlet problem (P_t). We will adapt the proof of [7, Theorem 2.2] to our situation.

We are going to use the notations of Definition 5.1. Let us recall that P is the geodesic two-plane containing the geodesic lines γ and L. Let $\varepsilon > 0$ and let us call $c_\varepsilon \subset P$ the intersection of the circle or radius ε centered at A_1 with the compact subset of P delimited by γ, L and the euclidean segment α. We denote by $C_\varepsilon \subset \mathbb{H}^n$ the compact hypersurface obtained by rotating c_ε with respect to γ. Let V_ε be the $n-1$ volume of C_ε. Observe that $V_\varepsilon \to 0$ when $\varepsilon \to 0$. From now the arguments follow as in [7], so we just sketch the proof.

For $N > 0$ large we define

$$\varphi = \begin{cases} N - \varepsilon & \text{if} \quad u - v \geqslant N \\ u - v - \varepsilon & \text{if} \quad \varepsilon < u - v < N \\ 0 & \text{if} \quad u - v \leqslant \varepsilon \end{cases}$$

Let us call $\mathcal{D}_\varepsilon$ the connected component of $\mathcal{D}_\Sigma \setminus C_\varepsilon$ containing A_0 (we have $\mathcal{D}_\varepsilon \to \mathcal{D}_\Sigma$ when $\varepsilon \to 0$). Observe that $\varphi \equiv 0$ along $\partial\mathcal{D}_\varepsilon \setminus C_\varepsilon$. So that, applying the divergence theorem and using the fact that u and v are solutions of the minimal graph equation, we obtain

$$\int_{C_\varepsilon} \varphi \langle \frac{\nabla u}{W_M u} - \frac{\nabla v}{W_M v}, \nu \rangle ds = \int_{\mathcal{D}_\varepsilon} \langle \nabla\varphi, \frac{\nabla u}{W_M u} - \frac{\nabla v}{W_M v} \rangle dV$$

where ν is the exterior normal to ∂C_ε. It is shown in [7, Lemma 2.1] that $\langle \nabla u - \nabla v, \frac{\nabla u}{W_M u} - \frac{\nabla v}{W_M v}\rangle \geqslant 0$ with equality at a point if, and only if, $\nabla u = \nabla v$. Therefore

$$0 \leqslant \int_{\mathcal{D}_\varepsilon} \langle \nabla\varphi, \frac{\nabla u}{W_M u} - \frac{\nabla v}{W_M v}\rangle dV = \int_{C_\varepsilon} \varphi\langle \frac{\nabla u}{W_M u} - \frac{\nabla v}{W_M v}, \nu\rangle ds$$
$$\leqslant 2NV_\varepsilon$$

Letting $\varepsilon \to 0$, we get that $\nabla u \equiv \nabla v$ in the set where $0 < u - v < N$. Letting $N \to +\infty$ we obtain that $\nabla u \equiv \nabla v$ in the set $\{u > v\}$. Assume that $\operatorname{int}\{u > v\} \neq \emptyset$, then there exists a constant $\lambda > 0$ such that $u = v + \lambda$ on an open subset of $\mathcal{D}_\Sigma$. By analyticity we deduce that $u = v + \lambda$ everywhere on $\mathcal{D}_\Sigma \setminus \partial\Sigma$, which is absurd since $u = v$ on $\partial\mathcal{D}_\Sigma \setminus \partial\Sigma$. Therefore we get that $\operatorname{int}\{u > v\} = \emptyset$, that is $u \leqslant v$ on $\mathcal{D}_\Sigma \setminus \partial\Sigma$. The same argument shows also that $v \leqslant u$ on $\mathcal{D}_\Sigma \setminus \partial\Sigma$. Therefore $u = v$ and the proof of the uniqueness of the solution of Dirichlet problem (P_t) is completed.

At last, let us prove that the family $\{v_t\}$ of the solutions of Dirichlet problem $(\mathcal{P}_t)$ is strictly increasing on t. We could adapt the same arguments of [7, Theorem 2.2] as before, but we will give another proof.

Let $0 < t_1 < t_2$ and let v_1 and v_2 be the solutions of the Dirichlet problems (P_{t_1}) and (P_{t_2}) respectively. Let p be a fixed arbitrary point in the interior of $\mathcal{D}_\Sigma$.

For ε small enough consider a ε-translated copy of the graph of v_1 along γ in the orientation $A_0 \to B$. This graph is given by a function v_1^ε over a translated copy $\mathcal{D}_\Sigma(\varepsilon)$ of $\mathcal{D}_\Sigma$. Taking into account the properties on Σ stated in Definition 5.1, we have $\mathcal{D}_\Sigma(\varepsilon) \cap \Sigma = \emptyset$. We may assume that ε is chosen small so that p belongs to $\operatorname{int}(\mathcal{D}_\Sigma(\varepsilon))$. Since $0 < v_1 < t_1$ on $\operatorname{int}\mathcal{D}_\Sigma$, we get that v_1^ε is less than v_2 along the boundary of $\mathcal{D}_\Sigma \cap \mathcal{D}_\Sigma(\varepsilon)$. Using maximum principle we deduce that $v_1^\varepsilon(p) < v_2(p)$, for ε small enough, since $v_1^\varepsilon < v_2$ along $\partial\big(\mathcal{D}_\Sigma \cap \mathcal{D}_\Sigma(\varepsilon)\big)$. Thus letting $\varepsilon \to 0$ we have therefore that $v_1(p) \leqslant v_2(p)$, this accomplishes the proof.

□

Theorem 5.1 (***Rotational Scherk hypersurface***). *Let* $\mathcal{D}_\Sigma \subset \mathbb{H}^n$ *be a special rotational domain. There is a unique solution* v *of the vertical minimal equation in* $\operatorname{int}(\mathcal{D}_\Sigma)$ *which extends continuously to* $\operatorname{int}(\Sigma)$, *taking prescribed zero boundary value data and taking boundary value* ∞ *for any approach to* U_Σ.

More precisely, the following Dirichlet problem (P) *admits a unique solution* v_∞*.*

$$(P)\begin{cases}\mathcal{M}(u)=0 \text{ in } \operatorname{int}(\mathcal{D}_\Sigma),\\ u=0 \text{ on } \operatorname{int}(\Sigma),\\ u=+\infty \text{ on } U_\Sigma,\\ u\in C^2\left(\operatorname{int}(\mathcal{D}_\Sigma)\right)\cap C^0\left(\mathcal{D}_\Sigma\setminus\overline{U}_\Sigma\right).\end{cases}$$

We call the graph of v *in* $\mathbb{H}^n\times\mathbb{R}$ *a* rotational Scherk hypersurface*.*

Proof. First, we will prove the existence part of the Theorem. We consider the family of functions v_t, $t>0$, given by Proposition 5.1. Recall that $\Pi\subset\mathbb{H}^n$ is the totally geodesic hyperplane containing U_Σ. We consider a suitable copy of M_1 (see section 2) as barrier as follows: choose M_1 such that M_1 is a graph of a function u_1 whose domain is the component of $\mathbb{H}^n\setminus\Pi$ that contains $\mathcal{D}_\Sigma$, with u_1 taking boundary value data $+\infty$ on Π and taking zero asymptotic boundary value data. By applying maximum principle we have that $u_1(p)>v_t(p)$ for all $p\in\mathcal{D}_\Sigma$ and all $t>0$.

Using *compactness principle* we obtain that a subsequence of the family converges uniformly on any compact subsets of $\operatorname{int}(\mathcal{D}_\Sigma)$ to a solution v_∞ of the minimal equation. Since the family is strictly increasing v_∞ takes the value $+\infty$ on U_Σ. That is, for any sequence (q_k) in $\operatorname{int}(\mathcal{D}_\Sigma)$ converging to some point of U_Σ we have $v_\infty(q_k)\to+\infty$.

Let $p\in\operatorname{int}(\Sigma)$, since $\partial\mathcal{D}_\Sigma$ is C^0 strictly convex at p, the hypersurfaces M_d, $d<1$, provide a barrier at p, see the proof of Theorem 4.1-(4). Consequently v_∞ extends continuously at p setting $v_\infty(p)=0$. Therefore v_∞ is a solution of the Dirichlet problem (P).

The proof of uniqueness of v_∞ proceeds in the same way as the proof of the monotonicity of the family $\{v_t\}$ in Proposition 5.1. This completes the proof of the Theorem. □

Theorem 5.2 (***Barrier at a*** C^0 ***convex point***). *Let* $\Omega\subset\mathbb{H}^n$ *be a domain and let* $p_0\in\partial\Omega$ *be a boundary point where* Ω *is* C^0 *convex. Then for any bounded data* $g:\partial\Omega\cup\partial_\infty\Omega\to\mathbb{R}$ *continuous at* p_0*, the family of rotational Scherk hypersurfaces provides a barrier at* p_0 *for the Dirichlet problem* (P)*. In particular, in* Theorem 4.1-(4) *the assumption* C^0 strictly convex *can be replaced by* C^0 convex*.*

Proof. We use the same notations as in the definition of a special rotational domain, Definition 5.1.

We will prove that the rotational Scherk hypersurfaces with $-\infty$ boundary data on the boundary part U_Σ provide an upper barrier at p_0. For the lower barrier the construction is similar.

Let $\mathcal{D}_\Sigma$ be a special rotational domain. Let ω be the height function of the rotational Scherk hypersurface S taking $-\infty$ boundary data on U_Σ and 0 boundary data on the interior of Σ, given by Theorem 5.1.

Claim 1. ω is decreasing along the oriented geodesic segment $[A_0, B] \subset \gamma$ (going from A_0 to B).

Claim 2. Let D be any point on the open geodesic segment (A_0, B), and let $\beta \subset \mathcal{D}_\Sigma$ be a geodesic segment issuing from D, ending at some point $C \in \operatorname{int}(\Sigma)$ and orthogonal to $[A_0, B]$ at D.

Then ω is increasing along $\beta = [D, C]$, oriented from D to C.

We first prove the theorem assuming that the two claims hold.

Let $D \in (A_0, B)$ and let $\Pi_D \subset \mathbb{H}^n$ be the geodesic hyperplane through D orthogonal to the geodesic segment $[A_0, B]$. Let $\mathcal{D}^+_\Sigma$ be the connected component of $\mathcal{D}_\Sigma \setminus \Pi_D$ containing the point A_0. Let q be any point belonging to the closure of $\mathcal{D}^+_\Sigma$. The claims ensure that $\omega(q) \geqslant \omega(D)$.

Let $p_0 \in \partial\Omega$ be a C^0 convex point and let g be a bounded data continuous at p_0. Let $M > 0$ be any positive real number. It suffices to show that for any $k \in \mathbb{N}^*$ there is an open neighborhood $\mathcal{N}_k$ of p_0 in $\mathbb{H}^n$ and a function ω_k^+ in $C^2(\mathcal{N}_k \cap \Omega) \cap C^0(\overline{\mathcal{N}_k \cap \Omega})$ such that

i) $\omega_k^+(x)_{|\partial\Omega\cap\mathcal{N}_k} \geqslant g(x)$ and $\omega_k^+(x)_{|\partial\mathcal{N}_k\cap\Omega} \geqslant M$,
ii) $\mathcal{M}(\omega_k^+) = 0$ in $\mathcal{N}_k \cap \Omega$,
iii) $\omega_k^+(p_0) = g(p_0) + 1/k$.

By continuity there exists $\varepsilon > 0$ such that for any $p \in \partial\Omega$ with $\operatorname{dist}(p, p_0) < \varepsilon$ we have $g(p) < g(p_0) + 1/k$.

By assumption there exist a geodesic hyperplane Π_{p_0} through p_0 and an open neighborhood $W \subset \Pi_{p_0}$ of p_0 such that $W \cap \Omega = \emptyset$. We set $\Omega_\varepsilon = \{p \in \Omega \mid \operatorname{dist}(p_0, p) < \varepsilon\}$. Up to choosing ε small enough, we can assume that Ω_ε is entirely contained in a component of $\mathbb{H}^n \setminus \Pi_{p_0}$. Let γ be the geodesic through p_0 orthogonal to Π_{p_0}.

We choose a special rotational domain $\mathcal{D}_\Sigma$ such that:

- the hyperplane Π is orthogonal to γ, (recall that $U_\Sigma \subset \Pi$)
- the diameter of $\mathcal{D}_\Sigma$ is lesser than $\frac{\varepsilon}{4}$,
- $\overline{\Omega} \cap U_\Sigma = \emptyset$,
- $A_0 \in \gamma$, $\operatorname{dist}(p_0, A_0) < \frac{\varepsilon}{8}$ and A_0 belongs to the same component of $\mathbb{H}^n \setminus \Pi_{p_0}$ than Ω_ε.

Let $M' > \max\{M, g(p_0) + 1/k\}$. We consider the rotational Scherk hypersurface (graph of ω) taking M' boundary value data on the interior of Σ and $-\infty$ on U_Σ. By continuity, there exists a point $p_1 \in \gamma$ where $\omega(p_1) = g(p_0)+1/k$. Up to a horizontal translation along γ sending p_1 to p_0, we may assume that $\omega(p_0) = g(p_0) + 1/k$. Then we set $\mathcal{N}_k = \mathrm{int}(\mathcal{D}_\Sigma) \cap \Omega$ and $\omega_k^+ = \omega_{|\mathcal{N}_k}$, the restriction of ω to $\mathcal{N}_k$. Therefore we have $\omega_k^+(x)_{|\partial \mathcal{N}_k \cap \Omega} = M' \geqslant M$, furthermore Claim 1 and Claim 2 show that $\omega_k^+(x)_{|\partial\Omega \cap \mathcal{N}_k} \geqslant g(p_0) + 1/k \geqslant g(x)$, as desired.

We now proceed to the proof of Claim 1. Let $p_1, p_2 \in (A_0, B)$ with $p_1 < p_2$, we want to show that $\omega(p_1) \geqslant \omega(p_2)$. Let $p_3 \in (p_1, p_2)$ be the middle point of p_1 and p_2 and let $\Pi_{p_3} \subset \mathbb{H}^n$ be the geodesic hyperplane through p_3 orthogonal to (A_0, B). We denote by σ the reflection in $\mathbb{H}^n$ with respect to Π_{p_3}. Let $\mathcal{D}_\Sigma^+$ be the connected component of $\mathcal{D}_\Sigma \setminus \Pi_{p_3}$ containing A_0 and let $\mathcal{D}_\Sigma^-$ be the other component. We denote by S^+ the part of the rotational Scherk hypersurface which is a graph over $\mathcal{D}_\Sigma^+$. Observe that the definition of a special rotational domain ensures that $\sigma(\mathcal{D}_\Sigma^+) \cap \Sigma = \emptyset$. Hence a part of $\sigma(S^+)$ is the graph of a function v over a part W of $\mathcal{D}_\Sigma^-$ such that $v \geqslant \omega$ on ∂W. We conclude therefore with the aid of the maximum principle that $v \geqslant \omega$ on W. This shows that $\omega(p_1) \geqslant \omega(p_2)$ as desired.

Now let us prove Claim 2. Let $q_1, q_2 \in [D, C]$ with $q_1 < q_2$, we want to show that $\omega(q_1) \leqslant \omega(q_2)$. Let $q_3 \in (q_1, q_2)$ be the middle point of q_1 and q_2 and let Π_{q_3} be the geodesic hyperplane through q_3 orthogonal to $[D, C]$. Let σ be the reflection in $\mathbb{H}^n$ with respect to Π_{q_3}. Let $\mathcal{D}_\Sigma^-$ be the connected component of $\mathcal{D}_\Sigma \setminus \Pi_{q_3}$ containing A_0 and let $\mathcal{D}_\Sigma^+$ be the other component.

Assertion. If $U_\Sigma \cap \Pi_{q_3} \neq \emptyset$ then there exists a point $X_0 \in U_\Sigma \cap \mathcal{D}_\Sigma^+$ such that $\sigma(X_0) \notin \mathcal{D}_\Sigma$.

We assume this assertion for a while. If $U_\Sigma \cap \Pi_{q_3} \neq \emptyset$ then for any $Z \in U_\Sigma \cap \mathcal{D}_\Sigma^+$, with $Z \notin \Pi_{q_3}$, we have $\sigma(Z) \notin \mathcal{D}_\Sigma$. Indeed, if not, since $\sigma(X_0) \notin \mathcal{D}_\Sigma$, we would find by continuity a point $Y \in U_\Sigma \cap \mathcal{D}_\Sigma^+$, with $Y \notin \Pi_{q_3}$, such that $\sigma(Y) \in \Pi$ and $\sigma(Y) \neq Y$. Therefore the geodesic segment $[Y, \sigma(Y)]$ is globally invariant with respect to σ. Thus $[Y, \sigma(Y)]$ is orthogonal to Π_{q_3} and therefore Π is also orthogonal to Π_{q_3}. Hence, we conclude that the whole hyperplane Π is invariant by the reflection σ, which contradicts the assertion.

We denote by Σ^- the connected component of $\Sigma \setminus \Pi_{q_3}$ which contains A_0 and we denote by Σ^+ the other component.

Observe that for any $p \in \Sigma^+$ we have $\sigma(p) \notin \Sigma^-$. Indeed, assume first that p lies in the euclidean segment $\alpha \subset P$ (see Definition 5.1). By construction, $\sigma(p)$ belongs to the equidistant curve $E_p \subset P$, passing through p, of the geodesic line Γ containing the segment $[D, C]$. Recall that Γ and E_p have the same asymptotic boundary. Furthermore, E_p is symmetric with respect to any geodesic hyperplane orthogonal to Γ. Since $\mathcal{D}_\Sigma$ is symmetric with respect to the geodesic hyperplane through D orthogonal to Γ, we have that $\sigma(p) \notin \Sigma^-$. Assume now that $p \in \Sigma^+ \setminus \alpha$. Let us denote by V the 3-dimensional geodesic submanifold of $\mathbb{H}^n$ containing p and the geodesic two-plane P. Let $H_D \subset \mathbb{H}^n$ be the geodesic hyperplane through D orthogonal to the geodesic Γ. Then the symmetric of p with respect to H_D, denoted by p^*, is the same than the symmetric of p in V with respect to the geodesic two-plane $V \cap H_D$. As before, $\sigma(p)$ belongs to the equidistant curve $E_p \subset P$, passing through p, of the geodesic line Γ. Furthermore E_p is symmetric with respect to the geodesic hyperplanes H_D and Π_{q_3}. Now E_p is an arc of circle passing through p with the same asymptotic boundary than Γ. As $\mathcal{D}_\Sigma \cap V$ is a compact part of an euclidean cone we get that $E_p \cap \Sigma = \{p, p^*\}$. Since $\sigma(p) \neq p^*$, we conclude that $\sigma(p) \notin \Sigma^-$.

Thus the reflected of $\partial \mathcal{D}_\Sigma^+$ by σ does not have any intersection with Σ^-. We denote by S^+ the part of the rotational Scherk hypersurface which is a graph over $\mathcal{D}_\Sigma^+$. Hence a part of $\sigma(S^+)$ is the graph of a function v over the domain $W = \sigma(\mathcal{D}_\Sigma^+) \cap \mathcal{D}_\Sigma^-$ such that $v \geqslant \omega$ on ∂W. We now are able to conclude the proof of Claim 2, assuming the assertion, by applying the maximum principle, to infer that $\omega(q_2) \geqslant \omega(q_1)$.

Finally, if $U_\Sigma \cap \Pi_{q_3} = \emptyset$ by a similar and simpler argument we complete the proof of Claim 2.

To prove the assertion, let us denote by $P_C \subset \mathbb{H}^n$ the geodesic two-plane containing the geodesic segments $[A_0, B]$ and $[D, C]$. Thus P_C is orthogonal to Π_{q_3}, since it contains $[D, C]$, and is orthogonal to Π, since it contains $[A_0, B]$. We consider the open geodesic segment $\gamma_1 = P_C \cap U_\Sigma$ and the geodesic line $\gamma_2 = P_C \cap \Pi_{q_3}$. Assume that $U_\Sigma \cap \Pi_{q_3} \neq \emptyset$. Then, since P_C is orthogonal to Π and to Π_{q_3} we have $\gamma_2 \cap U_\Sigma \neq \emptyset$. Therefore γ_2 intersects γ_1 at some point $\{z\} = \gamma_1 \cap \gamma_2$.

Observe that the points D, q_3, z and B define a geodesic quadrilateral $\mathcal{Q}$ in P_C with right angles at vertices B, D and q_3. Therefore the interior angle of $\mathcal{Q}$ at z is strictly smaller than $\pi/2$. Let us denote by $\gamma_1^+ \subset \gamma_1$ the connected component of $\gamma_1 \setminus \{z\}$ which does not contain B. Observe that $\gamma_1^+ \subset U_\Sigma \cap \mathcal{D}_\Sigma^+$. Let s be the reflection in P_C with respect to γ_2. Then $s(\gamma_1^+)$ does not have intersection with $\mathcal{D}_\Sigma$, $s(\gamma_1^+) \cap \mathcal{D}_\Sigma = \emptyset$. Since P_C is orthogonal to Π_{q_3} we have that $s(\gamma_1^+) = \sigma(\gamma_1^+)$. Therefore for any $X \in \gamma_1^+$ we have $\sigma(X) \notin \mathcal{D}_\Sigma$ as claimed, this completes the proof. $\square$

Definition 5.2 (***Independent points and admissible polyhedra***).

(1) *We say that* $n+1$ *points* $A_0,\dots,A_n$ *in* $\mathbb{H}^n$ *are* independent *if there is no geodesic hyperplane containing these points. If* $A_0,\dots,A_n$ *in* $\mathbb{H}^n$ *are independent then we remark that any choice of* n *points among them determines a unique geodesic hyperplane of* $\mathbb{H}^n$.

(2) *Let* $A_0,\dots,A_n$ *be* $n+1$ *independent points in* $\mathbb{H}^n$. *We call* Π_i *the geodesic hyperplane containing these points excepted* A_i, $i=0,\dots,n$ *and we call* Π_i^+ *the closed half-space bounded by* Π_i *and containing* A_i. *Then the intersection of these half-spaces is a polyhedron* $\mathcal{P}$*: the convex closure of* $A_0,\dots,A_n$. *The boundary of* $\mathcal{P}$ *consists of* $n+1$ *closed faces* $F_i\subset\Pi_i$, *the face* F_i *contains in its boundary all the points* $A_0,\dots,A_n$ *excepted* A_i. *We call such a polyhedron an* admissible polyhedron.

Corollary 5.1. *Let* $\mathcal{P}$ *be an admissible polyhedron. For any number* $t\in\mathbb{R}$, *there is a unique solution* v_t *of the vertical minimal equation in* $\mathrm{int}(\mathcal{P})$ *which extends continuously to* $\partial\mathcal{P}\setminus\partial F_0$, *taking prescribed zero boundary value data on* $F_1\setminus\partial F_0,\dots,F_n\setminus\partial F_0$ *and prescribed boundary value* t *on* $\mathrm{int}(F_0)$. *More precisely, for any* $t\in\mathbb{R}$, *the following Dirichlet problem* (P_t) *admits a unique solution* v_t.

$$(P_t)\begin{cases}\mathcal{M}(u)=0 \text{ in } \mathrm{int}(\mathcal{P}),\\ u=0 \text{ on } F_j\setminus\partial F_0,\ j=1,\dots,n,\\ u=t \text{ on } \mathrm{int}(F_0),\\ u\in C^2\left(\mathrm{int}(\mathcal{P})\right)\cap C^0\left(\mathcal{P}\setminus\partial F_0\right).\end{cases}$$

Furthermore, the solutions v_t *are strictly increasing with respect to* t *and satisfy* $0<v_t<t$ *on* $\mathrm{int}(\mathcal{P})$.

Proof. The existence part of the statement is a consequence of Theorem 5.2.

The uniqueness is proved in the same way as in Proposition 5.1.

To prove the monotonicity of the family $\{v_t\}$ we consider a point $q\in\mathrm{int}(F_0)$. Notice that $\partial\mathcal{P}$ is transversal to the Killing field generated by translations along the geodesic line γ containing A_0 and q. Then the proof proceeds as in the proof of Proposition 5.1. □

Using the above proposition we are able to construct a *Scherk type minimal hypersurface* in $\mathbb{H}^n\times\mathbb{R}$.

Theorem 5.3 (***First Scherk type hypersurface in*** $\mathbb{H}^n\times\mathbb{R}$). *Let* $\mathcal{P}$ *be an admissible convex polyhedron. There is a unique solution* v_∞ *of* *the minimal equation in* $\mathrm{int}(\mathcal{P})$ *extending continuously up to* $\partial\mathcal{P} \setminus F_0$, *taking prescribed zero boundary value data on* $F_1 \setminus \partial F_0, \dots, F_n \setminus \partial F_0$ *and prescribed boundary value* ∞ *for any approach to* $\mathrm{int}(F_0)$. *More precisely, we prove existence and uniqueness of the following Dirichlet problem* (P_∞)*:*

$$(P_\infty)\begin{cases}\mathcal{M}(u) = 0 \text{ in } \mathrm{int}(\mathcal{P}),\\ u = 0 \text{ on } F_j \setminus \partial F_0,\ j = 1, \dots, n,\\ u = \infty \text{ on } \mathrm{int}(F_0),\\ u \in C^2\left(\mathrm{int}(\mathcal{P})\right) \cap C^0\left(\mathcal{P} \setminus F_0\right).\end{cases}$$

Proof. With the aid of Theorem 5.2 we may use the rotational Scherk hypersurfaces as barrier. Therefore, we obtain for any $t \in \mathbb{R}$ a solution v_t of the vertical minimal equation in $\mathrm{int}(\mathcal{P})$ which extends continuously to $\partial\mathcal{P}\setminus\partial F_0$, taking prescribed zero boundary value data on $\partial\mathcal{P}\setminus F_0$ and prescribed boundary value t on $\mathrm{int}(F_0)$. Now letting $t \to \infty$ as in the proof of Theorem 5.1 we have that a subsequence of the family $\{v_t\}$ converges to a solution as desired, taking into account that the rotational Scherk hypersurfaces give a barrier at any point of $\mathcal{P}$.

The uniqueness is obtained as in the proof of the monotonicity of the family $\{v_t\}$ in Proposition 5.1, see also the proof of Corollary 5.1. □

Theorem 5.4 (***Second Scherk type hypersurface in*** $\mathbb{H}^n \times \mathbb{R}$). *For any* $k \in \mathbb{N}$, $k \geqslant 2$, *there exists a family of polyhedron* $\mathcal{P}_k$ *with* $2^{n-1}k$ *faces and a solution* w_k *of the vertical minimal equation in* $\mathrm{int}\,\mathcal{P}_k$ *taking alternatively infinite values* $+\infty$ *and* $-\infty$ *on adjacent faces of* $\mathcal{P}_k$. *Moreover, the polyhedron* $\mathcal{P}_k$ *can be chosen to be convex and can also be chosen to be non convex.*

Proof. Let us fix a point A_0 in $\mathbb{H}^n$. Let $\{e_1, \dots, e_n\}$ be a positively oriented orthornormal basis of $T_{A_0}\mathbb{H}^n$. For $k \geqslant 2$ we set $u := \sin(\pi/k)e_1 + \cos(\pi/k)e_2$. Let γ_j^+, $j = 2, \dots, n$ and γ_u^+ be the oriented half geodesics issuing from A_0 and tangent to $e_2, \dots, e_n$ and to u, respectively. Now we choose an interior point A_1 on γ_u^+ and an interior point A_j on γ_j^+, $j = 2, \dots, n$. Therefore, $A_0, A_1, \dots, A_n$ are independent points of $\mathbb{H}^n$. Let $\widetilde{\mathcal{P}}$ be the polyhedron determined by these points. The faces are denoted by $F_0, \dots, F_n$, with the convention that the face F_j does not contain the vertex A_j, $j = 0, \dots, n$.

Let Π_i the totally geodesic hyperplane containing the face F_i. Observe that:

(1) F_1 and F_2 make an interior angle equal to π/k.
(2) $F_j \perp F_1$, $F_j \perp F_2, j = 3, \dots, n$.
(3) $F_j \perp F_k, j, k = 3, \dots, n\,(j \neq k)$.

Therefore, the reflections in $\mathbb{H}^n$ with respect to the geodesic hyperplanes Π_1 and Π_2 leave the other geodesic hyperplanes $\Pi_j, j = 3, \ldots, n$ globally invariant. The first step of the construction of the polyhedron $\mathcal{P}_k$ is the following: Doing reflection about F_2 we obtain another polyhedron with faces F_1^* (the symmetric of F_1 about F_2), and faces $\widetilde{F_j}$ containing F_j, $\widetilde{F_j} \subset \Pi_j$, $j = 3, \ldots, n$. Notice that in the process the face F_2 disappears and the interior angle between the faces F_1 and F_1^* is $2\pi/k$. Furthermore, the reflection of F_0 about F_2 generates another face F_0^1.

Continuing this process doing reflections with respect to F_1^* and so on, we obtain a new polyhedron $\mathcal{P}^+$ with faces $\widehat{F_j} \subset \Pi_j$, $j = 3, \ldots, n$, $\widehat{F_j}$ containing $\widetilde{F}_j$, and $2k$ faces issuing from the successive reflections of F_0. Notice that both faces F_1 and F_2 disappear at the end of the process, that is $\mathcal{P}^+$ does not contain any face in the hyperplane Π_1 or Π_2.

Next, let us perform the reflections about Π_3. Doing this the face F_3 disappears and we get a new polyhedron with $2 \cdot 2k$ faces issuing from F_0 and a face in each Π_j, $j = 4, \ldots, n$, by Property (3). Each such face contains $\widehat{F_j}$, $j = 4, \ldots, n$. Continuing this process doing reflections on $\Pi_4, \ldots, \Pi_n$ we finally get a polyhedron $\mathcal{P}_k$ with $2^{n-1} \cdot k$ faces, each one issuing from F_0.

Now we discuss the convexity of $\mathcal{P}_k$. Let $P \subset \mathbb{H}^n$ be the geodesic two-plane containing the points A_0, A_1 and A_2. Let $\Gamma \subset P$ be the geodesic polygon obtained by the reflection of the segment $[A_0, A_1]$ with respect to $[A_0, A_2]$ and so on. Thus Γ is a polygon with $2k$ sides and $2k$ vertices, among them A_1 and A_2, and A_0 is an interior point of Γ. Then, the polyhedron $\mathcal{P}_k$ is convex if, and only if, the polygon Γ is convex too. For example, if $d(A_0, A_1) = d(A_0, A_2)$ we get that Γ is a regular polygon and then is convex. On the other hand, if $d(A_0, A_1)$ is much bigger than $d(A_0, A_2)$ then Γ is non convex.

Now, considering the polyhedron $\widetilde{\mathcal{P}}$ of the beginning, with the aid of Theorem 5.3, we are able to solve the Dirichlet problem of the minimal equation taking $+\infty$ value data on F_0 and zero value data on $F_j \setminus F_0$, $j = 1, \ldots, n$. Using the reflection principle on the faces, in each step of the preceding process, we obtain at the end of the process a solution of the minimal equation on $\operatorname{int} \mathcal{P}_k$, taking alternatively infinite values $+\infty$ and $-\infty$ on adjacent faces of $\mathcal{P}_k$, as desired. This accomplishes the proof of the theorem. $\square$

The following theorem are consequence of the previous results.

Theorem 5.5 (***Dirichlet problem for the minimal equation in*** $\mathbb{H}^n \times \mathbb{R}$ ***on a*** C^0 ***bounded convex domain taking continuous boundary data***).

Let Ω *be a* C^0 *bounded convex domain and let* $g : \partial\Omega \to \mathbb{R}$ *be a continuous function.*

Then, g *admits a unique continuous extension* $u : \Omega \cup \partial\Omega \to \mathbb{R}$ *satisfying the vertical minimal hypersurface equation* (4) *on* Ω.

Proof. The proof is a consequence of the Perron process (Theorem 4.1) and the construction of barriers at any convex point of a C^0 domain, using rotational Scherk hypersurfaces (Theorem 5.2). Uniqueness follows from the maximum principle. □

Theorem 5.6 (***Dirichlet problem for the minimal equation in*** $\mathbb{H}^n \times \mathbb{R}$ ***on a*** C^0 ***convex domain taking continuous finite and asymptotic boundary data***).

Let $\Omega \subset \mathbb{H}^n$ *be a* C^0 *convex domain and let* $g : \partial\Omega \cup \partial_\infty\Omega \to \mathbb{R}$ *be a continuous function.*

Then g *admits a unique continuous extension* $u : \Omega \cup \partial\Omega \cup \partial_\infty\Omega \to \mathbb{R}$ *satisfying the vertical minimal hypersurface equation* (4) *on* Ω.

Proof. Notice that working in the ball model of hyperbolic space, we have that g is a continuous function on a compact set, hence g is bounded. Therefore there exist supersolutions and subsolutions for the Dirichlet problem. The proof is a consequence of the Perron process (Theorem 4.1) and the constructions of barriers, using the rotational Scherk hypersurfaces (Theorem 5.2) at any point of $\partial\Omega$, and using M_1 at any point of $\partial_\infty\Omega$ (Theorem 4.1-(2)). Uniqueness follows from the maximum principle. □

6. Existence of minimal graphs over non convex admissible domains

We will establish some existence of minimal graphs on certain admissible domains and certain asymptotic boundary, in the same way as in [15, Theorem 5.1 and Theorem 5.2]. The proofs are the same as in the two-dimensional situation, using the n-dimensional catenoids and the n-dimensional translation hypersurfaces M_d obtained for $n \geqslant 3$ in [3]. Therefore we will just state the related definitions and the theorems without proof.

Definition 6.1 (***Admissible unbounded domains in*** $\mathbb{H}^n$). Let $\Omega \subset \mathbb{H}^n$ be an unbounded domain. We say that Ω is an *admissible domain* if each connected component C_0 of $\partial\Omega$ satisfies the *Exterior sphere of* (*uniform*) *radius* ρ *condition*, that is, at any point $p \in C_0$

there exists a sphere S_ρ of radius ρ such that $p \in C_0 \cap S_\rho$ and $\overline{\operatorname{int} S_\rho} \cap \Omega = \emptyset$.

If Ω is an unbounded admissible domain then we denote by ρ_Ω the supremum of the set of these ρ.

Let us write down a formula obtained in [3] that is useful in the sequel. Let $t = \lambda(a, \rho)$, $\rho \geqslant a$, be the height function of the upper half-catenoid in $\mathbb{H}^n \times \mathbb{R}$. Then as ρ goes to infinity $\lambda(a, \rho)$ goes to $R(a)$ where $R(a)$ is given by

$$R(a) := \sinh(a) \int_1^\infty \left(\sinh^2(a)s^2 + 1\right)^{-1/2} \left(s^{2n-2} - 1\right)^{-1/2} ds.$$

Furthermore, the function R increases from 0 to $\pi/(2n-2)$ when a increases from 0 to ∞. This means that the catenoids in the family have finite height bounded from above by $\pi/(n-1)$ ([3, Proposition 3.2]). We set $f(\rho) := R(\rho)$.

Theorem 6.1. *Let $\Omega \subset \mathbb{H}^n$ be an admissible unbounded domain. Let $g : \partial\Omega \cup \partial_\infty\Omega \to \mathbb{R}$ be a continuous function taking zero boundary value data on $\partial\Omega$. Let $\Gamma_\infty \subset \partial_\infty\mathbb{H}^n \times \mathbb{R}$ be the graph of g restricted to $\partial_\infty\Omega$. If the height function t of Γ_∞ satisfies $-f(\rho_\Omega) \leqslant t \leqslant f(\rho_\Omega)$, then there exists a vertical minimal graph over Ω with finite boundary $\partial\Omega$ and asymptotic boundary Γ_∞.*

Furthermore, there is no such minimal graph, if $\partial\Omega$ is compact and the height function t of Γ_∞ satisfies $|t| > \pi/(2n-2)$.

Definition 6.2 (***E-admissible unbounded domains in*** $\mathbb{H}^n$).

Let Ω be an unbounded domain in $\mathbb{H}^n$ and let $\partial\Omega$ be its boundary. We say that Ω is an *E-admissible domain* if there exists $r > 0$ such that each point of $\partial\Omega$ satisfies the *exterior equidistant hypersurface of (uniform) mean curvature* $\tanh r$ *condition*; that is, at any point $p \in \partial\Omega$ there exists an equidistant hypersurface E_r of a geodesic hyperplane, of mean curvature $\tanh r$ (with respect to the exterior unit normal to Ω at p), with $p \in \partial\Omega \cap E_r$ and $E_r \cap \Omega = \emptyset$.

If Ω is an unbounded E-admissible domain then we denote by $r_\Omega \geqslant 0$ the infimum of the set of these r. If Ω is a convex E-admissible domain then $r_\Omega = 0$.

Thus every E-admissible domain is an admissible domain.

If Ω is a convex domain then Ω is an E-admissible domain.

If each connected component C_0 of $\partial\Omega$ is an equidistant hypersurface then Ω is an E-admissible (maybe non convex) domain.

Let us write down again some formulas extracted from [3]. Up to a vertical translation, the height $t = \mu_+(a,\rho)$ of the translation hypersurface M_d, $d > 1$, is given by

$$\mu_+(a,\rho) = \cosh(a) \int_1^{\cosh(\rho)/\cosh(a)} (s^{2n-2} - 1)^{-1/2} \, (\cosh^2(a)s^2 - 1)^{-1/2} \, ds.$$

These integrals converge at $s = 1$ and when $\rho \to +\infty$, with limit value

$$T(a) := \cosh(a) \int_1^{\infty} (s^{2n-2} - 1)^{-1/2} \, (\cosh^2(a)s^2 - 1)^{-1/2} \, ds.$$

T is a decreasing function of a, which tends to infinity when a tends to zero (when $d > 1$ tends to 1) and to $\pi/(2n-2)$ when a (or d) tends to infinity ([3, Equations 3.55, 3.56, 3.57]).

We set $H(r) := T(r)$.

Theorem 6.2. *Let $\Omega \subset \mathbb{H}^n$ be an E-admissible unbounded domain. Let $g : \partial\Omega \cup \partial_\infty\Omega \to \mathbb{R}$ be a continuous function taking zero boundary value data on $\partial\Omega$. Let $\Gamma_\infty \subset \partial_\infty\mathbb{H}^n \times \mathbb{R}$ be the graph of g restricted to $\partial_\infty\Omega$. If the height function t of Γ_∞ satisfies $-H(r_\Omega) \leqslant t \leqslant H(r_\Omega)$, then there exists a vertical minimal graph over Ω with finite boundary $\partial\Omega$ and asymptotic boundary Γ_∞.*

7. Minimal graphs in $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$.

We will write-down in this section some natural extensions of the previous constructions to obtain minimal graphs in the $n+1$- Euclidean space. The proof of the related results for minimal graphs in $\mathbb{R}^{n+1}$ are *mutatis mutandis* the same as in $\mathbb{H}^n \times \mathbb{R}$, but simpler. So we will just summarize them.

The dictionary to perform the understanding of the structure of the proofs is as follows: The hypersurface corresponding to the family M_d ($d < 1$) to provide barriers at a strictly convex point for minimal solutions when the ambient space is $\mathbb{H}^n \times \mathbb{R}$ is the family of hyperplanes in $\mathbb{R}^{n+1}$. The hypersurface corresponding to M_1 to get height estimates at a compact set in the domain Ω is now the family of n-dimensional catenoids.

The reflection principle for minimal graphs in Euclidean space can be proved in the same way as in Lemma 3.1. Finally we note that the Perron process is classical in Euclidean space.

We now consider *special rotational domain* in $\mathbb{R}^n$. The definition is analogous to Definition 5.1. Now the curve γ is a straight line and we choose a smooth curve $\alpha \subset P$ joining A_0 and A_1 such that the hypersurface Σ generated by rotating α with respect to γ has the following properties.

(1) Σ is smooth except possibly at point A_0.
(2) Σ is strictly convex.
(3) $\mathrm{int}(\Sigma) \setminus \{A_0\}$ is transversal to the parallel lines to γ.

We recall the minimal equation in $\mathbb{R}^{n+1}$:

$$\operatorname{div}\left(\frac{\nabla u}{W(u)}\right) := \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{u_{x_i}}{\sqrt{1+\|\nabla u\|^2_{\mathbb{R}^n}}}\right) = 0$$

(just make $\lambda = 1$ and $H = 0$ in Equation (3)). Explicitly, we have that the minimal equation in $\mathbb{R}^{n+1}$ is given by

$$\sum_{i=1}^{n}\left(1 + (u^2_{x_1} + \cdots + \widehat{u^2_{x_i}} + \cdots + u^2_{x_n})\right) u_{x_i x_i} - 2\sum_{i<k} u_{x_i} u_{x_k} u_{x_i x_k} = 0$$

THEOREM 7.1 (***Rotational Scherk hypersurface***). *Let $\mathcal{D}_\Sigma \subset \mathbb{R}^n$ be a special rotational domain. There is a unique solution v of the vertical minimal equation in $\mathrm{int}(\mathcal{D}_\Sigma)$ which extends continuously to $\mathrm{int}(\Sigma)$, taking prescribed zero boundary value and taking prescribed boundary value ∞ for any approach to U_Σ.*

More precisely, the following Dirichlet problem admits a unique solution v.

$$\begin{cases} \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{u_{x_i}}{\sqrt{1+\|\nabla u\|^2_{\mathbb{R}^n}}}\right) = 0 \text{ on } \mathrm{int}(\mathcal{D}_\Sigma), \\ u = 0 \text{ on } \mathrm{int}(\Sigma), \\ u = +\infty \text{ on } U_\Sigma, \\ u \in C^2\left(\mathrm{int}(\mathcal{D}_\Sigma)\right) \cap C^0\left(\mathcal{D}_\Sigma \setminus \overline{U}_\Sigma\right). \end{cases}$$

We call the graph of v in $\mathbb{R}^{n+1}$ a rotational Scherk hypersurface*.*

Proof. We first solve the auxiliary Dirichlet problem (P_t) taking zero boundary value data on the interior of Σ and prescribed boundary value t on U_Σ, in the same way as in the Proposition 5.1. On account that the family of n-dimensional catenoids provides an upper and lower barrier to a solution over any compact set of $\mathrm{int}(\mathcal{D}_\Sigma)$, letting $t \to \infty$ we get the desired solution.

Uniqueness is shown in the same way as the proof of monotonicity in Proposition 5.1. $\square$

We observe that this result was also obtained by A. Coutant [5] using a different approach.

THEOREM 7.2 (***Barrier at a*** C^0 ***convex point***). *Let* $\Omega \subset \mathbb{R}^n$ *be a domain and let* $p_0 \in \partial\Omega$ *be a boundary point where* Ω *is* C^0 *convex. Then for any bounded data* $g : \partial\Omega \to \mathbb{R}$ *continuous at* p_0 *the family of rotational Scherk hypersurfaces provides a barrier at* p_0*.*

Proof. The proof is the same, but simpler, as the proof of Theorem 5.2. More precisely the proofs of the analogous of Claim 1 and 2 are simpler, passing first by the solution v_t of the related auxiliary Dirichlet problem (P_t). □

COROLLARY 7.1 (***Rotational Scherk hypersurface***). *Let* $\mathcal{D}_\Sigma \subset \mathbb{R}^n$ *be a special rotational domain generated by a segment* α *of a straight line. Then:*

1. *There is a unique solution* v *of the vertical minimal equation in* $\mathrm{int}(\mathcal{D}_\Sigma)$ *which extends continuously to* $\mathrm{int}(\Sigma) \cup U_\Sigma$*, taking prescribed zero boundary value data on the interior of* Σ *and prescribed boundary value* ∞ *on* U_Σ*.*

   We also call the graph of v *in* $\mathbb{R}^{n+1}$ *a* rotational Scherk hypersurface.
2. *Let* $\Omega \subset \mathbb{R}^n$ *be a domain and let* $p_0 \in \partial\Omega$ *be a boundary point where* Ω *is* C^0 *convex. Then for any bounded data* $g : \partial\Omega \to \mathbb{R}$ *continuous at* p_0 *the family of rotational Scherk hypersurfaces given in the first statement provides a barrier at* p_0*.*

We define the notion of *admissible polyhedron* in $\mathbb{R}^n$ in the same way as in hyperbolic space, see Definition 5.2. The following result is proved in the same way as in Theorem 5.3.

THEOREM 7.3 (***First Scherk type hypersurface in*** $\mathbb{R}^{n+1}$). *Let* $\mathcal{P}$ *be an admissible convex polyhedron in* $\mathbb{R}^n$*. There is a unique solution* v_∞ *of the vertical minimal equation in* $\mathrm{int}(\mathcal{P})$ *extending continuously to* $\partial\mathcal{P} \setminus F_0$*, taking prescribed zero boundary value data on* $F_1 \setminus \partial F_0, \dots, F_n \setminus \partial F_0$ *and prescribed boundary value* $+\infty$ *for any approach to* $\mathrm{int}(F_0)$*. More precisely, we prove existence and uniqueness of the following Dirichlet problem* (P_∞)*:*

$$(P_\infty)\begin{cases} \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{u_{x_i}}{\sqrt{1+\|\nabla u\|^2_{\mathbb{R}^n}}}\right) = 0 \text{ on } \mathrm{int}(\mathcal{P}), \\ u = 0 \text{ on } F_j \setminus \partial F_0,\ j = 1, \dots, n, \\ u = +\infty \text{ on } \mathrm{int}(F_0), \\ u \in C^2\left(\mathrm{int}(\mathcal{P})\right) \cap C^0\left(\mathcal{P} \setminus F_0\right). \end{cases}$$

We remark that the above result is also obtained by A. Coutant [5].

Next theorem can be proved exactly as in Theorem 5.4.

THEOREM 7.4 (***Second Scherk type hypersurface in*** $\mathbb{R}^{n+1}$). *For any $k \in \mathbb{N}$, $k \geqslant 2$, there exists a family of polyhedron $\mathcal{P}_k$ with $2^{n-1}k$ faces and a solution w_k of the vertical minimal equation in* $\operatorname{int}\mathcal{P}_k$ *taking alternatively infinite values $+\infty$ and $-\infty$ on adjacent faces of $\mathcal{P}_k$. Moreover, the polyhedron $\mathcal{P}_k$ can be chosen to be convex and can also be chosen to be non convex.*

REMARK 7.1. When the ambient space is $\mathbb{R}^4$ with the aid of Theorem 7.4 we have a solution of the minimal equation in the interior of an octahedron in $\mathbb{R}^3$ taking alternatively infinite values $+\infty$ and $-\infty$ on adjacent faces. Indeed, using the notations of the proof of Theorem 5.4, we set $k = 2$ and we choose A_1, A_2 and A_3 so that $d(A_1, A_2) = d(A_1, A_3) = d(A_2, A_3)$. Thus the polyhedron $\mathcal{P}_2$ obtained is an octahedron.

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