Minimal graphs in $\mathbb{H}^n \times \mathbb{R}$ and $\mathbb{R}^{n+1}$

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Let us consider the upper half-space model of hyperbolic space: $\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$. Then the height function u satisfies the following equation

$$\begin{aligned}
&\sum_{i=1}^{n} \left(1 + x_n^2 (u_{x_1}^2 + \cdots + \widehat{u_{x_i}^2} + \cdots + u_{x_n}^2)\right) u_{x_i x_i} \\
&\qquad + \frac{(2-n)\left(1 + x_n^2 (u_{x_1}^2 + \cdots + u_{x_n}^2)\right) u_{x_n}}{x_n} - 2x_n^2 \sum_{i<k} u_{x_i} u_{x_k} u_{x_i x_k} \\
&\qquad - x_n u_{x_n} \left(u_{x_1}^2 + \cdots + u_{x_n}^2\right) = 0 \qquad \text{(Minimal equation)}
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We wish to provide geometric barriers for the Dirichlet problem.

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    The asymptotic boundary of M_d is topologically an $(n-1)$-sphere that is homologically trivial in $\partial_\infty \mathbb{H}^n \times \mathbb{R}$. More precisely, the asymptotic boundary consists of the union of two copies of an hemisphere $S^{n-1}_+ \times \{0\}$ of $\partial_\infty \mathbb{H}^n \times \{0\}$ in parallel slices $t = \pm T(a)$, glued with the finite cylinder $\partial S^{n-1}_+ \times [-T(a), T(a)]$, where $d =: \cosh^{n-1}(a)$.

- The vertical height of M_d, $d > 1$ is $2T(a)$. The height of the family $\mathcal{M}_d$ is a decreasing function in d and varies from infinity (when $d \to 1$) to $\pi/(n-1)$ (when $d \to \infty$). It is therefore bounded from below by $\pi/(n-1)$,the upper bound of the heights of the family of catenoids.

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  The Scherk type hypersurface gives a barrier at a finite convex point.

Our Problem :

Let $g : \partial\Omega \cup \partial_\infty\Omega \to \mathbb{R}$ be a bounded function. We consider the Dirichlet problem, say problem (P), for the vertical minimal hypersurface equation taking at any point of $\partial\Omega \cup \partial_\infty\Omega$ prescribed boundary (finite and asymptotic) value data g. More precisely,

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$$(P)\begin{cases} u \in C^2(\Omega) \text{ and } \mathcal{M}(u) = 0 \text{ in } \Omega, \\ \text{for any } p \in \partial\Omega \cup \partial_\infty\Omega \text{ where } g \text{ is continuous, } u \text{ extends} \\ \text{ continuously at } p \text{ setting } u(p) = g(p). \end{cases}$$

Now, let $u : \Omega \cup \partial\Omega \to \mathbb{R}$ be a continuous function.
Let $U \subset \Omega$ be a closed round ball in $\mathbb{H}^n$. We then define the continuous function $M_U(u)$ on $\Omega \cup \partial\Omega$ by:

$$M_U(u)(x) \begin{cases} u(x) & \text{if } x \in \Omega \cup \partial\Omega \setminus U \\ \tilde{u}(x) & \text{if } x \in U \end{cases} \tag{0.1}$$

where $\tilde{u}$ is the minimal extension of $u_{|\partial U}$ on $\overline{U}$.

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We say that u is a subsolution (resp. supersolution) of (P) if:

i) For any closed round ball $U \subset \Omega$ we have $u \leqslant M_U(u)$ (resp. $u \geqslant M_U(u)$).

ii) $u_{|\partial\Omega} \leqslant g$ (resp. $u_{|\partial\Omega} \geqslant g$).

iii) We have $\limsup_{q \to p} u(q) \leqslant g(p)$ (resp. $\liminf_{q \to p} u(q) \geqslant g(p)$) for any $p \in \partial_\infty \Omega$.

Theorem (**Perron process**)

Let $\Omega \subset \mathbb{H}^n$ be a domain and let $g : \partial\Omega \cup \partial_\infty\Omega \to \mathbb{R}$ be a bounded function. Let ϕ be a bounded supersolution of the Dirichlet problem (P), for example the constant function $\phi \equiv \sup g$.
Set $\mathcal{S}_\phi = \{\varphi$, subsolution of (P), $\varphi \leqslant \phi\}$. We define for each $x \in \Omega$

$$u(x) = \sup_{\varphi \in \mathcal{S}_\phi} \varphi(x).$$

(Observe that $\mathcal{S}_\phi \neq \emptyset$ since the constant function $\varphi \equiv \inf g$ belongs to $\mathcal{S}_\phi$.)
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We have the following:

1. *The function u is C^2 on Ω and satisfies the vertical minimal equation.*

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2. If $\partial\Omega$ is C^0 strictly convex at p then u extends continuously at p setting $u(p) = g(p)$.

Scherk type minimal hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$

Definition (**Special rotational domain**)

Let $\gamma, L \subset \mathbb{H}^n$ be two complete geodesic lines with L orthogonal to γ at some point $B \in \gamma \cap L$. Using the half-space model for $\mathbb{H}^n$, we can assume that γ is the vertical geodesic such that $\partial_\infty \gamma = \{0, \infty\}$. We call $P \subset \mathbb{H}^n$ the geodesic two-plane containing L and γ. We choose $A_0 \in (0, B) \subset \gamma$ and $A_1 \in L \setminus \gamma$ and we denote by $\alpha \subset P$ the euclidean segment joining A_0 and A_1. Therefore the hypersurface Σ generated by rotating α with respect to γ has the following properties.

1. $\mathrm{int}(\Sigma)$ is smooth except at point A_0.
2. Σ is strictly convex in hyperbolic meaning and convex in euclidean meaning.
3. $\mathrm{int}(\Sigma) \setminus \{A_0\}$ is transversal to the Killing field generated by the translations along γ.

Consequently Σ lies in the mean convex side of the domain of $\mathbb{H}^n$ whose boundary is the hyperbolic cylinder with axis γ and passing through A_1. Let us call $\Pi \subset \mathbb{H}^n$ the geodesic hyperplane orthogonal to γ and passing through B. Observe that the boundary of Σ is a $n-2$ dimensional geodesic sphere of Π centered at B.

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We denote by $U_\Sigma \subset \Pi$ the open geodesic ball centered at B whose boundary is the boundary of Σ. We call $\mathcal{M}_\Sigma \subset \mathbb{H}^n$ the closed domain whose boundary is $U_\Sigma \cup \Sigma$. Observe that $\partial\mathcal{M}_\Sigma$ is strictly convex at any point of Σ and convex at any point of U_Σ. Such a domain will be called a special rotational domain.

Theorem (Rotational Scherk hypersurface)

Let $\mathcal{M}_\Sigma \subset \mathbb{H}^n$ be a special rotational domain. There is a unique solution v of the vertical minimal equation in $\mathrm{int}(\mathcal{M}_\Sigma)$ which extends continuously to $\mathrm{int}(\Sigma)$, taking prescribed zero boundary value data and taking boundary value ∞ for any approach to U_Σ.

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$$(P)\begin{cases} \mathcal{M}(u) = 0 \text{ in } \mathrm{int}(\mathcal{M}_\Sigma), \\ u = 0 \text{ on } \mathrm{int}(\Sigma), \\ u = +\infty \text{ on } U_\Sigma, \\ u \in C^2\left(\mathrm{int}(\mathcal{M}_\Sigma)\right) \cap C^0\left(\mathcal{M}_\Sigma \setminus \overline{U}_\Sigma\right). \end{cases}$$

We call the graph of v in $\mathbb{H}^n \times \mathbb{R}$ a rotational Scherk hypersurface.

Proof

First, we will prove the existence part of the Theorem. We consider the family of functions v_t, $t > 0$. Recall that $\Pi \subset \mathbb{H}^n$ is the totally geodesic hyperplane containing U_Σ. We consider a suitable copy of M_1 as barrier as follows: choose M_1 such that M_1 is a graph of a function u_1 whose domain is the component of $\mathbb{H}^n \setminus \Pi$ that contains $\mathcal{M}_\Sigma$, with u_1 taking boundary value data $+\infty$ on Π and taking zero asymptotic boundary value data. By applying maximum principle we have that $u_1(p) > v_t(p)$ for all $p \in \mathcal{M}_\Sigma$ and all $t > 0$.

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Using compactness principle we obtain that a subsequence of the family converges uniformly on any compact subsets of $\mathrm{int}(\mathcal{M}_\Sigma)$ to a solution v_∞ of the minimal equation. Since the family is strictly increasing v_∞ takes the value $+\infty$ on U_Σ. That is, for any sequence (q_k) in $\mathrm{int}(\mathcal{M}_\Sigma)$ converging to some point of U_Σ we have $v_\infty(q_k) \to +\infty$.

Let $p \in \mathrm{int}(\Sigma)$, since $\partial\mathcal{M}_\Sigma$ is C^0 strictly convex at p, the hypersurfaces M_d, $d < 1$, provide a barrier at p. Consequently v_∞ extends continuously at p setting $v_\infty(p) = 0$. Therefore v_∞ is a solution of the Dirichlet problem (P).

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The proof of uniqueness of v_∞ proceeds in the same way as the proof of the monotonicity of the family $\{v_t\}$. This completes the proof of the Theorem.

Theorem (Barrier at a C^0 convex point)

Let $\Omega \subset \mathbb{H}^n$ be a domain and let $p_0 \in \partial\Omega$ be a boundary point where Ω is C^0 convex. Then for any bounded data $g : \partial\Omega \cup \partial_\infty\Omega \to \mathbb{R}$ continuous at p_0, the family of rotational Scherk hypersurfaces provides a barrier at p_0 for the Dirichlet problem (P). In particular, in Theorem 1-(2) *the assumption C^0 strictly convex can be replaced by C^0 convex.*

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Proof. We will prove that the rotational Scherk hypersurfaces with $-\infty$ boundary data on the boundary part U_Σ provide an upper barrier at p_0. For the lower barrier the construction is similar.

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Claim 2. Let D be any point on the open geodesic segment (A_0, B), and let $\beta \subset \mathcal{M}_\Sigma$ be a geodesic segment issuing from D, ending at some point $C \in \mathrm{int}(\Sigma)$ and orthogonal to $[A_0, B]$ at D.

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Then ω is increasing along $\beta = [D, C]$, oriented from D to C.

We first prove the theorem assuming that the two claims hold.

Let $D \in (A_0, B)$ and let $\Pi_D \subset \mathbb{H}^n$ be the geodesic hyperplane through D orthogonal to the geodesic segment $[A_0, B]$. Let $\mathcal{M}_\Sigma^+$ be the connected component of $\mathcal{M}_\Sigma \setminus \Pi_D$ containing the point A_0. Let q be any point belonging to the closure of $\mathcal{M}_\Sigma^+$. The claims ensure that $\omega(q) \geqslant \omega(D)$.

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Let $p_0 \in \partial\Omega$ be a C^0 convex point and let g be a bounded data continuous at p_0. Let $M > 0$ be any positive real number. It suffices to show that for any $k \in \mathbb{N}^*$ there is an open neighborhood $\mathcal{N}_k$ of p_0 in $\mathbb{H}^n$ and a function ω_k^+ in $C^2(\mathcal{N}_k \cap \Omega) \cap C^0(\overline{\mathcal{N}_k \cap \Omega})$ such that

i) $\omega_k^+(x)_{|\partial\Omega\cap\mathcal{N}_k} \geqslant g(x)$ and $\omega_k^+(x)_{|\partial\mathcal{N}_k\cap\Omega} \geqslant M$,

ii) $\mathcal{M}(\omega_k^+) = 0$ in $\mathcal{N}_k \cap \Omega$,

iii) $\omega_k^+(p_0) = g(p_0) + 1/k$.

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By assumption there exist a geodesic hyperplane Π_{p_0} through p_0 and an open neighborhood $W \subset \Pi_{p_0}$ of p_0 such that $W \cap \Omega = \emptyset$. We set $\Omega_\varepsilon = \{p \in \Omega \mid \mathrm{dist}(p_0, p) < \varepsilon\}$. Up to choosing ε small enough, we can assume that Ω_ε is entirely contained in a component of $\mathbb{H}^n \setminus \Pi_{p_0}$. Let γ be the geodesic through p_0 orthogonal to Π_{p_0}.

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We choose a special rotational domain $\mathcal{M}_\Sigma$ such that:

- the hyperplane Π is orthogonal to γ, (recall that $U_\Sigma \subset \Pi$)
- the diameter of $\mathcal{M}_\Sigma$ is lesser than $\frac{\varepsilon}{4}$,
- $\overline{\Omega} \cap U_\Sigma = \emptyset$,
- $A_0 \in \gamma$, $\mathrm{dist}(p_0, A_0) < \frac{\varepsilon}{8}$ and A_0 belongs to the same component of $\mathbb{H}^n \setminus \Pi_{p_0}$ than Ω_ε.

Let $M' > \max\{M, g(p_0) + 1/k\}$. We consider the rotational Scherk hypersurface (graph of ω) taking M' boundary value data on the interior of Σ and $-\infty$ on U_Σ.
By continuity, there exists a point $p_1 \in \gamma$ where $\omega(p_1) = g(p_0) + 1/k$. Up to a horizontal translation along γ sending p_1 to p_0, we may assume that $\omega(p_0) = g(p_0) + 1/k$.
Then we set $\mathcal{N}_k = \mathrm{int}(\mathcal{M}_\Sigma) \cap \Omega$ and $\omega_k^+ = \omega_{|\mathcal{N}_k}$, the restriction of ω to $\mathcal{N}_k$. Therefore we have $\omega_k^+(x)_{|\partial\mathcal{N}_k \cap \Omega} = M' \geqslant M$, furthermore Claim 1 and Claim 2 show that
$\omega_k^+(x)_{|\partial\Omega \cap \mathcal{N}_k} \geqslant g(p_0) + 1/k \geqslant g(x)$, as desired.

Theorem (Dirichlet problem for the minimal equation in $\mathbb{H}^n \times \mathbb{R}$ on a C^0 bounded convex domain taking continuous boundary data)

Let Ω be a C^0 bounded convex domain and let $g : \partial\Omega \to \mathbb{R}$ be a continuous function.
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Proof.

The proof is a consequence of the Perron process and the construction of barriers at any convex point of a C^0 domain, using rotational Scherk hypersurfaces. Uniqueness follows from the maximum principle. $\square$

Theorem (Dirichlet problem for the minimal equation in $\mathbb{H}^n \times \mathbb{R}$ on a C^0 convex domain taking continuous finite and asymptotic boundary data)

Let $\Omega \subset \mathbb{H}^n$ be a C^0 convex domain and let $g : \partial\Omega \cup \partial_\infty\Omega \to \mathbb{R}$ be a continuous function.
Then g admits a unique continuous extension $u : \Omega \cup \partial\Omega \cup \partial_\infty\Omega \to \mathbb{R}$ satisfying the vertical minimal hypersurface equation on Ω.

Theorem (Dirichlet problem for the minimal equation in $\mathbb{H}^n \times \mathbb{R}$ on a C^0 convex domain taking continuous finite and asymptotic boundary data)

Let $\Omega \subset \mathbb{H}^n$ be a C^0 convex domain and let $g : \partial\Omega \cup \partial_\infty\Omega \to \mathbb{R}$ be a continuous function.
Then g admits a unique continuous extension $u : \Omega \cup \partial\Omega \cup \partial_\infty\Omega \to \mathbb{R}$ satisfying the vertical minimal hypersurface equation on Ω.

Proof.

Notice that working in the ball model of hyperbolic space, we have that g is a continuous function on a compact set, hence g is bounded. Therefore there exist supersolutions and subsolutions for the Dirichlet problem. The proof is a consequence of the Perron process and the constructions of barriers, using the rotational Scherk hypersurfaces at any point of $\partial\Omega$, and using M_1 at any point of $\partial_\infty\Omega$. Uniqueness follows from the maximum principle. $\square$

Definition (**Independent points and admissible polyhedra**)

1. We say that $n+1$ points $A_0, \ldots, A_n$ in $\mathbb{H}^n$ are independent if there is no geodesic hyperplane containing these points. If $A_0, \ldots, A_n$ in $\mathbb{H}^n$ are independent then we remark that any choice of n points among them determines a unique geodesic hyperplane of $\mathbb{H}^n$.

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2. Let $A_0, \ldots, A_n$ be $n+1$ independent points in $\mathbb{H}^n$. We call Π_i the geodesic hyperplane containing these points excepted A_i, $i = 0, \ldots, n$ and we call Π_i^+ the closed half-space bounded by Π_i and containing A_i. Then the intersection of these half-spaces is a polyhedron $\mathcal{P}$: the convex closure of $A_0, \ldots, A_n$. The boundary of $\mathcal{P}$ consists of $n+1$ closed faces $F_i \subset \Pi_i$, the face F_i contains in its boundary all the points $A_0, \ldots, A_n$ excepted A_i. We call such a polyhedron an admissible polyhedron.

Theorem (First Scherk type hypersurface in $\mathbb{H}^n \times \mathbb{R}$)

Let $\mathcal{P}$ be an admissible convex polyhedron. There is a unique solution v_∞ of the minimal equation in $\mathrm{int}(\mathcal{P})$ extending continuously up to $\partial\mathcal{P} \setminus F_0$, taking prescribed zero boundary value data on $F_1 \setminus \partial F_0, \ldots, F_n \setminus \partial F_0$ and prescribed boundary value ∞ for any approach to $\mathrm{int}(F_0)$. More precisely, we prove existence and uniqueness of the following Dirichlet problem (P_∞):

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$$(P_\infty) \begin{cases} \mathcal{M}(u) = 0 \text{ in } \mathrm{int}(\mathcal{P}), \\ u = 0 \text{ on } F_j \setminus \partial F_0,\ j = 1, \ldots, n, \\ u = \infty \text{ on } \mathrm{int}(F_0), \\ u \in C^2\left(\mathrm{int}(\mathcal{P})\right) \cap C^0\left(\mathcal{P} \setminus F_0\right). \end{cases}$$

Proof.

We may use the rotational Scherk hypersurfaces as barrier. Therefore, we obtain for any $t \in \mathbb{R}$ a solution v_t of the vertical minimal equation in $\mathrm{int}(\mathcal{P})$ which extends continuously to $\partial\mathcal{P} \setminus \partial F_0$, taking prescribed zero boundary value data on $\partial\mathcal{P} \setminus F_0$ and prescribed boundary value t on $\mathrm{int}(F_0)$. Now letting $t \to \infty$ we have that a subsequence of the family $\{v_t\}$ converges to a solution as desired, taking into account that the rotational Scherk hypersurfaces give a barrier at any point of $\mathcal{P}$.

The uniqueness is obtained as in the proof of the monotonicity of the family $\{v_t\}$ in the previous Proposition. $\square$

Theorem (Second Scherk type hypersurface in $\mathbb{H}^n \times \mathbb{R}$)

For any $k \in \mathbb{N}$, $k \geqslant 2$, there exists a family of polyhedron $\mathcal{P}_k$ with $2^{n-1}k$ faces and a solution w_k of the vertical minimal equation in $\operatorname{int} \mathcal{P}_k$ taking alternatively infinite values $+\infty$ and $-\infty$ on adjacent faces of $\mathcal{P}_k$. Moreover, the polyhedron $\mathcal{P}_k$ can be chosen to be convex and can also be chosen to be non convex.

Proof. Let us fix a point A_0 in $\mathbb{H}^n$. Let $\{e_1, \ldots, e_n\}$ be a positively oriented orthornormal basis of $T_{A_0}\mathbb{H}^n$. For $k \geqslant 2$ we set $u := \sin(\pi/k)e_1 + \cos(\pi/k)e_2$. Let γ_j^+, $j = 2, \ldots, n$ and γ_u^+ be the oriented half geodesics issuing from A_0 and tangent to $e_2, \ldots, e_n$ and to u, respectively. Now we choose an interior point A_1 on γ_u^+ and an interior point A_j on γ_j^+, $j = 2, \ldots, n$. Therefore, $A_0, A_1, \ldots, A_n$ are independent points of $\mathbb{H}^n$. Let $\widetilde{\mathcal{P}}$ be the polyhedron determined by these points. The faces are denoted by $F_0, \ldots, F_n$, with the convention that the face F_j does not contain the vertex A_j, $j = 0, \ldots, n$.
Let Π_i the totally geodesic hyperplane containing the face F_i. Observe that:

1. F_1 and F_2 make an interior angle equal to π/k.
2. $F_j \perp F_1$, $F_j \perp F_2, j = 3, \ldots, n$.
3. $F_j \perp F_k, j, k = 3, \ldots, n\,(j \neq k)$.

Therefore, the reflections in $\mathbb{H}^n$ with respect to the geodesic hyperplanes Π_1 and Π_2 leave the other geodesic hyperplanes $\Pi_j, j = 3, \ldots, n$ globally invariant. The first step of the construction of the polyhedron $\mathcal{P}_k$ is the following: Doing reflection about F_2 we obtain another polyhedron with faces F_1^* (the symmetric of F_1 about F_2), and faces $\widetilde{F}_j$ containing F_j, $\widetilde{F}_j \subset \Pi_j$, $j = 3, \ldots, n$. Notice that in the process the face F_2 disappears and the interior angle between the faces F_1 and F_1^* is $2\pi/k$. Furthermore, the reflection of F_0 about F_2 generates another face F_0^1.

Continuing this process doing reflections with respect to F_1^* and so on, we obtain a new polyhedron $\mathcal{P}^+$ with faces $\widehat{F}_j \subset \Pi_j$, $j = 3, \ldots, n$, $\widehat{F}_j$ containing $\widetilde{F}_j$, and $2k$ faces issuing from the successive reflections of F_0. Notice that both faces F_1 and F_2 disappear at the end of the process, that is $\mathcal{P}^+$ does not contain any face in the hyperplane Π_1 or Π_2.

Next, let us perform the reflections about Π_3. Doing this the face F_3 disappears and we get a new polyhedron with $2 \cdot 2k$ faces issuing from F_0 and a face in each Π_j, $j = 4, \ldots, n$. Each such face contains $\widehat{F}_j$, $j = 4, \ldots, n$. Continuing this process doing reflections on $\Pi_4, \ldots, \Pi_n$ we finally get a polyhedron $\mathcal{P}_k$ with $2^{n-1} \cdot k$ faces, each one issuing from F_0.

Now we discuss the convexity of $\mathcal{P}_k$. Let $P \subset \mathbb{H}^n$ be the geodesic two-plane containing the points A_0, A_1 and A_2. Let $\Gamma \subset P$ be the geodesic polygon obtained by the reflection of the segment $[A_0, A_1]$ with respect to $[A_0, A_2]$ and so on. Thus Γ is a polygon with $2k$ sides and $2k$ vertices, among them A_1 and A_2, and A_0 is an interior point of Γ. Then, the polyhedron $\mathcal{P}_k$ is convex if, and only if, the polygon Γ is convex too. For example, if $d(A_0, A_1) = d(A_0, A_2)$ we get that Γ is a regular polygon and then is convex. On the other hand, if $d(A_0, A_1)$ is much bigger than $d(A_0, A_2)$ then Γ is non convex.

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Now, considering the polyhedron $\widetilde{\mathcal{P}}$ of the beginning, we are able to solve the Dirichlet problem of the minimal equation taking $+\infty$ value data on F_0 and zero value data on $F_j \setminus F_0$, $j = 1, \ldots, n$. Using the reflection principle on the faces, in each step of the preceding process, we obtain at the end of the process a solution of the minimal equation on $\operatorname{int} \mathcal{P}_k$, taking alternatively infinite values $+\infty$ and $-\infty$ on adjacent faces of $\mathcal{P}_k$, as desired. This accomplishes the proof of the theorem.

Theorem

Let $\Omega \subset \mathbb{H}^n$ be an admissible unbounded domain. Let $g : \partial\Omega \cup \partial_\infty\Omega \to \mathbb{R}$ be a continuous function taking zero boundary value data on $\partial\Omega$. Let $\Gamma_\infty \subset \partial_\infty\mathbb{H}^n \times \mathbb{R}$ be the graph of g restricted to $\partial_\infty\Omega$. If the height function t of Γ_∞ satisfies $-f(\rho_\Omega) \leqslant t \leqslant f(\rho_\Omega)$, then there exists a vertical minimal graph over Ω with finite boundary $\partial\Omega$ and asymptotic boundary Γ_∞.

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Furthermore, there is no such minimal graph, if $\partial\Omega$ is compact and the height function t of Γ_∞ satisfies $|t| > \pi/(2n-2)$.

Theorem

Let $\Omega \subset \mathbb{H}^n$ be an E-admissible unbounded domain. Let $g : \partial\Omega \cup \partial_\infty\Omega \to \mathbb{R}$ be a continuous function taking zero boundary value data on $\partial\Omega$. Let $\Gamma_\infty \subset \partial_\infty\mathbb{H}^n \times \mathbb{R}$ be the graph of g restricted to $\partial_\infty\Omega$. If the height function t of Γ_∞ satisfies $-H(r_\Omega) \leqslant t \leqslant H(r_\Omega)$, then there exists a vertical minimal graph over Ω with finite boundary $\partial\Omega$ and asymptotic boundary Γ_∞.

Minimal graphs in $\mathbb{R}^{n+1}$

Theorem (Second Scherk type hypersurface in $\mathbb{R}^{n+1}$)

For any $k \in \mathbb{N}$, $k \geqslant 2$, there exists a family of polyhedron $\mathcal{P}_k$ with $2^{n-1}k$ faces and a solution w_k of the vertical minimal equation in $\operatorname{int} \mathcal{P}_k$ taking alternatively infinite values $+\infty$ and $-\infty$ on adjacent faces of $\mathcal{P}_k$. Moreover, the polyhedron $\mathcal{P}_k$ can be chosen to be convex and can also be chosen to be non convex.

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Remark

When the ambient space is $\mathbb{R}^4$ with the aid of the Theorem we have a solution of the minimal equation in the interior of an octahedron in $\mathbb{R}^3$ taking alternatively infinite values $+\infty$ and $-\infty$ on adjacent faces.

Definition (**Barriers**)

We consider the Dirichlet problem (P). Let $p \in \partial\Omega \cup \partial_\infty\Omega$ be a boundary point where g is continuous.

1. • Assume first that $p \in \partial\Omega$. Suppose that for any $M > 0$ and for any $k \in \mathbb{N}$ there is an open neighborhood $\mathcal{N}_k$ of p in $\mathbb{H}^n$ and a function ω_k^+ (resp. ω_k^-) in $C^2(\mathcal{N}_k \cap \Omega) \cap C^0(\overline{\mathcal{N}_k \cap \Omega})$ such that

   i) $\omega_k^+(x)_{|\partial\Omega\cap\overline{\mathcal{N}_k}} \geqslant g(x)$ and $\omega_k^+(x)_{|\partial\mathcal{N}_k\cap\Omega} \geqslant M$ (resp. $\omega_k^-(x)_{|\partial\Omega\cap\overline{\mathcal{N}_k}} \leqslant g(x)$ and $\omega_k^-(x)_{|\partial\mathcal{N}_k\cap\Omega} \leqslant -M$).

   ii) $\mathcal{M}(\omega_k^+) \leqslant 0$ (resp. $\mathcal{M}(\omega_k^-) \geqslant 0$) in $\mathcal{N}_k \cap \Omega$.

   iii) $\lim_{k\to+\infty} \omega_k^+(p) = g(p)$ (resp. $\lim_{k\to+\infty} \omega_k^-(p) = g(p)$).

   • If $p \in \partial_\infty\Omega$, then we choose for $\mathcal{N}_k$ an open set of $\mathbb{H}^n$ containing a half-space with p in its asymptotic boundary. We recall that a half-space is a connected component of $\mathbb{H}^n \setminus \Pi$ for any geodesic hyperplane Π. Then the functions ω_k^+ and ω_k^- are in $C^2(\mathcal{N}_k \cap \Omega) \cap C^0(\overline{\mathcal{N}_k \cap \Omega})$and satisfy:

i) $\omega_k^+(x)_{|\partial\Omega\cap\overline{\mathcal{N}_k}} \geqslant g(x)$ and $\omega_k^+(x)_{|\partial\mathcal{N}_k\cap\Omega} \geqslant M$ (resp. $\omega_k^-(x)_{|\partial\Omega\cap\overline{\mathcal{N}_k}} \leqslant g(x)$ and $\omega_k^-(x)_{|\partial\mathcal{N}_k\cap\Omega} \leqslant -M$).

ii) For any $x \in \partial_\infty(\Omega \cap \mathcal{N}_k)$ we have $\liminf_{y\to x} \omega_k^+(y) \geqslant g(x)$ (for $y \in \mathcal{N}_k \cap \Omega$) (resp. $\limsup_{y\to x} \omega_k^-(y) \geqslant g(x)$).

iii) $\mathcal{M}(\omega_k^+) \leqslant 0$ (resp. $\mathcal{M}(\omega_k^-) \geqslant 0$) in $\mathcal{N}_k \cap \Omega$.

iv) $\lim_{k\to+\infty}\big(\liminf_{q\to p} \omega_k^+(q)\big) = g(p)$ and $\lim_{k\to+\infty}\big(\limsup_{q\to p} \omega_k^-(q)\big) = g(p)$.

2. Suppose that $p \in \partial\Omega$ and that there exists a supersolution ϕ (resp. a subsolution η) in $C^2(\Omega) \cap C^0(\overline{\Omega})$ such that $\phi(p) = g(p)$ (resp. $\eta(p) = g(p)$).
In both cases 1 or 2 we say that p admits an upper barrier ($\omega_k^+, k \in \mathbb{N}$ or ϕ) (resp. lower barrier $\omega_k^-, k \in \mathbb{N}$ or η) for the problem (P). If p admits an upper and a lower barrier we say more shortly that p admits a barrier.