# Minimal hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$, total curvature and index

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**Abstract**

In this paper, we consider minimal hypersurfaces in the product space $\mathbb{H}^n \times \mathbb{R}$. We begin by studying examples of rotation hypersurfaces and hypersurfaces invariant under hyperbolic translations. We then consider minimal hypersurfaces with finite total curvature. This assumption implies that the corresponding curvature goes to zero uniformly at infinity. We show that surfaces with finite total intrinsic curvature have finite index. The converse statement is not true as shown by our examples which also serve as useful barriers.

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**Keywords**: Minimal hypersurfaces, stability, index.

# 1 Introduction

In this paper, we focus on complete oriented minimal hypersurfaces $M$ immersed in $\mathbb{H}^n \times \mathbb{R}$ equiped with the product metric.

In Section 3, we study the family $\{\mathcal{C}_a, a > 0\}$ of hypersurfaces invariant under rotations about the vertical geodesic $\{0\} \times \mathbb{R} \subset \mathbb{H}^n \times \mathbb{R}$ ("catenoids") and the family $\{\mathcal{M}_d, d > 0\}$ of hypersurfaces invariant under hyperbolic translations. These examples generalize to higher dimensions some of the minimal surfaces constructed in [25, 27, 28].

In particular, we prove that the $n$-dimensional catenoids $\mathcal{C}_a$ have vertical heights bounded from above by $\pi/(n-1)$ (Proposition 3.2). In Section 3.3, we describe the maximal stable rotationally invariant domains on $\mathcal{C}_a$ and we prove that the catenoids have index 1 (Theorem 3.5). We also give an interpretation in terms of the envelope of the family $\mathcal{C}_a$ (Corollary 3.7). Finally, we observe that the half-catenoid $\mathcal{C}_a \cap \left(\mathbb{H}^n \times \mathbb{R}_+\right)$ is not maximally stable.

We describe the minimal hypersurfaces invariant under hyperbolic translations in Theorem 3.9. In particular, we find a hypersurface $\mathcal{M}_1$ which is a complete non-entire vertical graph over a half-space bounded by some hyperplane $\Pi$ in $\mathbb{H}^n \times \{0\}$. It takes infinite value data on $\Pi$ and zero asymptotic boundary value data. When $d < 1$, the hypersurface $\mathcal{M}_d$ is an entire vertical graph. When $d > 1$, it is a bi-graph over the exterior of an equidistant hypersurface of $\mathbb{H}^n \times \{0\}$.

In Section 4, we consider the relationships between finiteness of the total curvature and finiteness of the index. In dimension 2, we consider the curvature integrals $\int_M |A_M|^2$ and $\int_M |K_M|$, where $A_M$ is the second fundamental form of the immersion and $K_M$ the Gauss curvature. Finiteness of these integrals implies that the corresponding curvatures tend to zero uniformly at infinity; finiteness of the latter implies finiteness of the index of the Jacobi (stability) operator (Theorem 4.1). The converse statements do not hold. For a related result see [31].
On the one hand, the catenoids $\mathcal{C}_a$ have finite index although they have infinite total intrinsic curvature. This is in contrast with the case of minimal surfaces in Euclidean 3-space ([15]) and with the case of surfaces with constant mean curvature 1 in hyperbolic 3-space ([13, 23]). Note that catenoids have finite total extrinsic curvature. On the other hand, the surfaces invariant under hyperbolic translations are stable graphs, their curvature goes to zero at infinity although they have infinite total curvature. The proof we give of Theorem 4.1 relies mainly on Simons' equation and the de Giorgi-Moser-Nash method which shows that finite total curvature implies that the curvature tends to zero uniformly at infinity. We point out that the finiteness of the intrinsic total curvature has deep consequences. Under this assumption on $M$, L. Hauswirth and H. Rosenberg ([17], Theorem 3.1) have indeed shown that the total intrinsic curvature is quantified, that the ends of $M$ are asymptotic to Scherk type surfaces and obtained a $C^2$-control on the curvature at infinity. In the paper by L. Hauswirth, B. Nelli, R. Sa Earp and E. Toubiana [18], one can find many details about how finite curvature ends behave.
In dimension $n \geq 3$, we give an upper bound of the index in terms of the total extrinsic curvature (Theorem 4.3).

In Section 5, using the catenoids $\mathcal{C}_a$ as barriers, we prove some symmetry and characterization results for minimal hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$ whose boundary consists of two congruent convex hypersurfaces in parallel slices (Theorem 5.1). A further characterization of the $n$-catenoids can be found in [26].
We point out that the hypersurfaces $\mathcal{M}_d$ ($d < 1$ and $d = 1$) have been used in [29, 30] as barriers for the Dirichlet problem and that they play a crucial role for some existence theorem for the vertical minimal surface equation. They have also been used in [26] for other applications.

Finally, we observe that Theorem 3.5 has been extended to minimal catenoids in Nil$(2n+1)$ equipped with a left-invariant metric in Bérard-Cavalcante [9], and that Bérard-Castillon-Cavalcante [8] improved 3 of Theorem 4.1, replacing intrinsic curvature by extrinsic curvature.

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# 2 General framework

## 2.1 Notations

We consider hypersurfaces $M$ immersed in the space $\widehat{M} := \mathbb{H}^n \times \mathbb{R}$ equiped with the product metric $\widehat{g} = g_{\mathbb{B}} + dt^2$, where $g_{\mathbb{B}}$ is the hyperbolic metric,

$$g_{\mathbb{B}} := \big(\frac{2}{1-|x|^2}\big)^2 \big(dx_1^2 + \cdots + dx_n^2\big). \tag{2.1}$$

We have chosen the ball model $\mathbb{B}$ for the $n$-dimensional hyperbolic space $\mathbb{H}^n$.

## 2.2 Jacobi operator, Index, Jacobi fields

Let $M^n \looparrowright \widehat{M}^{n+1}$ be an orientable minimal hypersurface in an oriented Riemannian manifold $\widehat{M}$ with metric $\widehat{g}$. Let $N_M$ be a unit normal field along $M$ and let $A_M$ be the second fundamental form of the immersion with respect to $N_M$. Let $\widehat{\mathrm{Ric}}$ be the normalized Ricci curvature of $\widehat{M}$. The second variation of the volume functional gives rise to the *Jacobi operator* (or stability operator) $J_M$ of $M$ (see [33, 22, 11]),

$$J_M := -\Delta_M - \big(|A_M|^2 + \widehat{\mathrm{Ric}}(N_M)\big), \tag{2.2}$$

where $\Delta_M$ is the (non-positive) Laplacian on $M$ (for the induced metric).

Given a relatively compact regular domain $\Omega$ on the hypersurface $M$, we let $\mathrm{Ind}(\Omega)$ denote the number of negative eigenvalues of $J_M$ for the Dirichlet problem on $\Omega$ (this is well defined because $\Omega$ is relatively compact). The *index* of $M$ is defined to be the supremum ($\le +\infty$)

$$\mathrm{Ind}(M) := \sup\{\mathrm{Ind}(\Omega) \mid \Omega \Subset M\}, \tag{2.3}$$

taken over all relatively compact regular domains.

Let $\lambda_1(\Omega)$ be the least eigenvalue of the operator $J_M$ with Dirichlet boundary conditions in $\Omega$. Recall that a relatively compact regular domain $\Omega$ is said to be *stable*, if $\lambda_1(\Omega) > 0$; *unstable*, if $\lambda_1(\Omega) < 0$; *stable-unstable*, if $\lambda_1(\Omega) = 0$. More generally, we say that a domain $\Omega$ is stable if any relatively compact subdomain is stable.

**Properties 2.1** *Recall the following properties.*

1. *Let $\Omega$ be a stable-unstable relatively compact domain. Then, any smaller domain is stable while any larger domain is unstable (monotonicity of Dirichlet eigenvalues).*

2. *Of particular interest are the solutions of the equation $J_M(u) = 0$. We call such functions* Jacobi fields *on $M$. Let $X_a : M^n \looparrowright (\widehat{M}^{n+1}, \widehat{g})$ be a one-parameter family of oriented minimal immersions, with variation field $V_a = \frac{\partial X_a}{\partial a}$ and unit normal $N_a$. Then, the function $\widehat{g}(V_a, N_a)$ is a Jacobi field on $M$ ([1], Theorem 2.7).*

3. *Let $\Omega$ be a relatively compact domain on a minimal manifold $M$. If there exists a positive function $u$ on $\Omega$ such that $J_M(u) \geq 0$, then $\Omega$ is stable ([16], Theorem 1).*

# 3 Examples of minimal hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$

In this section we give examples of minimal hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$. We use these examples as guidelines and counter-examples to study the relationships between index properties of the Jacobi operator and the finiteness of some total curvature of $M$, see Theorems 4.1 and 4.3. We also use them as barriers for a symmetry and characterization result in Section 5.

## 3.1 Rotational hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$

We first consider rotational hypersurfaces about a vertical geodesic axis in $\mathbb{H}^n \times \mathbb{R}$. Up to isometry, we can assume the rotation axis to be $\{0\} \times \mathbb{R}$. Recall that we take the ball model for $\mathbb{H}^n$.

Take the vertical plane $\mathbb{V} := \{(x_1, \dots, x_n, t) \in \widehat{M} \mid x_1 = \dots = x_{n-1} = 0\}$ and consider a generating curve $\big(\tanh(f(t)/2), t\big)$ for some positive function $f$ which represents the hyperbolic distance to the axis $\mathbb{R}$, at height $t$.

We define a *rotational hypersurface* $M \looparrowright \widehat{M}$ by the "parametrization"

$$X : \begin{cases} \mathbb{R}_+ \times S^{n-1} \to \widehat{M}, \\ (t, \xi) \mapsto \big(\tanh(f(t)/2)\xi, t\big), \end{cases} \tag{3.4}$$

where $\xi = (\xi_1, \dots, \xi_n)$ is a point in the unit sphere $S^{n-1}$ and $\tanh(\rho/2)\xi$ stands for the point $(\tanh(\rho/2)\xi_1, \dots, \tanh(\rho/2)\xi_n)$ in the ball $\mathbb{B}$.

The basic tangent vectors to the immersion $X$ are

$$T(t, \xi) := T_{t,\xi}X(\partial_t) = \big(\frac{f_t(t)}{2\cosh^2(f(t)/2)}\xi, 1\big),$$

where $f_t$ is the derivative of $f$ with respect to $t$, and

$$U(t, \xi, u) := T_{t,\xi}X(u) = \big(\tanh(f(t)/2)u, 0\big),$$

where $u \in T_\xi S^{n-1}$ is a unit vector.

We collect basic formulas in the next proposition whose proof is straightforward.

**Proposition 3.1** *Let $(M, g_M) \looparrowright (\widehat{M}, \widehat{g})$ be an isometric immersion. We have the following formulas in the parametrization $X$ on $\mathbb{R} \times S^{n-1}$.*

1. *The induced metric $g_M$ is given by*

$$g_M = \big(1 + f_t^2(t)\big)dt^2 + \sinh^2(f(t))g_S\,, \tag{3.5}$$

*where $g_S$ is the canonical metric on $S^{n-1}$.*

2. *The Riemannian measure $d\mu_M$ for the metric $g_M$ is given by*

$$d\mu_M = \big(1 + f_t^2(t)\big)^{1/2} \sinh^{n-1}(f(t))\, dt\, d\mu_S, \tag{3.6}$$

*where $d\mu_S$ is the canonical measure on the sphere.*

3. *The unit normal field to the immersion can be chosen to be*

$$N_M(t,\xi) = (1 + f_t^2(t))^{-1/2} \big( \frac{-1}{2\cosh^2(f(t)/2)} \xi, f_t(t) \big)\,. \tag{3.7}$$

*In particular, the vertical component of the unit normal field is given by*

$$v_M(t) := f_t(t)\big(1 + f_t^2(t)\big)^{-1/2}. \tag{3.8}$$

We now compute the mean curvature of $M$.
At the point $X(t,\xi)$, the principal directions of $M$ are

- the tangent to the meridian curve in the vertical 2-plane

$$\mathbb{V}_\xi = \{\big(\tanh(f(t)/2)\xi, t\big) \mid t \in \mathbb{R}\},$$

- the vectors tangent to the distance sphere $X(t, S^{n-1})$ at $\xi$ in the hyperbolic slice $\mathbb{H}^n \times \{t\}$, where the restriction of the second fundamental form $A_M$ is a scalar multiple of the identity.

The principal curvatures with respect to $N_M$ are

- $k_n(t)$, the principal curvature in the direction tangent to the meridian curve, given by

$$k_n(t) = -f_{tt}(t)(1 + f_t^2(t))^{-3/2}, \tag{3.9}$$

- the principal curvatures in the directions tangent to $X(t, S^{n-1})$ at $X(t,\xi)$,

$$k_1(t) = \cdots = k_{n-1}(t) = \coth(f(t))(1 + f_t^2(t))^{-1/2}. \tag{3.10}$$

We conclude that the mean curvature $H(t)$ of the rotational hypersurface $M \looparrowright \widehat{M}$ with respect to the unit normal $N_M$ is given by

$$nH(t) = -f_{tt}(t)(1 + f_t^2(t))^{-3/2} + (n-1)\coth(f(t))(1 + f_t^2(t))^{-1/2}, \tag{3.11}$$

or

$$n f_t(t)\, \sinh^{n-1}(f(t))\, H(t) = \partial_t\Big( \sinh^{n-1}(f(t))(1 + f_t^2(t))^{-1/2} \Big). \tag{3.12}$$

where $f_t$ and $f_{tt}$ are the first and second derivatives of $f$ with respect to $t$.

## 3.2 Catenoids in $\mathbb{H}^n \times \mathbb{R}$

In this Section, we describe the minimal rotational hypersurfaces about $\{0\} \times \mathbb{R}$, in $\mathbb{H}^n \times \mathbb{R}$. By analogy with the Euclidean case, we call them *catenoids*. For $n = 2$ the catenoids are studied in [28].
Given some $a > 0$, let $\big(I_a, f(a,\cdot)\big)$ denote the maximal solution of the Cauchy problem

$$\left\{ \begin{array}{lcl} f_{tt} & = & (n-1)\coth(f)(1 + f_t^2), \\ f(0) & = & a > 0, \\ f_t(0) & = & 0, \end{array} \right. \tag{3.13}$$

where $f_t$ and $f_{tt}$ are the first and second derivatives of $f$ with respect to $t$.

**Proposition 3.2** *For $a > 0$, the maximal solution $\big(I_a, f(a,\cdot)\big)$ gives rise to the generating curve $C_a$, $t \mapsto \big(\tanh(f(a,t)), t\big)$ (*catenary*), of a complete minimal rotational hypersurface $\mathcal{C}_a$ (*catenoid*) in $\mathbb{H}^n \times \mathbb{R}$, with the following properties.*

1. *The interval $I_a$ is of the form $I_a = ]-T(a), T(a)[$ for some finite positive number $T(a)$ and $f(a,\cdot)$ is an even function of the second variable.*
2. *For all $t \in I_a$, $f(a,t) \geq a$.*
3. *The derivative $f_t(a,\cdot)$ is positive on $]0, T(a)[$, negative on $]-T(a), 0[$.*
4. *The function $f(a,\cdot)$ is a bijection from $[0, T(a)[$ onto $[a, \infty[$, with inverse function $\lambda(a,\cdot)$ given by*

$$\lambda(a,\rho) = \sinh^{n-1}(a) \int_a^\rho \big(\sinh^{2n-2}(u) - \sinh^{2n-2}(a)\big)^{-1/2} du. \tag{3.14}$$

5. *The catenoid $\mathcal{C}_a$ has finite vertical height $h(a)$,*

$$h(a) = 2\sinh^{n-1}(a) \int_a^\infty \big(\sinh^{2n-2}(u) - \sinh^{2n-2}(a)\big)^{-1/2} du. \tag{3.15}$$

6. *The function $a \mapsto h(a)$ increases from $0$ to $\frac{\pi}{(n-1)}$ when $a$ increases from $0$ to infinity. Furthermore, given $a \neq b$, the generating catenaries $C_a$ and $C_b$ intersect at exactly two symmetric points.*

**Proof.** *Assertion 1* follows from the Cauchy-Lipschitz theorem for some positive $T(a)$ which is finite as we will see below.

*Assertion 2* follows from the fact that $\sinh^{n-1}(f(a,t))\big(1 + f_t^2(a,t)\big)^{-1/2} = \sinh^{n-1}(a)$ for all $t \in ]-T(a), T(a)[$ (see (3.12)).

*Assertion 3* is clear.

*Assertion 4.* According to Assertion 3, $t \mapsto f(a,t)$ is increasing on $[0, T(a)[$ so that it has a limit when $t$ tends to $T(a)$ and this limit must be infinite because we took a maximal solution. It follows that the inverse function $\lambda(a,\cdot)$ maps $[a, \infty[$ onto $[0, T(a)[$ . Moreover $\lambda_\rho(a, f(a,t)) f_t(a,t) \equiv 1$. That implies $\lambda_\rho(a,\rho) = \sinh^{n-1}(a)\big(\sinh^{2n-2}(\rho) - \sinh^{2n-2}(a)\big)^{-1/2}$ on $]a, \infty[$. The formula for $\lambda(a,\rho)$ follows because $f(a,0) = a$. Note that the integral (3.14) converges at $u = a$.

*Assertion 5.* We have that $h(a) = 2T(a)$, where

$$T(a) = \lim_{\rho \to \infty} \lambda(a,\rho) = \sinh^{n-1}(a) \int_a^\infty \big(\sinh^{2n-2}(u) - \sinh^{2n-2}(a)\big)^{-1/2} du,$$

where the integral converges at both $a$ and $\infty$.

*Assertion 6.* By a change of variables, we can write

$$T(a) = \sinh(a) \int_1^\infty \big(v^{2n-2} - 1\big)^{-1/2} \big(\sinh^2(a) v^2 + 1\big)^{-1/2} dv$$

and compute the derivative

$$T'(a) = \cosh(a) \int_1^\infty \big(v^{2n-2} - 1\big)^{-1/2} \big(\sinh^2(a) v^2 + 1\big)^{-3/2} dv > 0.$$

Note that

$$\sinh(a)\big(v^{2n-2}-1\big)^{-1/2}\big(\sinh^2(a)v^2+1\big)^{-1/2} \le v^{-1}\big(v^{2n-2}-1\big)^{-1/2}$$

and that the right-hand side of the last inequality is in $L^1([1,\infty[)$ for $n \ge 2$. So we can take the limits under the integral and obtain that $\lim_{a\to 0} T(a) = 0$ and $\lim_{a\to\infty} T(a) = \int_1^\infty v^{-1}\big(v^{2n-2}-1\big)^{-1/2}\,dv$. The last integral can be calculated explicitly because $\big(\arctan\sqrt{v^N-1}\big)' = \frac{N}{2\sqrt{v^N-1}}$. The last assertion follows by considering the function $\lambda(a,\rho)-\lambda(b,\rho)$ and by using the monotonicity of $T(a)$. □

**Remark**. The above proposition shows that the catenoids in $\mathbb{H}^n\times\mathbb{R}$ have uniformly bounded finite vertical height. This is in contrast with the Euclidean catenoids ($n \ge 3$) which have finite, yet unbounded, vertical heights.

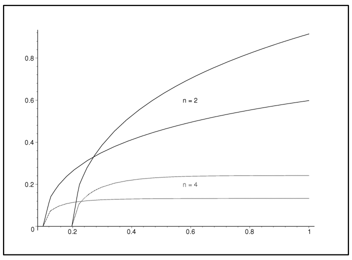


Figure 1: Catenaries $n = 2, 4$

**Notations**. Let $N_a$ denote the unit normal to the catenoid $\mathcal{C}_a$, let $A_a$ denote its second fundamental form relative to the normal $N_a$ and let $d\mu_a$ denote its Riemannian measure. When $n = 2$, let $K_a$ denote the Gauss curvature of $\mathcal{C}_a$. We state the following proposition for later purposes.

**Proposition 3.3** *For $a > 0$, the $n$-dimensional catenoid $\mathcal{C}_a$ in $\mathbb{H}^n\times\mathbb{R}$ has infinite volume and finite* total extrinsic curvature $\int_{\mathcal{C}_a}|A_a|^n\,d\mu_a$. *When $n = 2$, the catenoid $\mathcal{C}_a$ has infinite* total intrinsic curvature $\int_{\mathcal{C}_a}|K_a|\,d\mu_a$.

**Proof.** We can restrict to the upper half-catenoid, $\mathcal{C}_{a,+} = \mathcal{C}_a\cap(\mathbb{H}^n\times\mathbb{R}_+)$, which admits the parametrization

$$Y(a,\rho,\xi) := \big(\tanh(\rho/2)\xi, \lambda(a,\rho)\big), \quad \rho \ge a.$$

The geometric data of $\mathcal{C}_{a,+}$ are readily calculated. In particular,

$$|A_a|^2(\rho) = n(n-1)\big(\frac{\sinh^{n-1}(a)\cosh(\rho)}{\sinh^n(\rho)}\big)^2,$$

and

$$d\mu_a = \sinh^{2n-2}(\rho)\big(\sinh^{2n-2}(\rho)-\sinh^{2n-2}(a)\big)^{-1/2}\,d\rho\,d\mu_S.$$

The first assertion follows ($|A_a|^n d\mu_a$ tends to zero exponentially at infinity). For the second assertion, we use Gauss equation and minimality to get that

$$K_a = \widehat{K}_a - \frac{1}{2}|A_a|^2 = -v_a^2 - \frac{1}{2}|A_a|^2,$$

where $\widehat{K}_a$ is the sectional curvature of the 2-plane tangent to $\mathcal{C}_a$ in the ambient space $\mathbb{H}^2\times\mathbb{R}$ and where $v_a$ is the vertical component of the unit normal to $\mathcal{C}_a$,

$$v_a(\rho) = \widehat{g}(N_a,\partial_t) = \sinh^{1-n}(\rho)\big(\sinh^{2n-2}(\rho)-\sinh^{2n-2}(a)\big)^{1/2}.$$

Assertion 2 follows because $v_a$ tends to 1 at infinity on $\mathcal{C}_{a,+}$. $\square$

Recently, Elbert-Nelli-Santos [14] found rotationally invariant hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$ with r-mean curvature $H_{r+1} = 0$. Their behavior is very similar to that of the $n$-catenoids.

### 3.3 Catenoids in $\mathbb{H}^n \times \mathbb{R}$, stability properties

Recall that the catenoid $\mathcal{C}_a$ is generated by the curve $t \mapsto \big(\tanh(f(a,t)/2), t\big)$ in the vertical plane $\mathbb{V}$, where $f(a,\cdot)$ is the maximal solution of the Cauchy problem (3.13). This yields the parametrization

$$X(a,t,\xi) = \big(\tanh(f(a,t)/2)\xi, t\big) \tag{3.16}$$

for $\mathcal{C}_a$, with $t \in \mathbb{R}$ and $\xi \in S^{n-1}$. According to Property 2.1 (2), we have two Jacobi fields on the catenoid $\mathcal{C}_a$.

• The *vertical Jacobi field* $v(a,t)$ comes from the vertical translations $(x,t) \mapsto (x, t+\tau)$ in $\mathbb{H}^n \times \mathbb{R}$. It is given by $v(a,t) = \widehat{g}(N_a, \partial_t)$, where $N_a$ is the unit normal to $\mathcal{C}_a$. According to (3.7), it is given by the formula

$$v(a,t) = f_t(a,t)\big(1 + f_t^2(a,t)\big)^{-1/2}, \tag{3.17}$$

where $f_t$ stands for the derivative with respect to the variable $t$. Because $t \mapsto f(a,t)$ is even, the function $t \mapsto v(a,t)$ is odd.

• The *variation Jacobi field* $e(a,t)$ comes from the variations with respect to the parameter $a$. It is given by $e(a,t) = \widehat{g}(N_a, \frac{\partial X}{\partial a})$. According to (3.7) and (3.16), the function $e(a,t)$ is given

$$e(a,t) = -f_a(a,t)\big(1 + f_t^2(a,t)\big)^{-1/2}, \tag{3.18}$$

where $f_a$ stands for the derivative with respect to the variable $a$. Because $t \mapsto f(a,t)$ is even, the function $t \mapsto e(a,t)$ is even.

• The Jacobi fields $v(a,t)$ and $e(a,t)$ have nice expressions when restricted to the upper-half $\mathcal{C}_{a,+} = \mathcal{C}_a \cap (\mathbb{H}^n \times \mathbb{R}_+)$ of the catenoid $\mathcal{C}_a$. Indeed, recall that the function $f(a,\cdot) : [0, T(a)[ \to [0,\infty[$ has an inverse function $\lambda(a,\rho)$ given by (3.14). Using the relationships

$$\lambda\big(a, f(a,t)\big) \equiv t \quad \text{for} \quad t \geq 0$$

and

$$\lambda_\rho(a,f)f_t \equiv 1 \quad \text{and} \quad \lambda_a(a,f) + \lambda_\rho(a,f)f_a \equiv 0,$$

we get the following expressions for $v(a,t)$ and $e(a,t)$ for $t \geq 0$,

$$\left\{ \begin{array}{rcl} v(a,t) & = & \big(1 + \lambda_\rho^2(a, f(a,t))\big)^{-1/2}, \\ \\ e(a,t) & = & v(a,t)\lambda_a(a, f(a,t)). \end{array} \right. \tag{3.19}$$

For $\rho \geq a$, define the functions $v_1(a,\rho), A_1(a,\rho)$ and $B_1(a,\rho)$, by the following formulas

$$\left\{ \begin{array}{rcl} v_1(a,\rho) & = & \big(1 + \lambda_\rho^2(a,\rho)\big)^{-\frac{1}{2}} = \big(\dfrac{\sinh^{2n-2}(\rho) - \sinh^{2n-2}(a)}{\sinh^{2n-2}(\rho)}\big)^{\frac{1}{2}} \\ A_1(a,\rho) & = & \dfrac{\cosh(a)}{\cosh(\rho)} \Big(\dfrac{\sinh(a)}{\sinh(\rho)}\Big)^{n-2}, \\ B_1(a,\rho) & = & \cosh(a) \int_1^{\frac{\sinh(\rho)}{\sinh(a)}} (v^{2n-2} - 1)^{-\frac{1}{2}} (\sinh^2(a)v^2 + 1)^{-\frac{3}{2}}\, dv. \end{array} \right. \tag{3.20}$$

From (3.14), we can write

$$\lambda(a,\rho) = \sinh(a) \int_1^{\frac{\sinh(\rho)}{\sinh(a)}} \left(v^{2n-2}-1\right)^{-\frac{1}{2}} \left(\sinh^2(a)v^2+1\right)^{-\frac{1}{2}} dv \tag{3.21}$$

and compute $\lambda_a$,

$$\begin{aligned}\lambda_a(a,\rho) = \quad & -\cosh(a)\sinh^{n-2}(a)\tanh(\rho)\big(\sinh^{2n-2}(\rho)-\sinh^{2n-2}(a)\big)^{-\frac{1}{2}} + \\ & + \cosh(a)\textstyle\int_1^{\frac{\sinh(\rho)}{\sinh(a)}} (v^{2n-2}-1)^{-\frac{1}{2}} (\sinh^2(a)v^2+1)^{-\frac{3}{2}}\, dv.\end{aligned}$$

We obtain,

$$\lambda_a(a,\rho)v_1(a,\rho) = -A_1(a,\rho) + B_1(a,\rho)v_1(a,\rho). \tag{3.22}$$

We summarize the relevant properties in the following lemma whose proof is straightforward. We define the functions $A(a,t)$ and $B(a,t)$ for $t \geq 0$ by

$$A(a,t) = A_1(a, f(a,t)), \quad B(a,t) = B_1(a, f(a,t)), \tag{3.23}$$

see Formulas (3.20).

**Lemma 3.4** *Then,*

$$e(a,t) = -A(a,t) + B(a,t)v(a,t), \quad \textit{for} \quad t \geq 0. \tag{3.24}$$

*Furthermore, for $t \geq 0$,*

1. $A(a,t) > 0,\ A(a,0) = 1$ *and* $\lim_{t\to T(a)} A(a,t) = 0$;
2. $B(a,t) > 0,\ B(a,0) = 0$ *and* $\lim_{t\to T(a)} B(a,t) = C(a)$,
   *where* $C(a) = \cosh(a)\int_1^{\infty} (v^{2n-2}-1)^{-\frac{1}{2}} (\sinh^2(a)v^2+1)^{-\frac{3}{2}}\, dv$;
3. $v(a,t) = v_1(a, f(a,t))$ *for $t > 0$, so that*
   $v(a,t) > 0$ *for $t > 0$,* $v(a,0) = 0$ *and* $\lim_{t\to T(a)} v(a,t) = 1$.

**Notation**. For $\alpha < \beta \in [0, T(a)]$, let $\mathcal{D}(\alpha,\beta)$ denote the rotationally invariant domain

$$\mathcal{D}_a(\alpha,\beta) = X(a, ]\alpha,\beta[, S^{n-1}). \tag{3.25}$$

In particular, $\mathcal{D}_a(0,T(a))$ is the half-vertical catenoid $\mathcal{C}_{a,+} = \mathcal{C}_a \cap (\mathbb{H}^n \times \mathbb{R}_+)$.

**Theorem 3.5** *The stability properties of the rotationally invariant domains $\mathcal{D}_a(\alpha,\beta)$ on the catenoid $\mathcal{C}_a$ are as follows.*

1. *There exists some $\sigma(a) \in ]0, T(a)[$ such that the relatively compact domain $\mathcal{D}_a(-\sigma(a),\sigma(a))$ is stable-unstable. Hence, for any $\alpha \in ]0,\sigma(a)[$, the domain $\mathcal{D}_a(-\alpha,\alpha)$ is stable; for any $\alpha \in ]\sigma(a), T(a)[$, the domain $\mathcal{D}_a(-\alpha,\alpha)$ is unstable.*
2. *There exists some $\tau(a) \in ]0, T(a)[$ such that*
   (a) *the (non relatively compact) domain $\mathcal{D}_a(-\tau(a), T(a))$ is stable,*
   (b) *for any $\alpha \in ]\tau(a), T(a)[$, there exists some $\beta(\alpha) \in ]\tau(a), T(a)[$ such that the domain $\mathcal{D}_a(-\alpha,\beta(\alpha))$ is stable-unstable.*

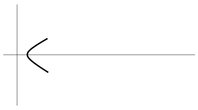

Figure 2: Case 1

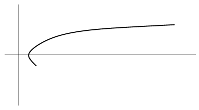

Figure 3: Case 2a

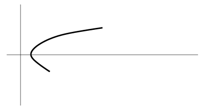

Figure 4: Case 2b

*3. The catenoid* $\mathcal{C}_a$ *has index* 1.

The above domains are generated by the portions of curves illustrated in Figures 2-4.
**Proof.** *Assertion 1.* Consider the function $e(a,t)$. According to Lemma 3.4, $e(a,0) = -1$ and $\lim_{t\to T(a)} e(a,t) = C(a) > 0$, so that it must vanish at least once on $]0,T(a)[$. It turns out (compare with Lemma 3.6 below) that $e(a,\cdot)$ has a unique positive zero $\sigma(a)$. Because $e(a,t)$ is even in $t$, it does not vanish in the open set $\mathcal{D}_a(-\sigma(a),\sigma(a))$ and satisfies $J_a(e) = 0$ in $\mathcal{D}_a(-\sigma(a),\sigma(a))$, and $e|\partial\mathcal{D}_a(-\sigma(a),\sigma(a)) = 0$. This means that $\mathcal{D}_a(-\sigma(a),\sigma(a))$ is a stable-unstable domain. The second assertion follows from Property 2.1 (1).

*Assertion 2.* Take any $\alpha \in ]0,T(a)[$ and define the function $w(a,\alpha,t)$ by

$$w(a,\alpha,t) = e(a,\alpha)v(a,t) + v(a,\alpha)e(a,t), \quad \text{for} \quad t \in ]-T(a),T(a)[. \tag{3.26}$$

This is a Jacobi field on $\mathcal{C}_a$ and furthermore $w(a,\alpha,-\alpha) = 0$, because $v$ is odd and $e$ is even with respect to $t$. Note also that $w(a,\alpha,0) = -v(a,\alpha) < 0$.

**Lemma 3.6** *The function* $w(a,\alpha,\cdot)$ *vanishes only once on* $]-T(a),0[$ *and vanishes at most once on* $]0,T(a)[$.

Let us prove the first assertion of the Lemma, the proof of the second assertion is similar. Assume that $w(a,\alpha,\cdot)$ has at least two consecutive zeroes $\alpha_1 < \alpha_2$ in the interval $]-T(a),0[$. The domain $\mathcal{D}_a(\alpha_1,\alpha_2)$ would then be stable-unstable because $J_a(w) = 0$ on $\mathcal{D}_a(\alpha_1,\alpha_2)$ and because $w$ vanishes on $\partial\mathcal{D}_a(\alpha_1,\alpha_2)$. On the other hand, the Jacobi field $v$ satisfies $J_a(v) = 0$ and $v < 0$ in $\mathcal{D}_a(\alpha_1,\alpha_2)$. By Property 2.1 (3), we have that $\lambda_1(\mathcal{D}_a(\alpha_1,\alpha_2)) > 0$ which contradicts the fact that this domain is stable-unstable. This proves the lemma.

In order to determine whether the function $w(a,\alpha,\cdot)$ vanishes on $]0,T(a)[$ or not, it is sufficient to look at the behaviour of $w(a,\alpha,t)$ when $t$ tends to $T(a)$ from below. For this purpose, we use the expression (3.24) for $e(a,t)$ and we write

$$w(a,\alpha,t) = -A(a,t)v(a,\alpha) + v(a,t)\big(e(a,\alpha) + B(a,t)v(a,\alpha)\big).$$

Using Lemma 3.4, we can write

$$W(a,\alpha) := \lim_{t\to T(a)} w(a,\alpha,t) = e(a,\alpha) + C(a)v(a,\alpha).$$

If $W(a,\alpha)\le 0$, then $w(a,\alpha,t)$ does not vanish on $]0,T(a)[$ and in fact on $]-\alpha,T(a)[$; if $W(a,\alpha)>0$, then $w(a,\alpha,t)$ has one and only one zero $\beta(\alpha)$on $]0,T(a)[$.

We now observe that $W(a,t):=e(a,t)+C(a)v(a,t)$ is a Jacobi field on $]0,T(a)[$ which takes the value $-1$ at 0 and the value $C(a)v(a,\sigma(a))>0$ at $\sigma(a)$. It follows from Lemma 3.6 that $W(a,\cdot)$ has one and only one positive zero $\tau(a)\in]0,\sigma(a)[$. We have that $W(a,t)\le 0$ on $]0,\tau(a)[$, so that for any $\alpha\in]0,\tau(a)]$, the function $w(a,\alpha,t)$ has only one zero $-\alpha$ on $]-T(a),T(a)[$. This proves the Assertion 2(a). On the other-hand, $W(a,t)>0$ on $]\tau(a),T(a)[$, so that for any $\alpha\in]\tau(a),T(a)[$, the function $w(a,\alpha,t)$ has a unique positive zero $\beta(\alpha)\in]0,T(a)[$. This proves the Assertion 2(b).

*Assertion 3.* Assertion 1 shows that $\mathcal{C}_a$ has index at least 1. In order to show that the index is at most one, we use Fourier decomposition with respect to the variable $\xi$ and an extra stability argument.

Recall that we work in the ball model for $\mathbb{H}^n$. Let $\gamma$ be a geodesic through 0 in $\mathbb{H}^n$. Up to a rotation, we may assume that $\gamma(s)=\big(\tanh(s/2),0,\cdots,0\big)$. Let $\mathbb{H}^n_+=\{(x_1,\cdots,x_n)\in\mathbb{B}\mid x_1>0\}$ and let $\mathcal{C}_{a,\gamma+}=\mathcal{C}_a\cap(\mathbb{H}^n_+\times\mathbb{R})$. We call this set a *half-horizontal catenoid.*

*Claim 1.* A half-horizontal catenoid $\mathcal{C}_{a,\gamma+}$ is stable.

To prove the claim, we shall find a positive Jacobi field on $\mathcal{C}_{a,\gamma+}$.
Let $z=x+iy$ denote the complex coordinate in $\mathbb{H}^2$ (ball model). We consider the group of hyperbolic isometries along the geodesic $\gamma$ and we extend these isometries slice-wise as isometries in $\mathbb{H}^2\times\mathbb{R}$. We then have the one-parameter group of isometries

$$(z;t)\mapsto\big(\frac{e^\tau(1+z)-(1-z)}{e^\tau(1+z)+(1-z)};t\big)\ \text{ in }\ \mathbb{H}^2\times\mathbb{R}.$$

The associated Killing vector-field in $\mathbb{H}^2\times\mathbb{R}$ is given by $\mathcal{K}_\gamma(z;t)=\big(\frac{1}{2}(1-z^2);0\big)$ or, in the $(x,y)$ coordinates, $\mathcal{K}_\gamma(x,y;t)=\big(\frac{1}{2}(1-x^2+y^2),-xy;0\big)$ which can be written as

$$\mathcal{K}_\gamma(x,y;t)=\frac{1}{2}\big(1+x^2+y^2)\big)(1,0;0)-x(x,y;0)$$

where $(1,0;0)$ and $(x,y;0)$ are seen as vectors in $\mathbb{R}^2\times\mathbb{R}=T_{(x,y;t)}\mathbb{H}^2\times\mathbb{R}$.

This formula can easily be generalized to higher dimensions as

$$\mathcal{K}_\gamma(x;t)=\frac{1}{2}(1+|x|^2)(e_1;0)-x_1(x;0),$$

where $x=(x_1,\cdots,x_n),e_1=(1,0,\cdots,0),|x|^2=x_1^2+\cdots+x_n^2$, and where $(e_1;0)$ and $(x;0)$ are seen as vectors in $\mathbb{R}^n\times\mathbb{R}=T_{(x,t)}\mathbb{H}^n\times\mathbb{R}$. Writing the point $x$ in the parametrization $X$ as $x=\tanh(f(a,t)/2)\xi$, we obtain that

$$\mathcal{K}_\gamma(\tanh(f(a,t)/2)\xi;t)=\frac{1}{2}\big(1+\tanh^2(f/2)\big)(e_1;0)-\tanh^2(f/2)\xi_1(\xi;0).$$

Using the fact that $(1+f_t^2)^{-1/2}=\big(\frac{\sinh(a)}{\sinh(f)}\big)^{n-1}$ on $\mathcal{C}_a$, we find that the Killing field $\mathcal{K}_\gamma$ gives rise to the horizontal Jacobi field

$$h_\gamma(a,t,\xi)=\big(\frac{\sinh(a)}{\sinh(f(a,t))}\big)^{n-1}\xi_1$$

which is positive on $\mathcal{C}_{a,\gamma+}$.

*Claim 2.* On $S^{n-1}$ equiped with the standard Riemannian metric, there exists an orthonormal basis of spherical harmonics $Y_k, k \geq 0$ with the property that the nodal domains of the $Y_k, k \geq 1$ are contained in hemispheres.

The property is clearly true on $S^1$ and can be proved by induction on the dimension, using polar coordinates centeblack at a given point on the sphere.

*Claim 3.* The Jacobi operator on $\mathcal{C}_a$ can be written as

$$J_a = L_{a,t} - q(a,t)\Delta_{S,\xi},$$

where $L_{a,t}$ is a Sturm-Liouville operator on the $t$ variable, with coefficients depending only on $a$ and $t$, where $q(a,t)$ is a positive function and where $\Delta_{S,\xi}$ is the Laplacian of the sphere $S^{n-1}$ acting on the $\xi$-variable. This claim follows immediately from Formulas (3.5) and (3.6) for the metric and the Riemannian measure on a rotational hypersurface and from the expression for the quadratic form associated with $J_a$.

Assume that the index of $\mathcal{C}_a$ is at least 2. Then, there exists some $S \in ]0, T(a)[$ such that $J_a$ has at least two negative eigenvalues $\lambda_1(S) < \lambda_2(S) < 0$ in $\mathcal{C}_a(-S, S)$ (we only consider Dirichlet boundary conditions). Because the least eigenvalue $\lambda_1(S)$ is simple, a corresponding eigenfunction $u$ must be invariant under rotations (*i.e.* only depends on the variable $t$) and say positive. Consider an eigenfunction $v$ associated with $\lambda_2(S)$. We claim that $v$ cannot be invariant under rotations. Indeed, $v$ would otherwise depend only on the variable $t$, it would be orthogonal to $u$ and hence it would have to vanish on $]-S, S[$. This would contradict the fact that the domains $\mathcal{C}_a(-S, 0)$ and $\mathcal{C}_a(0, S)$ are stable. Since $v$ is not rotationally invariant, there exists some $p \geq 1$ and some $v_p \neq 0$ in the decomposition into spherical harmonics with respect to the second variable, $v(t,\xi) = \sum_{k=0}^{\infty} v_k(t) Y_k(\xi)$. We would have $J_a(v_p Y_p) = \lambda_2(S) v_p Y_p$. Using Claim 2 and the fact that $\lambda_2(S) < 0$, this would mean that any nodal domain of $v_p Y_p$ is unstable, in contradiction with Claim 1.

Assuming that the index is at least 2 therefore yields a contradiction and hence the index of $\mathcal{C}_a$ is exactly one. $\square$

**Remark**. It follows from the positivity of the Jacobi field $v(a,t)$ for $t \in ]0, T(a)[$ that the upper half-catenoid $\mathcal{C}_{a,+}$ is stable (in the sense that any relatively compact domain $\Omega$ contained in $\mathcal{C}_{a,+}$ is stable, see Section 2.2). The second assertion in the preceding theorem says more. Indeed, there exists some $\tau(a) \in ]0, T(a)[$ such that the non-compact domain $\mathcal{D}_a(-\tau(a), T(a))$ is stable and stricly contains $\mathcal{C}_{a,+}$. This is different from what happens for Euclidean catenoids. Indeed, the half-catenoid $\mathcal{C}_{a,+}$ in $\mathbb{R}^3$ is a *maximal* stable domain ([24]). We study this phenomenon with more details in [5, 6].

**Geometric interpretation**. According to Proposition 3.2, Assertion (6), two distinct catenaries $C_a$ and $C_b$ meet at exactly two symmetric points, $m_{\pm}(a,b)$. Fixing $a$ and letting $b$ tend to $a$, the points $m_{\pm}(a,b)$ tend to limit points $m_{\pm}(a)$ which correspond to the points where the catenary $C_a$ touches the envelope of the family of catenaries $\{C_a\}_{a>0}$. According to [35], §58, page 127 ff, the condition defining the envelope of a family $\Gamma_a$, given by the parametrization $\big(x(a,t), y(a,t)\big)$, is the condition

$$\begin{vmatrix} x_a(a,t) & x_t(a,t) \\ y_a(a,t) & y_t(a,t) \end{vmatrix} = 0.$$

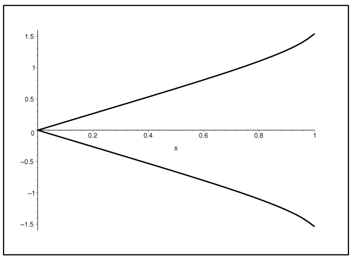

Figure 5: Envelope, $n = 2$

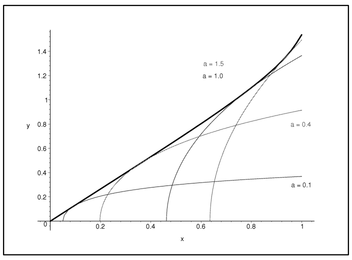

Figure 6: Catenaries, envelope

Specializing to catenaries, we find that the envelope condition is precisely the condition that $e(a, t) = 0$. Therefore, the value $\sigma(a)$ is precisely the value of $t$ at which the catenary $C_a$ touches the envelope of the family, see Figures 5 and 6.

**Corollary 3.7** *The stable-unstable domain* $\mathcal{D}_a(-\sigma(a), \sigma(a))$ *is precisely the symmetric, rotationally invariant compact domain bounded by the two spheres where the catenoid* $\mathcal{C}_a$ *touches the envelope of the family.*

**Remark**. Results about stability of higher dimensional catenoids in $\mathbb{R}^{n+1}$ can be found in [34].

## 3.4 Translationally invariant hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$

Let $\gamma$ be a complete geodesic through 0 in the ball model $\mathbb{B}$ of the hyperbolic space $\mathbb{H}^n$, parametrized by the signed distance $\rho$ to 0. Let $\mathbb{P}$ be the hyperbolic hyperplane orthogonal to $\gamma$ at 0. We consider the hyperbolic translations along the geodesics passing through 0 in $\mathbb{P}$. The image of a point of $\gamma$ under these translations is an equidistant hypersurface to $\mathbb{P}$ in $\mathbb{H}^n$. We can extend these translations "slice-wise" to give positive isometries of $\mathbb{H}^n \times \mathbb{R}$ which we call *hyperbolic translations.*

A generating curve $(\tanh(\rho/2), \mu(\rho))$ in the vertical Euclidean plane $\gamma \times \mathbb{R}$ gives rise, under the previous isometries, to a translationally invariant hypersurface $M \hookrightarrow \mathbb{H}^n \times \mathbb{R}$, whose intersection with the slice $\mathbb{H}^n \times \{\mu(\rho)\}$ is the equidistant hypersurface to $\mathbb{P} \times \{\mu(\rho)\}$ in the slice, at distance $\rho$.

The principal directions of curvature of $M$ are the tangent vector to the generating curve and the directions tangent to the equidistant hypersurface. The corresponding principal curvatures are given respectively by

$$k_G(\rho) = \ddot{\mu}(\rho)\big(1 + \dot{\mu}^2(\rho)\big)^{-3/2}, \tag{3.27}$$

and

$$k_E(\rho) = \dot{\mu}(\rho)\big(1 + \dot{\mu}^2(\rho)\big)^{-1/2} \tanh(\rho). \tag{3.28}$$

It follows that the mean curvature of $M$ is given by

$$nH(\rho) = \ddot{\mu}(\rho)\big(1 + \dot{\mu}^2(\rho)\big)^{-3/2} + (n-1)\dot{\mu}(\rho)\big(1 + \dot{\mu}^2(\rho)\big)^{-1/2} \tanh(\rho)$$

or, equivalently, by

$$nH(\rho)\cosh^{n-1}(\rho) = \partial_\rho\Big(\dot{\mu}(\rho)\big(1 + \dot{\mu}^2(\rho)\big)^{-1/2} \cosh^{n-1}(\rho)\Big). \tag{3.29}$$

This formula allows us to study constant mean curvature hypersurfaces invariant by hyperbolic translations. In this paper we only consider the case $H = 0$ and we refer to [7] for the case $H \neq 0$.

### 3.5 Translation invariant minimal hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$

In this section, we establish the following theorem which generalizes the 2-dimensional result of [29]. We first introduce the notion of *horizontal graph* which is essentially the same as in [31].

**Definition 3.8** *Let $\mathbb{P} \subset \mathbb{H}^{n+1}$ be the totally geodesic $n$-space, orthogonal to the geodesic $\gamma$ at $\gamma(0)$. Let $\{F(t), t \in \mathbb{R}\}$ be the one-parameter family of hyperbolic translations along $\gamma$. For $x \in \mathbb{P}$, the trajectory $\{F(t) \cdot x, t \in \mathbb{R}\}$ is an equidistant curve to $\gamma$. Let $\Omega \subset \mathbb{P}$ be a domain and let $u : \Omega \to \mathbb{R}$ be a smooth function. The* horizontal graph with respect to a geodesic $\gamma$,*or simply horizontal graph, of $u$ above $\Omega$ is the set of points $\{F(u(x)) \cdot x \mid x \in \Omega\}$. Horizontal graphs are transversal to the trajectories of the hyperbolic translations along geodesics. In the half-space model, a horizontal graph with respect to the vertical axis is a radial graph in each horizontal slice.*
*In $\mathbb{H}^n \times \mathbb{R}$, we give a similar definition for a horizontal graph over a domain in an $n$-space of the form $\mathbb{P} \times \mathbb{R}$, where $\mathbb{P}$ is a totally geodesic $(n-1)$-space in $\mathbb{H}^n$.*

**Theorem 3.9** *There exists a 1-parameter family $\{\mathcal{M}_d,\ d > 0\}$ of complete embedded minimal hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$ invariant under hyperbolic translations. The hypersurfaces $\mathcal{M}_d$ are stable (in the sense of the Jacobi operator), their principal curvatures go uniformly to zero at infinity, but they have infinite total curvatures.*
*More precisely,*

1. *If $d > 1$, the hypersurface $\mathcal{M}_d$ consists of the union of two symmetric vertical graphs over the exterior of an equidistant hypersurface in the slice $\mathbb{H}^n \times \{0\}$. It is also a horizontal graph, and hence stable. Furthermore, the asymptotic boundary of $\mathcal{M}_d$ consists of the union of two copies of an hemisphere $S^{n-1}_+ \times \{0\}$ of $\partial_\infty \mathbb{H}^n \times \{0\}$ in parallel slices $t = \pm S(d)$, glued with the finite cylinder $\partial S^{n-1}_+ \times [-S(d), S(d)]$.*

2. *If $d = 1$, the hypersurface $\mathcal{M}_1$ is a complete stable vertical graph over a half-space in $\mathbb{H}^n \times \{0\}$, bounded by a totally geodesic hyperplane $P$. It takes infinite boundary value data on $P$ and constant asymptotic boundary value data $c$. Furthermore, the asymptotic boundary of $\mathcal{M}_1$ is the union of a spherical cap $S$ in $\partial_\infty \mathbb{H}^n \times \{c\}$ with a half-vertical cylinder over $\partial S$.*

3. *If $d < 1$, the hypersurface $\mathcal{M}_d$ is an entire stable vertical graph with finite vertical height. Furthermore, its asymptotic boundary consists of a homologically non-trivial $(n-1)$-sphere in $\partial_\infty \mathbb{H}^n \times \mathbb{R}$.*

**Remark:** If $d > 1$, the family $\mathcal{M}_d$ has finite vertical height $h_T(d)$, a function which decreases from infinity to $\pi/(n-1)$. In particular, it is bounded from below by $\pi/(n-1)$, the upper bound of the heights of the family of catenoids.

**Proof.** In the minimal case, Equation (3.29) can be written

$$\dot{\mu}(\rho)\big(1 + \dot{\mu}^2(\rho)\big)^{-1/2} \cosh^{n-1}(\rho) = d, \tag{3.30}$$

for some constant $d$ which satisfies $d \le \cosh^{n-1}(\rho)$ for all $\rho$ for which the solution exists. Changing $\mu$ to $-\mu$ if necessary, we may assume that $d$ is non-negative and hence, $\dot{\mu}(\rho) \ge 0$ and $\dot{\mu}(\rho) = d\big(\cosh^{2n-2}(\rho) - d^2\big)^{-1/2}$ whenever the square root exists. We have to consider three cases, $d > 1$, $d = 1$ and $d < 1$.

$\boxed{d > 1}$ Let $d =: \cosh^{n-1}(a)$, with $a > 0$. It follows from Equation (3.30) that

$$\dot{\mu}(\rho) = \cosh^{n-1}(a)\big(\cosh^{2n-2}(\rho) - \cosh^{2n-2}(a)\big)^{-1/2}.$$

Up to a vertical translation, the solution $\mu_+(a,\rho)$ of Equation (3.30) is given by

$$\mu_+(a,\rho) = \cosh^{n-1}(a)\int_a^{\rho}\big(\cosh^{2n-2}(r) - \cosh^{2n-2}(a)\big)^{-1/2}\,dr \tag{3.31}$$

or, making $\cosh(r) = \cosh(a)\,t$,

$$\mu_+(a,\rho) = \cosh(a)\int_1^{\frac{\cosh(\rho)}{\cosh(a)}}(t^{2n-2}-1)^{-\frac{1}{2}}\,(\cosh^2(a)t^2-1)^{-\frac{1}{2}}\,dt. \tag{3.32}$$

These integrals converge at $\rho = a$ (*resp.* at $t = 1$) and at infinity and

$$h_T(d) := 2\cosh(a)\int_1^{\infty}(t^{2n-2}-1)^{-1/2}\,(\cosh^2(a)t^2-1)^{-1/2}\,dt \tag{3.33}$$

is the height of the hypersurface $\mathcal{M}_{\cosh^{n-1}(a)}$. The function $h_T(d)$ is decreasing in $d$, tends to infinity when $d$ tends to $1_+$ and to $\pi/(n-1)$ when $d$ tends to infinity. (Hints. When $a$ tends to zero, use the fact that (3.33) is bigger than some constant times the integral $\int_1^2\big((t-1)(\cosh(a)t-1)\big)^{-1/2}\,dt$ which can be computed explicitly. When $a$ tends to infinity, use the fact that $\int(t^N-1)^{-1/2}t^{-1}\,dt = \frac{2}{N}\arctan\sqrt{t^N-1}$.) The assertions about the asymptotic boundary are clear.

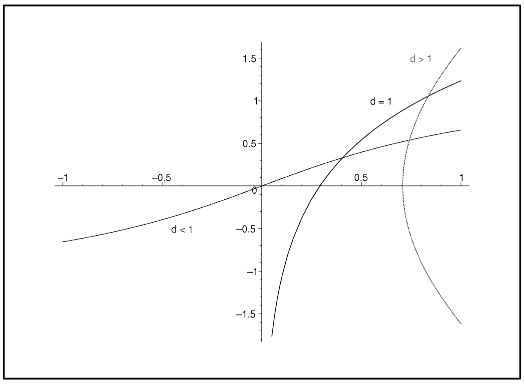


Figure 7: Generatrices for translationally invariant hypersurfaces, $n = 2$

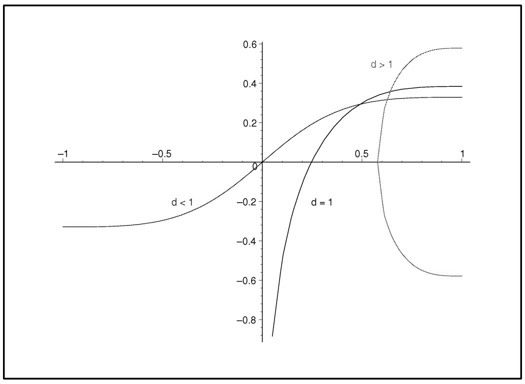


Figure 8: Generatrices for translationally invariant hypersurfaces, $n = 4$

$\boxed{d = 1}$ It follows from Equation (3.30) that

$$\dot{\mu}(\rho) = \big(\cosh^{2n-2}(\rho) - 1\big)^{-1/2},$$

so that, when $d = 1$, the solution is given by

$$\mu_0(\rho) = \int_b^{\rho}\big(\cosh^{2n-2}(r) - 1\big)^{-1/2}\,dr, \tag{3.34}$$

for some constant $b > 0$, and $\mu_0(\rho)$ tends to $-\infty$ when $\rho$ tends to zero and to a finite value when $\rho$ tends to infinity. The corresponding hypersurface is complete. It is a vertical graph so that it is stable. The assertion about the asymptotic boundary is clear.

$\boxed{0 < d < 1}$ In this case, Equation (3.30) gives the following solution (up to a vertical translation),

$$\mu_-(d,\rho) = d \int_0^\rho \big(\cosh^{2n-2}(r) - d^2\big)^{-1/2}\, dr. \tag{3.35}$$

The corresponding curve can be extended by symmetry and we get a complete hypersurface in a vertical slab with finite height. This surface is an entire vertical graph (hence stable). The assertion about the asymptotic boundary is clear. □

The generating curves for translationally invariant minimal hypersurfaces are given in Fig. 7 and 8. Note that they cannot meet tangentially at finite distance.

**Remark.** Using the catenoids and the minimal translational hypersurfaces, the second author and E. Toubiana have extended the 2-dimensional results of their paper [29] to higher dimensions, see [30].

# 4 Index and total curvature for minimal hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$

## 4.1 Dimension two, $M^2 \looparrowright \mathbb{H}^2 \times \mathbb{R}$

For oriented minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ we have the following general theorem in which we consider two possible notions of total curvature.

**Theorem 4.1** *Let $M \looparrowright \mathbb{H}^2 \times \mathbb{R}$ be a complete oriented minimal immersion with unit normal field $N_M$. Let $v_M := \hat{g}(N_M, \partial_t)$ be the vertical component of $N_M$, let $A_M$ be the second fundamental form of $M$ and let $K_M$ be the intrinsic curvature of $M$.*

1. *If the total curvature $\int_M |A_M|^2\, d\mu_M$ is finite, then $A_M$ tends to zero uniformly at infinity.*
2. *If the intrinsic total curvature $\int_M |K_M|\, d\mu_M$ is finite, then $A_M, v_M$ and $K_M$ tend to zero uniformly at infinity.*
3. *If the intrinsic total curvature $\int_M |K_M|\, d\mu_M$ is finite, then the Jacobi operator of $M$ has finite index.*

**Remarks.**

1. For complete orientable minimal surfaces in $\mathbb{R}^3$, finiteness of the index is equivalent to finiteness of the intrinsic total curvature (see [15, 13, 23]). No such statement can hold in $\mathbb{H}^2 \times \mathbb{R}$. Indeed, the surfaces $\mathcal{M}_d$ ([29] and Section 3.5) are stable complete minimal surfaces, invariant under a group of hyperbolic translations. Their total curvatures are infinite, so that the converse to Assertion 3 is false.
2. The assumption $\int_M |K_M|\, d\mu_M$ finite is natural in view of Huber's theorem. In [17], L. Hauswirth and H. Rosenberg show that this assumption actually implies that the total intrinsic curvature is an integer multiple of $2\pi$. There are actually many examples of such surfaces ([12, 17]).

3. In the paper by L. Hauswirth, B. Nelli, R. Sa Earp and E. Toubiana [18], a geometric description of minimal ends of finite total curvature in $\mathbb{H}^2 \times \mathbb{R}$ is given. The authors also proved that a minimal complete end E with finite total curvature is properly immersed and that the Gaussian curvature of E is locally bounded in terms of the geodesic distance to its boundary.

4. As pointed out in the introduction, Assertion 2 is contained in [17], Theorem 3.1 whose proof actually gives a $C^2$-control on the curvature at infinity. We provide a simple proof of Assertion 2 for completeness.

In the following proposition we give a slight improvement of the previous theorem.

**Proposition 4.2** *The notations are the same as in Theorem 4.1.*

1. *Assume that $5v_M^2 \leq 1$. Then there exists a universal constant $C$ such that if the integral $\int_M |A_M|^2 \, d\mu_M$ is less than $C$ then $M$ is a vertical plane.*

2. *Assume that the integral $\int_M |A_M|^2 \, d\mu_M$ is finite and that there exists a compact set $\Omega \subset M$ and a positive constant $c$ such that $v_M^2 \leq 1 - c < 1$ on $M \setminus \Omega$. Then the Jacobi operator of the immersion has finite index.*

**Remarks.**

1. Assertion 1 generalizes the following facts : (i) A minimal surface whose intrinsic curvature $K_M$ is zero is part of a vertical plane $\gamma \times \mathbb{R}$ (where $\gamma$ is a geodesic in $\mathbb{H}^2$). Indeed, we have $K_M = -\frac{1}{2}|A_M|^2 - v_M^2$ and hence $M$ is totally geodesic with horizontal normal vector. (ii) A complete minimal surface whose total intrinsic curvature is less than $2\pi$ is a vertical plane (see [17]).

2. We do not know whether the sole assumption $\int_M |A_M|^2 \, d\mu_M$ finite is sufficient to insure the finiteness of the index of the Jacobi operator of $M$.

**Sketch of the proof of Theorem 4.1 and Proposition 4.2.**

*Fact 1.* The function $u := |A_M|$ satisfies the non-linear elliptic inequality

$$-u \, \Delta_M u \leq u^4 - (5\widehat{K}_M + 1)u^2 \leq u^4 + 4u^2, \tag{4.36}$$

where $\widehat{K}_M$ is the sectional curvature of the 2-plane $TM$ in $\mathbb{H}^2 \times \mathbb{R}$.

This equation follows from J. Simons' equation for minimal submanifolds ([33]), applied to our context. To establish (4.36) we use the fact that $\mathbb{H}^2 \times \mathbb{R}$ is locally symmetric (hence the covariant derivative of its curvature tensor vanishes), that we work in codimension 1 (thus the term $u^4$). The term $(5\widehat{K}_M + 1)u^2$ comes from explicit curvature computations in $\mathbb{H}^2 \times \mathbb{R}$.

*Fact 2.* The surface $M$ satisfies the Sobolev inequality

$$\|f\|_2^2 \leq S(M) \|df\|_1^2 \tag{4.37}$$

for some positive constant $S(M)$ and all $C^1$ functions $f$ with compact support. This follows from [19, 20] and the fact that $\mathbb{H}^2 \times \mathbb{R}$ is simply-connected and non-positively curved.

*Fact 3.* Curvature computations in $\mathbb{H}^2 \times \mathbb{R}$ give

$$\widehat{K}_M = -v_M^2 \quad \text{and} \quad \widehat{\mathrm{Ric}}(N_M, N_M) = -(1 - v_M^2). \tag{4.38}$$

The fact that $v_M$ is a Jacobi field implies that

$$-\Delta_M v_M = v_M^3 + (|A_M|^2 - 1)v_M \tag{4.39}$$

and a similar equality for $|v_M|$.

*Theorem, Assertion 1.* Following the general ideas of [32], we use (4.36) and (4.37), to estimate the $L^p$-norms of $u$ and the classical de Giorgi-Moser-Nash method to estimate $\|u\|_\infty$ outside big balls. The details appear in the proof of Theorem 4.1, p. 282 of [4], where it is observed that the proof only uses the facts that $u$ satisfies Simons' inequality and $M$ a Sobolev inequality.
*Theorem, Assertion 2.* By Gauss equation, the Gauss curvature $K_M$ of $M$ satisfies

$$K_M = -\frac{1}{2}|A_M|^2 - v_M^2. \tag{4.40}$$

The assumption implies that both integrals $\int_M |A_M|^2 \, d\mu_M$ and $\int_M v_M^2 \, d\mu_M$ are finite. By Assertion 1, we already know that $|A_M|$ tends to zero at infinity, and hence that it is bounded. Equation (4.39) then tells us that $|v_M|$ satisfies an elliptic inequality similar to (4.36) and we can again apply the de Giorgi-Moser-Nash method to conclude.

*Theorem, Assertion 3 and Proposition, Assertion 2.* According to Section 2.2 and to the above curvature calculations, the Jacobi operator can be written as $J_M = -\Delta_M + 1 - |A_M|^2 - v_M^2$. We now follow [3], Section 2. It follows from Assertion 2 in the Theorem that $J_M$ is bounded from below, essentially self-adjoint and that its essential spectrum lies above 1. As the eigenvalues below the essential spectrum can only accumulate at $-\infty$ or at the bottom of the essential spectrum, it follows that $J_M$ has finite index.

*Proposition, Assertion 1.* By (4.38), we have that $5\widehat{K}_M + 1 = 1 - 5v_M^2$. Using (4.36) and the assumption on $v_M$, we obtain that

$$(a) \quad - u\,\Delta_M u \le u^4.$$

Multiply equation (a) by $\xi^2$ for some function $\xi$ with compact support (to be chosen later on) and integrate by parts to obtain,

$$\int_M \xi^2 |du|^2 + 2\int_M \xi u \langle du, d\xi\rangle \le \int_M \xi^2 u^4.$$

Using Cauchy-Schwarz inequality, we obtain

$$(b) \quad \int_M \xi^2 |du|^2 \le 2\int_M \xi^2 u^4 + 4\int_M u^2 |d\xi|^2.$$

Plug the function $f = \xi u^2$ into Sobolev inequality (4.37) to obtain

$$\int_M \xi^2 u^4 \le S\big(\int_M |d(\xi u^2)|\big)^2 \le 2S\big(\int_M u^2 |d\xi|\big)^2 + 8S\big(\int_M u\xi |du|\big)^2,$$

where we have noted $S$ for $S(M)$. Using the fact that $\int_M u^2$ is finite and Cauchy-Schwarz, we find

$$(c) \quad \int_M \xi^2 u^4 \le 2S\big(\int_M u^2 |d\xi|\big)^2 + 8S\big(\int_M u^2\big)\int_M \xi^2 |du|^2.$$

Plug (c) into (a) to get

$$(1 - 16S \int_M u^2) \int_M \xi^2 |du|^2 \leq 4S\big(\int_M u^2 |d\xi|\big)^2 + 4 \int_M u^2 |d\xi|^2.$$

We now assume that $16S \int_M u^2 < 1$ and we choose a family of functions $\xi_R$ such that $\xi_R$ is equal to 1 in $B(x_0, R)$ (the ball with radius $R$ centeblack at some $x_0 \in M$), $\xi_R$ is equal to 0 outside $B(x_0, 2R)$ and $|d\xi_R| \leq 2/R$. Letting $R$ tend to infinity and using the fact that $\int_M u^2$ is finite, we obtain that $du = 0$. Since $M$ has infinite volume, it follows that $u = 0$. □

**Remark.** The reader is referblack to Simon's type inequality in [21, Corollary 3.2], related to Theorem 4.1 and Proposition 4.2.

## 4.2 Higher dimension, $M^n \looparrowright \mathbb{H}^n \times \mathbb{R}, n \geq 3$

Recall the formula for the Jacobi operator,

$$J_M := -\Delta_M - \big(|A_M|^2 + \widehat{\mathrm{Ric}}(N_M)\big)$$

where $N_M$ is a unit normal field along $M$ and $A_M$ the second fundamental form of $M$ with respect to $N_M$ (Section 2.2).

Let $v_M := \hat{g}(N_M, \partial_t)$ be the vertical component of the unit normal vector $N_M$. A simple computation gives that $\widehat{\mathrm{Ric}}(N_M) = -(n-1)(1 - v_M^2)$. It follows that the Jacobi operator of $M$ is given by

$$J_M := -\Delta_M + (n-1)(1 - v_M^2) - |A_M|^2. \tag{4.41}$$

We have the following theorem.

**Theorem 4.3** *Let $M^n \looparrowright \mathbb{H}^n \times \mathbb{R}$ a complete oriented minimal immersion. Assume that $M$ has* finite total curvature, i.e. $\int_M |A_M|^n \, d\mu_M < \infty$.

1. *For $n \geq 2$, the second fundamental form $A_M$ tends to zero uniformly at infinity.*
2. *For $n \geq 3$, the Jacobi operator of the immersion has finite index and, more precisely, there exists a universal constant $C(n)$ such that*

$$\mathrm{Ind}(J_M) \leq C(n) \int_M |A_M|^n \, d\mu_M. \tag{4.42}$$

**Remarks.**
(i) The examples $\mathcal{M}_d$ prove that the converse statements in the previous theorems are not true in general, see Section 3.5.
(ii) Note that we state the second assertion of Theorem 4.3 only for $\dim(M) \geq 3$ (our proof does not apply in dimension 2, see [2]).

**Sketch of the proof**. As in the proof of Theorem 4.1, the manifold $M$ satisfies a Sobolev inequality of the form (4.37), namely

$$\|f\|_{n/(n-1)} \leq S(M) \|df\|_1 \quad \text{for all} \quad f \in C_0^1(M)$$

for some constant $S(M)$. Furthermore, the second fundamental form $A_M$ satisfies the following Simons' equation (compare with (4.36)),

$$-\Delta |A_M| \leq |A_M|^3 + C(n) |A_M|,$$

for some constant $C(n)$ which only depends on the dimension (this follows from the expression of the term $\widehat{R}(A)$ as given in [33].

The de Giorgi-Moser-Nash technique applies (see [4], Theorem 4.1) and it follows that $|A_M|$ tends to zero uniformly at infinity.

Since $|v_M| \leq 1$, the operator $J_M$ is bounded from below and essentially self-adjoint. Furthermore, its index is less than or equal to the index of the operator $-\Delta - |A_M|^2$ which is also bounded from below and essentially self-adjoint. The estimate (4.42) then follows by applying Theorem 39 in [2]. $\square$

**Remark.** The reader is referblack to Simon's type inequality in [21, Corollary 3.2], related to Theorem 4.3.

# 5 Applications

In this section, we use the examples constructed in Section 3 as barriers to prove some general results on minimal hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$. These results generalize results obtained in [25] for dimension 2. Similar results hold for $H$-hypersurfaces as well, see [25, 7].

**Theorem 5.1** *Let $\Gamma \subset \mathbb{H}^n$ be a connected compact embedded hypersurface and consider two copies of $\Gamma$ in different slices, $\Gamma_- = \Gamma \times \{-a\}$ and $\Gamma_+ = \Gamma \times \{a\} \subset \mathbb{H}^n \times \mathbb{R}$, for some $a > 0$. Assume that $\Gamma$ is convex.*

*Let $M \subset \mathbb{H}^n \times \mathbb{R}$ be a connected compact immersed minimal hypersurface such that $\partial M = \Gamma_- \cup \Gamma_+$. Then,*

$$2a < \frac{\pi}{n-1} \quad \textit{(the height of the family of catenoids).}$$

*Furthermore, if $M$ is embedded and is a vertical graph in a neighborhood of $\partial M$,*

1. *$M$ is symmetric with respect to the slice $\mathbb{H}^n \times \{0\}$.*
2. *The parts of $M$ above and below the slice of symmetry are vertical graphs.*
3. *If $\Gamma$ is symmetric with respect to a hyperbolic hyperplane $P$, and is a horizontal graph in each side of $P$, then $M$ is a horizontal graph in each side of the hyperplane $P$ and symmetric with respect to the vertical hyperplane $P \times \mathbb{R}$. In particular, if $\Gamma$ is an $(n-1)$-sphere, then $M$ is part of a catenoid.*

**Proof.** We reason ad absurdo. Suppose that the height of $M$ is greater than or equal to $\dfrac{\pi}{n-1}$, that is $2a \geq \dfrac{\pi}{n-1}$. We recall that the height of the family of $n$-dimensional catenoids $\{\mathcal{C}_\rho,\ \rho \in (0,\infty)\}$ is bounded from above by $\dfrac{\pi}{n-1}$, but each catenoid $\mathcal{C}_\rho$ has height strictly less than $\dfrac{\pi}{n-1}$. Now as $M$ is compact, there is a (hyperbolic) radius $\rho_0$ big enough such that $M$ is strictly contained inside the vertical cylinder $\mathbb{M}_{\rho_0}$ of radius $\rho_0$ (where $\mathbb{M}_{\rho_0}$ is a cylinder over a $n-1$ sphere $\mathbb{S}_{\rho_0} \subset \mathbb{H}^n \times \{0\}$ of radius $\rho_0$) containing $M$ in its mean convex side. Recall that, by the geometry of the catenoids, the catenoid $\mathcal{C}_{\rho_0}$ whose distance to the $t$-axis is $\rho_0$ is contained in the closure of the non mean convex side of $\mathbb{M}_{\rho_0}$ touching $\mathbb{M}_{\rho_0}$ just along the $n-1$ sphere $\mathbb{S}_{\rho_0}$. Hence, $M$ is strictly contained in the connected component of $\mathbb{H}^2 \times \mathbb{R} \setminus \mathcal{C}_{\rho_0}$ that contains the $t$ axis of $\mathcal{C}_{\rho_0}$. Notice that the whole family of catenoids $\mathcal{C}_\rho$ is strictly contained in the slab of $\mathbb{H}^n \times \mathbb{R}$ with boundary $\Gamma_- \cup \Gamma_+$. Starting from

$\rho = \rho_0$, making $\rho \to 0$, that is moving the family of catenoids $\{\mathcal{C}_\rho,\, \rho \leq \rho_0\}$ towards $M$, we will find a first interior point of contact with some $\mathcal{C}_\rho$ and $M$, since the family of catenoid cannot touch the boundary of $M$. We arrive at a contradiction, by the the maximum principle. The proof of the first part of the statement is completed.

Now using the family of slices $\mathbb{H}^n \times \{t\} \cup \mathbb{H}^n \times \{-t\}$ coming from the infinity towards $M$ we get, by the maximum principle, that $M$ is entirely contained in the closed slab whose boundary consists of the slices $\mathbb{H}^n \times \{a\} \cup \mathbb{H}^n \times \{-a\}$ and $(\mathbb{H}^n \times \{a\} \cup \mathbb{H}^n \times \{-a\}) \cap M = \partial M$.

In the same way, considering the family of vertical hyperplanes, we get that $M$ is contained in the mean convex side of the vertical cylinder $\mathbb{M}_\Gamma$ over $\Gamma$ and $\mathbb{M}_\Gamma \cap M = \partial M$. Now using Alexandrov Reflection Principle on the slices, moving the slices from $t = a$ towards $t = 0$, by vertical reflections, we get that the reflection of $M^+ = M \cap \{t \geqslant 0\}$ with respect to the horizontal slice $t = 0$, is above $M^- = M \cap \{t \leqslant 0\}$. Moreover, we find that $M^+$ is a vertical graph. Notice that we may start the reflections since $M$ is a vertical graph in a neighborhood of $\partial M$. In the same way, moving the slices from $t = -a$ to $t = 0$, doing vertical reflections, we get that the reflection of $M^- = M \cap \{t \leqslant 0\}$ with respect to the horizontal slice $t = 0$, is below $M^+ = M \cap \{t \geq 0\}$. We conclude that $M^- = M^+$, hence both $M^+$ and $M^-$ are vertical graphs and $M$ is symmetric with respect to the slice $\mathbb{H}^n \times \{0\}$. The proof of the second part of the statement is completed.

Let us assume now that $P \subset \mathbb{H}^n \times \{0\}$ is a hyperplane of symmetry of $\Gamma$. Consider the vertical hyperplane $\mathbb{P} = P \times \mathbb{R}$ and the family of hyperplanes $\mathbb{P}_t$ at signed distance $t$ from $\mathbb{P}$ obtained from $\mathbb{P}$ by horizontal translations along an oriented geodesic $\gamma$ orthogonal to $\mathbb{P}$ at the origin. Choosing $|t|$ big enough, we move the family $\mathbb{P}_t$ towards $\mathbb{P}$ (in the two sides of $\mathbb{H}^n \times \mathbb{R} \setminus \mathbb{P}$), doing Alexandrov Reflection Principle on $\mathbb{P}_t$, taking into account that $\Gamma$ is a horizontal graph in both sides of $P$, and that the symmetric of $\partial M$ on $\mathbb{P}_t$ stays on the slices $t = \pm a$, so that it does not touch the interior of $M$. We can argue as before to conclude that $\mathbb{P}$ is a hyperplane of symmetry of $M$. Of course, if $\Gamma$ is rotationally symmetric then $M$ is a minimal hypersurface of revolution. Henceforth, by the classification theorem, $M$ is part of a catenoid. This completes the proof of the theorem. $\square$

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