Recent researches

Comments about the results obtained in the period 2012- 2017:

Summary: We are interested in the study of minimal and constant mean curvature surfaces in three-dimensional homogeneous manifolds. Mainly, when the ambient space is the product space $H^2 \times R$, where H^2 is the hyperbolic plane or when the ambient is the Heisenberg space. We focus the following geometric phenomena: the maximum principle, the a priori estimates, the Dirichlet problems, the symmetry and uniqueness, the stability, the embeddedness and the finite or infinite total curvature We also study properties of some non linear elliptic P. D. E. arising from geometry; e. g. the minimal surfaces equation. Finally, we study general principles for minimal hypersurfaces in Riemannian manifolds.

Minimal ends in $H^2 \times R$

We studied in cooperation with Laurent Hauswirth, Barbara Nelli and Eric Toubiana, the minimal ends of finite total curvature in $H^2 \times R$. We have established the behavior of such an end by making a complete geometric description, determining the horizontal section of this end, by intercepting it with a slice. We showed that a horizontal section of an end of finite total curvature converges geometrically to a horizontal geodesic [H-N-SE-T]. Moreover, we also proved that a minimal complete end *E* with finite total curvature is properly immersed and that the Gaussian curvature of *E* is locally bounded in terms of the geodesic distance to its boundary. The proof uses some basic hyperbolic geometry and classical surfaces theory [SE-T8], [SE-T9]. This work was based in previous works as the pioneer study by L. Hauswirth and H. Rosenberg on minimal surfaces of finite total curvature [H-R]. It was also based on the theory of harmonic maps developed by Z. Han , L. Tan , A. Treibergs and T. Wan [H -T -T- W] and Y. N. Minsky [My]. Using the results mentioned above, and other types of arguments, such as the Alexandrov reflection principle, based on the maximum principle, we deduced an uniqueness Schoen type theorem [S] in the context of finite total curvature [H-N- SE- T, Main Theorem]. This result characterizes immersed minimal surfaces in $H^2 \times R$ with two distinct complete ends of finite total curvature, each end asymptotic to a vertical plane as the models surfaces discovered independently by J. Pyo [P] and F. Morabito , M. Rodriguez [M -R]; now called *the horizontal catenoids*.

–On the other hand, in collaboration with Barbara Nelli and Eric Toubiana, we studied the influence of the behavior of the asymptotic boundary on the geometry of minimal hypersurface of $H^n \times R$ in several geometric situations. The definition of the asymptotic bord of $H^2 \times R$ can be found in [SE-T7] or

[SE-T10]. In particular, we obtained a characterization of the *n*-catenoid in $H^n \times R$ in terms of the asymptotic bord [N-SE-T, Theorem 2.1, 4.2]. The n-catenoid were constructed in [B-SE1, Proposition 3.2], when n≥3. In fact, this result is a Schoen type theorem [S] in the context of infinite total curvature. We established also a maximum principle for minimal surfaces contained in a closed half-space [N-SE-T, Theorem 3.1, 4.4].

We notice that the techniques of the paper [N-SE-T] has been applied in others papers, see, for instance, [SE-T7] and [SE-T10].

─In a project with collaboration of Eric Toubiana [SE-T7], we have developed further these studies leading to the investigation of the properties of immersed minimal annulus in $H^2 \times R$ with prescribed asymptotic boundary. We showed in this work that an oriented stable minimal annulus end *M* whose finite asymptotic boundary is contained in two vertical lines and converges to a vertical plane, has finite total curvature. If the end is embedded we showed that, up to a compact part, it is a *horizontal minimal graph with respect to a geodesic*, or simply, *a horizontal minimal graph*, see the definition in [H-N-SE-T]. There is another notion of "horizontal graph", see below. Notice that a horizontal minimal graph is stable. This is well-known and follows from the classical criterion of stability for minimal surfaces: Let *M* be an oriented connected minimal surface immersed into $H^2 \times R$. If there exists a positive smooth function *u* on a bounded domain Ω of *M* satisfying $Lu = 0$, where *L* is the stability operator [B-SE1, Section 2.2], then Ω is stable [C-M, Lemma 1.36]. Classic examples show that our hypothesis on the asymptotic boundary is necessary [SE1], [SE-T5].

─We point out that in the Euclidean space a famous result of D. Fisher-Colbrie [F-C] states that a complete oriented minimal surface has finite index, if and only if it has finite total curvature. Finite index of a complete minimal surface immersed in a tridimensional manifold means stability of the surface outside a compact subset [F-C]. Observe that if the ambient space $H^2 \times R$, in a joint work with Pierre Bérard, we showed that finite total curvature of a complete oriented minimal surface implies finite index [B-SE1], but the converse does not hold. There are many examples of oriented complete stable minimal surfaces with infinite total curvature. Indeed, there are families of oriented complete stable minimal surfaces invariant by a nontrivial group of screw-motions [SE1]. The catenoid in $H^2 \times R$ is another counter-example: It has infinite total curvature and index one [B-SE1 Proposition 3.3 and Theorem 3.5].

─We pursue further the research about the geometry of minimal surfaces in $H^2 \times R$ with finite or infinite total curvature, see [SE-T10]. In [SE-T10] we define the finite asymptotic boundary. In this paper we prove a *phenomenon of concentration of total curvature* in $H^2 \times R$. Under some geometric conditions on the finite asymptotic boundary of an oriented stable minimal surface *M* immersed in $H^2 \times R$ (not necessarily complete nor properly immersed), *M* has infinite total curvature. The result in [SE-T10] is "local" in nature: If a part of the finite asymptotic boundary of a minimal surface *M* is a graph of a sufficient regular function over an arc of the asymptotic boundary of H^2 and if *M* extends regularly up to the asymptotic boundary (as a subset of the Euclidean three space), then *M* has infinite total curvature.

In fact, we prove that if the finite asymptotic boundary is "nice" and "non vertical", then *M* has infinite total curvature. In particular, we infer that a minimal graph *M* in $H^2 \times R$ whose finite asymptotic boundary is a graph over an arc of the asymptotic boundary of H^2, different from the finite asymptotic boundary of the boundary of M, has infinite total curvature. Thus, a consequence of the theorem is the following: Let *M* be a stable minimal surface immersed into $H^2 \times R$ with compact boundary (e. g. a minimal graph with compact boundary), whose asymptotic boundary is a graph over the whole $\partial_\infty H^2 \times \{0\}$. Then *M* has infinite total curvature. We refer to [SE-T5] for an existence result of such a graph. In the classical case of the end of the catenoid, this result follows from an explicit computation carry out in [B-SE1, Proof of the Proposition 3.3]. The main result in [SE-T10] is in certain sense a counterpart of the main result in [SE-T7]. The techniques applied in [SE-T10], are a combination of the techniques used in some previous work [SE-T5], [H-N-SE-T], [N-SE-T], [SE-T7] and [SE-T10], with the a priori curvature estimates obtained in [R-S-T].

—We exhibit a remarkable example of a minimal graph such that in a neighborhood whose asymptotic boundary is a compact vertical segment the total curvature is finite, but the total curvature of the graph is infinite, by the theorem cited before.

—We give also many simple examples, of complete and non complete, proper and non proper, embedded and non embedded minimal surfaces. We also exhibit their asymptotic boundary pointing out some peculiar behaviors.

We are working now in a new project on the *horizontal minimal equation* (with respect to a geodesic) in $H^2 \times R$ motivated by our results cited before. Recall that there is another notion of "horizontal equation" studied below in $H^n \times R$.

We are planning to study the following problems (among others):

- The Dirichlet problem over bounded and unbounded domains: existence and uniqueness.
- Construction of new minimal complete examples.
- Characterizations of classical examples.

Horizontal minimal graphs in $H^n \times R$.

We study in the individual paper [S-E3] a class of horizontal minimal equations in $H^n \times R$, involving a family of second order elliptic PDE's indexed by a parameter ε in [0, 1].

$$\sum_{k=1}^{n-1}\left[g^2\left(1+g_{x_1}^2+\cdots+g_{x_k}^2+\cdots g_{x_{n-1}}^2\right)+g_t^2+\frac{\varepsilon}{n-1}\right]g_{x_k x_k}+\left(1+\sum_{k=1}^{n-1}g_{x_k}^2\right)g_{tt}-2\sum_{k=1}^{n-1}g_{x_k}g_t g_{x_k t}-$$
$$2g^2\sum_{1\le j<k\le n-1}g_{x_j}g_{x_k}g_{x_j x_k}+(n-1)g\left(1+\sum_{k=1}^{n-1}g_{x_k}^2\right)+(n-2)\frac{g_t^2}{g}=0$$

When ε = 0, we recover the horizontal minimal equation which is not a strictly elliptic EDP in general. When ε > 0, we obtain a strictly elliptic PDE that we call the ε- horizontal minimal equation. We infer a priori estimates for the horizontal length and a priori boundary gradient estimates that are quite general and quite natural as we explained in the text. We also obtain a priori global gradient estimates in the presence of a strong constraint on the horizontal length, which seems to be natural for this kind of PDE. This fact is somehow related to the following phenomenon: There are no solutions to the horizontal minimal equation over a bounded strictly convex domain, which vanishes on the boundary of this domain and that are continuous up to the boundary. This rather surprising phenomenon, in dimension 2, follows from the asymptotic theorem deduced in [SE-T7]. In arbitrary dimension it follows from the generalization accomplished in [N-ET]. Furthermore, we deduce an existence result for the ε-horizontal minimal equation in the two-dimensional case, that combined together with our uniform a priori estimates and elliptic theory yields an existence result for the horizontal minimal equation (ε=0). The uniqueness of the solutions obtained for the horizontal minimal equation is shown for admissible boundary data satisfying an admissible bounded slope condition. This follows from the Radó type theorem mentioned above.

The techniques are non linear P.D.E. elliptic theory [GT] and Differential Geometry (minimal surfaces techniques). See also, [SE4], "*Lista de Teoria do Grau e Aplicações Analíticas, Topológicas e Geométricas*".

We set in the context of elliptic quasilinear EDP´s several new (we believe interesting) open problems.

Minimal surfaces in Heisenberg space.

We studied the minimal surface equation in the Heisenberg space, Nil_3 [N-SE-T2]: A geometric proof of nonexistence of minimal graphs over non convex bounded and unbounded domains taking certain prescribed continuous data is achieved. Our proof holds in the Euclidean space as well. We solved the Dirichlet problem for the minimal surface equation over bounded and unbounded convex domains, taking bounded, piecewise continuous boundary value. In fact, we were able to construct a Scherk type minimal surface and we use it as a barrier to construct non trivial minimal graphs over a wedge of angle between $\pi/2$ e π taking non negative continuous boundary data, having at least quadratic growth. In the case of a half-plane, we were also able to give solutions (with either linear or quadratic growth), provided some geometric hypothesis on the boundary data.

Finally, some open problems arising from our work, are posed in the paper. We are planning to study these open problems in a future project.

Hypersurfaces with constant r-mean curvature in $H^n \times R$

It is known that the hypersurfaces of constant mean curvature in the ambient space $H^n \times R$ has been studied in recent years by several researchers for example, ([B-SE2] , [E-SE], [SE-T6], [Sp]). The idea now is to explore the geometry of hypersurfaces in $H^n \times R$ with some symmetric function of curvature S_r constant (S_r=cst), or say constant r-mean curvature H_r, where $S_r=\binom{n}{r}H_r$. We pause momentarily to remember that properties of hypersurfaces with some S_r=cst in spaces of constant curvature (*space form*) have been extensively studied by many researchers in the late twentieth century. Among others, we find the following researchers: Hilário Alencar , Manfredo do Carmo Lucas Barbosa , Gervásio Colares, Louis Nirenberg , Luis Caffarelli , Joel Spruck , Jorge Hounie , Maria Luiza Leite , Sebastian Montiel , Antonio Ros , Robert Reilly and Maria Fernanda Elbert [A-Doc-C] [B -C1], [C2-B], [CNS], [L1-H], [H-L2] , [L1] [L2] , [M- Ros] , [Re] , [Ro] , [CR] , [El] . In the case of the product $M^n \times R$, we highlight the work done by Xu Cheng and Harold Rosenberg. They obtained some a priori height estimates for hypersurfaces with some r -th symmetric function of constant- positive curvature. Moreover, they did some applications [C -R]. One goal of our work is intended also to fill the gap on the theory by providing several model examples in $H^n \times R$. Some of these examples are useful as important *barriers*. We deduced some uniqueness results that characterize these barriers for constant r-mean curvature. Initially working in collaboration with Maria Fernanda Elbert UFRJ [E- SE2] , we initiated the study of the construction of hypersurfaces in $H^n \times R$ with S_r=cte, having various interesting geometric properties . We could infer several basic formulas for the equation of the hypersurface with S_r=cte for a vertical graph in $M^n \times R$, applying them to $H^n \times R$. We deduced a suitable divergence form for the r-mean curvature of a vertical graph in $M^n \times R$. In the case where the ambient is $H^n \times R$, the r-mean equation reads:

$$(r+1)S_{r+1} = F^2 div\left(P_r \frac{\nabla u}{W}\right) + \frac{(2-n)F < P_r \nabla u, \nabla F >}{W} - F^2(n-r) < P_{r-1}\left(\frac{\nabla u}{W}\right), \frac{\nabla u}{W} >.$$

where $F=x_n$ e *div* e ∇ denote the divergence and the gradient in R^n, respectively . The symbol P_k denotes the k th Newton tensor associate.

Making use of these formulas, following techniques and methods developed in [L2], [El], [E- SE], we have established a first integral for hypersurfaces invariant by parabolic screw motions in $H^n \times R$ with prescribed S_2 . From this point, we have obtained explicit examples of entire graphs and other examples of complete hypersurfaces (which are complete horizontal graphs) when $S_2 = 0$.

— We found that there is a single family consisting of entire vertical graphs extrinsic curvature $K_{ext} = S_2 = 0$ (n=2) , invariant by parabolic translations . Interestingly, each member of the family has constant mean curvature 0 < H < 1/2 and each such *H* can be realized.

Came the open question whether there are other vertical graphs with zero extrinsic curvature and non-zero constant mean curvature in $H^2 \times R$?
— We also obtained examples of surfaces with prescribed (non constant) extrinsic curvature invariant by parabolic screw motions.

— We also obtained examples of hypersurfaces not complete when S_2 = cst $\neq 0$. For n=2, we got a " cylinder " of zero extrinsic curvature revolution, which is an entire vertical graph , which is not too surprising .
It should be noted that this first work on hypersurfaces with S_r =cst in $H^n \times R$, with the collaboration of Maria Fernanda Elbert, we found the critical value $\frac{n-r}{n}$ which appears in the theory. Moreover, we found explicit formulas for the rotational S_r =cst with numerous examples that are used in the text as geometric barriers:
— Among these important barriers include compact , embedded and strictly convex hypersurfaces of revolution , when $H_r > \frac{n-r}{n}$, which are classified .
— Further, when $0 < H \leq \frac{n-r}{n}$, we constructed barriers that are strictly convex entire vertical graphs of revolution generated by a strictly convex curve in a vertical plane

On the other hand, for $H_r > \frac{n-r}{n}$, we proved that there is no entire rotational H_r-graph . It is interesting to investigate the complete H_r -hypersurface for this case. We asked, for instance: Is there a noncompact complete embedded H_r -hypersurface in $H^n \times R$, $H_r > \frac{n-r}{n}$, with only one end ?
If n=2, in [E-G-R, Theorem 7.2], the authors proved that if $K_{ext} > 0$ (or H > 1/2), there is no properly embedded K_{ext}-surface (or H-surface) in $H^n \times R$ with finite topology and one end.

In short, we have obtained several explicit examples having interesting geometric properties both embedded rotational and entire graphs that support the theory. Using these geometric barriers, we established results of symmetry, uniqueness and a priori estimates . Among the a priori estimates obtained we include the a priori height estimates for a compact graph, whose boundary is contained in a slice, with prescribed *r- mean* curvature satisfying $0 < H_r(p) \leq \frac{n-r}{n}$.

Among the results of uniqueness we highlight an Alexandrov type theorem classifying the compact rotational constant *r - mean* curvature mentioned above.
We observe that the Alexandrov theorem for H_r–hypersurfaces in Euclidean space, hyperbolic space and half-sphere was proved, independently, by Korevaar [K] and Montiel-Ros [M-R].

We have then seen that for $H_r > \frac{n-r}{n}$, the only compact embedded immersion is a rotational n-sphere and that there exist no entire rotational *Hr*-graph. We ask if for the particular case r = n we have the same behavior of the case r = n = 2, that is:

Is a complete immersion in $H^n \times R$, with H_r = constant > 0 and r = n a rotational n-sphere ?

It will also be interesting to establish sharp a priori height estimates of time sharp for constant $H_r > \frac{n-r}{n}$, bearing in mind the construction of compact embedded rotational hypersurfaces .

Geometric Partial Differential Equations

Barbara Nelli and I are working to develop our project of writing a survey/book about geometric partial differential equations in the hyperbolic space. This would be an important step of our long-standing collaboration. The main goals of the survey are to show how to apply the maximum principle to study minimal and constant mean curvature hypersurfaces and to use classic elliptic theory and differential geometry to solve problems about geometric equations. More precisely, we focus the constant mean curvature equations in the hyperbolic space.
We plan to develop some relevant methods that arise in many published journal articles about the subject. We aim at explaining some basic tools and the most current techniques in geometric partial differential equations. We mainly deal with results by the authors and co-authors. Moreover the reader will be guided
through many timely related results.
Our target is both researchers and graduate students interested in elliptic partial differential equations from a geometric viewpoint.

Our strategy is as follows.
We start by giving an overview of classic results about the minimal equation in the Euclidean space.
We state the maximum principle and we apply it in several geometric configurations.
Then, we explain how to use elliptic theory and geometric barriers to get estimates for solutions of many Dirichlet problems in the hyperbolic space.
We notice that, the estimates are a crucial step towards the existence of solutions.
Finally we discuss some open problems and we are planning to prove new theorems on the matter.

Bibliography:

[A-doC-C] H. Alencar, M. do Carmo, G. Colares. *Stable hypersurfaces with constant scalar curvature.* Mat. Z. 213, 117-131, 1993.

[B-C1] J. L. Barbosa, G. Colares. *Soap bubbles in space forms.* Mat. Contemporânea, 9, 23-40, 1995.

[B-C2] J. L. Barbosa, G. Colares. *Stability of hypersurfaces with constant r-mean curvature.* Ann. Global Ana. Geom. 15, 277-297, 1997.

[Br] R. L. Bryant. *Complex analysis and a class of Weingarten surfaces.* Arxiv: 1105.5589v1 [DG], 2011.

[B-SE1] P. Bérard and R. Sa Earp. *Minimal hypersurfaces in $H^n \times R$, total curvature and index*. Bollettino dell'Unione Matematica Italiana, 9 (3), 341-362, 2016. DOI: 10.1007/s40574-015-0050-0. arXiv: 0808.3838v3 [Math. DG].

[B-SE2] P. Bérard and R. Sa Earp. *Examples of H-hypersurfaces in $H^n \times R$ and geometric applications.* Matemática Contemporânea, 34, 19-51, 2008. Escrito em homenagem aos oitenta anos de Manfredo do Carmo

[C-M] T.H. Colding, W.P. Minicozzi *A course in minimal surfaces*, Graduate Studies in Mathematics, 121. American Mathematical Society, Providence, RI, 2011.

[C-N-S] L. Caffarelli, L. Nirenberg and J. Spruck. *Nonlinear second-order elliptic equations V. The Dirichlet problem for Weingarten hypersurfaces.* Comm. Pure and Appl. Math., XLI, 47-70, 1988.

[C-R] X. Cheng and H. Rosenberg. *Embedded positive constant r-mean curvature hypersurfaces in $H^n \times R$.* Anais da Academia Brasileira de Ciências 77, 2, 183-199, 2005.

[El] M. F. Elbert. *On complete graphs with negative r-mean curvature.* Proceedings American Mathematical Society, 67, 1443-1450, 2000.

[E-SE] M. F. Elbert and R. Sa Earp. *All solutions of the CMC-equation in $H^n \times R$ invariant by parabolic screw motion.* Annali di Matematica Pura ed Applicata, 2014, 193, 1, 103-114, 2014.

[E-SE2] M. F. Elbert and R. Sa Earp. *Constructions of H_r-hypersurfaces, barriers and Alexandrov theorem in $H^n \times R$.* Annali di Matematica Pura ed Applicata, 194:1809–1834, 2015.

[F-C] D. Fisher-Colbrie. *On complete minimal surfaces with finite Morse index in three manifolds*. *Inventiones Mathematicae* **82,** 1, 121-132, 1985.

[H-R] L. Hauswirth and H. Rosenberg, *Minimal Surfaces of Finite Total Curvature in $H^2 \times R$*, Matemática Contemporânea 31, 65-80, 2006.

[G-T] D. Gilbarg and N.S. Trudinger. Elliptic partial differential equations of second order, Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001.

[H-SE-T] L. Hauswirth, R. Sa Earp and E. Toubiana. *Associate and conjugate minimal immersions in $M^2 \times R$.* Tohôku Math J, 60, 267-286, 2008.

[H-N-SE-T] L. Hauswirth, B. Nelli, R. Sa Earp and Eric Toubiana. *A Schoen theorem for minimal surfaces in $H^2 \times R$.* Advances in Mathematics, 274, 199-240, 2015.

[H-T-T-W] Z.C. Han, L.F. Tam, A. Treibergs, T. Wan. *Harmonic maps from the complex plane into surfaces with nonpositive curvature*, Communications in Analysis and Geometry, Vol 3, N. 1, 85–114, 1995.

[H-L1] J. Hounie and M. L. Leite. *The maximum principle for hypersurfaces with vanishing curvature functions.* J. Diff. Geom. 41, 247-258, 1995.

[H-L2] J. Hounie and M. L. Leite. *Two-ended hypersurfaces with zero scalar curvature.* Indiana Univ. Math. J. 48, 3, 867-882, 1999.

[K] N. Korevaar. *Sphere theorems via Alexandrov for constant Weingarten curvature hypersurfaces-appendix to a note of A. Ros.* J. Differential Geometry 27, 221-223, 1988.

[L1] M. L. Leite. *Rotational hypersurfaces of space forms with constant scalar curvature.* Manuscripta Math. 67, 285-304, 1990.

[L2] M. L. Leite. *The tangent principle for hypersurfaces with null intermediate curvature.* XI Escola de Geometria Diferencial- UFF Brazil, (2000).

[M-Ros] S. Montiel, A. Ros. *Compact hypersurfaces ; the Alexandrov theorem for higher mean curvature,* Differential Geometry, a Symposium in honor of Manfredo do Carmo. Pitman monogr. 52, 279-296, Longman Scientific and Technical, 1991.

[M-R] F. Morabito, M. Rodriguez: *Saddle towers and minimal k-noids in* $\mathrm{H}^2 \times \mathrm{R}$*;* Journal of the Institute of Mathematics of Jussieu, 11 (2): 333-349 (2012).

[My] Y. N. Minsky. *Harmonic maps, length, and energy in Teichmüller space*, J. Differential Geom. 35, No. 1, 151–217, 1992.

[N-SE-T] B. Nelli, R. Sa Earp and E. Toubiana. *Maximum Principle and Symmetry for Minimal Hypersurfaces in* $H^n \times R$. Annali della Scuola Normale Superiore di Pisa, Classe di Scienze, (2), XIV, 387-400, 2015.

[N-SE-T2] B. Nelli, R. Sa Earp and E. Toubiana. *Minimal Graphs in* Nil_3 *: existence and non-existence results*. Calculus of Variations & PDE, 56, 2. Doi: 10.1007/s00526-017-1123-y.

[P] J. Pyo. *New complete embedded minimal surfaces in* $H^2 \times R$. Ann. Global Anal. Geom. 40, 2, 167–176, 2011.

[Re] R. Reilly. *Variational properties of the mean curvature for hypersurfaces in space forms.* J. Diff. Geom. 8, 465-477, 1973.

[Ro] H. Rosenberg. *Hypersurfaces of constant curvature in space forms.* Bull. Sci. Math. 117, 211-239, 1993.

[R-S-T] H. Rosenberg, R. Souam and E. Toubiana, *General curvature estimates for stable H-surfaces in 3-manifolds and applications,* Journal of Differential Geometry 84 (2010), 623–648.

[SE-T1] R. Sa Earp and E. Toubiana. *A note on special Weingarten surfaces.* Matemática Contemporânea 4, 108-118, 1993.

[SE-T2] R. Sa Earp and E. Toubiana. *Sur les surfaces de Weingarten spéciales de type minimale*. Boletim da Sociedade Brasileira de Matemática, 26, No. 2, 129-148, 1995.

[SE-T3] R. Sa Earp and E. Toubiana. *Classification des surfaces spéciales de révolution de type Delaunay.* American Journal of Mathematics, 121, No. 3, 671-700, 1999.

[SE-T4] R. Sa Earp and E. Toubiana. *Screw motion surfaces in* $H^2 \times R$ *and* $S^2 \times R$. Illinois J. Math., 49, No. 3, 1323-1362, 2005.

[SE-T5] R. Sa Earp and E. Toubiana. *An asymptotic theorem for minimal surfaces and existence results for minimal graphs in* $H^2 \times R$. Mathematische Annalen, 342, No. 2, 309-331, 2008.

[SE-T6] R. Sa Earp and E. Toubiana. *Minimal graphs in* $H^n \times R$ and R^{n+1}. Annales Inst. Fourier 60, 2373-2402, 2010.

[SE-T7] R. Sa Earp and E. Toubiana. *A minimal stable vertical planar end in* $H^2 \times R$ *has finite total curvature.* Journal of the London Mathematical Society, 92 (3), 712-723, 2015.

[SE-T8] R. Sa Earp and E. Toubiana. *Introduction à la géométrie hyperbolique et aux surfaces de Riemann*, 2009, Cassini.

[SE-T9] R. Sa Earp and E. Toubiana. *Topologie, courbure et structure conforme sur les surfaces*. R^G, 2015 (eBook-**open access**). Doi: 10.13140/RG.2.1.3623.1769. https://www.researchgate.net/publication/288516126_Topologie_courbure_et_structure_conforme_sur_les_surfaces

[SE-T10]. R. Sa Earp and E. Toubiana. *Concentration of total curvature of minimal surfaces in* $H^2 \times R$. Mathematische Annalen, 2016. arXiv:1603.03335 [Math.DG].

[SE1] R. Sa Earp. *Parabolic and hyperbolic screw motion surfaces in* $H^2 \times R$. Journal of the Australian Math. Society, 85, 113-143, 2008.

[SE2] R. Sa Earp. *Uniqueness of minimal surfaces whose boundary is a horizontal graph and some Bernstein problems in* $H^2 \times R$. Mathematische Zeitschrift, 273, 1, 211-217, 2013.

[SE3] R. Sa Earp. *Uniform a priori estimates for a class of horizontal minimal equations*, Mathematische Nachrichten, 290, 2-3, 415-434, 2017. Doi:10.1002/mana.201500090.

[SE4] R. Sa Earp. *Lista de Teoria do Grau e Aplicações Analíticas, Topológicas e Geométricas*. Doi: 10.13140/RG.2.1.2696.4244. https://www.researchgate.net/publication/294261135_Lista_de_Teoria_do_Grau_e_Aplicacoes_Analiticas_Topologicas_e_Geometricas

[S] R. Schoen. *Uniqueness, symmetry and embeddedness of minimal surfaces.* Jour. of Diff. Geom. 18, 791-809, 1983.

[Sp] J. Spruck. *Interior gradient estimates and existence theorems for constant mean curvature graphs in* $M^n \times R$. Pure Appl. Math. Q. 3, No. 3, part 2, 785-800, 2007.