Symmetry of properly embedded special Weingarten surfaces in $\mathbf{H^3}$

Ricardo SA EARP*
Departamento de Matemática
Pontifícia Universidade Católica
Rua Marquês de São Vicente, 225
22453-900 Gávea RIO DE JANEIRO
E-mail: earp@mat.puc-rio.br
BRESIL

Eric TOUBIANA*
Département de Mathématiques
Université Paris VII
2, Place Jussieu
75251 PARIS - Cedex 05
E-mail: toubiana@math.jussieu.fr
FRANCE

Abstract: In this paper we prove some existence and uniqueness results about special Weingarten surfaces in hyperbolic space.

1. Introduction

In this paper we consider C^2 surfaces M in $\mathbf{H^3}$ oriented by a global unit normal field N whose mean curvature H and extrinsic Gaussian curvature K_e, satisfy a Weingarten relation of the form:

$$H = f(H^2 - K_e). \tag{1}$$

We shall require that f is a C^1 function defined on $[0, +\infty[$, satisfying:

$$\forall t \in [0, +\infty[,\ 4tf'^2(t) < 1. \tag{2}$$

We will say that f is elliptic if f satisfies inequality (2). If M satisfies relation (1) for f elliptic, we will call M either a *special Weingarten surface* or a $f - surface$. In euclidean space they have been studied by Hopf [10], Hartman and Wintner [9], Chern [5]. More recently, there has been main progress, as much in euclidean case as in hyperbolic case, done by Bryant [4], Braga Brito [3], Rosenberg [15]. In section 2 we will briefly describe this and we will develop some fundamental basis of the theory. Both authors have constructed a family of embedded (and immersed) complete rotational Weingarten surfaces in $\mathbf{R^3}$. They have also proved uniqueness of such surfaces (see [16], [17] and [18]).

An interesting question in this subject is whether or not a f-surface inherits the symmetries of its boundary: When $f = H$ (constant) and $\mid H \mid \leq 1$, Nelli and Rosenberg [14] solved the boundary value problem for compact embedded surfaces and spherical boundary. The sharp related theorem for M *immersed* was proved by Barbosa and the first author in [2]. Here, we shall generalize Rosenberg and Nelli's result (cited above) for f-surfaces when $f^2 \leq 1$.

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In [1], it is stated that when the boundary consists of two circles invariant by a 1-parameter group of rotations of $\mathbf{H^3}$, then stability implies the surface is also invariant by rotations. Barbosa and the first author have asked if the assumption of stability could be exchanged by embeddedness [1]. In this direction, we are able to obtain the following result: *Assume $\partial M = C_1 \cup C_2$, where C_1 and C_2 are circles of same radius. Suppose that ∂M is invariant by a rotation. Then there is a positive real number d_0, such that if $\mid H \mid = 1$ and $dist(C_1, C_2) \geq 2d_0$, M is a piece of a Catenoid Cousin.* In fact, we prove an analogous result for a wider class of f-surfaces (see Theorem 1, section 3).

We notice that symmetry results proved by several authors concerning H-surfaces whose asymptotic boundary consists of a point, a circle or the union of two circles [6], [7], [13], are valid for f-surfaces, since the maximum principle still holds (see section 2). On the other hand, others characterizations of f-surfaces with boundary a circle, proved in [3] when the ambient space is euclidean space, are also stated in section 2.

Finally, we will give existence and uniqueness results of rotational f-surfaces in section 4, under further hypothesis on f. Moreover, we will prove that the geometric behaviour of such surfaces is the same as the related well-known rotational H-surfaces in $\mathbf{H^3}$ (see Theorem 2 and 3).

2. Some basic properties: the structure of principal lines and the maximum principle. Immediate consequences and generalizations.

In this section we shall establish some basic properties of special Weingarten surfaces. We will state some results of Bryant's theory [4] on the structure of the principal line distribution and we will write-down the maximum principle. We will point out that one can apply Alexandrov techniques to obtain immediate consequences generalizing well-known results for the case $H = cst$.

2.1. Principal lines distribuition (R.Bryant's theory)

Let M be an immersed special Weingarten surface satisfying (1) with respect to a global unit normal field N.

In a pioneer paper R.Bryant had an amazing idea to construct a new metric ds^2 by means of the function f given in the Weingarten relation (1) to obtain constant mean curvature type results, as follows:

Proposition 1 ([4])
Let z be an isothermal coordinate of the Bryant's metric ds^2 in U. If $ds^2 = \lambda^2 \mid dz \mid^2$ then the function $\phi(z) = \lambda^2(z)\left(\frac{l-n}{2} - im\right)$ is holomorphic, where l, m, and n are the coefficients of the second fundamental form with respect to a local oriented orthonormal frame field. Furthermore, the holomorphic quadratic form $\Phi(z) = \phi(z)(dz)^2$ is defined globally on M.

Proposition 2 ([4])
Either M is totally umbilic or else the umbilic locus consists entirely of isolated points of strictly negative index.

We remark that when f is analytic Proposition 2 follows from the proof of theorem 3.2, pg 142, of Hopf's famous book (see [10]). Notice that z_0 is an umbilic point of M if and only if $\phi(z_0) = 0$. Thus, in a neighborhood of $z_0 = 0$ the holomorphic quadratic form is $z^n(dz)^2$, hence the index at z_0 is $\frac{-n}{2}$.

We emphasize that R. Bryant proved (in view of Propositions 1, 2) the following generalization of Hopf's theorem: *A closed genus zero special Weingarten surface satisfying (1) is a round sphere.*

Remark 1

We can derive from Bryant's construction and straightforward computations that the principal line distribution is given by $Im\Phi = 0$.

2.2 Maximum principle

In this section we will establish interior and boundary maximum principle for special Weingarten surfaces in hyperbolic space. We shall apply maximum principle in subsequent sections to derive several uniqueness and symmetry results.

Lemma 1 (Interior maximum principle)

Suppose M_1, M_2 are C^2 surfaces in $\mathbf{H^3}$ which are given locally as graphs of C^2 functions u, v. Suppose the tangent planes of both M_1, M_2 agree at a point p.

Let $H(N_1)$ and $H(N_2)$ be the mean curvature functions of u and v with respect to unit normals N_1 and N_2 that agree at p. Let K_i be the extrinsic Gaussian curvature of $M_i, i = 1, 2$. Suppose M_i satisfy

$$H(N_i) = f({H_i}^2 - K_i), i = 1, 2$$

for f elliptic.

Consequently, if $u \leq v$ near p then $M_1 = M_2$ near p.

Lemma 2 (Boundary maximum principle)

Consider M_1, M_2 as in the statement of the interior maximum principle with C^2 boundaries B_1, B_2 given by restritions of u and v.

Suppose $T_pM_1 = T_pM_2$ and $T_pB_1 = T_pB_2$ with p in the interior of both B_1 and B_2.

Suppose M_1, M_2 satisfy (1) and (2) with respect to the same normal N at p.

Then if $u \leq v$ near p we have $M_1 = M_2$ near p.

The analogous statement in euclidean space is proved in [3], so we omit the proof here.

2.3. Immediate consequences

In this paragraph we will derive some applications extending known results of the constant mean curvature theory.

Proposition: *Alexandrov theorem for special Weingarten surfaces in hyperbolic space.*

If M is a closed embedded special Weingarten surface in $\mathbf{H^3}$ satisfying (1) for f elliptic then M is a sphere.

The proof makes use of Alexandrov reflection principle, with totally geodesic hyperbolic planes in the place of ordinary planes in the Euclidean case. For Alexandrov techniques the reader is referred to [3], [13] and [15].

The Alexandrov reflection principle is a beautiful and fundamental tool for several nice theorems in the constant mean curvature surfaces theory. It provides several analogous results for special Weingarten surfaces. At this point, we will make use of the unit ball model for the hyperbolic space $\mathbf{H^3}$. Thus, $\overline{\mathbf{H^3}}$ will denote the closed unit ball and $S^2(\infty)$ will denote the unit sphere. We will recall some basic definitions given in [7]. Let M be an embedded surface in $\mathbf{H^3}$. We will denote $\partial_\infty M$ the intersection of the closure of M with $S^2(\infty)$, and we will call $\partial_\infty M$ the asymptotic boundary of M. We will say M is $C^2 - regular$ at infinity if $\overline{M} \subset \overline{\mathbf{H^3}}$ is a C^2 surface with boundary in $\overline{\mathbf{H^3}}$, and $\partial_\infty M$ is a C^2 curve of $S^2(\infty)$.

Now we set up the following symmetry statements: *Let M be a connected complete properly embedded surface in $\mathbf{H^3}$. Suppose M is a special Weingarten surface satisfying (1) for f elliptic. Assume M is $C^2 - regular$ at infinity. Then*
(1) If the asymptotic boundary is one point then M is a horosphere.
(2) If the asymptotic boundary is one circle then M is a geodesic plane or a equidistant surface.
(3) If the asymptotic boundary is the union of two disjoint circles then M is a surface of revolution.

The proof is an application of Alexandrov principle as in the constant mean curvature case, so we omit the proof here. Let us give the references of the above statements in the case $H = cst$. If $H = 0$ and M is immersed, statement 3 was proved by Levitt and Rosenberg [13]. Statements 1 and 2 was derived by Do Carmo and Lawson [6], while statement 3 for $H = cst \neq 0$ was deduced by Do Carmo, Gomes and Thorbergsson [7].

Another amazing symmetry result for $H = cst$, is the Hsiang theorem which implies that if M is cylindrically bounded, complete and embedded then M has rotational symmetry [11]. We now assert the following: *Let M be a connected properly embedded complete special Weingarten surface in $\mathbf{H^3}$ satisfying (1) for f elliptic. If M is cylindrically bounded then M is a surface of revolution.*

The proof is similar to the constant mean curvature case.

3. **Symmetry arising from the boundary**

Is this section we shall establish several symmetry results for special Weingarten surfaces arising from the symmetry of the boundary. Some of them are not known even in the constant mean curvature case. Others are generalizations of well-know results for $H = cst$ or simple extensions of results obtained recently in euclidean space.

Let C_1, C_2 be two circles in $\mathbf{H^3}$ with same radius. Suppose $C_1 \cup C_2$ is invariant by a rotation. That is, C_1, C_2 are on a same cylinder C. We will denote by $int(C)$ the component of $\mathbf{H^3}$-C containing the axis of C. The distance $d(C_1, C_2)$ between C_1 and C_2 is the distance between the "parallel" geodesic planes determined by C_1 and C_2.

It is well-known (see [11]) that if $C_1 \cup C_2$ is the boundary of a minimal surface of revolution then $d(C_1, C_2) \leq 2d_0$, where d_0 is the maximum value of the function $x(y_0)$, given by:

$$x(y_0) = \int_{y_0}^{+\infty} \frac{sinhy_0 coshy_0}{coshy} \sqrt{\frac{1}{sinh^2 y cosh^2 y - sinh^2 y_0 cosh^2 y_0}} dy.$$

Clearly, the above function is bounded and numerical computations shows that $d_0 \approx 0.5$ (The above formula follows from the first integral of the second order differential equation satisfied by the generating curve [11]).

From now on, we will denote by d_0 the positive real number defined above.

We shall need the following lemma:

Lemma 3

Let M be a compact connected embedded special Weingarten surface in $\mathbf{H^3}$ satisfying (1) for f elliptic. Assume ∂M is the union of two distint circles C_1, C_2 lying in a cylinder C of axis γ. Then if $M \subset \overline{int(C)}$ and $M \cap C = \partial M$, M has a plane of symmetry.

Moreover, if f does not change sign and if $d(C_1, C_2) \geq 2d_0$ then M lies between the parallel planes containing C_1 and C_2.

Proof

We denote by P the plane of symmetry of $C_1 \cup C_2$, that is the plane such that C_1 is the symmetric image of C_2 with respect to P. We will prove P is a plane of symmetry of M as follows: We start by moving P along γ by doing hyperbolic translations. This movement gives rise to a 1-parameter family $\{P_t\}$ of geodesic planes starting from $P = P_0$ and cutting ortogonally γ. We may choose a parameter t such that t is the oriented distance between P_t and P. We claim that our assumptions in the first statement allow us to apply Alexandrov reflection, by the means of the family $\{P_t\}$, to conclude M inherits the symmetry of ∂M: Actually, to explain how to do this, we will denote by ${M_t}^*$ the reflection on P_t for $t \neq 0$ of the components of $M - P_t$ lying in the connected component of $\mathbf{H^3} - P_t$ not containing P. We begin the standard procedure by moving P until P_t is disjoint of M, *i.e* $P_t \cap M = \emptyset$. Then moving back P_t towards P, doing Alexandrov reflection during this movement, we find a first point of contact between M and ${M_t}^*$. Since C is invariant by reflection on P_t then, if by absurd $P_t \neq P$, this first point of contact cannot occurs at a boundary point of ∂M, because $\partial M \subset C$ and $M \cap C = \partial M$. So, if $t \neq 0$ we get a tangent point of contact, both M and ${M_t}^*$ have the correct orientation at this point. We arrive to a contradiction for the maximum principle yields $P_t, t \neq 0$, is a plane of symmetry of M. Then $P_t = P$, that is P is a plane of symmetry of M, as required.

To prove the second statement we will proceed as follows: Let P_{d_1} and P_{-d_1}, $d_1 \geq d_0$, be the geodesic planes passing through C_1 and C_2. We will denote by $\mathcal{H}$ the region bounded by P_{d_1} and P_{-d_1}. Also we will denote by ${P_t}^+$ the connected component of $\mathbf{H^3} - P_t$ not containing P. Under those conventions, notice that $M \cup (P^+_{d_1} \cap C) \cup (P^+_{-d_1} \cap C)$ is the boundary of a region V in $\mathbf{H^3}$; V is contained in $\overline{int(C)}$, ∂V is not smooth over ∂M. Note that, since f does not change sign, either the mean curvature vector $\overrightarrow{H}$ of M is pointing into the interior of V or else $\overrightarrow{H}$ is pointing towards the exterior of V. We claim $\overrightarrow{H}$ is an

inward pointing normal vector. Indeed, set $p = \gamma \cap P$ and let $\alpha \subset P$ be a geodesic line cutting γ orthogonally at p. Let $\{\zeta_t\}$ be the 1-parameter family of minimal surfaces of revolution with axis γ and generating curve c_t such that α is the symmetry line of c_t and $t = d(\zeta_t, \gamma)$. Note that, as $d(C_1, C_2) \geq 2d_0$, the family $\{\zeta_t\}$ is inside $\mathcal{H}$. To prove the claim, we move ζ_t coming from the infinity towards M making $t \to 0$ to reach a first interior point of contact. By the comparison principle $\overrightarrow{H}$ is pointing into V, as required. To conclude the proof, if $M \cap ext(\mathcal{H}) \neq \emptyset$, one derives a contradiction by the same argument as in the preceding paragraph, making use of the family $\{P_t\}$ coming from the infinity towards $\partial\mathcal{H}$.

We remark that there always exists an umbilic surface in the class of any Weingarten surface satisfying (1) for f elliptic: If $f(0) = 0$ any geodesic plane is in the class. If $f^2(0) = 1$, any horosphere is in the class. If $f^2(0) < 1$ any equidistant surface with mean curvature $f(0)$ is in the class. If $f^2(0) > 1$ any sphere of radius ρ with $\coth \rho =\mid f(0) \mid$ is in the same class of M.

Theorem 1

Let M be a connected properly embedded special Weingarten surface in $\mathbf{H^3}$ satisfying (1) for f elliptic. Assume $\partial M \neq \emptyset$. Suppose $f^2 \leq 1$, we have the following:

(1) Let us suppose that $\partial_\infty M = \emptyset$. If ∂M is a circle, then M is totally umbilic.

(2) Let us assume that M is compact and ∂M is the union of two circles C_1, C_2 of same radius with $C_1 \cup C_2$ invariant by a rotation. If we suppose that $d(C_1, C_2) \geq 2d_0$, $|f(0)| = 1$, and f does not change sign, then M is part of a complete embedded special surface of revolution. Furthermore, the generating curve attains one and only one global minimum and each end is asymptotically umbilic in the sense of Theorem 2, section 4.

Proof of Statement 1

We suppose $\partial_\infty M = \emptyset$, and then M is compact:

Let P be the geodesic plane in $\mathbf{H^3}$ containing ∂M. We consider the halfspace model $\mathbf{H^3} = \{(x, y, z), z > 0\}$. Without loss of generality, we can suppose $P = \{x = 0\}$. Let H_0 and ${H_0}^*$ be the unique horospheres in $\mathbf{H^3}$ such that $H_0 \cap P = {H_0}^* \cap P = \partial M$. Denote by $D_0,\ {D_0}^*$,
$D_0 \subset H_0,\ {D_0}^* \subset {H_0}^*$ the geodesic disks with boundary ∂M. It follows from [2] that M is inside the region of $\mathbf{H^3}$ with boundary $D_0 \cup {D_0}^*$. Now, we claim M is contained in one side of P. Indeed, if $f(0) = 0$, then maximum principle shows M is the geodesic disk $D \subset P$ whose boundary is equal to ∂M. Now, if $f(0) \neq 0$, we will arrive to a contradition by the following argument: Take the family $\{S\}$ of umbilic surfaces contained in $\mathbf{H^3} - P$ which is in the same classe of M, *i.e* $H = f(0)$, by choosing an appropriate normal field N. We require further the mean curvature vector of each surface S is pointing into the region of $\mathbf{H^3} - S$ not containing P (this makes sense since $f(0) \neq 0$). Now it is clear that if $M \cap \{x < 0\} \neq \emptyset$ and $M \cap \{x > 0\} \neq \emptyset$, we may make a comparison of M with a member S of the family in a suitable way to obtain a contradiction by the maximum principle. This proves the previous claim and we conclude M is in one side of P. Finally,

standard Alexandrov principle using the family of geodesic planes ortogonal to P, allow us to conclude M inherits the symmetry of ∂M, *i.e* M is invariant by a 1-parameter group of rotations. Clearly, the curvature lines are meridians and parallels. Note that either M is totally umbilic or else the singularities of the lines of curvature are isolated, by Bryant's theorem (see Proposition 2). Also note that the only possible singularities of the lines of curvatures are the points on the rotation axis, and there the singularity has non negative index. Consequently M is totally umbilic.

Proof of Statement 2

Let C be the cylinder with axis γ containing $C_1 \cup C_2$. Let $\{C_t\},\ t \geq 0$ be the 1-parameter family of cylinder of same axis γ with $C_0 = C$ and $t = d(C_t, \gamma)$. We assert that $M \subset \overline{int(C)}$ and $M \cap C = \partial M$. Indeed, if this is not the case, then as $t \to +\infty$ one may find a "last" cylinder C_{t_1} such that $C_{t_1} \cap M \neq \emptyset$ and $C_t \cap M = \emptyset$, for $t > t_1$. If $M \cap (\mathbf{H^3} - \overline{int(C)}) \neq \emptyset$ then $t_1 > 0$, hence we get a contradiction with the standard comparison principle (because the mean curvature of M is not greater than 1). For the same reasons, $M \cap C = \partial M$. It can be inferred from a straightforward application of Alexandrov principle and Lemma 3 that M is a surface of revolution whose generating curve attains a local minimum. Suppose without loss of generality that γ is the z-axis (in upper halfspace model). Assume $C_1,\ C_2$ belong to the horospheres $\Theta_{z_1} = \{z = z_1\}$ and $\Theta_{z_2} = \{z = z_2\}$, respectively, with $z_1 < z_2$. Consider the 1 - parameter family of horospheres $\Theta_t = \{z = t\}$ for $t > z_2$. Recall that we may move this family along γ by making use of hyperbolic translations. Now, doing $t \downarrow z_2$ coming from the infinity towards M it follows from the maximum principle and from the proof of Lemma 3 that M is entirely contained inside the region $\{z \leq z_2\}$ of $\mathbf{H^3}$. By taking into account again Lemma 3, one may conclude M is still inside the region $\{z \geq z_1\}$. Finallly, doing Alexandrov reflection in the same way as explained in 2.3., on may derive our previous claim, as desired. Recall that it follows from the proof of Lemma 3 that $\overrightarrow{H}$ is an inward pointing normal vector. Thus we have $H = f \leq 0$ on M with respect to the outward normal orientation. Therefore we are able to apply Theorem 2, section 4, to conclude that the generating curve attains one and only one global minimum since $f(0) = -1$. Furthermore in Theorem 2, section 4, it is proved that each end of M is asymptotically umbilic. This concludes the proof of Theorem 1.

Remark 2

Statement 1 of Theorem 1 when $H = cst,\ H^2 \leq 1$ is obtained in [14], the sharp result (assuming M immersed) is succeeded in [2]. The general situation for $H = cst,\ H^2 > 1$, as in the euclidean case, is still not known. Partial results was obtained in [2].

The following corollary is immediate:

Corollary 1

Let M be a compact embedded surface in $\mathbf{H^3}$ with constant mean curvature 1. Suppose ∂M is the union of two circles C_1, C_2 of same radius with $C_1 \cup C_2$ invariant by a rotation.

If $d(C_1, C_2) \geq 2d_0$ then M is a piece of a Catenoid Cousin.

Proposition 3

Let M be a compact embedded special Weingarten surface in $\mathbf{H^3}$ satistying (1) for f elliptic. Assume there exists a cylinder C such that $\partial M \subset C$ and such that $M \subset \overline{int(C)}$. Suppose f satisfies, $f < 0$, $f(t) + \sqrt{t+1}$ is an increasing function for $t \geq 0$ and $f(c) + \sqrt{c+1} = 0$, where $H = -\sqrt{c+1}$ is the mean curvature of C with respect to the outward normal orientation (that is C is f-special, see Remark 4).

Then if ∂M is the union of two circles C_1, C_2 invariant by the group of rotations of C, and if $d(C_1, C_2) \geq 2d_0$, M is part of a complete embedded periodic special surface of revolution.

Proof

Let us consider the half-space model of $\mathbf{H^3}$. First, notice that our assumptions give C is the *unique* cylinder in the same class of M, *i.e* C satisfies the Weingarten relation (1). Thus, ellipticity yields $M \cap C = \partial M$. Since our hypothesis imply $f(0) < -1$, we may use the spheres of the same class than M, above M, to show as in Theorem 1 that M lies between the parallel euclidean planes containing C_1 and C_2 respectively (which are hyperbolic horospheres, denoted by Θ_{z_1} and Θ_{z_2} in the proof of Theorem 1). So we may apply Lemma 3 and the maximum principle to derive as in statement 2 of Theorem 1, that M is invariant by the group of rotations keeping C invariant. Furthermore since Lemma 3 shows that M has a plane of symmetry we deduce that the graph generating M has a local minimum. With Remark 6, section 4, we conclude M is part of a complete embedded periodic special surface of revolution.

To conclude this section we shall figure out a characterization of geodesic disks of spheres proved by Braga Brito and the first author in [3], when the ambient space is euclidean space. The proof is the same in both euclidean and hyperbolic space. So, we will omit it here: *Let M be an immersed disk type special Weingarten surface in $\mathbf{H^3}$ satisfying (1) for f elliptic. Assume ∂M is a circle of radius ρ. Suppose f is analytic with $f(0) > 0$. Then*

a) $f(0) \leq coth\rho$

b) If $f(0) = coth\rho$, M is a half sphere.

Remark 3

There is another characterization with same statement and proof in both euclidean and hyperbolic space derived in [3]: *If M is a disk type embedded special Weingarten surface in $\mathbf{H^3}$ satisfying (1) for f elliptic, if ∂M is a circle of radius ρ, if f is positive $f > 0$, and if M cuts transversely, along ∂M, the geodesic plane P containing ∂M, then M is a geodesic disk of a sphere.*

4. Rotational special Weingarten surfaces. Existence and Uniqueness.

In this section, we are interested on special Weingarten surfaces of revolution in the hyperbolic space $\mathbf{H^3}$. We choose the model of the 3-ball, hence our surfaces are generated by a curve of the 2-hyperbolic disc $\mathbf{D} = \{(u,v) \in \mathbf{R^2} \ / \ u^2 + v^2 < 1\}$. We choose the following coordinates (x,y), x, $y \in \mathbf{R}$, of $\mathbf{D}$: Let $p \in \mathbf{D}$, let γ be the unique geodesic passing by p and orthogonal to the horizontal geodesic $\{v = 0\}$. Let q be the intersection between the two geodesics. Then x is equal to the oriented hyperbolic distance between q and 0 and y is the oriented hyperbolic lenght of γ between p and q. This means that $x \geq 0$ (resp. $y \geq 0$) if and only if $u \geq 0$ (resp. $v \geq 0$), where (u,v) are the euclidean coordinates of p. Note that the coordinate curves $\{x = cst\}$ are the geodesics orthogonal to the u-axis and $\{y = y_0\}$ are the equidistant-curves of the u-axis, namely the arcs of circles passing through the points $(-1,0)$ and $(1,0)$ making angle α with $\partial\mathbf{D}$ such that $\cos(\alpha) = \tanh(\mathrm{y}_0)$. It is well-know that the metric of $\mathbf{D}$ with (x,y)-coordinates is:

$$ds^2 = \cosh^2(\mathrm{y})\mathrm{dx}^2 + \mathrm{dy}^2.$$

Now, let $\gamma \subset \mathbf{D}$ be a curve and let us call M the revolution surface generated by γ, where the revolution axis is always the x-axis (which is also the u-axis). Suppose that γ is the graph of a positive C^2 function $y = y(x)$. We call *outward normal orientation* on M the orientation given on γ by the unit normal field pointing toward the direction of increasing y. Straightforward computations show that principal curvatures of M with respect to the outward normal orientation are:

$$\lambda_1(x) = \frac{y^{"}\cosh(\mathrm{y}) - 2\sinh(\mathrm{y})\mathrm{y}^{'2} - \sinh(\mathrm{y})\cosh^2(\mathrm{y})}{(\cosh^2(\mathrm{y}) + \mathrm{y}'^2)^{3/2}}, \quad \lambda_2(x) = -\frac{\cosh^2(\mathrm{y})}{\sinh(\mathrm{y})(\cosh^2(\mathrm{y}) + \mathrm{y}'^2)^{1/2}}.$$

Note that $\lambda_1(x)$ is the hyperbolic curvature of γ as a planar curve. From now on we always will assume that f is an elliptic function (see equation 2). Let M be a surface of $\mathbf{H^3}$ and let N be a normal unit field over M. Recall that M is a f-surface (with respect to N) if and only if the principal curvatures of M satisfy the relation (see equation 1):

$$\frac{\lambda_1 + \lambda_2}{2}(N) = f([\frac{\lambda_1 - \lambda_2}{2}]^2).$$

Recall that for every elliptic function f any totally umbilic hypersurfaces with mean curvature $f(0)$ are f-surfaces. Namely: if $\mid f(0) \mid > 1$ they are the compact spheres of hyperbolic radius r with $\coth(r) = \mid f(0) \mid$ and if $\mid f(0) \mid \leq 1$ they are the intersections of spheres with the 3-ball which intersect the boundary $\partial\mathbf{H^3}$ with angle α such that $\cos(\alpha) = f(0)$.

Now, let M be a surface of revolution of $\mathbf{H^3}$ and let γ be the plane curve generating M. Let us assume that γ is the graph of a positive C^2 function y. Then M is a f-surface if and only if f satisfies the relation $F(y, y^{'}, y^{"}) = 0$, where:

$$F(y, y^{'}, y^{"}) = \frac{y^{"}\cosh(\mathrm{y})\sinh(\mathrm{y}) - (2\sinh^2(\mathrm{y}) + \cosh^2(\mathrm{y}))\mathrm{y}^{'2} - \cosh^2(\mathrm{y})(\sinh^2(\mathrm{y}) + \cosh^2(\mathrm{y}))}{2\sinh(\mathrm{y})(\cosh^2(\mathrm{y}) + \mathrm{y}'^2)^{3/2}}$$
$$- f([\frac{y^{"}\cosh(\mathrm{y})\sinh(\mathrm{y}) + (1 - \sinh^2(\mathrm{y}))\mathrm{y}^{'2} + \cosh^2(\mathrm{y})}{2\sinh(\mathrm{y})(\cosh^2(\mathrm{y}) + \mathrm{y}'^2)^{3/2}}]^2).$$

From the ellipticity of f, we deduce that $\frac{\partial F}{\partial y^{"}} > 0$, so F is a strictly increasing function of the third variable.

Proposition 4

Let f be an elliptic function. Then there exists a f-surface of revolution, possibly non complete, if and only if f satisfies the following condition:

$$\lim_{t\to+\infty}(t - f(t^2)) > 0.$$

Proof

Suppose there exists a f-surface of revolution M. We can assume that part of M is generated by the graph of a strictly positive function $y(x)$. It follows that there exists real numbers $y_0 > 0$, $y_0^{'}$ and $y_0^{"}$ such that $F(y_0, y_0^{'}, y_0^{"}) = 0$. But this is equivalent to

$$t - f(t^2) = \frac{\cosh^2(\mathrm{y}_0)}{\sinh(\mathrm{y}_0)(\cosh^2(\mathrm{y}_0) + \mathrm{y}_0^{'2})^{1/2}}$$

where

$$t = \frac{y_0^{"}\cosh(\mathrm{y}_0)\sinh(\mathrm{y}_0) + (1 - \sinh^2(\mathrm{y}_0))\mathrm{y}_0^{'2} + \cosh^2(\mathrm{y}_0)}{2\sinh(\mathrm{y}_0)(\cosh^2(\mathrm{y}_0) + \mathrm{y}_0^{'2})^{3/2}}.$$

As f is elliptic we know that $(t - f(t^2))$ is a strictly increasing function. Whence

$$\lim_{t\to+\infty}(t - f(t^2)) > \frac{\cosh^2(\mathrm{y}_0)}{\sinh(\mathrm{y}_0)(\cosh^2(\mathrm{y}_0) + \mathrm{y}_0^{'2})^{1/2}}$$

thereby, $\lim_{t\to+\infty}(t - f(t^2)) > 0$.

Conversely, assume that $\lim_{t\to+\infty}(t - f(t^2)) > 0$. Note that we have $\lim_{t\to-\infty}(t-f(t^2)) < \lim_{t\to+\infty}(t-f(t^2))$. Clearly, there exist strictly positive real numbers y_0, $y_0^{'} > 0$ such that

$$\lim_{t\to-\infty}(t - f(t^2)) < \frac{\cosh^2(\mathrm{y}_0)}{\sinh(\mathrm{y}_0)(\cosh^2(\mathrm{y}_0) + \mathrm{y}_0^{'2})^{1/2}} < \lim_{t\to+\infty}(t - f(t^2)).$$

On the other hand, $F(y_0, y_0^{'}, .)$ is an increasing function. Therefore, we conclude from the ellipticity that

$$\lim_{s\to-\infty} F(y_0, y_0^{'}, s) < 0 \quad \text{and} \quad \lim_{s\to+\infty} F(y_0, y_0^{'}, s) > 0.$$

This allows to conclude that there exists a real number $y_0^{"}$ such that $F(y_0, y_0^{'}, y_0^{"}) = 0$. Then, using the implicit function theorem we can prove that there exists a real C^1 function h defined on a neighborhood of $(y_0, y_0^{'})$ satisfying:

$$F(y, y^{'}, y^{"}) = 0 \Leftrightarrow y^{"} = h(y, y^{'}), \quad \text{and} \quad h(y_0, y_0^{'}) = y_0^{"}.$$

Now Picard's theorem shows that there exists an unique function $y(x)$ satisfying the last ODE with the initial conditions $y(0) = y_0$ and $y^{'}(0) = y_0^{'}$. Clearly, the graph of this function generates a f-surface as desired.

From now on, we only consider the outward normal orientation for a graph of $\mathbf{D}$ and we will always suppose that $f(0) \leq 0$. For later use, we shall state the following lemmas.

Lemma 4

Let f be an elliptic function and let y be a C^2 real function whose graph γ generates a f-surface M. Suppose that $y^{'}(x_0) = 0$ for a real number x_0. Then γ is symmetric with respect to the geodesic orthogonal to γ at point $(x_0, y(x_0))$ (hence M is symmetric with respect to the totally geodesic hyperplane of $\mathbf{H^3}$ orthogonal to $\mathbf{D}$ which contains this geodesic).

Proof

Observe that the functions y and $z(x) = y(2x_0 - x)$ satisfy the same second order differential equation with the same initial conditions at x_0, since $y(x_0) = z(x_0)$ and $y^{'}(x_0) = 0 = z^{'}(x_0)$. Actually (1) can be write locally in the form $y^{"} = h(y, y^{'})$, see the proof of Proposition 4. So we have $y = z$.

Lemma 5

Let f be an elliptic function and let y be a C^2 real function whose graph γ generates a f-surface M. Suppose that M has an umbilic point. Then M is totally umbilic.

Proof

This is a direct consequence of Proposition 2, section 2. If M admits an umbilic point outside the revolution axis then M have at least a circle of umbilic point. Since Proposition 2, section 2, shows that M is totally umbilic or umbilic points are isolated we conclude in this case that M is totally umbilic. If the umbilic point stays on the x-axis, note that the index of this umbilic point (if it is isolated) is $+1$, but the Proposition 2, section 2, shows that this index should be negative. We also conclude that M is totally umbilic.

We remark that we can give a direct proof of Lemma 5. For this, it suffices to show that there is a C^2 curve tangent to γ at the umbilic point generating a totally umbilic surface with constant mean curvature $f(0)$ with respect to outward normal orientation.

Lemma 6

Let f be an elliptic function and let y be a C^2 positive monotonous function whose graph generates a non-totally umbilic f-surface M of $\mathbf{H^3}$. Then we have:

$$\lambda_2^{'} = y^{'} \coth(y)(\lambda_1 - \lambda_2).$$

Consequently, the principal curvatures of M also are monotonous and λ_1 is increasing if and only if λ_2 is decreasing.

Proof

A simple computation shows that the principal curvatures of M satisfy the formula in the statement. Then, Lemma 5 shows that λ_2 is a monotonous function. Moreover, by differentiating the relation (1) we obtain:

$$\lambda_1^{'} + \lambda_2^{'} = (\lambda_1^{'} - \lambda_2^{'}).(\lambda_1 - \lambda_2).f^{'}([\frac{\lambda_1 - \lambda_2}{2}]^2).$$

Using ellipticity of f we infer:

$$\mid \lambda_1^{'} + \lambda_2^{'} \mid < \mid \lambda_1^{'} - \lambda_2^{'} \mid,$$

which achieves the proof.

Now, recall that a cylinder of $\mathbf{H^3}$ is a complete surface of revolution generated by the graph of a constant function $y(x) = \tau > 0$, which will be noted C_τ. Note that the principal curvatures of C_τ are $\lambda_1 = -\tanh(\tau)$ and $\lambda_2 = -\coth(\tau)$. Then, if f is an elliptic function the cylinder C_τ is a f-surface if and only if:

$$\frac{\tanh(\tau) + \coth(\tau)}{2} + f([\frac{\tanh(\tau) - \coth(\tau)}{2}]^2) = 0.$$

Proposition 5

Let f be an elliptic function with $f^{'} \geq 0$. Then there exists a f-special cylinder if and only if $f(0) < -1$. Furthermore, this cylinder is unique.

Proof

As f is elliptic the function $t + f(t^2)$ is increasing. Moreover, as f is increasing the function

$$g(t) = \frac{\tanh(\mathrm{t}) + \coth(\mathrm{t})}{2} + f([\frac{\tanh(\mathrm{t}) - \coth(\mathrm{t})}{2}]^2)$$

is decreasing. Let us assume first that $f(0) < -1$. Note that $\lim_{t\to+\infty} g(t) = 1 + f(0) < 0$ and $\lim_{t\to 0} g(t) = \lim_{t\to+\infty}(t + f(t^2)) = +\infty$. So, there exists an unique positive real number τ satisfying $g(\tau) = 0$, *i.e* there exists an unique f-special cylinder.

Conversely, suppose there exists a f-special cylinder C_τ. Consequently $g(\tau) = 0$ and we deduce (as f is increasing):

$$f(0) \leq -\frac{\tanh(\tau) + \coth(\tau)}{2} < -1.$$

As before we may conclude that C_τ is the unique f-special cylinder.

Remark 4

Note that in the euclidean case ellipticity of f ensures uniqueness of f-special cylinder (see [16]). In the hyperbolic case, if we do not assume f increasing, it may exist many and

even infinitely many f-special cylinders. To see this, set $X = X(t) = [\frac{\tanh(t)-\coth(t)}{2}]^2$. We have then $g(t) = h(X) = \sqrt{X+1} + f(X)$ with $X \in [0, +\infty[$. Observe that each zero of h gives a f-special cylinder. Moreover, we can write h as an integral:

$$h(X) = \int_0^X (f^{'}(t) + \frac{1}{2\sqrt{t+1}})dt + f(0) + 1.$$

Then

$$h(X) = 0 \Leftrightarrow \int_0^X (f^{'}(t) + \frac{1}{2\sqrt{t+1}})dt = -(f(0)+1).$$

Now observe that the last integral is the (oriented) aire of the finite plane region limited by the graph of the two functions $f^{'}(t)$ and $-\frac{1}{2\sqrt{t+1}}$ for t between 0 and X. Also f elliptic means only that the graph of $f^{'}$ stays between the graph of the two functions $\frac{1}{2\sqrt{t}}$ and $-\frac{1}{2\sqrt{t}}$. So it is not difficult to find elliptic functions f such that the associated function h has as many (and even infinitely many) zeros we want. Finally, observe that the weaker condition $f^{'}(t) \geq -\frac{1}{2\sqrt{t+1}}$ ensures uniqueness of f-special cylinder, since the function h is increasing.

Theorem 2

Let f be an elliptic function with $f(0) \leq 0$ and $\lim_{t\to+\infty}(t + f(t^2)) > 0$. Let $\tau > 0$ be a positive real number satisfying:

$$\frac{\tanh(\tau)+\coth(\tau)}{2} + f([\frac{\tanh(\tau)-\coth(\tau)}{2}]^2) > 0.$$

$$\coth(\tau) < \lim_{t\to+\infty}(t - f(t^2)).$$

Then there exists a unique complete curve γ_τ which is the graph of a C^2 function $y_\tau(x)$, admitting a minimum at $(0,\tau)$, symmetric with respect to the y-axis and generating a complete embedded f-surface of revolution M_τ.

Moreover if $f^{'} \geq 0$ the curve γ_τ has the following behaviour.

1) If $-1 < f(0) \leq 0$, y_τ is defined on an open interval $]-x_\tau, x_\tau[$, with $0 < x_\tau < +\infty$, y is increasing on $[0, x_\tau[$ and satisfies $\lim_{x\to x_\tau} y_\tau = +\infty$. Also γ_τ has a limit direction tangent and makes an angle $\theta \in]0, \pi/2]$ with $\partial\mathbf{D}$ such that $\cos(\theta) =\mid f(0) \mid$ (see figure 1). Furthermore both of principal curvatures of M_τ go to $f(0)$ when x goes to $\pm x_\tau$. Consequently M_τ is asymptotically umbilic.

2) If $f(0) = -1$, y_τ is defined on $]-\infty, +\infty[$, y_τ is increasing on $[0, +\infty[$ and satisfies $\lim_{x\to\infty} y_\tau = +\infty$. Also γ_τ has a limit tangent direction and is tangent to $\partial\mathbf{D}$ (see figure 2). Furthermore both of principal curvatures of M_τ go to -1 when x goes to $\pm\infty$. It follows that M_τ is asymptotically umbilic.

3) If $f(0) < -1$, y_τ is defined on $]-\infty, +\infty[$ and is periodic (see figure 3).

In the same way, if $-1 \leq f \leq 0$ (without suppose $f^{'} \geq 0$) then cases 1) and 2) still hold.

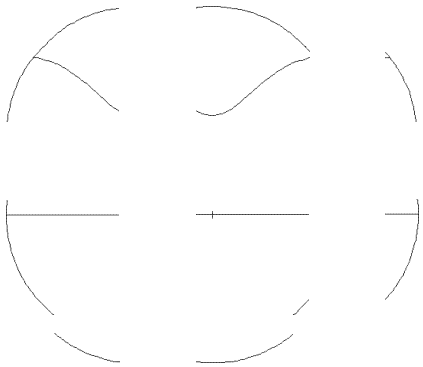

figure 1

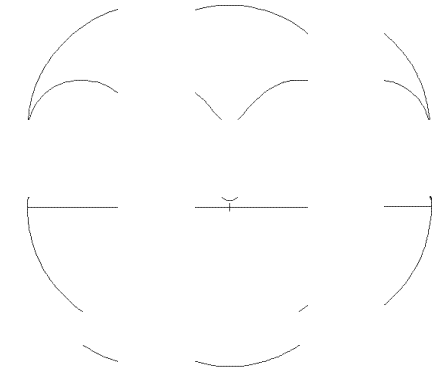

figure 2

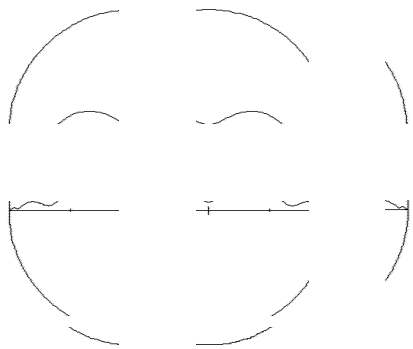

figure 3

Remark 5

(a) Note that in each case γ_τ has the same geometrical behaviour than the curve generating a complete embedded surface of $\mathbf{H^3}$ with constant mean curvature $f(0)$.

(b) We deduce that if $-1 < f(0) \leq 0$ and $f^{'} \geq 0$ (or $-1 \leq f \leq 0$) each end of the f-surfaces given by Theorem 2 is C^1 asymptotic to an end of a totally umbilic surface of $\mathbf{H^3}$ with constant mean curvature $f(0)$.

Proof of Theorem 2

We have seen that the graph of a positive function y_τ generates a f-surface if and only if $F(y_\tau, y_\tau^{'}, y_\tau^{"}) = 0$. Also, f elliptic implies that F is strictly increasing with respect to $y_\tau^{"}$.

A computation shows that

$$F(\tau, 0, 0) = -\frac{\tanh(\tau) + \coth(\tau)}{2} - f([\frac{\tanh(\tau) - \coth(\tau)}{2}]^2) < 0,$$

and

$$\lim_{t \to +\infty} F(\tau, 0, t) = \lim_{t \to +\infty} (t - f(t^2)) - \coth(\tau) > 0.$$

Therefore, we derive that there is an unique positive real number $y_0^{"} > 0$ satisfying $F(\tau, 0, y_0") = 0$. Now, the implicit function theorem shows that there exists a C^1 real function h defined in a neighborhood of $(\tau, 0)$ satisfying:

$$F(y_\tau, y_\tau^{'}, y_\tau^{"}) = 0 \Leftrightarrow y_\tau^{"} = h(y_\tau, y_\tau^{'}), \quad \text{and} \quad h(\tau, 0) = y_0^{"}.$$

Furthermore, Picard's theorem shows that the above differential equation has an unique solution y_τ satisfying $y_\tau(0) = \tau$ and $y_\tau^{'}(0) = 0$. Lemma 4 shows that $y_\tau(-x) = y_\tau(x)$, so we may suppose y_τ is defined on an interval $]-x_1, x_1[$. If $x_1 = +\infty$ we are done, so suppose $0 < x_1 < +\infty$. Note that $y_\tau^{'}(x) > 0$ for $x > 0$ near of 0. If $y_\tau^{'}$ had another zero after 0, y_τ would have another symmetry but then y_τ would be a periodic function and its graph, γ_τ would be complete as we wish. So suppose that $y_\tau^{'}(x) > 0$ for $x > 0$, subsequently $y_\tau(x)$ has a positive limit y_1 as x goes to x_1. If $y_1 = +\infty$ the graph γ_τ is complete, so suppose that $0 < y_1 < +\infty$.

Observe that $\lambda_2(0) = -\coth(\tau)$ and $\lambda_1(0) = \frac{y_0^{"}}{\cosh^2(\tau)} - \tanh(\tau)$, thereby, $\lambda_2(0) < \lambda_1(0)$. Since the f-surface generated by γ_τ is not totally umbilic, Lemma 5 shows that $\lambda_2(x) < \lambda_1(x)$ for every x. Moreover, Lemma 6 says that $\lambda_2(x)$ is strictly increasing and $\lambda_1(x)$ is strictly decreasing on $[0, x_1[$, so they have finite limit when x goes to x_1, hence $y_\tau^{'}$ has a positive limit $y_1^{'}$.

Note also that, as f is elliptic, the function

$$G(\lambda_1, \lambda_2) = \frac{\lambda_1 + \lambda_2}{2} - f([\frac{\lambda_1 - \lambda_2}{2}]^2)$$

is strictly increasing with respect to λ_1 and λ_2 and $G(f(0), f(0)) = 0$. We get from this and from the above observations that $\lambda_2(x) < f(0) < \lambda_1(x)$. If $f(0) < 0$ the first inequality shows that $y_1^{'} < +\infty$. If $f(0) = 0$ we also have $y_1^{'} < +\infty$ for, in the contrary, the graph γ_τ would be tangent and into one side of the geodesic $\{x = x_1\}$. However, those two curves generate two f-surfaces of revolution. This last situation gives a contradiction with the maximum principle with boundary.

Thus in all cases we have $y_1^{'} < +\infty$ and as λ_1 has a finite limit we derive that $y_\tau^{"}$ also has a finite limit $y_1^{"}$. As F is a continuous function we have $F(y_1, y_1^{'}, y_1^{"}) = 0$. So using the implicit function theorem as above we can extend y_τ beyond x_1. Repeting this argument we see that we obtain a complete symmetric curve γ_τ generating a f-surface, which is the graph of a C^2 function y_τ defined on $]-x_\tau, x_\tau[$, $0 < x_\tau \leq +\infty$, as claimed. This completes the proof of the first assertion.

Proof of Statement (3)

Now let us assume $f^{'} \geq 0$. Suppose first $f(0) < -1$. Note that in this case, we have $x_\tau = +\infty$. Otherwise, we could find a f-special sphere with mean curvature $f(0)$ tangent and on one side of M_τ, which gives a contradiction with the maximum principle. Then y_τ is defined for all real number $x \in \mathbf{R}$. Now we are going to show that y_τ is periodic. In the contrary, assume that y_τ is an increasing function on $[0, +\infty[$ and consequently has a positive limit $c > 0$ when x goes to $+\infty$. The last argument with f-special spheres shows that $c \neq +\infty$. Hence, $0 < c < +\infty$. Recall that λ_2 is strictly increasing and λ_1 is strictly decreasing and that $\lambda_2(x) < f(0) < \lambda_1(x)$. Hence, λ_2 has a strictly negative limit and we deduce that $y_\tau^{'}(x) \to 0$. Furthermore, since λ_1 has a finite limit we get that $y_\tau^{"}$ also has a limit. Then $y_\tau^{"}(x) \to 0$. It follows that:

$$\lim_{x \to +\infty} \lambda_1 = -\tanh(\mathrm{c}), \qquad \lim_{\mathrm{x} \to +\infty} \lambda_2 = -\coth(\mathrm{c}).$$

We conclude that the cylinder C_c is a f-surface. Observe that, until now, we did not use the hypothesis $f^{'} \geq 0$. As f is increasing we know that the function

$$g(t) = \frac{\tanh(\mathrm{t}) + \coth(\mathrm{t})}{2} + f([\frac{\tanh(\mathrm{t}) - \coth(\mathrm{t})}{2}]^2)$$

is decreasing. Consequently, $t < c$ implies $-f([\frac{\tanh(\mathrm{t})-\coth(\mathrm{t})}{2}]^2) < \frac{\tanh(\mathrm{t})+\coth(\mathrm{t})}{2}$. Also let us note that the above observations about the principal curvatures show that $\lambda_1 - \lambda_2$ is a decreasing function for $x > 0$. We deduce that $[\frac{\lambda_1-\lambda_2}{2}]^2$ is also a decreasing function. Then, for every $x > 0$ we have $[\frac{\lambda_1-\lambda_2}{2}]^2 > [\frac{\tanh(\mathrm{c})-\coth(\mathrm{c})}{2}]^2$, from which we derive:

$$-f([\frac{\lambda_1 - \lambda_2}{2}]^2) < \sqrt{[\frac{\lambda_1 - \lambda_2}{2}]^2 + 1}$$

for $x > 0$, since $\frac{\tanh(\mathrm{t})+\coth(\mathrm{t})}{2} = \sqrt{[\frac{\tanh(\mathrm{t})-\coth(\mathrm{t})}{2}]^2 + 1}$. The last inequality implies $0 < \frac{\lambda_1+\lambda_2}{2} + \sqrt{[\frac{\lambda_1-\lambda_2}{2}]^2 + 1}$. A computation shows that this is equivalent, for x big enough, to $\lambda_1.\lambda_2 < 1$. Substituting in the last inequality λ_1 and λ_2 by their expression in function of y_τ and doing further simplifications we get for x big enough:

$$(*) \qquad 0 < y_\tau^{"}\cosh^3(\mathrm{y}_\tau) + \mathrm{y}_\tau^{'4}\sinh(\mathrm{y}_\tau).$$

On the other hand, note that $y_\tau^{'}(x)\cosh(\mathrm{y}_\tau(\mathrm{x})) > 0$ for every $x > 0$, moreover

$$y_\tau^{'}(0)\cosh(\mathrm{y}_\tau(0)) = 0 \quad \text{and} \quad \lim_{\mathrm{x} \to +\infty} \mathrm{y}_\tau^{'}(\mathrm{x})\cosh(\mathrm{y}_\tau(\mathrm{x})) = 0.$$

From the above, we derive that there is a sequence $x_n > 0$, with $\lim_{n \to +\infty} x_n = +\infty$, such that:

$$(y_\tau^{'}\cosh(\mathrm{y}_\tau))^{'}(\mathrm{x_n}) < 0.$$

This means $(\mathrm{y}_\tau^{"}\cosh(\mathrm{y}_\tau))(\mathrm{x_n}) < -(\mathrm{y}_\tau^{'2}\sinh(\mathrm{y}_\tau))(\mathrm{x_n})$. Then, for every n:

$$(\mathrm{y}_\tau^{"}\cosh^3(\mathrm{y}_\tau) + \mathrm{y}_\tau^{'4}\sinh(\mathrm{y}_\tau))(\mathrm{x_n}) < (\sinh(\mathrm{y}_\tau)\mathrm{y}_\tau^{'2}(\mathrm{y}_\tau^{'2} - \cosh^2(\mathrm{y}_\tau))(\mathrm{x_n}) < 0,$$

the last inequality is true for n big enough. But this gives a contradiction with $(*)$. We conclude that y_τ has to be periodic, as desired.

For the remaining cases, $-1 \leq f(0) \leq 0$ we need to consider the angle $\sigma \in]-\pi/2, \pi/2[$ between γ_τ and the coordinate curves $\{y = cst\}$. Notice that the orientation chosen is such that $\cos\sigma = \frac{\cosh(\mathrm{y})}{(\cosh^2(\mathrm{y})+\mathrm{y}'^2)^{1/2}}$. It follows that

$$\lambda_2(x) = \frac{-\cos\sigma}{\tanh(\mathrm{y})}.$$

Proof of Statement (2)

Let us suppose now that $f(0) = -1$ (and $f' \geq 0$). Therefore $x_\tau = +\infty$ or equivalently y_τ is defined on $]-\infty, +\infty[$. For in the other case, using the family of horocycles issue from the point $(u, v) = (1, 0)$ in $\partial\mathbf{D}$, we may reach a tangent point between γ_τ and one of those horocycles with γ_τ on one side of the horocycle. But those two curves generate two f-surfaces, so it would give a contradiction with the maximum principle. Observe also that if $\lim_{x\to+\infty} y_\tau(x) = c < +\infty$, the cylinder C_c should be a f-surface. This gives a contradiction since the mean curvature of C_c is strictly less than -1 (with respect to the exterior normal orientation) and $f \geq -1$ for f is increasing and $f(0) = -1$. So this allows to conclude that $\lim_{x\to+\infty} y_\tau(x) = +\infty$. As $\lambda_2(x)$ is a strictly increasing function and bounded from above by -1 we deduce that $\lim_{x\to+\infty} \lambda_2 = \lim_{x\to+\infty} -\cos\sigma \leq -1$; hence we have $\lim_{x\to+\infty} \cos\sigma = 1$. This means that γ_τ is tangent to $\partial\mathbf{D}$ when x goes to $\pm\infty$. We also have that $\lim_{x\to+\infty} \lambda_2 = -1$. As the function $G(\lambda_1, \lambda_2) = \frac{\lambda_1+\lambda_2}{2} - f([\frac{\lambda_1-\lambda_2}{2}]^2)$ is increasing with respect to λ_1 and λ_2 with $G(f(0), f(0)) = 0$ we also deduce that $\lim_{x\to+\infty} \lambda_1 = -1$. We conclude that M_τ is asymptotically umbilic.

Proof of Statement (1)

Let us assume $-1 < f(0) \leq 0$ (and $f' \geq 0$). Let us consider the family of umbilic f-surfaces of $\mathbf{H^3}$ invariant by rotation about the y-axis. Namely this is the family of pieces of sphere making angle $\alpha \in]0, \pi/2]$ with $\partial\mathbf{H^3}$ such that $\cos(\alpha) = -f(0)$ and each spherical piece stands above the totally geodesic plane with the same boundary. Suppose that $x_\tau = +\infty$. Assume first that $\lim_{x\to+\infty} y_\tau(x) = +\infty$. Then we could find one of the previous spherical piece tangent and above M_τ, which gives a contradiction with the maximum principle. If $\lim_{x\to+\infty} y_\tau(x) = c$ with $c < +\infty$ we would conclude as before that the cylinder C_c is a f-surface. But any cylinder has mean curvature strictly less than -1 (with respect to the outward orientation) and f is bigger than -1. So we deduce that $x_\tau < +\infty$. As we know that γ_τ is a complete curve we have $\lim_{x\to x_\tau} y_\tau = +\infty$. As before, as λ_2 has a finite limit at x_τ, we conclude that $\cos\sigma$ also has a limit at x_τ. Let us call σ_τ the limit angle. This means that γ_τ has a limit tangent direction and that the angle between γ_τ and $\partial\mathbf{D}$ is σ_τ.

Using the family of umbilic f-surfaces (with respect to the outward normal orientation) invariant by rotation about the x-axis, the maximum principle shows that $\sigma_\tau \leq \alpha$. Moreover, using the first family of umbilic f-surface (namely, invariant by rotation about the y-axis), the maximum principle shows that $\sigma_\tau \geq \alpha$. We obtain that $\sigma_\tau = \alpha$, hence $\cos(\sigma_\tau) = -f(0)$. As λ_2 goes to $-\cos(\sigma_\tau)$ when x goes to x_τ we deduce that $\lim_{x\to x_\tau} \lambda_2 = f(0)$. The same argument as in the case $f(0) = -1$ shows that we also have $\lim_{x\to x_\tau} \lambda_1 = f(0)$. Thereby, M_τ is again asymptotically umbilic.

Finally, assume that $-1 \leq f \leq 0$. Observe that in this case there is no f-special cylinder since the mean curvature of any cylinder is strictly less than -1. Therefore, we can follow the same proof as before for the cases $f(0) = -1$ and $-1 < f(0) \leq 0$. This achieves the proof of Theorem 2.

Remark 6

(a) It is easy to show that if a plane curve which attains a minimum $(0, \tau)$, $\tau > 0$, generates a f-surface of revolution M then τ satisfies the conditions of Theorem 2 (so our hypothesis are necessary). Therefore, the proof of the first part of Theorem 2 also shows that M can be extended as a complete embedded f-surface of revolution. Namely, M is part of one surface M_τ given by Theorem 2.

(b) In the case $f(0) < -1$, for the geometrical description of γ_τ, note that we only used the fact that g is a decreasing function. It follows that we only need the hypothesis $f^{'}(t) > \frac{-1}{2\sqrt{t+1}}$ to show γ_τ is periodic.

In order to prove uniqueness of f-surfaces given by Theorem 2, we shall need the following lemma.

Lemma 7

Let f be an elliptic function with $f^{'} \geq 0$. Let $y = y(x)$ be a C^2 decreasing and strictly positive function defined on an interval $]x_0, +\infty[$. Then the surface of revolution M generated by the graph of $y(x)$ is not a f-surface.

Proof

Let us label c the limit of y: $c = \lim_{x\to+\infty} y(x)$. We have $0 \leq c < +\infty$. Let us suppose that M is a f-surface. Observe that Lemma 6 implies that principal curvatures of M are monotonous functions and that λ_1 is increasing if and only if λ_2 is decreasing.

Let us first suppose $c = 0$. Therefore $y^{'}$ has not $-\infty$ as limit when x goes to $+\infty$. We deduce that $\lim_{x\to+\infty} \lambda_2 = -\infty$. It follows that λ_2 is decreasing, so $\lim_{x\to+\infty}(\lambda_2 - \lambda_1) = -\infty$. Note that $-\lambda_1 = \frac{\lambda_2-\lambda_1}{2} - f([\frac{\lambda_2-\lambda_1}{2}]^2)$. Then

$$\lim_{x\to+\infty} \lambda_1 = \lim_{t\to+\infty}(t + f(t^2)).$$

As f is increasing it follows that $\lim_{x\to+\infty} \lambda_1(x) = +\infty$. Consequently the expression of λ_1 in terms of y implies that $\lim_{x\to+\infty} y^{"}(x) = +\infty$. But the last fact yields a contradiction for $y^{'}$ is negative. Hence, $c \neq 0$.

Let us assume now that $c > 0$. As $\lim_{x\to+\infty} y^{'} \neq -\infty$ and λ_2 has a limit we conclude that $y^{'}$ has a finite limit and then $\lim_{x\to+\infty} y^{'} = 0$. Similarly, we can show that

$\lim_{x\to+\infty} y^{"}(x) = 0$. It follows that:

$$\lim_{x\to+\infty} \lambda_1(x) = -\tanh(\mathrm{c}), \qquad \lim_{\mathrm{x}\to+\infty} \lambda_2(\mathrm{x}) = -\coth(\mathrm{c}).$$

Consequently, we have $\lambda_1(x) > \lambda_2(x)$ and Lemma 6 shows that λ_2 is decreasing and $\lambda_1(x)$ is increasing. Hence $[\lambda_1(x) - \lambda_2(x)]^2$ is an increasing function and we get $[\lambda_1(x) - \lambda_2(x)]^2 < [\tanh(\mathrm{c}) - \coth(\mathrm{c})]^2$. As f is increasing we have

$$f([\frac{\lambda_1(x) - \lambda_2(x)}{2}]^2) < f([\frac{\tanh(\mathrm{c}) - \coth(\mathrm{c})}{2}]^2).$$

On the other hand, an argument of continuity shows that the cylinder C_c is a f-surface. This allows to conclude that:

$$\frac{\lambda_1(x) + \lambda_2(x)}{2} < \frac{-\tanh(\mathrm{c}) - \coth(\mathrm{c})}{2}.$$

Whence, the mean curvature of M is smaller than the mean curvature of C_c, with respect to the outward normal orientation. But it is well-know (see [8] and [11]) that for every $H < -1$ there exists a family of complete embedded Delaunay type surfaces of revolution with constant mean curvature H. This means that those surfaces are periodic and vary continuously. Now, it is also well-know (see [12] Lemma 6.4 or [15] Corollary 4.1.1) that doing a little perturbation of C_c in this family, we can get a surface with same constant mean curvature than C_c which is tangent and stays under M. As the mean curvature of M is smaller this gives a contradiction with the usual maximum principle (that is the maximum principle concerning the mean curvature). Thus M cannot be a f-surface.

Remark 7

Observe that Lemma 7 remains true if we replace the hypothesis $f^{'} \geq 0$ by the assumption $\mid f \mid \leq 1$. Indeed we would have again $\lim_{t\to+\infty}(t + f(t^2)) = +\infty$, so this eliminates the case $c = 0$ (keeping same notations of the proof). Observe also that, in this case, the mean curvature of any f-surface is always bigger than -1 and smaller than 1. Then there is no f-special cylinder. This eliminates the case $c > 0$.

Let M be a complete embedded non-compact surface of revolution in $\mathbf{H^3}$ which does not intersects the axis of rotation (x-axis). Hence, $\mathbf{H^3} - M$ has two connected components, one of them contains the x axis. Then we call *exterior normal orientation* the unit normal field along M pointing toward the component which does not contain the x-axis. Observe that this definition coincides with the outward normal orientation in case where M is generated by a graph.

Theorem 3

Let f be an elliptic function satisfying one of the two following conditions:
(1) $f^{'} \geq 0$ and $f(0) \leq 0$.
(2) $-1 \leq f \leq 0$.
Let M be a complete and embedded f-surface of revolution non-totally umbilic.

Then M is one of the surfaces given by Theorem 2.

Proof

Let us assume first that f satisfies conditions (1).

Let $\gamma \subset \mathbf{D}$ be the curve generating M. Observe that if γ intersects the x-axis then, for regularity, this intersection must be orthogonal. But this point would be an umbilic point. Hence, as M is not a totally umbilic surface, Lemma 5 shows this situation is impossible. So we can suppose that γ stays in the part $\{y > 0\}$ of $\mathbf{D}$.

Let us prove that γ is a graph. In the other case, γ should have a vertical point p (this means the tangent of γ at p is vertical). Let us call γ^- the component of $\gamma - p$ which begins under p. Up to a symmetry, we can suppose that a neighborhood of γ^- near p stays in the region $\{x \geq x(p)\}$. Observe that the y-coordinate of γ^- is decreasing near p. Suppose that γ^- has an horizontal point (this means a point where the tangent is horizontal) and let q be the first one. Lemma 4 implies that γ is symmetric with respect to the vertical geodesic $\{x = x(q)\}$ and then q is a local minimum. Observe that, if γ^- had another horizontal point, then the next such point after q would be a local maximum. Hence γ should be a periodic curve (see Lemma 4) and q a global minimum for γ. This implies that the exterior normal orientation of γ (or M) at q is pointing in the direction of the increasing y. But then near q the curve γ is a graph with q as minimum. As γ generates a f-surface M, the proof of Theorem 2 shows that M should be one of the surfaces given there (see Remark 6-(a)). But this is absurd since no surfaces given by Theorem 2 is generated by a curve which admits vertical point. So γ^- has no other horizontal point after q. Hence we deduce that q is a global minimum for γ which is absurd as we have seen before. It follows that γ^- cannot have horizontal point after p.

Now if γ^- where a graph, this graph should be decreasing and defined on the interval $]x(p), +\infty[$ (as γ is complete). But Lemma 7 shows this is not possible. Then, we conclude γ^- is not a graph. This allows to deduce that γ^- has another vertical point after p. Let $p^{'}$ be the first one. Note hence that γ^- is a decreasing graph between p and $p^{'}$. Combining this with Lemma 6 we derive that λ_2 is strictly monotonous between those two vertical points. Now as $y(p),\ y(p^{'}) > 0$ we have $\lambda_2(p) = \lambda_2(p^{'}) = 0$ which gives a contradiction.

Thus γ is a complete graph. If γ had not horizontal points then, up to a symmetry, we could assume that γ is a decreasing graph defined on an interval $]x_0, +\infty[,\ -\infty \leq x_0$. But Lemma 7 shows this is impossible. If γ had an unique horizontal point, γ should be symmetric (see Lemma 4), then this point should be a global maximum or minimum. Lemma 7 shows that this point cannot be a global maximum. We derive that γ admits a global minimum, hence M is one of the surfaces given by Theorem 2 (see Remark 6-(a)). At last, if γ has many horizontal points, Lemma 4 shows that γ is periodic and then admits a global minimum, which conclude the proof in the first case.

In the second case the proof is analogous since the principal tool, Lemma 7, again is true in this new context, see Remark 7.

Remark 8

In the euclidean case we do not need to assume $f(0) \leq 0$. This is induced by the other hypothesis stated in Theorem 3, see [16]. In the hyperbolic case we must assume this hypothesis since J.Gomes [8] has showed existence of embedded complete surface

of revolution in $\mathbf{H^3}$ with constant and strictly positive mean curvature (with respect to exterior normal orientation).

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