Minimal graphs in $\mathbb{H}^n \times \mathbb{R}$ and $\mathbb{R}^{n+1}$

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Joint work with Eric Toubiana, Univ. Paris VII

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We consider **vertical graphs** in $\mathbb{H}^n \times \mathbb{R}$. That is a set $G = \{(x, u(x)) \in \mathbb{H}^n \times \mathbb{R}, x \in \Omega\}$, where $\Omega \subset \mathbb{H}^n \times \{0\}$ is a domain. We call $t = u(x)$ the **height function**.
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$$\mathcal{M}(u) := \text{div}_{\mathbb{H}} \left( \frac{\nabla_{\mathbb{H}} u}{W_{\mathbb{H}} u} \right) = 0$$
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We consider vertical graphs in $\mathbb{H}^n \times R$. That is a set $G = \{(x, u(x)) \in \mathbb{H}^n \times R, x \in \Omega\}$, where $\Omega \subset \mathbb{H}^n \times \{0\}$ is a domain. We call $t = u(x)$ the height function. The minimal equation is given by

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Let us consider the upper half-space model of hyperbolic space: $\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$. Then the height function $u$ satisfies the following equation

$$\sum_{i=1}^{n} \left( 1 + x_n^2 \left( u_{x_1}^2 + \cdots + \widehat{u_{x_i}^2} + \cdots + u_{x_n}^2 \right) \right) u_{x_i x_i}$$

$$+ \frac{(2 - n) \left( 1 + x_n^2 \left( u_{x_1}^2 + \cdots + u_{x_n}^2 \right) \right)}{x_n} u_{x_n} - 2 x_n^2 \sum_{i < k} u_{x_i} u_{x_k} u_{x_i x_k}$$

$$- x_n u_{x_n} \left( u_{x_1}^2 + \cdots + u_{x_n}^2 \right) = 0 \quad \text{(Minimal equation)}$$
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$$+ \frac{(2 - n)(1 + x_n^2 (u_{x_1}^2 + \cdots + u_{x_n}^2))}{x_n} u_x - 2x_n^2 \sum_{i<k} u_x u_{x_k} u_{x_i x_k}$$

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We wish to provide geometric barriers for the Dirichlet problem.
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  - $\mathcal{M}_d$, $d < 1$. If $d < 1$ then $\mathcal{M}_d$ is an entire (stable) vertical graph with finite vertical height.
    
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  - $\mathcal{M}_d$, $d > 1$. $\mathcal{M}_d$ consists of the union of two symmetric vertical graphs over the exterior of an equidistant hypersurface in the slice $\mathbb{H}^n \times \{0\}$. Furthermore $\mathcal{M}_d$ is a horizontal graph, and hence it is stable.
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  - $\mathcal{M}_d$, $d > 1$. $\mathcal{M}_d$ consists of the union of two symmetric vertical graphs over the exterior of an equidistant hypersurface in the slice $\mathbb{H}^n \times \{0\}$. Furthermore $\mathcal{M}_d$ is a horizontal graph, and hence it is stable.
    The asymptotic boundary of $\mathcal{M}_d$ is topologically an $(n - 1)$-sphere that is homologically trivial in $\partial_\infty \mathbb{H}^n \times \mathbb{R}$. More precisely, the asymptotic boundary consists of the union of two copies of an hemisphere $S_{+}^{n-1} \times \{0\}$ of $\partial_\infty \mathbb{H}^n \times \{0\}$ in parallel slices $t = \pm T(a)$, glued with the finite cylinder $\partial S_{+}^{n-1} \times [-T(a), T(a)]$, where $d =: \cosh^{n-1}(a)$. 
The vertical height of $M_d$, $d > 1$ is $2T(a)$. The height of the family $\mathcal{M}_d$ is a decreasing function in $d$ and varies from infinity (when $d \to 1$) to $\pi/(n-1)$ (when $d \to \infty$). It is therefore bounded from below by $\pi/(n-1)$, the upper bound of the heights of the family of catenoids.
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$\mathcal{M}_1$ consists of a complete (stable, non-entire) vertical graph over a half-space in $\mathbb{H}^n \times \{0\}$, bounded by a totally geodesic hyperplane $P$. It takes infinite boundary value data on $P$ and constant asymptotic boundary value data.
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Our Problem:

Let $g : \partial \Omega \cup \partial_{\infty} \Omega \to \mathbb{R}$ be a bounded function. We consider the **Dirichlet problem**, say problem $(P)$, for the vertical minimal hypersurface equation taking at any point of $\partial \Omega \cup \partial_{\infty} \Omega$ prescribed boundary (finite and asymptotic) value data $g$. More precisely,
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$$
\begin{cases}
    u \in C^2(\Omega) \text{ and } \mathcal{M}(u) = 0 \text{ in } \Omega, \\
    \text{for any } p \in \partial \Omega \cup \partial_\infty \Omega \text{ where } g \text{ is continuous, } u \text{ extends continuously at } p \text{ setting } u(p) = g(p).
\end{cases}
$$
Now, let $u : \Omega \cup \partial \Omega \to \mathbb{R}$ be a continuous function. Let $U \subset \Omega$ be a closed round ball in $\mathbb{H}^n$. We then define the continuous function $M_U(u)$ on $\Omega \cup \partial \Omega$ by:

$$M_U(u)(x) = \begin{cases} 
    u(x) & \text{if } x \in \Omega \cup \partial \Omega \setminus U \\
    \tilde{u}(x) & \text{if } x \in U 
\end{cases} \quad (0.1)$$

where $\tilde{u}$ is the minimal extension of $u|_{\partial U}$ on $\overline{U}$.
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We say that $u$ is a subsolution (resp. supersolution) of $(P)$ if:

i) For any closed round ball $U \subset \Omega$ we have $u \leq M_U(u)$ (resp. $u \geq M_U(u)$).

ii) $u|_{\partial \Omega} \leq g$ (resp. $u|_{\partial \Omega} \geq g$).

iii) We have $\limsup_{q \to p} u(q) \leq g(p)$ (resp. $\liminf_{q \to p} u(q) \geq g(p)$) for any $p \in \partial_{\infty} \Omega$. 


Theorem (Perron process)

Let $\Omega \subset H^n$ be a domain and let $g : \partial \Omega \cup \partial_{\infty} \Omega \to \mathbb{R}$ be a bounded function. Let $\phi$ be a bounded supersolution of the Dirichlet problem $(P)$, for example the constant function $\phi \equiv \sup g$.

Set $S_\phi = \{ \varphi, \text{ subsolution of } (P), \varphi \leq \phi \}$. We define for each $x \in \Omega$

$$u(x) = \sup_{\varphi \in S_\phi} \varphi(x).$$

(Observe that $S_\phi \neq \emptyset$ since the constant function $\varphi \equiv \inf g$ belongs to $S_\phi$.)

We have the following:
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We have the following:

1. The function $u$ is $C^2$ on $\Omega$ and satisfies the vertical minimal
equation.
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1. Let $p \in \partial_\infty \Omega$ be an **asymptotic boundary point** where $g$ is continuous. Then $p$ admits a **barrier** and therefore $u$ extends continuously at $p$ setting $u(p) = g(p)$; that is, if $(q_m)$ is a sequence in $\mathbb{H}^n$ such that $q_m \to p$, then $u(q_m) \to g(p)$.
Minimal graphs in $\mathbb{H}^n \times \mathbb{R}$ and $\mathbb{R}^{n+1}$

1. Let $p \in \partial_\infty \Omega$ be an asymptotic boundary point where $g$ is continuous. Then $p$ admits a barrier and therefore $u$ extends continuously at $p$ setting $u(p) = g(p)$; that is, if $(q_m)$ is a sequence in $\mathbb{H}^n$ such that $q_m \to p$, then $u(q_m) \to g(p)$. In particular, if $g$ is continuous on $\partial_\infty \Omega$ then the asymptotic boundary of the graph of $u$ is the restriction of the graph of $g$ to $\partial_\infty \Omega$.

2. If $\partial \Omega$ is $C^0$ strictly convex at $p$ then $u$ extends continuously at $p$ setting $u(p) = g(p)$. 
Scherk type minimal hypersurfaces in $H^n \times \mathbb{R}$

**Definition (Special rotational domain)**

Let $\gamma, L \subset H^n$ be two complete geodesic lines with $L$ orthogonal to $\gamma$ at some point $B \in \gamma \cap L$. Using the half-space model for $H^n$, we can assume that $\gamma$ is the vertical geodesic such that $\partial_{\infty} \gamma = \{0, \infty\}$. We call $P \subset H^n$ the geodesic two-plane containing $L$ and $\gamma$. We choose $A_0 \in (0, B) \subset \gamma$ and $A_1 \in L \setminus \gamma$ and we denote by $\alpha \subset P$ the euclidean segment joining $A_0$ and $A_1$. Therefore the hypersurface $\Sigma$ generated by rotating $\alpha$ with respect to $\gamma$ has the following properties.

1. $\text{int}(\Sigma)$ is smooth except at point $A_0$.

2. $\Sigma$ is strictly convex in hyperbolic meaning and convex in euclidean meaning.

3. $\text{int}(\Sigma) \setminus \{A_0\}$ is transversal to the Killing field generated by the translations along $\gamma$. 
Consequently $\Sigma$ lies in the mean convex side of the domain of $\mathbb{H}^n$ whose boundary is the hyperbolic cylinder with axis $\gamma$ and passing through $A_1$. Let us call $\Pi \subset \mathbb{H}^n$ the geodesic hyperplane orthogonal to $\gamma$ and passing through $B$. Observe that the boundary of $\Sigma$ is a $n - 2$ dimensional geodesic sphere of $\Pi$ centered at $B$. 
Consequently $\Sigma$ lies in the mean convex side of the domain of $\mathbb{H}^n$ whose boundary is the hyperbolic cylinder with axis $\gamma$ and passing through $A_1$. Let us call $\Pi \subset \mathbb{H}^n$ the geodesic hyperplane orthogonal to $\gamma$ and passing through $B$. Observe that the boundary of $\Sigma$ is a $n-2$ dimensional geodesic sphere of $\Pi$ centered at $B$.

We denote by $U_\Sigma \subset \Pi$ the open geodesic ball centered at $B$ whose boundary is the boundary of $\Sigma$. We call $M_\Sigma \subset \mathbb{H}^n$ the closed domain whose boundary is $U_\Sigma \cup \Sigma$. Observe that $\partial M_\Sigma$ is strictly convex at any point of $\Sigma$ and convex at any point of $U_\Sigma$. Such a domain will be called a special rotational domain.
Theorem (Rotational Scherk hypersurface)

Let $\mathcal{M}_\Sigma \subset \mathbb{H}^n$ be a special rotational domain. There is a unique solution $v$ of the vertical minimal equation in $\text{int}(\mathcal{M}_\Sigma)$ which extends continuously to $\text{int}(\Sigma)$, taking prescribed zero boundary value data and taking boundary value $\infty$ for any approach to $U_\Sigma$. 
Theorem (Rotational Scherk hypersurface)

Let $\mathcal{M}_\Sigma \subset \mathbb{H}^n$ be a special rotational domain. There is a unique solution $v$ of the vertical minimal equation in $\text{int}(\mathcal{M}_\Sigma)$ which extends continuously to $\text{int}(\Sigma)$, taking prescribed zero boundary value data and taking boundary value $\infty$ for any approach to $U_\Sigma$. More precisely, the following Dirichlet problem $(P)$ admits a unique solution $v_\infty$.

$$
\begin{aligned}
(P) & \begin{cases}
\mathcal{M}(u) = 0 \text{ in } \text{int}(\mathcal{M}_\Sigma), \\
u = 0 \text{ on } \text{int}(\Sigma), \\
u = +\infty \text{ on } U_\Sigma, \\
u \in C^2(\text{int}(\mathcal{M}_\Sigma)) \cap C^0(\mathcal{M}_\Sigma \setminus \overline{U_\Sigma}).
\end{cases}
\end{aligned}
$$

We call the graph of $v$ in $\mathbb{H}^n \times \mathbb{R}$ a rotational Scherk hypersurface.
Proof
First, we will prove the existence part of the Theorem. We consider the family of functions $\nu_t$, $t > 0$. Recall that $\Pi \subset \mathbb{H}^n$ is the totally geodesic hyperplane containing $U_\Sigma$. We consider a suitable copy of $M_1$ as barrier as follows: choose $M_1$ such that $M_1$ is a graph of a function $u_1$ whose domain is the component of $\mathbb{H}^n \setminus \Pi$ that contains $M_\Sigma$, with $u_1$ taking boundary value data $+\infty$ on $\Pi$ and taking zero asymptotic boundary value data. By applying maximum principle we have that $u_1(p) > \nu_t(p)$ for all $p \in M_\Sigma$ and all $t > 0$. 
Proof

First, we will prove the existence part of the Theorem. We consider the family of functions $v_t$, $t > 0$. Recall that $\Pi \subset \mathbb{H}^n$ is the totally geodesic hyperplane containing $U_\Sigma$. We consider a suitable copy of $M_1$ as barrier as follows: choose $M_1$ such that $M_1$ is a graph of a function $u_1$ whose domain is the component of $\mathbb{H}^n \setminus \Pi$ that contains $M_\Sigma$, with $u_1$ taking boundary value data $+\infty$ on $\Pi$ and taking zero asymptotic boundary value data. By applying maximum principle we have that $u_1(p) > v_t(p)$ for all $p \in M_\Sigma$ and all $t > 0$.

Using compactness principle we obtain that a subsequence of the family converges uniformly on any compact subsets of int($M_\Sigma$) to a solution $v_\infty$ of the minimal equation. Since the family is strictly increasing $v_\infty$ takes the value $+\infty$ on $U_\Sigma$. That is, for any sequence $(q_k)$ in int($M_\Sigma$) converging to some point of $U_\Sigma$ we have $v_\infty(q_k) \to +\infty$. 


Let $p \in \text{int}(\Sigma)$, since $\partial M_\Sigma$ is $C^0$ strictly convex at $p$, the hypersurfaces $M_d, d < 1$, provide a barrier at $p$. Consequently $\nu_\infty$ extends continuously at $p$ setting $\nu_\infty(p) = 0$. Therefore $\nu_\infty$ is a solution of the Dirichlet problem $(P)$. 
Let $p \in \text{int}(\Sigma)$, since $\partial M_\Sigma$ is $C^0$ strictly convex at $p$, the hypersurfaces $M_d$, $d < 1$, provide a barrier at $p$. Consequently $\nu_\infty$ extends continuously at $p$ setting $\nu_\infty(p) = 0$. Therefore $\nu_\infty$ is a solution of the Dirichlet problem ($P$).

The proof of uniqueness of $\nu_\infty$ proceeds in the same way as the proof of the monotonicity of the family $\{\nu_t\}$. This completes the proof of the Theorem.
Theorem (Barrier at a $C^0$ convex point)

Let $\Omega \subset \mathbb{H}^n$ be a domain and let $p_0 \in \partial \Omega$ be a boundary point where $\Omega$ is $C^0$ convex. Then for any bounded data $g : \partial \Omega \cup \partial_\infty \Omega \rightarrow \mathbb{R}$ continuous at $p_0$, the family of rotational Scherk hypersurfaces provides a barrier at $p_0$ for the Dirichlet problem $(P)$. In particular, in Theorem 1-(2) the assumption $C^0$ strictly convex can be replaced by $C^0$ convex.
Theorem (Barrier at a $C^0$ convex point)

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Proof. We will prove that the rotational Scherk hypersurfaces with $-\infty$ boundary data on the boundary part $U_{\Sigma}$ provide an upper barrier at $p_0$. For the lower barrier the construction is similar.
Let $\mathcal{M}_\Sigma$ be a special rotational domain. Let $\omega$ be the height function of the rotational Scherk hypersurface $S$ taking $-\infty$ boundary data on $U_\Sigma$ and $0$ boundary data on the interior of $\Sigma$. 

Claim 1. $\omega$ is decreasing along the oriented geodesic segment $[A_0, B] \subset \gamma$ (going from $A_0$ to $B$).

Claim 2. Let $D$ be any point on the open geodesic segment $(A_0, B)$, and let $\beta \subset \mathcal{M}_\Sigma$ be a geodesic segment issuing from $D$, ending at some point $C \in \text{int}(\Sigma)$ and orthogonal to $[A_0, B]$ at $D$.

Then $\omega$ is increasing along $\beta = [D, C]$, oriented from $D$ to $C$.

We first prove the theorem assuming that the two claims hold.
Let $\mathcal{M}_\Sigma$ be a special rotational domain. Let $\omega$ be the height function of the rotational Scherk hypersurface $S$ taking $-\infty$ boundary data on $U_\Sigma$ and 0 boundary data on the interior of $\Sigma$.

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We first prove the theorem assuming that the two claims hold.
Let $D \in (A_0, B)$ and let $\Pi_D \subset \mathbb{H}^n$ be the geodesic hyperplane through $D$ orthogonal to the geodesic segment $[A_0, B]$. Let $\mathcal{M}_\Sigma^\pm$ be the connected component of $\mathcal{M}_\Sigma \setminus \Pi_D$ containing the point $A_0$. Let $q$ be any point belonging to the closure of $\mathcal{M}_\Sigma^\pm$. The claims ensure that $\omega(q) \geq \omega(D)$. 
Let $D \in (A_0, B)$ and let $\Pi_D \subset \mathbb{H}^n$ be the geodesic hyperplane through $D$ orthogonal to the geodesic segment $[A_0, B]$. Let $\mathcal{M}_\Sigma^+$ be the connected component of $\mathcal{M}_\Sigma \setminus \Pi_D$ containing the point $A_0$. Let $q$ be any point belonging to the closure of $\mathcal{M}_\Sigma^+$. The claims ensure that $\omega(q) \geq \omega(D)$.

Let $p_0 \in \partial \Omega$ be a $C^0$ convex point and let $g$ be a bounded data continuous at $p_0$. Let $M > 0$ be any positive real number. It suffices to show that for any $k \in \mathbb{N}^*$ there is an open neighborhood $N_k$ of $p_0$ in $\mathbb{H}^n$ and a function $\omega_k^+$ in $C^2(N_k \cap \Omega) \cap C^0(\overline{N_k \cap \Omega})$ such that

i) $\omega_k^+(x)|_{\partial \Omega \cap N_k} \geq g(x)$ and $\omega_k^+(x)|_{\partial N_k \cap \Omega} \geq M$,

ii) $\mathcal{M}(\omega_k^+) = 0$ in $N_k \cap \Omega$,

iii) $\omega_k^+(p_0) = g(p_0) + 1/k$. 
By continuity there exists $\varepsilon > 0$ such that for any $p \in \partial \Omega$ with $\text{dist}(p, p_0) < \varepsilon$ we have $g(p) < g(p_0) + 1/k$. 
By continuity there exists $\varepsilon > 0$ such that for any $p \in \partial \Omega$ with $\text{dist}(p, p_0) < \varepsilon$ we have $g(p) < g(p_0) + 1/k$.

By assumption there exist a geodesic hyperplane $\Pi_{p_0}$ through $p_0$ and an open neighborhood $\mathcal{W} \subset \Pi_{p_0}$ of $p_0$ such that $\mathcal{W} \cap \Omega = \emptyset$. We set $\Omega_{\varepsilon} = \{p \in \Omega \mid \text{dist}(p_0, p) < \varepsilon\}$. Up to choosing $\varepsilon$ small enough, we can assume that $\Omega_{\varepsilon}$ is entirely contained in a component of $\mathbb{H}^n \setminus \Pi_{p_0}$. Let $\gamma$ be the geodesic through $p_0$ orthogonal to $\Pi_{p_0}$.  


By continuity there exists \( \varepsilon > 0 \) such that for any \( p \in \partial \Omega \) with \( \text{dist}(p, p_0) < \varepsilon \) we have \( g(p) < g(p_0) + 1/k \).

By assumption there exist a geodesic hyperplane \( \Pi_{p_0} \) through \( p_0 \) and an open neighborhood \( W \subset \Pi_{p_0} \) of \( p_0 \) such that \( W \cap \Omega = \emptyset \). We set \( \Omega_\varepsilon = \{ p \in \Omega \mid \text{dist}(p_0, p) < \varepsilon \} \). Up to choosing \( \varepsilon \) small enough, we can assume that \( \Omega_\varepsilon \) is entirely contained in a component of \( \mathbb{H}^n \setminus \Pi_{p_0} \). Let \( \gamma \) be the geodesic through \( p_0 \) orthogonal to \( \Pi_{p_0} \).

We choose a special rotational domain \( \mathcal{M}_\Sigma \) such that:

- the hyperplane \( \Pi \) is orthogonal to \( \gamma \), (recall that \( U_\Sigma \subset \Pi \))
- the diameter of \( \mathcal{M}_\Sigma \) is lesser than \( \varepsilon/4 \),
- \( \overline{\Omega} \cap U_\Sigma = \emptyset \),
- \( A_0 \in \gamma, \text{dist}(p_0, A_0) < \varepsilon/8 \) and \( A_0 \) belongs to the same component of \( \mathbb{H}^n \setminus \Pi_{p_0} \) than \( \Omega_\varepsilon \).
Let $M' > \max\{M, g(p_0) + 1/k\}$. We consider the rotational Scherk hypersurface (graph of $\omega$) taking $M'$ boundary value data on the interior of $\Sigma$ and $-\infty$ on $U_\Sigma$.

By continuity, there exists a point $p_1 \in \gamma$ where $\omega(p_1) = g(p_0) + 1/k$. Up to a horizontal translation along $\gamma$ sending $p_1$ to $p_0$, we may assume that $\omega(p_0) = g(p_0) + 1/k$.

Then we set $N_k = \text{int}(M_\Sigma) \cap \Omega$ and $\omega_k^+ = \omega|_{N_k}$, the restriction of $\omega$ to $N_k$. Therefore we have $\omega_k^+(x)|_{\partial N_k \cap \Omega} = M' \geq M$, furthermore Claim 1 and Claim 2 show that $\omega_k^+(x)|_{\partial \Omega \cap N_k} \geq g(p_0) + 1/k \geq g(x)$, as desired.
Theorem (Dirichlet problem for the minimal equation in $\mathbb{H}^n \times \mathbb{R}$ on a $C^0$ bounded convex domain taking continuous boundary data)

Let $\Omega$ be a $C^0$ bounded convex domain and let $g : \partial \Omega \to \mathbb{R}$ be a continuous function.
Then, $g$ admits a unique continuous extension $u : \Omega \cup \partial \Omega \to \mathbb{R}$ satisfying the vertical minimal hypersurface equation on $\Omega$. 
Theorem (Dirichlet problem for the minimal equation in $\mathbb{H}^n \times \mathbb{R}$ on a $C^0$ bounded convex domain taking continuous boundary data)

Let $\Omega$ be a $C^0$ bounded convex domain and let $g : \partial \Omega \to \mathbb{R}$ be a continuous function. Then, $g$ admits a unique continuous extension $u : \Omega \cup \partial \Omega \to \mathbb{R}$ satisfying the vertical minimal hypersurface equation on $\Omega$.

Proof.

The proof is a consequence of the Perron process and the construction of barriers at any convex point of a $C^0$ domain, using rotational Scherk hypersurfaces. Uniqueness follows from the maximum principle. □
Theorem (Dirichlet problem for the minimal equation in $\mathbb{H}^n \times \mathbb{R}$ on a $C^0$ convex domain taking continuous finite and asymptotic boundary data)

Let $\Omega \subset \mathbb{H}^n$ be a $C^0$ convex domain and let $g : \partial\Omega \cup \partial\infty \Omega \to \mathbb{R}$ be a continuous function.
Then $g$ admits a unique continuous extension $u : \Omega \cup \partial\Omega \cup \partial\infty \Omega \to \mathbb{R}$ satisfying the vertical minimal hypersurface equation on $\Omega$. 
Theorem (Dirichlet problem for the minimal equation in $\mathbb{H}^n \times \mathbb{R}$ on a $C^0$ convex domain taking continuous finite and asymptotic boundary data)

Let $\Omega \subset \mathbb{H}^n$ be a $C^0$ convex domain and let $g : \partial \Omega \cup \partial_{\infty} \Omega \rightarrow \mathbb{R}$ be a continuous function. Then $g$ admits a unique continuous extension $u : \Omega \cup \partial \Omega \cup \partial_{\infty} \Omega \rightarrow \mathbb{R}$ satisfying the vertical minimal hypersurface equation on $\Omega$.

Proof.
Notice that working in the ball model of hyperbolic space, we have that $g$ is a continuous function on a compact set, hence $g$ is bounded. Therefore there exist supersolutions and subsolutions for the Dirichlet problem. The proof is a consequence of the Perron process and the constructions of barriers, using the rotational Scherk hypersurfaces at any point of $\partial \Omega$, and using $M_1$ at any point of $\partial_{\infty} \Omega$. Uniqueness follows from the maximum principle.
Definition (Independent points and admissible polyhedra)

1. We say that \( n + 1 \) points \( A_0, \ldots, A_n \) in \( \mathbb{H}^n \) are independent if there is no geodesic hyperplane containing these points. If \( A_0, \ldots, A_n \) in \( \mathbb{H}^n \) are independent then we remark that any choice of \( n \) points among them determines a unique geodesic hyperplane of \( \mathbb{H}^n \).
Definition (Independent points and admissible polyhedra)

1. We say that $n + 1$ points $A_0, \ldots, A_n$ in $\mathbb{H}^n$ are independent if there is no geodesic hyperplane containing these points. If $A_0, \ldots, A_n$ in $\mathbb{H}^n$ are independent then we remark that any choice of $n$ points among them determines a unique geodesic hyperplane of $\mathbb{H}^n$.

2. Let $A_0, \ldots, A_n$ be $n + 1$ independent points in $\mathbb{H}^n$. We call $\Pi_i$ the geodesic hyperplane containing these points excepted $A_i$, $i = 0, \ldots, n$ and we call $\Pi_i^+$ the closed half-space bounded by $\Pi_i$ and containing $A_i$. Then the intersection of these half-spaces is a polyhedron $\mathcal{P}$: the convex closure of $A_0, \ldots, A_n$. The boundary of $\mathcal{P}$ consists of $n + 1$ closed faces $F_i \subset \Pi_i$, the face $F_i$ contains in its boundary all the points $A_0, \ldots, A_n$ excepted $A_i$. We call such a polyhedron an admissible polyhedron.
Theorem (First Scherk type hypersurface in $\mathbb{H}^n \times \mathbb{R}$)

Let $\mathcal{P}$ be an admissible convex polyhedron. There is a unique solution $v_\infty$ of the minimal equation in $\text{int}(\mathcal{P})$ extending continuously up to $\partial \mathcal{P} \setminus F_0$, taking prescribed zero boundary value data on $F_1 \setminus \partial F_0, \ldots, F_n \setminus \partial F_0$ and prescribed boundary value $\infty$ for any approach to $\text{int}(F_0)$. More precisely, we prove existence and uniqueness of the following Dirichlet problem $(P_\infty)$:
Theorem (First Scherk type hypersurface in $\mathbb{H}^n \times \mathbb{R}$)

Let $\mathcal{P}$ be an admissible convex polyhedron. There is a unique solution $v_\infty$ of the minimal equation in $\text{int}(\mathcal{P})$ extending continuously up to $\partial \mathcal{P} \setminus F_0$, taking prescribed zero boundary value data on $F_1 \setminus \partial F_0, \ldots, F_n \setminus \partial F_0$ and prescribed boundary value $\infty$ for any approach to $\text{int}(F_0)$. More precisely, we prove existence and uniqueness of the following Dirichlet problem $(P_\infty)$:

\[
(P_\infty) \begin{cases}
M(u) = 0 \text{ in } \text{int}(\mathcal{P}), \\
u = 0 \text{ on } F_j \setminus \partial F_0, j = 1, \ldots, n, \\
u = \infty \text{ on } \text{int}(F_0), \\
u \in C^2(\text{int}(\mathcal{P})) \cap C^0(\mathcal{P} \setminus F_0).
\end{cases}
\]
Proof.
We may use the rotational Scherk hypersurfaces as barrier. Therefore, we obtain for any $t \in \mathbb{R}$ a solution $v_t$ of the vertical minimal equation in $\text{int}(P)$ which extends continuously to $\partial P \setminus \partial F_0$, taking prescribed zero boundary value data on $\partial P \setminus F_0$ and prescribed boundary value $t$ on $\text{int}(F_0)$. Now letting $t \to \infty$ we have that a subsequence of the family $\{v_t\}$ converges to a solution as desired, taking into account that the rotational Scherk hypersurfaces give a barrier at any point of $P$.

The uniqueness is obtained as in the proof of the monotonicity of the family $\{v_t\}$ in the previous Proposition.
Theorem (Second Scherk type hypersurface in $\mathbb{H}^n \times \mathbb{R}$)

For any $k \in \mathbb{N}$, $k \geq 2$, there exists a family of polyhedron $\mathcal{P}_k$ with $2^{n-1}k$ faces and a solution $w_k$ of the vertical minimal equation in $\text{int} \, \mathcal{P}_k$ taking alternatively infinite values $+\infty$ and $-\infty$ on adjacent faces of $\mathcal{P}_k$. Moreover, the polyhedron $\mathcal{P}_k$ can be chosen to be convex and can also be chosen to be non convex.
Proof. Let us fix a point $A_0$ in $\mathbb{H}^n$. Let \( \{e_1, \ldots, e_n\} \) be a positively oriented orthornormal basis of $T_{A_0} \mathbb{H}^n$. For $k \geq 2$ we set \( u := \sin(\pi/k)e_1 + \cos(\pi/k)e_2 \). Let $\gamma_j^+, j = 2, \ldots, n$ and $\gamma_u^+$ be the oriented half geodesics issuing from $A_0$ and tangent to $e_2, \ldots, e_n$ and to $u$, respectively. Now we choose an interior point $A_1$ on $\gamma_u^+$ and an interior point $A_j$ on $\gamma_j^+$, $j = 2, \ldots, n$. Therefore, $A_0, A_1, \ldots, A_n$ are independent points of $\mathbb{H}^n$. Let $\tilde{P}$ be the polyhedron determined by these points. The faces are denoted by $F_0, \ldots, F_n$, with the convention that the face $F_j$ does not contain the vertex $A_j$, $j = 0, \ldots, n$. Let $\Pi_i$ the totally geodesic hyperplane containing the face $F_i$. Observe that:

1. $F_1$ and $F_2$ make an interior angle equal to $\pi/k$.
2. $F_j \perp F_1, F_j \perp F_2, j = 3, \ldots, n$.
3. $F_j \perp F_k, j, k = 3, \ldots, n (j \neq k)$. 
Therefore, the reflections in $\mathbb{H}^n$ with respect to the geodesic hyperplanes $\Pi_1$ and $\Pi_2$ leave the other geodesic hyperplanes $\Pi_j, j = 3, \ldots, n$ globally invariant. The first step of the construction of the polyhedron $\mathcal{P}_k$ is the following: Doing reflection about $F_2$ we obtain another polyhedron with faces $F_1^*$ (the symmetric of $F_1$ about $F_2$), and faces $\tilde{F}_j$ containing $F_j$, $\tilde{F}_j \subset \Pi_j, j = 3, \ldots, n$. Notice that in the process the face $F_2$ disappears and the interior angle between the faces $F_1$ and $F_1^*$ is $2\pi/k$. Furthermore, the reflection of $F_0$ about $F_2$ generates another face $F_0^1$.

Continuing this process doing reflections with respect to $F_1^*$ and so on, we obtain a new polyhedron $\mathcal{P}^+$ with faces $\hat{F}_j \subset \Pi_j, j = 3, \ldots, n$, $\hat{F}_j$ containing $\tilde{F}_j$, and $2k$ faces issuing from the successive reflections of $F_0$. Notice that both faces $F_1$ and $F_2$ disappear at the end of the process, that is $\mathcal{P}^+$ does not contain any face in the hyperplane $\Pi_1$ or $\Pi_2$. 
Next, let us perform the reflections about $\Pi_3$. Doing this the face $F_3$ disappears and we get a new polyhedron with $2 \cdot 2k$ faces issuing from $F_0$ and a face in each $\Pi_j$, $j = 4, \ldots, n$. Each such face contains $\hat{F}_j$, $j = 4, \ldots, n$. Continuing this process doing reflections on $\Pi_4, \ldots, \Pi_n$ we finally get a polyhedron $\mathcal{P}_k$ with $2^{n-1} \cdot k$ faces, each one issuing from $F_0$. 
Now we discuss the convexity of $\mathcal{P}_k$. Let $P \subset \mathbb{H}^n$ be the geodesic two-plane containing the points $A_0, A_1$ and $A_2$. Let $\Gamma \subset P$ be the geodesic polygon obtained by the reflection of the segment $[A_0, A_1]$ with respect to $[A_0, A_2]$ and so on. Thus $\Gamma$ is a polygon with $2k$ sides and $2k$ vertices, among them $A_1$ and $A_2$, and $A_0$ is an interior point of $\Gamma$. Then, the polyhedron $\mathcal{P}_k$ is convex if, and only if, the polygon $\Gamma$ is convex too. For example, if $d(A_0, A_1) = d(A_0, A_2)$ we get that $\Gamma$ is a regular polygon and then is convex. On the other hand, if $d(A_0, A_1)$ is much bigger than $d(A_0, A_2)$ then $\Gamma$ is non convex.

Now, considering the polyhedron $\overline{\mathcal{P}}_0$ of the beginning, we are able to solve the Dirichlet problem of the minimal equation taking $+\infty$ value data on $F_0$ and zero value data on $F_j \setminus F_0$, $j = 1, \ldots, n$. Using the reflection principle on the faces, in each step of the preceding process, we obtain at the end of the process a solution of the minimal equation on $\text{int} \mathcal{P}_k$, taking alternatively infinite values $+\infty$ and $-\infty$ on adjacent faces of $\mathcal{P}_k$, as desired. This accomplishes the proof of the theorem.
Now we discuss the convexity of $\mathcal{P}_k$. Let $P \subset \mathbb{H}^n$ be the geodesic two-plane containing the points $A_0, A_1$ and $A_2$. Let $\Gamma \subset P$ be the geodesic polygon obtained by the reflection of the segment $[A_0, A_1]$ with respect to $[A_0, A_2]$ and so on. Thus $\Gamma$ is a polygon with $2k$ sides and $2k$ vertices, among them $A_1$ and $A_2$, and $A_0$ is an interior point of $\Gamma$. Then, the polyhedron $\mathcal{P}_k$ is convex if, and only if, the polygon $\Gamma$ is convex too.

For example, if $d(A_0, A_1) = d(A_0, A_2)$ we get that $\Gamma$ is a regular polygon and then is convex. On the other hand, if $d(A_0, A_1)$ is much bigger than $d(A_0, A_2)$ then $\Gamma$ is non convex.

Now, considering the polyhedron $\tilde{\mathcal{P}}$ of the beginning, we are able to solve the Dirichlet problem of the minimal equation taking $+\infty$ value data on $F_0$ and zero value data on $F_j \setminus F_0$, $j = 1, \ldots, n$. Using the reflection principle on the faces, in each step of the preceding process, we obtain at the end of the process a solution of the minimal equation on $\text{int} \, \mathcal{P}_k$, taking alternatively infinite values $+\infty$ and $-\infty$ on adjacent faces of $\mathcal{P}_k$, as desired. This accomplishes the proof of the theorem.
Theorem
Let $\Omega \subset \mathbb{H}^n$ be an admissible unbounded domain. Let $g : \partial \Omega \cup \partial_\infty \Omega \to \mathbb{R}$ be a continuous function taking zero boundary value data on $\partial \Omega$. Let $\Gamma_\infty \subset \partial_\infty \mathbb{H}^n \times \mathbb{R}$ be the graph of $g$ restricted to $\partial_\infty \Omega$. If the height function $t$ of $\Gamma_\infty$ satisfies $-f(\rho_\Omega) \leq t \leq f(\rho_\Omega)$, then there exists a vertical minimal graph over $\Omega$ with finite boundary $\partial \Omega$ and asymptotic boundary $\Gamma_\infty$. 
Theorem

Let $\Omega \subset \mathbb{H}^n$ be an admissible unbounded domain. Let $g : \partial \Omega \cup \partial_\infty \Omega \rightarrow \mathbb{R}$ be a continuous function taking zero boundary value data on $\partial \Omega$. Let $\Gamma_\infty \subset \partial_\infty \mathbb{H}^n \times \mathbb{R}$ be the graph of $g$ restricted to $\partial_\infty \Omega$. If the height function $t$ of $\Gamma_\infty$ satisfies $-f(\rho_\Omega) \leq t \leq f(\rho_\Omega)$, then there exists a vertical minimal graph over $\Omega$ with finite boundary $\partial \Omega$ and asymptotic boundary $\Gamma_\infty$.

Furthermore, there is no such minimal graph, if $\partial \Omega$ is compact and the height function $t$ of $\Gamma_\infty$ satisfies $|t| > \pi/(2n-2)$. 
Theorem

Let $\Omega \subset \mathbb{H}^n$ be an $E$-admissible unbounded domain. Let $g : \partial \Omega \cup \partial_\infty \Omega \to \mathbb{R}$ be a continuous function taking zero boundary value data on $\partial \Omega$. Let $\Gamma_\infty \subset \partial_\infty \mathbb{H}^n \times \mathbb{R}$ be the graph of $g$ restricted to $\partial_\infty \Omega$. If the height function $t$ of $\Gamma_\infty$ satisfies $-H(r_\Omega) \leq t \leq H(r_\Omega)$, then there exists a vertical minimal graph over $\Omega$ with finite boundary $\partial \Omega$ and asymptotic boundary $\Gamma_\infty$. 
Minimal graphs in $\mathbb{R}^{n+1}$

**Theorem (Second Scherk type hypersurface in $\mathbb{R}^{n+1}$)**

For any $k \in \mathbb{N}$, $k \geq 2$, there exists a family of polyhedron $P_k$ with $2^{n-1}k$ faces and a solution $w_k$ of the vertical minimal equation in $\text{int} \ P_k$ taking alternatively infinite values $+\infty$ and $-\infty$ on adjacent faces of $P_k$. Moreover, the polyhedron $P_k$ can be chosen to be convex and can also be chosen to be non convex.
Theorem (Second Scherk type hypersurface in \( \mathbb{R}^{n+1} \))

For any \( k \in \mathbb{N}, k \geq 2 \), there exists a family of polyhedron \( \mathcal{P}_k \) with \( 2^{n-1}k \) faces and a solution \( w_k \) of the vertical minimal equation in \( \text{int} \mathcal{P}_k \) taking alternatively infinite values \( +\infty \) and \( -\infty \) on adjacent faces of \( \mathcal{P}_k \).
Moreover, the polyhedron \( \mathcal{P}_k \) can be chosen to be convex and can also be chosen to be non convex.

Remark

When the ambient space is \( \mathbb{R}^4 \) with the aid of the Theorem we have a solution of the minimal equation in the interior of an octahedron in \( \mathbb{R}^3 \) taking alternatively infinite values \( +\infty \) and \( -\infty \) on adjacent faces.
Definition (Barriers)

We consider the Dirichlet problem \((P)\). Let \(p \in \partial \Omega \cup \partial_\infty \Omega\) be a boundary point where \(g\) is continuous.

1. • Assume first that \(p \in \partial \Omega\). Suppose that for any \(M > 0\) and for any \(k \in \mathbb{N}\) there is an open neighborhood \(\mathcal{N}_k\) of \(p\) in \(\mathbb{H}^n\) and a function \(\omega_k^+(\text{resp. } \omega_k^-)\) in \(C^2(\mathcal{N}_k \cap \Omega) \cap C^0(\overline{\mathcal{N}_k \cap \Omega})\) such that

   i) \(\omega_k^+(x)|_{\partial \Omega \cap \overline{\mathcal{N}_k}} \geq g(x)\) and \(\omega_k^+(x)|_{\partial \mathcal{N}_k \cap \Omega} \geq M\)
   (resp. \(\omega_k^-(x)|_{\partial \Omega \cap \overline{\mathcal{N}_k}} \leq g(x)\) and \(\omega_k^-(x)|_{\partial \mathcal{N}_k \cap \Omega} \leq -M\)).

   ii) \(M(\omega_k^+) \leq 0\) (resp. \(M(\omega_k^-) \geq 0\)) in \(\mathcal{N}_k \cap \Omega\).

   iii) \(\lim_{k \to +\infty} \omega_k^+(p) = g(p)\) (resp. \(\lim_{k \to +\infty} \omega_k^-(p) = g(p)\)).

• If \(p \in \partial_\infty \Omega\), then we choose for \(\mathcal{N}_k\) an open set of \(\mathbb{H}^n\) containing a half-space with \(p\) in its asymptotic boundary. We recall that a half-space is a connected component of \(\mathbb{H}^n \setminus \Pi\) for any geodesic hyperplane \(\Pi\). Then the functions \(\omega_k^+\) and \(\omega_k^-\) are in \(C^2(\mathcal{N}_k \cap \Omega) \cap C^0(\overline{\mathcal{N}_k \cap \Omega})\) and satisfy:
i) $\omega_k^+(x)|_{\partial_\Omega \cap \overline{\mathcal{N}_k}} \geq g(x)$ and $\omega_k^+(x)|_{\partial_{\mathcal{N}_k} \cap \Omega} \geq M$
(resp. $\omega_k^-(x)|_{\partial_\Omega \cap \overline{\mathcal{N}_k}} \leq g(x)$ and $\omega_k^-(x)|_{\partial_{\mathcal{N}_k} \cap \Omega} \leq -M$).

ii) For any $x \in \partial_\infty (\Omega \cap \mathcal{N}_k)$ we have $\liminf_{y \to x} \omega_k^+(y) \geq g(x)$ (for $y \in \mathcal{N}_k \cap \Omega$) (resp. $\limsup_{y \to x} \omega_k^-(y) \geq g(x)$).

iii) $\mathcal{M}(\omega_k^+) \leq 0$ (resp. $\mathcal{M}(\omega_k^-) \geq 0$) in $\mathcal{N}_k \cap \Omega$.

iv) $\lim_{k \to +\infty} \left( \liminf_{q \to p} \omega_k^+(q) \right) = g(p)$ and
$\lim_{k \to +\infty} \left( \limsup_{q \to p} \omega_k^-(q) \right) = g(p)$.

2. Suppose that $p \in \partial \Omega$ and that there exists a supersolution $\phi$ (resp. a subsolution $\eta$) in $C^2(\Omega) \cap C^0(\overline{\Omega})$ such that $\phi(p) = g(p)$ (resp. $\eta(p) = g(p)$).
In both cases 1 or 2 we say that $p$ admits an upper barrier ($\omega_k^+, k \in \mathbb{N}$ or $\phi$) (resp. lower barrier $\omega_k^-, k \in \mathbb{N}$ or $\eta$) for the problem $(P)$. If $p$ admits an upper and a lower barrier we say more shortly that $p$ admits a barrier.