A NOTE ON SPECIAL SURFACES IN $\mathcal{R}^3$

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INTRODUCTION

In this paper we shall review some recent results on the new theory of special surfaces in $\mathcal{R}^3$. Our approach is *geometrical* (non variational) in nature (see [1],[6],[7],[8]). We shall show that the main global structure of constant mean curvature surfaces hold for a more general class of *special* surfaces satisfying a certain Weingarten relation. This has been studied by Hopf [5], Chern [3], Bryant [2], Hartman and Wintner [4]. More recently by Braga Brito, Rosenberg and the authors (see the references above).

We shall consider surfaces M immersed in $\mathcal{R}^3$, which are oriented by a unit normal vector field N and whose mean curvature $\mathrm{H} = \mathrm{H}(N)$ and Gaussian curvature K, satisfying a Weingarten relation of the form

$$\mathrm{H} = f(\mathrm{H}^2 - K) \tag{1}$$

where f is a C^1 function defined in the interval $[0, +\infty]$. From now on, we will only consider functions that verify the relation:

$$4t(f'(t))^2 < 1, \tag{2}$$

for $t \in [0, +\infty)$. We say that f is *elliptic.* We call such a surface M *special* if H, K satisfy (1) for f elliptic.

We shall establish that, if there is a plane in the class of M, i.e. $f(0) = 0$, then the theory is "minimal type" (see [7]). Otherwise, if there is a sphere in the class of M, i.e. $f(0) \neq 0$, then the theory is constant (non zero) mean curvature type (see [8]). In both cases, ellipticity ensures that M satisfies a maximum principle (see [6]). This allows us to apply Alexandrov reflection principle techniques (see [1], [6]). For instance, by using elementary geometrical arguments and by applying ellipticity (relation (2)) it is not hard to show that the following basic statements hold.

Basic Statements for a Special Minimal Type Surface $(f(0) = 0)$:

a) The Gaussian curvature K of M is non positive;
b) The zeros of K are isolated;
c) M is contained inside the convex hull of the boundary;
d) If M is complete with zero Gaussian curvature, i.e. $K \equiv 0$, then M is a plane.

Basic Statements for a Special Constant (non zero) Mean Curvature Type Surface $(f(0) \neq 0)$:

a) *Alexandrov's theorem* hold: If M is closed (compact without boundary) and embedded then M is a round sphere;
b) *Hopf's theorem* hold: If M is closed immersed with genus zero then M is a round sphere (here, ellipticity is not required, see [2]);
c) If M is complete with zero Gaussian curvature then M is a right cylinder;
d) If M is embedded, and if the boundary ∂M is a round circle contained in a plane $\mathcal{H}$, such that $M \cap \mathcal{H} = \partial$M then M is a round sphere.

In order to illustrate our main point, we will discuss the following example: We consider a *parallel surface* to M defined by $M_t = M + tN$ (N is the unit normal vector field). It is a classical fact that if M has constant mean curvature H equal to $\frac{1}{2a}$ then the parallel surface M_t to M at a distance $t = a$ has constant Gaussian curvature $\frac{1}{a^2}$. Clearly M_t has singularities where the Gaussian curvature K of M is zero. More generally we remark that, if H is a positive constant, it is easy to show that M_t satisfies a linear relation

$$2a\mathrm{H}_t + K_t = b \tag{3}$$

where $a > 0, b \geq 0(b = 0$ if $\mathrm{H} = 0)$, for t small enough. We call such an M_t a $M(a,b)$ surface (see[6]). Precisely, if H is constant and if the Gaussian curvature of M is bounded from below ($K >> -\infty$) then M_t is immersed for t sufficiently small (it is sufficient to prove $1-2t\mathrm{H}+t^2K \neq 0$). A straightforward computation shows that an arbitrarily $M(a,b)$ surface is special for $a > 0, b \geq 0$: just take $f(t) = -a + \sqrt{a^2 + b + t}$. Notice that H is never zero on $M(a,b)$.

Moreover, notice that if $M_t(a,b)$ is a parallel surface to a $M(a,b)$ surface (t small enough) with $a > 0$ and $b \geq 0$, then M_t still has the same structure preserving the minimal type if $b = 0$ and the constant (non zero) mean curvature type if $b \neq 0$. Indeed, $M_t = M(a_t, b_t)$ where $a_t = (a-bt)/(1-t^2b+2ta)$ and $b_t = b/(1-t^2b+2ta)$.

On the other hand, we claim that giving an arbitrarily $M(a,b)$ surface with positive mean curvature H, $a > 0, b \geq 0$, there is a 1-parameter family of immersed special parallel surfaces $M_t(a,b), \tau \leq t < 0$, τ chosen later, in such a way that M_τ has constant mean curvature: M_τ is minimal if and only if $b = 0$. For the proof of the claim, we proceed as follows:

Suppose $b > 0$ (if $b = 0$, see [6]). We will prove that $1-2t\mathrm{H}+t^2K > 0$. Setting

$$\alpha = \frac{2a}{b}, \beta = \frac{1}{b}$$

we may assume that M satisfies $\alpha\mathrm{H} + \beta K = 1$. Let

$$\tau = \frac{\alpha - \sqrt{\alpha^2 + 4\beta}}{2}$$

Then $M_\tau(a,b)$ has constant mean curvature

$$\mathrm{H}_\tau = \frac{1}{\sqrt{\alpha^2 + 4\beta}}.$$

Also, it is clear that if $K(p) > 0, p \in M$, then $1 - 2t\mathrm{H} + t^2K > 0$ at p, for $t \leq 0$. So then, assume $K \leq 0$: a simple compulation shows that $\alpha(1-2t\mathrm{H}+t^2K) = \sqrt{\alpha^2 + 4\beta} + Kt(2\beta+\alpha t)$. Then, to accomplish the proof of the above claim, notice that the last expression is positive for $\tau \leq t \leq 0$, since $2\beta + \alpha t > 0$ if $t \geq \tau$.

To conclude this elementary discussion, we emphasize one non elementary implication: if M is a properly embedded $M(a, b)$ surface $(a > 0, b \geq 0)$ then each of its annular ends converges exponentially to a parallel end of a Delaunay end (see [6]).

Now, we point out that we need a certain operator L_f to develop the theory of constant mean type special surface. In fact the study of L_f is useful to guarantee *apriori height estimates.* This yields extended Meeks theory (see[6]). We define L_f by

$$L_f = \frac{1 - 2ff'}{2} \triangle + f'L$$

where $\triangle$ is the Laplacian operator, $L = div(T\bigtriangledown), T = 2\mathrm{H}I - A$ (A is the shape operator of M. See [6]).

Notice that ellipticity of L_f is equivalent to equation (2), if $f' \neq 0$ then we may consider the normalized operator $L_F = F \triangle + L$, where

$$F = \frac{1}{2f'} - f.$$

Setting $F = cf + a$ one obtains

$$f(t) = -\frac{a + \sqrt{a^2 + (c+1)(t+b)}}{c+1}$$

as a positive solution.

Hence, a surface M with positive mean curvature H, satisfying

$$cH^2 + 2a\mathrm{H} + K = b \qquad (4),$$

$a > 0, b > 0, c \geq 0$ is a special surface (see[6]).

In the next two sections, we shall give a list of statements (without proofs), concerning the classification and application of rotationally symmetric special surfaces. In particular, we shall see that deforming the unique cylinder satisfying equation (4) with $a, b, c > 0$ one may figure out a 1-parameter family of Delaunay type surface (see theorem 1 in section 2 considering $-f$ instead of f in the above formulae).

Finally, we give a selected number of open question at the end of each section.

1. Minimal type special surfaces in $\mathcal{R}^3$ (see[7] for proofs and further details)

Theorem 1: Existence of Special Cateroids

Let f be an elliptic function with $f(0) = 0$ and let $\tau > 0$ be a real number that verifies

$$\frac{1}{\tau} < \lim_{t \to +\infty} (t - f(t^2))$$

Then there is a unique complete rotational symmetric special surface M_τ such that its generating curve is a graph of a C^2 convex, strictly positive function $y = y(x)$, which attains a minimum at 0 and is symmetric respect to the y axis, i.e., y verifies:

$$y > 0, y(0) = \tau, y'(0) = 0, y'' > 0 \, \text{and} \; y(x) = y(-x).$$

Remark 1: One may show that the condition

$$\frac{1}{\tau} < \lim_{t \to +\infty} (t - f(t^2))$$

required by Theorem 1 is necessary for our purpose (see [7]).

Theorem 2: The geometry of special catenoids.

Let f be a C^2 elliptic function near 0 with $f(0) = 0$. Let M be a rotational symmetric surface given by Theorem 1.

Then the x coordinate over M is a propre function.

Remark 2: The differentiability hypothesis for f in Theorem 2 is necessary, a counter example may be constructed considering $f_{\alpha}(t) = \alpha\sqrt{t}, t \geq 0, 0 < \alpha < 1$ (see [7]).

We remark that if f is non negative with $f(0) = 0$ and $\lim_{t\to+\infty}(t - f(t^2)) = +\infty$ then there is a 1-parameter family $M_\tau, \tau > 0$, of special catenoids converging to a vertical plane as $\tau \to 0$ (see [7]). This yields the following generalization of the well-known "Halfspace Theorem" for minimal type special surfaces.

Theorem 3: "Halfspace Theorem"

Let f be a elliptic non negative ($f \geq 0$) function. Assume f is C^2 in a neighborhood of the origin. Suppose f verifies

$$f(0) = 0, \lim_{t\to+\infty}(t - f(t^2)) = +\infty$$

If M is a complete connected properly immersed special surface (respect to f) contained inside a halfspace of $\mathcal{R}^3$, then M is a plane.

Theorem 4: Classification of Rotational Symmetric Minimal Type Special Surfaces.

a) Given a elliptic function f, then a complete rotational minimal type special surface M is equal to a special catenoid M_τ obtained by Theorem 1.

b) Assume f is non negative ($f \geq 0$) and assume $lim_{t\to+\infty}(t - f(t^2) = +\infty$.Then a non planar rotational minimal type special surface S is part of a special catenoid M_τ obtained by Theorem 1.

Remark 3: Notice that if $f(0) = 0$ and if

$$\lim_{t\to+\infty}(t - f(t^2)) \neq +\infty$$

then there is special surfaces M which are C^2 on $M - \partial M$ and C^1 up to the boundary ∂M, such that the Gaussian curvature $K(p)$ converges to $-\infty$ as $p \to \partial M$ (see[7]).

Finally, we point out that there is examples (see[7]) of special catenoids having *real* catenoid ends and a piece of *real* Delaunay surface (*nodoid*).

Selected Open Questions:

1. Is there a minimal type theory for complete minimal type special surfaces with finite total curvature? Is there an analogous Bernstein and Schoen Theorems?
2. Is there a variational approach for minimal type special surfaces?

2. Constant (non zero) mean curvature type special surfaces (See [6], [8] for proofs and further details).

Theorem 1: Existence of special onduloids .

Let f be an elliptic function satisfying

$$f(0) < 0 \text{ and } \lim_{t \to -\infty} (t - f(t^2)) < 0$$

Let $\tau > 0$ be a real number verifying:

$$\frac{1}{\tau} < \lim_{t \to +\infty} (t - f(t^2)) \text{and} \frac{-1}{2\tau} - f\big((\frac{1}{2\tau})^2\big) < 0.$$

Then there is a unique complete rotational symmetric embedded special surface such that its generating curve γ_τ is a graph of a C^2 function $y = y(x)$ which attains a minimum at 0. Furthermore M_τ is analogous to the embedded Delaunay surface. More precisely, there is a real number $T_\tau > 0$ such that

a) $\forall x \in \mathcal{R}, \mathrm{y}_\tau(\mathrm{x} + \mathrm{T}_\tau) = \mathrm{y}_\tau(\mathrm{x})$.

b) γ_τ is symmetric with respect to the vertical lines $\{x = 0\}$ and $\{x = \frac{T_\tau}{2}\}$.

c) Between 0 and $\frac{T_\tau}{2}, \gamma_\tau$ is strictly increasing with a maximum R_τ at $\frac{T_\tau}{2}$.

d) Between 0 and $\frac{T_\tau}{2}, \gamma_\tau$ has a unique inflexion point x_τ.

e) The inflexion point x_τ and the maximum $\mathcal{R}_\mathcal{T}$ verify

$$0 < y_\tau(x_\tau) < r, r < \mathcal{R}_\tau < \frac{-1}{\mathrm{f(0)}}$$

where r is the radius of the unique special cylinder.

Theorem 2: Classification of Special Onduloids.

Let f be an elliptic function and let M be a rotational complete embedded constant (non zero) mean curvature type special surface. If M is non compact then M is a special onduloid given by Theorem 1.

The above theorem together with Theorem 3.3 in [6], yield the following consequence:

Corollary 1: A characterization of special ondoloids.

Let f be a elliptic function verifying

$$f \geq \lambda > 0$$

$$f'(1 - 2ff') \geq 0$$

Let M be a properly embedded special surface with two annulus end. Then M is a special ondoloid given by Theorem 1.

The interested lector will find in [8] others results concerning the classification, geometrical characterization and application of special ondoloids. Moreover, we remark that if $\lim_{t\to-\infty}(t - f(t^2)) = 0$ then there is rotational special surfaces analogous to rotational constant (positive) Gauss curvature surfaces.

To conclude, we state briefly the following theorem concerning special nodoids which are analogous to Delaunay's nodoids (see [8] for details):

Theorem 3: Existence of special nodoids.

Let f be a elliptic function verifying

$$f(0) > 0 \text{ and } \lim_{t\to+\infty}(t - f(t^2)) = +\infty$$

Let $\tau > 0$ be a real number verifying

$$\frac{1}{\tau} < \lim_{t \to +\infty} (t - f(t^2))$$

then there is a unique complete rotational symmetric non embedded special surface N_τ.

Selected Open Questions

1. Is there a Kapouleas theory for special surfaces?
2. Is there a Korevaar, Kusner and Solomon theory for special surfaces (see [6])?

References

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