CLASSICAL SCHWARZ REFLECTION PRINCIPLE FOR JENKINS-SERRIN TYPE MINIMAL SURFACES

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Abstract. We give a proof of the classical Schwarz reflection principle for Jenkins-Serrin type minimal surfaces in the homogeneous three manifolds $\mathbb{E}(\kappa, \tau)$ for $\kappa < 0$ and $\tau \geqslant 0$. In our previous paper we proved a reflection principle in Riemannian manifolds. The statements and techniques in the two papers are distinct.

1. Introduction

In this paper we focus the classical Schwarz reflection principle across a geodesic line in the boundary of a minimal surface in $\mathbb{R}^3$ and more generally in three dimensional homogeneous spaces $\mathbb{E}(\kappa, \tau)$ for $\kappa < 0$ and $\tau \geqslant 0$.

The Schwarz reflection principle was shown in some special cases. One kind of examples arise for the solutions of the classical Plateau problem in $\mathbb{R}^3$ containing a segment of a straight line in the boundary, see Lawson [9, Chapter II, section 4, Proposition 10]. Another kind occur for vertical graphs in $\mathbb{R}^3$ and $\mathbb{H}^2 \times \mathbb{R}$ containing an arc of a horizontal geodesic, see [17, Lemma 3.6].

On the other hand, there is no proof of the reflection principle for general minimal surfaces in $\mathbb{R}^3$ containing a straight line in its boundary.

The goal of this paper is to provide a proof of the reflection principle about vertical geodesic lines for Jenkins-Serrin type minimal surfaces in $\mathbb{R}^3$ and other three dimensional homogeneous manifolds such as, for example, $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{\mathrm{PSL}}_2(\mathbb{R}, \tau)$ and $\mathbb{S}^2 \times \mathbb{R}$, see Theorem 4.1. The proof also holds for horizontal geodesic lines.

We observe that this classical Schwarz reflection principle was used by many authors, including the present authors, in $\mathbb{R}^3$ and $\mathbb{H}^2 \times \mathbb{R}$.

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We recall that the authors proved another reflection principle for minimal surfaces in general three dimensional Riemannian manifold with quite different statement and techniques, see [18].

2. A brief description of the three dimensional homogeneous manifolds $\mathbb{E}(\kappa,\tau)$

For any $r>0$ we denote by $\mathbb{D}(r)\subset\mathbb{R}^2$ the open disc of $\mathbb{R}^2$ with center at the origin and with radius r (for the Euclidean metric).

For any $\kappa\leqslant 0$ and $\tau\geqslant 0$ we consider the model of $\mathbb{E}(\kappa,\tau)$ given by $\mathbb{D}(\dfrac{2}{\sqrt{-\kappa}})\times\mathbb{R}$ equipped with the metric

$$\nu_\kappa^2(dx^2+dy^2)+\left(\tau\nu_\kappa(ydx-xdy)+dt\right)^2. \tag{1}$$

where $\nu_\kappa=\dfrac{1}{1+\kappa\frac{x^2+y^2}{4}}$. We observe that $\mathbb{E}(-1,\tau)=\widetilde{\mathrm{PSL}}_2(\mathbb{R},\tau)$. By abuse of notations we set $\mathbb{D}(\frac{1}{0})=\mathbb{D}(+\infty)=\mathbb{R}^2$. Thus $\mathbb{E}(0,\tau)=\mathrm{Nil}_3(\tau)$. Also, $\mathbb{R}^3$ equipped with the Euclidean metric is a model of $\mathbb{E}(0,0)$.

We denote by $\mathbb{M}(\kappa)$ the complete, connected and simply connected Riemannian surface with constant curvature κ. Notice that for $\kappa<0$ a model of $\mathbb{M}(\kappa)$ is given by the disc $\mathbb{D}(\dfrac{2}{\sqrt{-\kappa}})$ equipped with the metric $\nu_\kappa^2(dx^2+dy^2)$.

We recall that $\mathbb{E}(\kappa,\tau)$ is a fibration over $\mathbb{M}(\kappa)$, and the projection $\Pi:\mathbb{E}(\kappa,\tau)\longrightarrow\mathbb{M}(\kappa)$ is a Riemannian submersion, see for example [3]. Moreover the unit vertical field $\frac{\partial}{\partial t}$ is a Killing field generating a one-parameter group of isometries given by the vertical translations.

We have seen in [18, Example 2.2-(2)] that the horizontal geodesics and the vertical geodesics of $\mathbb{E}(\kappa,\tau)$ admit a reflection. That is, for any such a geodesic L, there exists a non trivial isometry I_L of $\mathbb{E}(\kappa,\tau)$ satisfying

- I_L is orientation preserving,
- $I_L(p)=p$ for any $p\in L$,
- $I_L\circ I_L=\mathrm{Id}$.

Let Ω be any domain of $\mathbb{M}(\kappa)$ and let $u:\Omega\longrightarrow\mathbb{R}$ be a C^2-function. We say that the set $\Sigma:=\{(p,u(p)),\ p\in\Omega\}\subset\mathbb{D}(\dfrac{2}{\sqrt{-\kappa}})\times\mathbb{R}$ is a *vertical graph*. Note that the Killing field $\frac{\partial}{\partial t}$ is transverse to Σ. Thus, by the well-known criterium of stability, if Σ is a minimal surface then Σ is stable.

Consider some arbitrary local coordinates (x_1, x_2, x_3) of $\mathbb{E}(\kappa, \tau)$. Let u be a C^2 function defined on a domain Ω contained in the x_1, x_2 plane of coordinates. Let $S \subset \mathbb{E}(\kappa, \tau)$ be the graph of u. Then S is a minimal surface if u satisfies an elliptic PDE (called *minimal surface equation*)

$$F(x, u, u_1, u_2, u_{11}, u_{12}, u_{22}) = 0,$$

see [18, Equation (13)]. Furthermore, if u has bounded gradient then the PDE is uniformly elliptic.

3. Jenkins-Serrin type minimal surfaces

The original Jenkins-Serrin's theorem was conceived in $\mathbb{R}^3$, see [8, Theorems 1, 2 and 3]. It was extended in $\mathbb{H}^2 \times \mathbb{R}$ by B. Nelli and H. Rosenberg [10, Theorem 3] and in $\mathbb{M}^2 \times \mathbb{R}$ by A.L. Pinheiro [13, Theorem 1.1] where $\mathbb{M}^2$ is a complete Riemannian surface. Later on it was established in $\widetilde{\mathrm{PSL}}_2(\mathbb{R})$ by R. Younes [23, Theorem 1.1]. As a matter of fact the same proof also works in the homogeneous spaces $\mathbb{E}(\kappa, \tau)$ for any $\kappa < 0$ and $\tau \geqslant 0$.

We state briefly below the Jenkin-Serrin type theorem in the homogeneous spaces $\mathbb{E}(\kappa, \tau)$ for $\kappa < 0$ and $\tau \geqslant 0$ (same statement holds in $\mathbb{R}^3$ and in $\mathbb{M}^2 \times \mathbb{R}$).

Let $\Gamma \subset \mathbb{M}^2(\kappa)$ be a convex Jordan curve constituted of two families of open geodesic arcs $A_1, \dots, A_a$, $B_1, \dots, B_b$ and a family of C^1 convex open arcs $C_1, \dots, C_c$ with their endpoints. We assume that no two A_i and no two B_j have a common endpoint. We denote by Ω the bounded convex domain in $\mathbb{M}^2(\kappa)$ with boundary Γ.

On each open arc C_k we assign a continuous boundary data g_k.

Let $P \subset \overline{\Omega}$ be any polygon whose vertices are chosen among the endpoints of the open geodesic arcs A_i, B_j, we call P an *admissible polygon*. We set

$$\alpha(P) = \sum_{A_i \subset P} \|A_i\|, \ \beta(P) = \sum_{B_j \subset P} \|B_j\|, \ \gamma(P) = \text{perimeter of } \ P.$$

With the above notations the Jenkins-Serrin's theorem asserts the following:

If the family $\{C_k\}$ is not empty then there exists a function $u : \Omega \longrightarrow \mathbb{R}$ whose graph is a minimal surface in $\mathbb{E}(\kappa, \tau)$ and such that

$$u_{|A_i} = +\infty, \ u_{|B_j} = -\infty, \ u_{|C_k} = g_k$$

if and only if

$$2\alpha(P) < \gamma(P), \quad 2\beta(P) < \gamma(P) \tag{2}$$

for any admissible polygon P. In this case the function u is unique.

If the family $\{C_k\}$ is empty such a function u exists if and only if $\alpha(\Gamma) = \beta(\Gamma)$ and condition (2) holds for any admissible polygon $P \neq \Gamma$. In this case the function u is unique up to an additive constant.

We denote by $\Sigma \subset \mathbb{E}(\kappa, \tau)$ the graph of u over Ω and we call such a surface a *Jenkins-Serrin type minimal surface.*

Remark 3.1. We observe that when the family $\{C_k\}$ is empty, the boundary of Σ is the union of vertical geodesic line $\{q\} \times \mathbb{R}$ for any common endpoint q between geodesic arcs A_i and B_j.

Suppose that the family $\{C_k\}$ is not empty and let x_0 be a common vertex between A_i and C_k, if any. If g_k has a finite limit at x_0, say α, then the half vertical line $\{x_0\} \times [\alpha, +\infty[$ lies in the boundary of Σ. Now if x_0 is a common vertex between B_j and C_k and if g_k has a finite limit at x_0, say β, then the half vertical line $\{x_0\} \times]-\infty, \beta]$ lies in the boundary of Σ. At last, if x_0 is a common vertex between C_i and C_k and if g_i and g_k have different finite limits at x_0, say $\alpha < \beta$, then the vertical segment $\{x_0\} \times [\alpha, \beta]$ lies in the boundary of Σ.

4. Main theorem

For any vertical geodesic line L of $\mathbb{E}(\kappa, \tau)$, we denote by I_L the reflection about the line L.

Theorem 4.1. *Using the notations of section 3 and under the assumptions of remark 3.1, let $\gamma \subset L := \{x_0\} \times \mathbb{R} \subset \mathbb{E}(\kappa, \tau)$ be a vertical component of the boundary of the minimal vertical graph $\Sigma \subset \mathbb{E}(\kappa, \tau)$, where $\kappa < 0$ and $\tau \geqslant 0$.*

Then, we can extend minimaly Σ by reflection about L. More precisely, $S := \Sigma \cup \gamma \cup I_L(\Sigma)$ is a smooth minimal surface invariant by the reflection about Γ, containing γ in its interior.

Furthermore the same statement and proof hold for $\Sigma \subset \mathbb{R}^3$ or $\Sigma \subset \mathbb{S}^2 \times \mathbb{R}$.

Observe that the possible cases for γ are the following: the whole line L, a half line of L or a closed geodesic arc of L.

Remark 4.2. We use the same notations as in Theorem 4.1. Suppose that the boundary of Σ contains an open arc δ (graph over an arc C_k) of an horizontal geodesic line Υ of $\widetilde{\mathrm{PSL}}_2(\mathbb{R}, \tau)$.

We denote by I_Υ the reflection in $\widetilde{\mathrm{PSL}}_2(\mathbb{R}, \tau)$ about Υ.

We can prove as in [17, Lemma 3.6] that we can extend Σ by reflection about Υ: $\Sigma \cup \delta \cup I_\Upsilon(\Sigma)$ is a connected smooth minimal surface containing δ in its interior.

On the other hand, we can verify that the proof of Theorem 4.1 also works for reflection about horizontal geodesic lines.

Proof. For the sake of clarity and simplicity of notations, we provide the proof in $\widetilde{\mathrm{PSL}}_2(\mathbb{R},\tau) = \mathbb{E}(-1,\tau)$. Nevertheless, all arguments and constructions hold in $\mathbb{E}(\kappa,\tau)$ for any $\kappa < 0$ and $\tau \geqslant 0$, in $\mathbb{R}^3$, that is for $\kappa = \tau = 0$ and in $\mathbb{S}^2 \times \mathbb{R}$, that is for $\kappa = 1$ and $\tau = 0$.

We assume that the family C_k is not empty. The other situation can be handle in a similar way.

We suppose also that all functions g_k admit a limit at the endpoints of C_k. It is possible to carry out a proof without this assumption but the details are cumbersome and we will not writedown it.

Let $n_0 \in \mathbb{N}$ be such that $n_0 > \max_k \sup_{x\in C_k} |g_k(x)|$.

For any integer $n \geqslant n_0$ we consider the Jordan curve Γ_n obtained by the union of the geodesic arcs A_i at height n, the geodesic arcs B_j at height $-n$, the graphs of functions g_k over the open arcs C_k and the vertical segments necessary to form a Jordan curve. Thus Γ is the projection of Γ_n on $\mathbb{H}^2$.

Let $\Sigma_n \subset \widetilde{\mathrm{PSL}}_2(\mathbb{R},\tau)$ be a solution of the classical Plateau problem for the Jordan curve Γ_n. Since $\Gamma_n \subset \Gamma \times \mathbb{R}$ and Γ is convex, we obtain that Σ_n is an embedded area minimizing disc in $\overline{\Omega} \times \mathbb{R}$.

The surface $\Sigma_n \cap (\Omega \times \mathbb{R})$ is a graph over Ω, see [1, Theorem 1]. Furthermore, by a general maximum principle adapted to $\widetilde{\mathrm{PSL}}_2(\mathbb{R},\tau)$, see for example [13, Theorem 1.3], we get that Σ_n is the unique disc type minimal surface with boundary Γ_n. We set $\mathring{\Sigma}_n = \Sigma_n \setminus \Gamma_n$.

Let $u_n : \Omega \longrightarrow \mathbb{R}$ be the function whose the graph is $\mathring{\Sigma}_n$. Thus u_n extends continuously by n on the edges $\mathrm{int}A_i$, by $-n$ on the edges $\mathrm{int}B_j$ and by g_k over the open arcs C_k. Using the lemmas derived in [23], following the original proof of [8, Theorem 3], it can be proved that the sequence of functions (u_n) converges to a function $u : \Omega \longrightarrow \mathbb{R}$ in the C^2-topology, uniformly over any compact subset of Ω.

We set $\gamma_n := \Sigma_n \cap L$, thus $\gamma_n \subset \gamma$ for any n. Due to the fact that Σ_n is area minimizing we can apply the reflection principle about the vertical line L, see [18, Remark 3.4]. That is, $S_n := \Sigma_n \cup I_L(\Sigma_n)$ is an embedded minimal surface containing $\mathrm{int}\gamma_n$ in its interior. By construction S_n is invariant under the reflection I_L and is orientable.

Let d_n be the intrinsic distance on S_n. For any $p \in S_n$ and any $r > 0$ we denote by $B_n(p,r) \subset S_n$ the geodesic disc of S_n centered at p with radius r. By construction, for any $p \in \mathrm{int}\gamma$ there exist $n_0 \in \mathbb{N}$ and a real number $c_p > 0$ such that for any integer $n \geqslant n_0$ we have $p \in \mathrm{int}\gamma_n \subset \mathrm{int}S_n$ and $d_n(p, \partial S_n) > 2c_p$.

We assert that the Gaussian curvature K_n of the surfaces S_n is uniformly bounded in the neighborhood of each point of $\text{int}\gamma$, independently of n.

Proposition 4.3. *For any $p \in \text{int}\gamma$ there exist $R_p, K_p > 0$, and there exists $n_p \in \mathbb{N}$ satisfying $p \in \text{int}\gamma_{n_p} \subset S_{n_p}$ and $d_{n_p}(p, \partial S_{n_p}) > 2R_p$, such that for any integer $n \geqslant n_p$ we have $p \in \text{int}\gamma_n \subset S_n$ and*

$$|K_n(x)| \leqslant K_p,$$

for any $x \in B_n(p, R_p)$,

Proof of the Proposition. We argue by absurd.

Suppose by contradiction that there exists $p \in \text{int}\gamma$ such that for any $k \in \mathbb{N}^*$ there exist an integer $n_k > k$ and $x_k \in B_{n_k}(p, \frac{1}{k})$ such that $|K_{n_k}(x_k)| > k^2$.

There exist $c > 0$ and $k_0 \in \mathbb{N}^*$ such that for any integer $k \geqslant k_0$ we have $p \in \text{int}\gamma_{n_k}$ and $d_{n_k}(p, \partial S_{n_k}) > 2c$. Thus $\overline{B}_{n_k}(p, c)) \subset \text{int}S_{n_k}$.

Moreover there exists an integer $k_1 > k_0$ such that for any integer $k \geqslant k_1$ we have $d_{n_k}(x_k, \partial B_{n_k}(p, c)) > c/2$.

From now on, we are going to use classical blow-up techniques.

Define the continuous function $f_k : \overline{B}_{n_k}(p, c)) \longrightarrow [0, +\infty[$ for any $k \geqslant k_1$, setting: $f_k(x) = \sqrt{|K_{n_k}(x)|}\, d_{n_k}(x, \partial B_{n_k}(p, c))$.

Clearly $f_k \equiv 0$ on $\partial B_{n_k}(p, c)$ and

$$f_k(x_k) = \sqrt{|K_{n_k}(x_k)|}\, d_{n_k}(x_k, \partial B_{n_k}(p, c)) \geqslant k\frac{c}{2}.$$

We fix a point $p_k \in B_{n_k}(p, c)$ where the function f_k attains its maximum value, hence

$$f_k(p_k) \geqslant k\frac{c}{2}. \tag{3}$$

We deduce therefore

$$\sqrt{|K_{n_k}(p_k)|} \geqslant \frac{kc}{2d_{n_k}(p_k, \partial B_{n_k}(p, c))} \geqslant \frac{kc}{2c} = \frac{k}{2}. \tag{4}$$

Definition 4.4. We set $\rho_k = d_{n_k}(p_k, \partial B_{n_k}(p, c))$ and we denote by $D_k \subset B_{n_k}(p, c) \subset S_{n_k}$ the geodesic disc with center p_k and radius $\rho_k/2$. Notice that D_k is embedded.

For further purpose we emphasize that D_k is an orientable minimal surface of $\widetilde{\text{PSL}}_2(\mathbb{R}, \tau)$.

For any integer $k \geqslant k_1$ we set

$$\lambda_k := \sqrt{|K_{n_k}(p_k)|} \geqslant k/2.$$

Let us consider the model of $\widetilde{\mathrm{PSL}}_2(\mathbb{R},\tau)=\mathbb{E}(-1,\tau)$ given by (1) for $\kappa=-1$, that is the product set $\mathbb{D}(2)\times\mathbb{R}$ equipped withe the metric

$$ds^2 := \mu^2(dx^2+dy^2)+\big(\tau\mu(ydx-xdy)+dt\big)^2 \tag{5}$$

where $\mu=\mu(x,y)=\dfrac{1}{1-\frac{x^2+y^2}{4}}$.

For any integer $k\geqslant k_1$ we set $\mu_k=\mu_k(u,v)=\dfrac{1}{1-\frac{u^2+v^2}{4\lambda_k^2}}$. We consider, as in the Nguyen's thesis [11, Section 2.2.3], the product set $\mathbb{D}(2\lambda_k)\times\mathbb{R}$ equipped with the metric

$$ds_k^2 := \mu_k^2(du^2+dv^2)+\left(\frac{\tau}{\lambda_k}\,\mu_k(vdu-udv)+dw\right)^2. \tag{6}$$

Thus $(\mathbb{D}(2\lambda_k)\times\mathbb{R},\, ds_k^2)$ is a model of $\mathbb{E}(\frac{-1}{\lambda_k^2},\frac{\tau}{\lambda_k})$.

For any integer $k\geqslant k_1$, we denote by T_k an isometry of $\widetilde{\mathrm{PSL}}_2(\mathbb{R},\tau)$ which sends p_k to the origine $0_3:=(0,0,0)$, see for example [12, Chapter 5] or [21].

Let us consider the homothety

$$\begin{aligned} H_k : \mathbb{D}(2)\times\mathbb{R} &\longrightarrow \mathbb{D}(2\lambda_k)\times\mathbb{R}\\ (x,y,t) &\longmapsto (u,v,w)=\lambda_k\,(x,y,t). \end{aligned}$$

We have $H_k^*(ds_k^2)=\lambda_k^2\,ds^2$, see (5) and (6). Then, it follows that $\widetilde{D}_k := (H_k\circ T_k)(D_k)$ is an embedded minimal surface of $(\mathbb{D}(2\lambda_k)\times\mathbb{R},\, ds_k^2)$.

By construction, $\widetilde{D}_k$ is a geodesic disc with center the origine 0_3 of $\mathbb{D}(2\lambda_k)\times\mathbb{R}$: $0_3\in\widetilde{D}_k\subset\mathbb{D}(2\lambda_k)\times\mathbb{R}$. Moreover the radius of $\widetilde{D}_k$ is $\widetilde{\rho}_k=\lambda_k\cdot$(radius of D_k), that is $\widetilde{\rho}_k=\lambda_k\,\rho_k/2$.

Using the estimate (3) we get

$$\widetilde{\rho}_k=\lambda_k\,\rho_k/2=\sqrt{|K_{n_k}(p_k)|}\;d_{n_k}(p_k,\partial B_{n_k}(p,c))/2=\frac{f_k(p_k)}{2}\geqslant\frac{kc}{4}, \tag{7}$$

thus $\widetilde{\rho}_k\to\infty$ if $k\to\infty$.

Let $g_{\mathrm{euc}}=du^2+dv^2+dw^2$ be the Euclidean metric of $\mathbb{R}^3$. We observe that $(\mathbb{D}(2\lambda_k)\times\mathbb{R},\, ds_k^2)$ converges to $(\mathbb{R}^2\times\mathbb{R},\, g_{\mathrm{euc}})$ for the C^2-topology, uniformly on any compact subset of $\mathbb{R}^3$.

We denote by $\widetilde{K}_{n_k}$ the Gaussian curvature of $\widetilde{D}_k$. For any $x\in D_k\subset\mathbb{D}(2)\times\mathbb{R}$, setting $X=(H_k\circ T_k)(x)\in\widetilde{D}_k\subset\mathbb{D}(2\lambda_k)\times\mathbb{R}$, we get

$\widetilde{K}_{n_k}(X) = \frac{K_{n_k}(x)}{\lambda_k^2}$. Hence for any $X \in \widetilde{D}_k$ we obtain

$$\sqrt{|\widetilde{K}_{n_k}(X)|} = \frac{\sqrt{|K_{n_k}(x)|}}{\lambda_k} \leqslant \frac{f_k(p_k)}{\lambda_k \, d_{n_k}(x, \partial B_{n_k}(p,c))} = \frac{d_{n_k}(p_k, \partial B_{n_k}(p,c))}{d_{n_k}(x, \partial B_{n_k}(p,c))} < 2, \tag{8}$$

since $d_{n_k}(x, \partial B_{n_k}(p,c)) > \dfrac{\rho_k}{2}$.

Furthermore, for any integer $k \geqslant k_1$ we have

$$\sqrt{|\widetilde{K}_{n_k}(0_3)|} = \frac{\sqrt{|K_{n_k}(p_k)|}}{\lambda_k} = 1. \tag{9}$$

We summarize some facts derived before:

- each $\widetilde{D}_k$ is an embedded and orientable minimal surface of $(\mathbb{D}(2\lambda_k) \times \mathbb{R}, \, ds_k^2) = \mathbb{E}(-\frac{1}{\lambda_k^2}, \frac{\tau}{\lambda_k})$,
- there is uniform estimate of Gaussian curvature, see (8),
- the radius $\widetilde{\rho}_k$ of the geodesic disc $\widetilde{D}_k$ go to $+\infty$ if $k \to \infty$,
- the metrics ds_k^2 converge to g_{euc} for the C^2-topology, uniformly on any compact subset of $\mathbb{R}^3$.

Therefore it can be proved, as in [16, Lemma 2.4] and the discussion that follows, that up to considering a subsequence, the $\widetilde{D}_k$ converge for the C^2-topology to a complete, connected and orientable minimal surface $\widetilde{S}$ of $\mathbb{R}^3$.

Remark 4.5. From the construction described in [16], the surface $\widetilde{S}$ has the following properties.

There exist $r, r_0 > 0$ such that for any $q \in \widetilde{S}$, a piece $\widetilde{G}(q)$ of $\widetilde{S}$, containing the geodesic disc with center q and radius r_0, is a graph over the open disc $D(q,r)$ of $T_q\widetilde{S}$ with center q and radius r (for the Euclidean metric of $\mathbb{R}^3$). Furthermore:

- for k large enough, a piece $\widetilde{G}_k(q)$ of $\widetilde{D}_k$ is also a graph over $D(q,r)$ and the surfaces $\widetilde{G}_k(q)$ converge for the C^2-topology to $\widetilde{G}(q)$,
- for any $y \in \widetilde{G}(q)$ there exists $k_y \in \mathbb{N}$ such that for any $k \geqslant k_y$ we can choose the piece $\widetilde{G}_k(y)$ of $\widetilde{D}_k$ such that $\widetilde{G}_k(q) \cup \widetilde{G}_k(y)$ is connected.

By construction we have $0_3 \in \widetilde{S}$ and, denoting by $\widetilde{K}$ the Gaussian curvature of $\widetilde{S}$ in $(\mathbb{R}^3, g_{\text{euc}})$, we deduce from (9)

$$|\widetilde{K}(0_3)| = 1. \tag{10}$$

For any integer $k \geqslant k_1$ we set $\widetilde{L}_k := (H_k \circ T_k)(L)$. Thus, $\widetilde{L}_k$ is a vertical straight line of $\mathbb{R}^3$.

Definition 4.6. Let δ_k be the distance in $\mathbb{D}(2\lambda_k) \times \mathbb{R}$ induced by the metric ds_k^2.

We say that the sequence of vertical lines $(\widetilde{L}_k)$ in $\mathbb{R}^3$ *vanishes at infinity* if $\delta_k(0_3, \widetilde{L}_k) \to +\infty$ when $k \to +\infty$

There are two possibilities: the sequence $(\widetilde{L}_k)$ vanishes or not at infinity. We are going to show that either case cannot occur, we will find therefore a contradiction.

First case: $(\widetilde{L}_k)$ vanishes at infinity.

Observe that, by construction, the geodesic discs $B_{n_k}(p, c)$ are invariant under the reflection I_L and $f_k(q) = f_k(I_L(q))$ for any $q \in B_{n_k}(p, c)$. So we can assume that $p_k \in \Sigma_{n_k} \subset S_{n_k}$ for any $k \geqslant k_1$.

Let $q \in \widetilde{S}$, and consider a minimizing geodesic arc $\delta \subset \widetilde{S}$ joining 0_3 to q. It follows from Remark 4.5 that there exist a finite number of points $q_1 = 0_3, \ldots, q_n = q$ and there exists $k_q \in \mathbb{N}$ such that:

- for any integer $k \geqslant k_q$ the subset $\cup_j \widetilde{G}_k(q_j) \subset \widetilde{D}_k$ is connected and converges for the C^2-topology to the subset $\cup_j \widetilde{G}(q_j) \subset \widetilde{S}$,
- for any integer $k \geqslant k_q$ we have $\big(\cup_j \widetilde{G}_k(q_j)\big) \cap \widetilde{L}_k = \emptyset$.

Thus for any integer $k \geqslant k_q$ we obtain that $(H_k \circ T_k)^{-1}(\cup_j \widetilde{G}_k(q_j)) \cap L = \emptyset$, that is $(H_k \circ T_k)^{-1}(\cup_j \widetilde{G}_k(q_j)) \subset D_k \cap \Sigma_{n_k}$.

Setting $\widehat{D}_k := (H_k \circ T_k)(D_k \cap \Sigma_{n_k})$ we deduce that the sequence $(\widehat{D}_k)$ converges to $\widetilde{S}$ for the C^2-topology too. Furthermore any minimal surface $\widehat{D}_k \setminus \widetilde{L}_k$ is a Killing graph and thus $\widehat{D}_k$ is a stable minimal surface of $\mathbb{E}(\frac{-1}{\lambda_k^2}, \frac{\tau}{\lambda_k})$.

Therefore it can be proved as in the discussion following Lemma 2.4 in [16] that $\widetilde{S}$ is a connected, complete, orientable and stable minimal surface of $\mathbb{R}^3$. Thanks to results of do Carmo-Peng [4], Fischer-Colbrie and Schoen [5] and Pogorelov [14], $\widetilde{S}$ is a plane. But this gives a contradiction with the relation (10).

Second case: $(\widetilde{L}_k)$ does not vanish at infinity.

We will prove that the Gauss map of $\widetilde{S}$ omits infinitely many points, hence $\widetilde{S}$ would be a plane contradicting the relation (10).

Let $\alpha \in (0, \pi]$ be the interior angle of Γ at vertex x_0. Observe that the case where $\alpha = \pi$ is under consideration.

Since Ω is convex, there exists a geodesic line $C_{x_0} \subset \mathbb{H}^2$ at x_0 such that $C_{x_0} \cap \Omega = \emptyset$. Let Π be the product $C_{x_0} \times \mathbb{R}$ in $(\mathbb{D}(2) \times \mathbb{R}, ds^2) =$

$\mathbb{E}(-1,\tau)$. When $\tau=0$ notice that Π, is a vertical totally geodesic plane in $\mathbb{H}^2\times\mathbb{R}$. We recall that there are no totally geodesic surfaces in $\mathbb{E}(-1,\tau)$ if $\tau\neq 0$, see [20, Theorem 1].

Under our assumption, up to considering a subsequence, we can assume that the sequence $(\widetilde{L}_k)$ converges to a vertical straight line $\widetilde{L}\subset\mathbb{R}^3$ and that $\big((H_k\circ T_k)(\Pi)\big)$ converges to a vertical plane $\widetilde{\Pi}\subset\mathbb{R}^3$ containing $\widetilde{L}$. Let us denote by $\widetilde{\Pi}^+$ and $\widetilde{\Pi}^-$ the two open halfspaces of $\mathbb{R}^3$ bounded by $\widetilde{\Pi}$.

Claim 1. We have $(\widetilde{S}\cap\widetilde{\Pi})\setminus\widetilde{L}=\emptyset$.

Otherwise assume there exists a point $q\in\widetilde{S}\cap\widetilde{\Pi}$ such that $q\notin\widetilde{L}$. We can suppose that $\widetilde{\Pi}$ is transverse to $\widetilde{S}$ at q. Thus there is an open piece $\widetilde{F}(q)$ of $\widetilde{S}$ containing q which is transverse to the plane $\widetilde{\Pi}$. Hence, for any integer k large enough, a piece $\widetilde{F}_k(q)$ of $\widetilde{S}_k$ is so close to $\widetilde{F}(q)$ that it is transverse to $\widetilde{\Pi}$ too. Consequently we would have $\mathrm{int}\widetilde{D}_k\cap\big((H_k\circ T_k)(\Pi)\setminus\widetilde{L}_k\big)\neq\emptyset$, that is $\mathrm{int}D_k\cap(\Pi\setminus L)\neq\emptyset$. But by construction we have $\mathrm{int}S_{n_k}\cap(\Pi\setminus L)=\emptyset$, which leads to a contradiction since $D_k\subset S_{n_k}$.

Claim 2. We have $\widetilde{S}\cap\widetilde{\Pi}=\widetilde{L}$.

Assume first that $\widetilde{S}\cap\widetilde{\Pi}=\emptyset$. Hence $\widetilde{S}$ stay in an open halfspace, say $\widetilde{\Pi}^+$, of $\mathbb{R}^3$ bounded by $\widetilde{\Pi}$. Observe that the halfspace $\widetilde{\Pi}^+$ is the limit of open subspaces $(H_k\circ T_k)(\Pi^+)$ of $\mathbb{D}(2\lambda_k)\times\mathbb{R}$ where Π^+ is one of the two open halfspaces of $\mathbb{D}(2)\times\mathbb{R}$ bounded by Π. Consequently $\widetilde{S}$ is the limit of the graphs $\widetilde{D}_k\cap(H_k\circ T_k)(\Pi^+)$. Therefore, as in the first case, we obtain that $\widetilde{S}$ is stable and thus is a plane, giving a contradiction with (10). We obtain therefore $\widetilde{S}\cap\widetilde{\Pi}\neq\emptyset$

Let $q\in\widetilde{S}\cap\widetilde{\Pi}$. By Claim 1 we have $q\in\widetilde{L}$. If $\widetilde{\Pi}$ were the tangent plane of $\widetilde{S}$ at q, then the intersection $\widetilde{S}\cap\widetilde{\Pi}$ would consist in a even number $\geqslant 4$ of arcs issued from q. Then we infer that $\widetilde{S}\cap(\widetilde{\Pi}\setminus\widetilde{L})\neq\emptyset$ which is not possible due to the Claim 1.

Thus $\widetilde{\Pi}$ is transverse to $\widetilde{S}$ at q. Since $\widetilde{S}\cap\widetilde{\Pi}\subset\widetilde{L}$ by Claim 1, we deduce that $\widetilde{S}\cap\widetilde{\Pi}$ contains an open arc of $\widetilde{L}$ containing q. This proves that $\widetilde{S}\cap\widetilde{\Pi}$ contains a segment of $\widetilde{L}$. It is well known that if a complete minimal surface of $\mathbb{R}^3$ contains a segment of a straight line then it contains the whole straight line, see Proposition 5.1 in the Appendix. We conclude that $\widetilde{S}\cap\widetilde{\Pi}=\widetilde{L}$ as desired.

Remark 4.7. To prove that $\widetilde{S}\cap\widetilde{\Pi}\neq\emptyset$ we can alternatively argue as follows. Assume that $\widetilde{S}\cap\widetilde{\Pi}=\emptyset$. By construction $\widetilde{S}$ is a complete and connected minimal surface in $\mathbb{R}^3$ without self-intersection. Furthermore we deduce from the estimates (8) that $\widetilde{S}$ has bounded curvature. It follows from [15, Remark] that $\widetilde{S}$ is properly embedded. Since $\widetilde{S}$ lies in a halfspace, we deduce from the halfspace theorem [7, Theorem 1] that $\widetilde{S}$ is a plane, which gives a contradiction with (10). Thus $\widetilde{S}\cap\widetilde{\Pi}\neq\emptyset$

We deduce from Claim 2 that $\widetilde{S}\setminus\widetilde{\Pi}=\widetilde{S}\setminus\widetilde{L}$ has two connected components, say $\widetilde{S}^-\subset\widetilde{\Pi}^-$ and $\widetilde{S}^+\subset\widetilde{\Pi}^+$. In the same way we denote by Π^+ and Π^- the two open halfspaces of $\mathbb{D}(2)\times\mathbb{R}$ bounded by Π. We can assume that $\widetilde{\Pi}^+$ (resp. $\widetilde{\Pi}^-$) is the limit of $(H_k\circ T_k)(\Pi^+)$ (resp. $(H_k\circ T_k)(\Pi^-)$).

We set $D_k^{\pm}:=D_k\cap\Pi^{\pm}$ and $\widetilde{D}_k^{\pm}:=(H_k\circ T_k)(D_k^{\pm})=\widetilde{D}_k\cap\Pi^{\pm}$. We observe that $\widetilde{D}_k^+$ and $\widetilde{D}_k^-$ are vertical graphs and that $\widetilde{S}^+$ (resp. $\widetilde{S}^-$) is the limit of $\widetilde{D}^+$ (resp. $\widetilde{D}^-$) for the C^2-topology.

For any integer $k\geqslant k_1$ we denote by $\widetilde{N}^k$ a smooth unit normal vector field on $\widetilde{D}_k$ with respect to the metric ds_k^2, see (6). Let $\widetilde{N}_3^k$ be the *vertical component* of $\widetilde{N}^k$, this means that $\widetilde{N}^k-\widetilde{N}_3^k\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial t}$ are orthogonal vector fields along $\widetilde{D}_k$.

Since $\widetilde{S}$ is the limit of $\widetilde{D}_k$ for the C^2-topology, we can define a unit normal field $\widetilde{N}$ on $\widetilde{S}$ as the limit of the fields $\widetilde{N}^k$.

Claim 3. We have $\widetilde{N}_3\neq 0$ on $\widetilde{S}^+\cup\widetilde{S}^-$. Furthermore $\widetilde{S}^+$ and $\widetilde{S}^-$ are vertical graphs.

Indeed, we know that $\widetilde{D}_k^+$ is a vertical graph. So we can assume that $\widetilde{N}_3^k>0$ along $\widetilde{D}_k^+$ for any $k\geqslant k_1$. By considering the limit of the fields $\widetilde{N}^k$ we get that $\widetilde{N}_3\geqslant 0$ on $\widetilde{S}^+$.

Let $q\in\widetilde{S}^+$ be a point such that $\widetilde{N}_3(q)=0$, if any. Recall that the Gauss map of a non planar minimal surface of $\mathbb{R}^3$ is an open map. Therefore, in any neighborhood of q in $\widetilde{S}^+$ it would exist points $y\in\widetilde{S}^+$ such that $\widetilde{N}_3(y)<0$, which leads to a contradiction.

Thus we have $\widetilde{N}_3\neq 0$ on $\widetilde{S}^+$. We prove in the same way that $\widetilde{N}_3\neq 0$ on $\widetilde{S}^-$ too.

Assume by contradiction that $\widetilde{S}^+$ is not a vertical graph. Then there exist two points $q,\overline{q}\in\widetilde{S}^+$ lying to same vertical straight line. As the tangent planes of $\widetilde{S}^+$ at q and $\overline{q}$ are not vertical, there exists a real number $\delta>0$ such that a neighborhood $V_{\overline{q}}\subset\widetilde{S}^+$ of $\overline{q}$ and a neighborhood $V_q\subset\widetilde{S}^+$ of q are vertical graphs over an Euclidean disc of radius δ in the (u,v)-plane.

But, by construction, for k large enough a piece $U_{\overline{q}}$ of $\widetilde{D}_k^+$ is C^2-close of $V_{\overline{q}}$ and a piece U_q of $\widetilde{D}_k^+$ is C^2-close of V_q. Clearly this would imply that the vertical projections of $U_{\overline{q}}$ and U_q on the (u,v)-plane have non empty intersection. But this is not possible since $\widetilde{D}_k^+$ is a vertical graph. This shows that $\widetilde{S}^+$ is a vertical graph.

We can prove in the same way that $\widetilde{S}^-$ is a vertical graph.

End of the proof of the proposition

Let $P \subset \mathbb{R}^3$ be any vertical plane verifying $\widetilde{L} \subset P$ and $P \neq \widetilde{\Pi}$. We deduce from Claims 2 and 3 that $(\widetilde{S} \cap P) \setminus \widetilde{L}$ is a vertical graph. Therefore, the structure of the intersection of two minimal surfaces tangent at a point, see [2, Theorem 7.3] or [19, Lemma, p. 380], shows that there cannot be two distinct points of $\widetilde{L}$ where the tangent plane of $\widetilde{S}$ is P.

Let ν and $-\nu$ be the two unit vectors orthogonal to P. Since $\widetilde{N}_3 \neq 0$ on $\widetilde{S} \setminus \widetilde{L}$ we deduce that ν and $-\nu$ are not both assumed by the Gauss map of $\widetilde{S}$. By varying the vertical planes P, we obtain that the Gauss map of $\widetilde{S}$ omits infinitely many points (belonging to the equator of the 2-sphere). Then $\widetilde{S}$ must be a plane, see [22, Theorem] or [6, Theorem I]. On account of (10) we arrive to a contradiction. This accomplishes the proof of the proposition. $\square$

End of the proof of the theorem

Assuming Proposition 4.3 we will prove that for any $p \in \text{int}\gamma$ there is a minimal disc $D(p)$, containing p in its interior, such that $D(p) \subset \Sigma \cup \gamma \cup I_L(\Sigma)$, this will prove that $\Sigma \cup \gamma \cup I_L(\Sigma)$ is a minimal surface, that is smooth along $\text{int}\gamma$.

Let $p \in \text{int}\gamma$, we deduce from Proposition 4.3 that there exist real numbers $R_p, K_p > 0$ and $n_p \in \mathbb{N}$ such that for any integer $n \geqslant n_p$ and for any point $x \in B_n(p, R_p)$ we have $|K_n(x)| \leqslant K_p$.

Using the same arguments applied in the proof of the Proposition 4.3 (see [16, Lemma 2.4] and the discussion that follows), we can show that, up to taking a subsequence, the geodesic discs $B_n(p, R_p)$ converge for the C^2-topology to a minimal disc $D(p) \subset \mathbb{R}^3$ containing p in its interior. We recall that each geodesic disc $B_n(p, R_p)$ contains an open subarc $\gamma(p)$ of γ (which does not depend on n) passing through p and $B_n(p, R_p)$ is invariant under the reflection I_L. Thereby the minimal disc $D(p)$ also contains the subarc $\gamma(p)$ and inherits the same symmetry.

We set $S := \Sigma \cup \gamma \cup I_L(\Sigma)$.

By construction the surfaces $\text{int}S_n \setminus L$ converge to $\Sigma \cup I_L(\Sigma)$. We observe that $B_n(p, R_p) \setminus L \subset S_n \setminus L$ and then $D(p) \setminus L \subset \Sigma \cup I_L(\Sigma)$. Then we have $D(p) \subset S$. We conclude henceforth that S is a smooth minimal surface invariant under the reflection I_L, this accomplishes the proof of the theorem. $\square$

Remark 4.8. We don't know if the Jenkins-Serrin type theorem was established in the Heisenberg spaces $Nil_3(\tau) = \mathbb{E}(0,\tau)$ for $\tau > 0$. Assuming the Jenkins-Serrin type theorem, the proof of Theorem 4.1 works to establish the same reflection principle in $Nil_3(\tau)$ for vertical and horizontal geodesic lines.

5. Appendix

Proposition 5.1. *Let $M \subset \mathbb{R}^3$ be a complete minimal surface containing a segment of a straight line D. Then the whole line D belongs to M: $D \subset M$.*

Proof. We denote by x, y, z the coordinates on $\mathbb{R}^3$. Up to an isometry of $\mathbb{R}^3$ we can assume that D is the x-axis: $D = \{(x,0,0),\ x \in \mathbb{R}\}$.

By assumption there exist real numbers $a < b$ such that $(x,0,0) \in M$ for any $x \in [a,b]$.

We set

$$B := \sup\{t > a,\ (x,0,0) \in M \text{ for any } x \in [a,t]\},$$
$$A := \inf\{t < b,\ (x,0,0) \in M \text{ for any } x \in [t,b]\}.$$

We are going to prove that $A = -\infty$ and $B = +\infty$ to conclude that $D \subset M$.

We have $B \geqslant b$. Assume by contradiction that $B \neq +\infty$, hence $(B,0,0) \in M$. Let $P \subset \mathbb{R}^3$ be the plane containing D and the orthogonal direction of M at $(B,0,0)$.

Since the surfaces M and P are transverse at $(B,0,0)$, their intersection in a neighborhood of $(B,0,0)$ is an analytic arc γ. Furthermore, up to choose a smaller arc, we can assume that γ is the graph of an analytic function f over the interval $[B-\varepsilon, B+\varepsilon]$ for $\varepsilon > 0$ small enough. Since f is an analytic function satisfying $f(x) = 0$ for any $x \in [B-\varepsilon, B]$, we deduce that $f(x) = 0$ for any $x \in [B-\varepsilon, B+\varepsilon]$. Therefore we have $(x,0,0) \in M$ for any $x \in [a, B+\varepsilon]$, contradicting the definition of B.

Thus we have $B = +\infty$. We prove in the same way that $A = -\infty$, concluding the proof. □

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