Summary of recent research papers

Comments about the results obtained in the period 2009-2012:

- Firstly, we are going to summarize the principal existence results obtained in the period concerning vertical minimal graphs in the product space $H^n \times R$, where H^n is the n-dimensional hyperbolic space. The height function u satisfies the minimal vertical equation:

$$\operatorname{diver}_{\mathbb{R}^n}\left(\frac{\nabla_{\mathbb{R}^n} u}{\sqrt{1+x_n^2(u_{x_1}^2+\cdots+u_{x_n}^2)}}\right)+\frac{(2-n)u_{x_n}}{x_n\sqrt{(1+x_n^2(u_{x_1}^2+\cdots+u_{x_n}^2)}}=0$$

  This research is a continuation of the study carried out in a joint paper with Eric Toubiana, University of Paris VII about minimal surfaces in the three dimensional space $H^2 \times R$ [SE-T7]. We were content to find suitable barriers for the Dirichlet problem. We remark that the starting point of this work is the discovery of the n-dimensional minimal Scherk type graphs [SE-T8]. It is worth noticing that the Scherk type surfaces (n=2) were constructed by Barbara Nelli and Harold Rosenberg in their fundamental paper [N-R]. Actually, we built with Eric Toubiana, a minimal vertical graph in $H^n \times R$ on the inside of a certain polyhedron admissible in H^n, which we call the first Scherk type minimal hypersurface taking infinite value on certain face of a polyhedron and zero on the other faces. Furthermore, we construct a second Scherk type minimal graph which is a minimal vertical graph in $H^n \times R$ over the inside of a polyhedron with 2k sides in H^n, taking values $+\infty$ or $-\infty$ on adjacent faces [SE-T8]. Besides, we develop other related results in the same paper [SE-T8]. For instance, using geometric barriers we obtain the solution of the Dirichlet problem for the minimal equation in $H^n \times R$ on a C^0 convex domain of H^n, taking continuous boundary value data on the finite boundary and continuous boundary value data on the asymptotic boundary. In fact, to obtain this result we make use of a rotational Scherk hypersurface as barrier at a finite point. At a point of the asymptotic boundary if the dimension is two we use a surface written in [S-E] and in arbitrary dimension we use a hypersurface obtained with Perre Bérard [B-SE1]. We also some obtain existence results for minimal graphs on certain admissible non convex domains [SE-T8]. We note that the same idea of these geometric constructions can be applied to the situation where the ambient space is R^{n+1}, leading to Scherk hypersurfaces and the related solution of the Dirichlet problem in Euclidean space [SE-T8]. Recall that when the domain is of class C^2, Jenkins and Serrin [J-S2] showed that given a C^2 boundary value data the mean convex condition is the necessary and sufficient condition for the solvability of the Dirichlet problem for the minimal equation in Euclidean space. When the environment is the product space $M^n \times R$ where M is a Riemannian manifold, if the domain is of classe C^2 if the mean curvature of the boundary of the domain is bounded from below by a positive constant, then given continuous boundary data, the Dirichlet problem for the minimal equation was solved by J. Spruck [Sp]. Summarizing: when the environment is $H^n \times R$, given continuous boundary value data by using geometric barriers-Scherk hypersurfaces, we solve the Dirichlet problem for the minimal equation in C^0 convex domains [SE-T8].

- We prove in a joint work with Barbara Nelli, University of Aquila, Italy, a vertical half-space theorem for mean curvature ½ surfaces in $H^2 \times R$ [N-SE]: We show that a complete surface with mean curvature ½, properly immersed in a mean convex side of a simply connected rotational surface with mean curvature ½ is rotational. In fact, when the environment is $H^2 \times R$, the mean curvature is ½ and the end is an annulus of revolution, it is known that this end has an asymptotic development. This implies that it has an exponential growth [SE-T3], [N-ST-UP]. Our result is somehow an extension of the known *half-space theorem* of Hoffman and Meeks [H-T2] in the context of surfaces with constant mean curvature ½ in $H^2 \times R$. The main idea in the proof is quite simple and geometric. We argue by contradiction using a one parameter family of rotational surfaces as barriers to ensure the result, by applying the maximum principle on account of the knowledge of the geometric behavior of the rotational ends (growth).

— We pause momentarily to point out two older works related with the theme:

1. A half- space theorem in the context of special minimal type Weingarten surfaces in Euclidean space was accomplished by Toubiana and me [SE-T]. The proof arises also from the idea of Hoffman and Meeks. It is an application of the maximum principle working with a one parameter family of special

rotational surfaces as barriers. Since the family of rotational special surfaces has a nice geometric behavior and since the mean curvature vector of the special rotational surface has the "good" normal orientation, one can argue by contradiction to get the result. To download an old version click here.

2. We remark that the author and Rosenberg working with the one parameter family of embedded Deluaunay surfaces proved a maximum principle inside a Delaunay surface in Euclidean space that yields uniqueness and other applications and generalizations [SE-R].

- We generalize with Pierre Bérard, University of Grenoble, France, a well-known theorem of Lindelöf, investigating the maximum domain of stability of minimal hypersurfaces of revolution, considering other environments different from the Euclidean space. In the Euclidean space R^3, the vertical half-catenoids are maximum domains of stability (Lindelöf's Property). This is the Lindelöf theorem [Li]. We outline a generalization and reinterpretation of this theorem with Pierre Bérard in the papers [B-SE3], [B-SE4]. In fact, we obtain in R^{n+1} a generalization of Lindelöf's theorem in the sense that we determine the maximum symmetric domains of stability. We also determine the maximum symmetric domains of stability when the ambient space is $H^2\times R$ or H^3. Surprisingly, we deduce that in R^{n+1}($n \geq 3$) the half-catenoids are not maximum domains of stability. Furthermore, we conclude that also in $H^2 \times R$ and in H^3, the half-catenoids are not maximum domains of stability. However, an embedded *catenoid cousin* in H^3 satisfies the Lindelöf's Property [B-SE4]. If the ambient is $H^n\times R$, these results are also established in arbitrary dimension. In the case of the hyperbolic space H^3, we get an improvement of the related results proved by H. Mori [M] and by M. Do Carmo-M.Dacjzer [DoC-D] about the index and stability of the family of catenoids in term of the parameter [B-SE3]. In summary: In the case of R^3 the half-catenoids are maximum domains of stability (Lindelöf theorem), but in the case of $H^2\times R$ or H^3 the half-catenoids are not maximum domain of stability.

- Moreover, we study with Pierre Bérard some properties of minimal hypersurfaces in the product space $H^n\times R$ [B-SE1]. In this paper we propose a notion of total curvature in this environment relying with the index of the Jacobi (stability) operator. We deduce roughly speaking that "total finite total curvature implies finite index."

However, the converse is not true, as shown by the examples giving in the same paper. Particularly, we show that certain problems are naturally posed and investigated in arbitrary dimensions. In fact, in a paper with Eric Toubiana we studied among other phenomena the minimal ('catenoids'), and constant mean curvature surfaces of revolution in $H^2 \times R$, exhibiting an explicit formula that has been very useful in the development of the theory [SE-T3]. In the joint work with Pierre Bérard [B-SE1], using the description of [SE-T3] we show that the index (number of negative eigenvalues of the Jacobi operator) is 1 and we describe certain domains of stability of the Jacobi operator, generalizing classical results for the classical catenoids in R^3. We establish the following general result: Let M be a complete minimal surface in $H^2 \times R$. If the integral of the intrinsic curvature of M is finite, then the index M is finite. The converse is not true, due to the existence of translational stable surfaces (that are vertical graphs) [S-E] [SE-T7]. It turns out that is quite natural to study this class of surfaces (finite total curvature) because of the results obtained in [HR] (and, recently, also because the results in [H-N-SE-T]). When $n \geq 3$ we deduce that the hypothesis of finiteness of the integral of $|A_M|$ (complete minimal hypersurfaces in $H^n \times R$) implies that the index of M is finite.

– It is worth noticing that together with P. Bérard we have studied hypersurfaces in $H^n \times R$ of constant (non zero) mean curvature H [B-SE2], constructing new examples and doing some geometric applications. For instance, we construct in these paper examples of hypersurfaces of revolution and translation hypersurfaces with non vanishing constant mean curvature H. Among them, we get entire vertical graphs and therefore stable hypersurfaces. We find examples of hypersurfaces of constant mean curvature $0 < H < (n-1)/n$, which are complete vertical graphs over the exterior of an equidistant hypersurface of H^n taking infinite boundary value data (on the equidistant hypersurface) and taking infinite asymptotic value data.

- In a joint work with Maria Fernanda Elbert (UFRJ) and Barbara Nelli, we construct examples of vertical graphs of constant mean curvature $H = ½$in $H^2 \times R$ over admissible exterior domains in H^2 [E-N-SE]. Such embedded examples are vertical graphs having a *weak growth* of a rotational end. The tools of this paper are a combination of geometric barriers (rotational surfaces of mean curvature ½) and elliptic theory, using the maximum principle.
- We built with Maria Fernanda Elbert all minimal hypersurfaces of constant mean curvature in $H^n \times R$, invariant by parabolic screw

motions [E-SE]. Among these examples we find several model stable hypersurfaces that are entire vertical graphs, and other invariant graphs which are not vertical but are complete horizontal graphs of arbitrary dimension. Some of these horizontal graphs are stable.

- We study in the individual article [S-E2] the horizontal minimal equation in $H^2 \times R$ [S-E]:

$$g_{xx}(g^2+g_t^2)+g_{tt}(1+g_x^2)-2g_x g_t g_{xt}+g(1+g_x^2)=0$$

We deduce a Bernstein type theorem and we set an open Bernstein type problem in the context of constant mean curvature ≤ ½. Moreover, we deduce for this equation a Radó type result.

- We study together with Laurent Hauswirth, Barbara Nelli and Eric Toubiana minimal ends of finite total curvature immersed in $H^2 \times R$ [HN-IF-T]. We establish the behavior of such an end, making a full geometric description, determining the horizontal section of this end by intercepting it with a slice of $H^2 \times R$. This work is based on earlier works as the pioneering study done by L. Hauswirth and H. Rosenberg on finite total curvature minimal surfaces [HR]. It is also aided by the theory of harmonic applications developed by Z. Han, L. Tan, A. Treiberg and T. Wan [H-T-T-W] and Y. N. Minsky [My]. Using the results mentioned before and other types of arguments, such as the Alexandrov reflection principle, based on the maximum principle, one can deduce a uniqueness Schoen type theorem in the context of finite total curvature surfaces in $H^2 \times R$. This result characterizes the complete finite total curvature minimal surface immersed in $H^2 \times R$, with two different ends, each end asymptotic to a vertical plane, as the model minimal surface independently discovered by J. Pyo [P] and F. Morabito, M. Rodriguez [M-R].

- We study in the individual paper [S-E3] a class of horizontal minimal equations in $H^n \times R$, involving a family of second order elliptic PDE's indexed by a parameter ε in [0, 1

$$\sum_{k=1}^{n-1}\left[g^2\left(1+g_{x_1}^2+\cdots+g_{x_k}^2+\cdots g_{x_{n-1}}^2\right)+g_t^2+\frac{\varepsilon}{n-1}\right]g_{x_k x_k}+\left(1+\sum_{k=1}^{n-1}g_{x_k}^2\right)g_{tt}-2\sum_{k=1}^{n-1}g_{x_k}g_t g_{x_k t}-$$
$$2g^2\sum_{1\le j<k\le n-1}g_{x_j}g_{x_k}g_{x_j x_k}+(n-1)g\left(1+\sum_{k=1}^{n-1}g_{x_k}^2\right)+(n-2)\frac{g_t^2}{g}=0$$

When $\varepsilon = 0$, we recover the horizontal minimal equation which is not a strictly elliptic EDP in general. When $\varepsilon > 0$, we obtain a strictly elliptic PDE that we call the ε- horizontal minimal equation. We infer a priori estimates for the horizontal length and a priori boundary gradient estimates that are quite general and quite natural as we explained in the text. We also obtain a priori global gradient estimates in the presence of a strong constraint on the horizontal length, which seems to be natural for this kind of PDE. This fact is somehow related to the following phenomenon: There are no solutions to the horizontal minimal equation over a bounded strictly convex domain, which vanishes on the boundary of this domain and that are continuous up to the boundary. This rather surprising phenomenon, in dimension 2, follows from the asymptotic theorem deduced in [SE-T7]. In arbitrary dimension it follows from the generalization accomplished in [N-ET]. Furthermore, we deduce an existence result for the ε- horizontal minimal equation in the two-dimensional case, that combined together with our uniform a priori estimates and elliptic theory yields an existence result for the horizontal minimal equation (ε=0). The uniqueness of the solutions obtained for the horizontal minimal equation is shown for admissible boundary data satisfying an admissible bounded slope condition. This follows from the Radó type theorem mentioned above. We set in the context of elliptic quasilinear EDP´s several new (we believe interesting) open problems.

– We point out that in a joint work with Elias Marion Guio, we establish a priori estimates for a prescribed mean curvature equation in hyperbolic space. In fact, this paper is based on Elias Doctoral Thesis PUC-Rio, April 2003, under my supervision. Click here.

- Finally, in a joint work with Barbara Nelli and Eric Toubiana, we obtain a characterization of the n-catenoid in H^n R [N-SE-T]. In fact ,we prove a Schoen type theorem [S] in the context of infinite total curvature. We remark that the n-catenoid in $H^n\times R$ were constructed in [B-SE1], when $n \geq 3$. We also establish a maximum principle for minimal surfaces lying in a closed half space. Moreover, we infer a generalization of the Asymptotic Theorem proved if the dimension is two in the joint work with Eric Toubiana already cited above [SE-T7]. Finally, we draw several conclusions from these results that suggest the strong influence of the asymptotic boundary in the geometry of the minimal surface or minimal hypersurface in $H^n\times R$.

– By the way, we would like to point out that we wrote two texts in collaboration with Eric Toubiana about applications of the classical maximum principle to the theory of minimal and constant mean curvature. The first text consists of several applications to minimal and constant mean curvature in both Euclidean and hyperbolic space. For instance, we solve an exterior Dirichlet problem for the minimal equation in the Euclidean space. The construction uses some geometric estimates together with the Perron process. We also prove some existence results for minimal graphs over a bounded annulus in the hyperbolic space. The assumptions lead to a geometric C^1a priori estimates to ensure the result by applying the elliptic theory. Click here to open the file.

— The second is an expository text in which we discuss several analytic and geometric applications of the maximum principle in the hyperbolic space. We infer symmetry and half-space results in the hyperbolic space. Notably, we demonstrate in the text the famous theorem of Alexandrov and we explain in detail the so-called Alexandrov Reflection Principle. We carry out a Molzon-Serrin type theorem for a classical overdetermined elliptic problem in the hyperbolic space. We also discuss the Perron process for vertical minimal graphs in the hyperbolic space. Click here to open the file.

— On the other hand, we refer to a paper written in a joint work with Lucas Barbosa in which we apply geometric and PDE methods to study constant mean curvature hypersurfaces in the hyperbolic space. It is published in Sémin. Théor. Spectr. Géom. 16, 43-79, 1998. We explain throughout these paper the quasilinear PDE techniques involved to obtain the existence and the uniqueness results. We also give the geometric knowledge of the model surfaces in hyperbolic space used as barriers to get the required a priori estimates. Click here.

The papers cited above can be downloaded in Ricardo Sa Earp-Preprints

References:

· [B-SE1] P. Bérard and R. Sa Earp. Minimal hypersurfaces in $H^n \times R$, *total curvature and index,* 2009. arXiv: 0808.3838v3 [Math. DG].

· [B-SE2] P. Bérard and R. Sa Earp. *Examples of H-hypersurfaces in $H^n \times R$ and geometric applications.* Matemática Contemporânea, 34, 19-51, 2008. Escrito em homenagem aos oitenta anos de Manfredo do Carmo.

· [B-SE3] P. Bérard and R. Sa Earp. *Lindelöf's theorem for catenoids revisited*. hal-00407395v1, arXiv-0907.4294v1.

· [B-SE4] P. Bérard and R. Sa Earp. *Lindelöf's theorem for hyperbolic catenoids.* hal-00429404v1. Proceedings of the American Mathematical Society. 138, 3657-3657, 2010.

· [DoC-D] M. do Carmo and M. Dajczer. *Hypersurfaces in spaces of constant curvature.* Transactions of the American Mathematical Society, 277, 685–709, 1983.

[E-N-SE] M. F. Elbert, N. Nelli and R. Sa Earp. *Existence of vertical ends of mean curvature ½ in* $H^2 \times R$. Transactions of the American Mathematical Society, 364, 3, 1179-119, 2012. DOI S0002-9947 (2011)05361-4.

· [E-SE] M. F. Elbert and R. Sa Earp. *All solutions of the CMC-equation in* $H^n \times R$ *invariant by parabolic screw motion*. Annali di Matematica Pura ed Appplicata 193, 1, 103-114, 2014. DOI 10.1007/s10231-012-0268-8.

· [G-T] D. Gilbarg e N.S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Springer- Verlag, 1983.

· [H-M2] D. Hoffman and W. Meeks III. *The Strong Halfspace Theorem for Minimal Surfaces,* Invent. Math. 101, No.1 373-377, 1990.

· [H-R] L. Hauswirth e H. Rosenberg, *Minimal Surfaces of Finite Total Curvature in* $H^2 \times R$, Matemática Contemporânea 31, 65-80, 2006.

· [H-SE-T] L. Hauswirth, R. Sa Earp and E. Toubiana. *Associate and conjugate minimal immersions in* $M^2 \times R$. Tohôku Math J, 60, 267-286, 2008.

· [H-N-SE-T] L. Hauswirth, B. Nelli, R. Sa Earp e Eric Toubiana. *A Schoen theorem for minimal surfaces in* $H^2 \times R$. ArXiv: 1111 0851.

· [H-T-T-W] Z.C. Han, L.F. Tam, A. Treibergs, T. Wan. *Harmonic maps from the complex plane into surfaces with nonpositive curvature*, Communications in Analysis and Geometry, Vol 3, No. 1, 85–114, 1995.

· [J-S2] H. Jenkins e J. Serrin. *The Dirichlet problem for the minimal surface equation in higher dimensions*. Journal fur die reine und angewandte Mathematik, 223, 170-187, 1968.

· [Li] L. Lindelöf. *Sur les limites entre lesquelles le caténoïde est une surface minimale.* Math. Annalen, 2, 160-166, 1870.

· [LW] L. L. Lima e W. Rossman. *On the índex of Constant mean curvature 1- surfaces in hyperbolic space.* Indiana Univ. Math. J.. 47, 1998.

· [M] H. Mori. *Minimal surfaces of revolution in H^3 and their stability properties*. Indiana Univ. Math. J., 30:787–794, 1981.

· [M-W] W. Meeks III e B. White. *Minimal surfaces bounded by convex curves in parallel planes.* Comment. Math. Helv. 66 , 263-278, 1991.

· [M-R] F. Morabito, M. Rodriguez: *Saddle towers and minimal k-noids in* $\mathrm{H}^2 \times \mathrm{R}$*;* Journal of the Institute of Mathematics of Jussieu, 1-17, 2011.

· [My] Y. N. Minsky. *Harmonic maps, length, and energy in Teichmüller space*, J. Differential Geom. 35 , No. 1, 151–217, 1992.

·[N-R] B. Nelli and H. Rosenberg. *Minimal surfaces in* $\mathrm{H}^2 \times \mathrm{R}$. Bull. Braz. Math. Soc. 33, 263-292, 2002.

· [N-SE-S-T]. B. Nelli, R. Sa Earp, W. Santos e E. Toubiana. *Uniqueness of H-surfaces in $H^2 \times R$, $H \leq ½$, with boundary one or two parallel horizontal circles*. Annals of Global Analysis and Geometry. 33, No. 4, 307-321, 2008.

· [N-SE] B. Nelli and R. Sa Earp. *A half-space theorem for Mean Curvature H=1/2 in $H^2 \times R$.* J. Math. Anal. Appl. 365, 167-170, 2010.

· [N-SE-T] B. Nelli, R. Sa Earp and E. Toubiana. *Maximum Principle and Symmetry for Minimal Hypersurfaces in $H^n \times R$.* ArXiv: 12112439, 2012. To appear in Annali della Scuola Normale Superiore di Pisa, Classe di Scienze. DOI: 10.2422/2036-2145.201211_004.

· [P] J. Pyo. *New complete embedded minimal surfaces in $H^2 \times R$.* Ann. Global Anal. Geom. 40, 2, 167–176, 2011.

· [SE-R] R. Sa Earp and H. Rosenberg. Differential Geometry, Pitman Monographs and Surveys in Pure and Applied Mathematics. K. Tenenblat; B. Lawson, Eds 52, 123-148, 1991.

· [SE-T] R. Sa Earp and E. Toubiana. *Sur les surfaces de Weingarten spéciales de type minimale*. Boletim da Sociedade Brasileira de Matemática, 26, No. 2, 129-148, 1995.

· [SE-T3] R. Sa Earp and E. Toubiana. *Screw motion surfaces in $H^2 \times R$ and $S^2 \times R$*. Illinois J. Math., 49, No. 3, 1323-1362, 2005.

· [SE-T5] R. Sa Earp and E. Toubiana. *Existence and uniqueness of minimal surfaces.* Asian J. Math., 4, No. 3, 669-694, 2000.

· [SE-T7] R. Sa Earp and E. Toubiana. *An asymptotic theorem for minimal surfaces and existence results for minimal graphs in $H^2 \times R$.* Mathematische Annalen, 342, No. 2, 309-331, 2008.

· [SE-T8] R. Sa Earp and E. Toubiana. *Minimal graphs in $H^n \times R$* and R^{n+1}. Annales Institut Fourier 60, 2373-2402, 2010.

· [SE-T9] R. Sa Earp and E. Toubiana. Introduction à la Geométrie Hyperbolique et aux Surfaces de Riemann. Segunda edição, Cassini Eds, Paris (com Eric Toubiana). Edição revista e ampliada. Enseignement des mathématiques, No 27. ISBN 2842250850, 2009.

· [S-E] R. Sa Earp. *Parabolic and hyperbolic screw motion surfaces in $H^2 \times R$*. J. Austr. Math. Soc. 85, 113-143, 2008.

· [S-E2] R. Sa Earp. *Uniqueness of minimal surfaces whose boundary is a horizontal graph and some Bernstein problems in $H^2 \times R$.* Mathematische Zeitschrift, 273, 1, 211-217, 2013. DOI: 10.1007/s00209-012-1001-4.

· [S-E3] R. Sa Earp. *Uniform a priori estimates for a class of horizontal minimal equation*. hal-00699216, arXiv: 1205.4375, 2012.

· [S] R. Schoen. *Uniqueness, symmetry and embeddedness of minimal surfaces*. Journal of Differential Geometry 18, 791-809, 1983.

· [Sp] J. Spruck. *Interior gradient estimates and existence theorems for constant mean curvature graphs in $M^n \times R$.* Pure Appl. Math. Q. 3, No. 3, part 2, 785-800, 2007.