Existence Serrin type results for the Dirichlet problem for the prescribed mean curvature equation in Riemannian manifolds

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Abstract

Given a complete Riemannian manifold M of dimension n, we study the existence of vertical graphs in $M \times \mathbb{R}$ with prescribed mean curvature $H = H(x, z)$. Precisely, we prove that such a graph exists over a smooth bounded domain Ω in M for arbitrary smooth boundary data, if $\operatorname{Ricc}_x \geq n \sup_{z\in\mathbb{R}} \|\nabla_x H(x, z)\| - \dfrac{n^2}{n-1} \inf_{z\in\mathbb{R}} (H(x, z))^2$ for each $x \in \Omega$ and $(n-1)\mathcal{H}_{\partial\Omega}(y) \geq n \sup_{z\in\mathbb{R}} |H(y, z)|$ for each $y \in \partial\Omega$. We also establish another existence result in the case where $M = \mathbb{H}^n$ if $\sup_{\Omega\times\mathbb{R}} |H(x, z)| \leq \frac{n-1}{n}$ in the place of the condition involving the Ricci curvature. Finally, we have a related result when M is a Hadamard manifold whose sectional curvature K satisfies $-c^2 \leq K \leq -1$ for some $c > 1$. We generalize classical results of Serrin and Spruck.

1 Introduction

Let M be a complete Riemannian manifold of dimension $n \geq 2$. Given a smooth bounded domain Ω in M, we ask if for a given smooth function φ and a prescribed smooth function $H = H(x, z)$ non-decreasing in the variable z, there exists a smooth up to the boundary function u satisfying

$$\begin{cases} \operatorname{div}\left(\dfrac{\nabla u}{W}\right) = nH(x, u) \text{ in } \Omega, \\ u = \varphi \text{ in } \partial\Omega, \end{cases} \tag{P}$$

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where $W = \sqrt{1 + \|\nabla u(x)\|^2}$ and the quantities involved are calculated with respect to the metric of M. If u satisfies the equation

$$\operatorname{div}\left(\frac{\nabla u}{W}\right) = nH(x,u), \tag{1}$$

then its vertical graph,

$$\operatorname{Gr}(u) = \{(x, u(x)); x \in \Omega\} \subset M \times \mathbb{R},$$

is an hypersurfaces in $M \times \mathbb{R}$ of mean curvature $H(x, u(x))$ at each point $(x, u(x))$.

In a coordinates system $(x_1, \ldots, x_n)$ in M equation (1) can be written in non-divergence form as

$$\mathcal{M}u := \sum_{i,j=1}^{n} \left(W^2\sigma^{ij} - u^i u^j\right) \nabla^2_{ij} u = nH(x,u)W^3, \tag{2}$$

where (σ^{ij}) is the inverse of the metric (σ_{ij}) of M, $u^i = \sum_{j=1}^{n} \sigma^{ij}\partial_j u$ are the coordinates of ∇u and $\nabla^2_{ij}u(x) = \nabla^2 u(x)\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$. We also define the operator $\mathfrak{Q}$ by

$$\mathfrak{Q}u = \mathcal{M}u - nH(x,u)W^3.$$

The matrix of the operator $\mathcal{M}$ (and $\mathfrak{Q}$) is given by $A = W^2 g$, where g is the induce metric on the graph of u. This implies that the eigenvalues of A are positive and depends on x and on ∇u. Hence, $\mathcal{M}$ is locally uniformly elliptic. Furthermore, if Ω is bounded and $u \in \mathscr{C}^1(\overline{\Omega})$, then $\mathcal{M}$ is uniformly elliptic in $\overline{\Omega}$ (see [19] for more details).

We recall that the Dirichlet problem (P) is a classical problem in the intersection between Differential Geometry and Partial Differential Equations. First steps were given by Bernstein [6], Douglas [10] and Radó [17, p. 795] in domains of $\mathbb{R}^2$ for the minimal case. In 1966 Jenkins-Serrin [13, Th. 1 p. 171] derived related results in higher dimensions.

Later on, Serrin [18] devoted his attention to study Dirichlet problems for a class of more general elliptic equations within which is the prescribed mean curvature equation. Specifically related to our work, he obtained the following result.

Theorem 1 (Serrin [18, Th. p. 484]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain whose boundary is of class $\mathscr{C}^2$. Let $H(x) \in \mathscr{C}^1(\overline{\Omega})$ and suppose that*

$$|\nabla H(x)| \leq \frac{n}{n-1}(H(x))^2 \;\; \forall\; x \in \Omega. \tag{3}$$

Then the Dirichlet problem in Ω for surfaces having prescribed mean curvature $H(x)$ is uniquely solvable for arbitrarily given $\mathscr{C}^2$ boundary values if, and only if,

$$(n-1)\mathcal{H}_{\partial\Omega}(y) \geq n\,|H(y)| \;\; \forall\; y \in \partial\Omega. \tag{4}$$

We note that in Serrin condition (4), $\mathcal{H}_{\partial\Omega}(y)$ denotes the inward mean curvature of $\partial\Omega$ at $y \in \partial\Omega$. A direct consequence of theorem 1 is the following sharp result.

Theorem 2 (Serrin sharp solvability criterion [18, p. 416]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain whose boundary is of class $\mathscr{C}^2$. Then the Dirichlet problem for the mean curvature equation has a unique solution for every constant H and arbitrary $\mathscr{C}^2$ boundary data if, and only if, $(n-1)\mathcal{H}_{\partial\Omega} \geq n\,|H|$.*

Joel Spruck [19] is the pioneer in the study of the Dirichlet problem (P) in the $M \times \mathbb{R}$ setting. Spruck established a priori estimates for this problem that led to several existence results when H is a positive constant. More specifically related with our work is the theorem stated below.

Theorem 3 (Spruck [19, T 1.4 p. 787]). *Let $\Omega \subset M$ be a bounded domain whose boundary is of class $\mathscr{C}^{2,\alpha}$ for some $\alpha \in (0,1)$. Let $H \in \mathbb{R}_+$ and suppose that*

$$(n-1)\mathcal{H}_{\partial\Omega}(y) \geq nH. \tag{5}$$

Suppose also that

$$\mathrm{Ricc}_x \geq -\frac{n^2}{n-1}H^2 \quad \forall x \in \Omega. \tag{6}$$

Then the Dirichlet problem (P) *is uniquely solvable for arbitrary continuous boundary data φ.*

Above, Ricc_x is the Ricci curvature of M at x. The notation $\mathrm{Ricc}_x \geq f(x)$ means that the Ricci curvature evaluated in any unitary tangent vector at x is bounded below by the function $f(x)$. The definition of the Ricci curvature we use throughout the text follows [16].

We note that, condition (6) is trivially satisfy for any constant H if $M = \mathbb{R}^n$. So, theorem 3 of Spruck is a generalization of the sufficient part of theorem 2 of Serrin.

On the other hand, in our previous work [3, Th. 1 p. 3] we proved that the *strong Serrin condition*,

$$(n-1)\mathcal{H}_{\partial\Omega}(y) \geq n \sup_{z\in\mathbb{R}} |H(y,z)| \;\; \forall\, y \in \partial\Omega, \tag{7}$$

is necessary for the solvability of problem (P) in a large class of Riemannian manifolds. As an examples are the Hadamard manifolds [3, Corollary 2 p. 3] and the simply connected and compact manifolds whose sectional curvature satisfies $0 < \frac{1}{4}K_0 < K \leq K_0$ provided $\mathrm{diam}(\Omega) < \frac{\pi}{2\sqrt{K_0}}$ [3, Corollary 3 p. 4].

In the present paper, our goal is to study under which conditions on the function H the strong Serrin condition (7) is also sufficient. The main theorem of this paper is the following.

Theorem 4 (main theorem). *Let $\Omega \subset M$ be a bounded domain with $\partial\Omega$ of class $\mathscr{C}^{2,\alpha}$ for some $\alpha \in (0,1)$. Let $H \in \mathscr{C}^{1,\alpha}(\overline{\Omega} \times \mathbb{R})$ satisfying $\partial_z H \geq 0$ and*

$$\mathrm{Ricc}_x \geq n \sup_{z\in\mathbb{R}} \|\nabla_x H(x,z)\| - \frac{n^2}{n-1} \inf_{z\in\mathbb{R}} \left(H(x,z)\right)^2 \;\; \forall\, x \in \Omega. \tag{8}$$

If

$$(n-1)\mathcal{H}_{\partial\Omega}(y) \geq n \sup_{z\in\mathbb{R}} |H\,(y,z)| \;\; \forall\, y \in \partial\Omega, \tag{9}$$

then for every $\varphi \in \mathscr{C}^{2,\alpha}(\overline{\Omega})$ there exists a unique solution $u \in \mathscr{C}^{2,\alpha}(\overline{\Omega})$ of the Dirichlet problem (P)*.*

Notice that assumptions (3) and (6) are particular cases of (8). Hence, theorem 4 generalizes the existence part in theorem 1 of Serrin and theorem 3 of Spruck. We also highlight that the combination of the non-existence results mentioned above with theorem 4 gives Serrin type solvability criteria for the Dirichlet problem (P) (see [3, Thms. 8 and 9]).

On the other hand, notice that, from the combination of theorem 3 of Spruck and our non-existence result [3, Corollary 2 p. 3] for Hadamard manifolds, we can deduce that the Serrin condition (5) is necessary and sufficient for the solvability of problem (P) for every constant H satisfying (6). In the case where $M = \mathbb{H}^n$ we see that condition (6) is satisfied for every constant $H \geq \frac{n-1}{n}$. In the opposite case, $H \in \left[0, \frac{n-1}{n}\right)$, Spruck [19, Th. 5.4 p. 797] obtained an existence result assuming the strict inequality in the Serrin condition.

In this paper we also extend this result of Spruck [19, Th. 5.4 p. 797] in the hyperbolic space by deriving the following theorem.

Theorem 5. *Let $\Omega \subset \mathbb{H}^n$ be a bounded domain with $\partial\Omega$ of class $\mathscr{C}^{2,\alpha}$ for some $\alpha \in (0,1)$ and $\varphi \in \mathscr{C}^{2,\alpha}(\overline{\Omega})$. Let $H \in \mathscr{C}^{1,\alpha}(\overline{\Omega} \times \mathbb{R})$ satisfying $\partial_z H \geq 0$ and $\sup\limits_{\Omega\times\mathbb{R}} |H| \leq \frac{n-1}{n}$. If*

$$(n-1)\mathcal{H}_{\partial\Omega}(y) \geq n \sup_{z\in\mathbb{R}} |H\,(y,z)| \;\; \forall\, y \in \partial\Omega,$$

then for every $\varphi \in \mathscr{C}^{2,\alpha}(\overline{\Omega})$ there exists a unique solution $u \in \mathscr{C}^{2,\alpha}(\overline{\Omega})$ of the Dirichlet problem (P)*.*

Putting together theorem 5 and our non existence result for Hadamard manifolds [3, Cor. 1 p. 3] with theorem 3 of Spruck, one can deduce: the Serrin sharp solvability criterion for arbitrary constant H as stated in theorem 2 above also holds in the hyperbolic case [3, Th. 7 p. 5].

At last, we use the barriers constructed by Galvez-Lozano [11, Th. 6 p. 12] to prove the following result in Hadamard manifolds.

Theorem 6. *Let M be a Hadamard manifold such that $-c^2 \leq K \leq -1$, for some $c > 1$. Let $\Omega \subset M$ be a bounded domain with $\partial\Omega$ of class $\mathscr{C}^{2,\alpha}$ for some $\alpha \in (0,1)$ and whose principal curvatures are greater than c. Let $\varphi \in \mathscr{C}^{2,\alpha}(\overline{\Omega})$ and $H \in \mathscr{C}^{1,\alpha}(\overline{\Omega} \times \mathbb{R})$ satisfying $\partial_z H \geq 0$ and $\sup\limits_{\Omega\times\mathbb{R}} |H| \leq \frac{n-1}{n}$. Then problem* (P) *has a unique solution $u \in \mathscr{C}^{2,\alpha}(\overline{\Omega})$.*

2 The a priori estimates

Firstly, we establish a lemma that will help us to obtain a priori height and boundary gradient estimates.

Lemma 7. *Let Γ be an embedded and oriented $\mathscr{C}^2$ hypersurface of M and Γ_t parallel to Γ for each $t \in [0, \tau)$. Assume that for some fix $y \in \Gamma$, $\mathcal{H}_\Gamma(y) \geq 0$ with respect to a normal field N. Suppose also that there exists a function $h \in \mathscr{C}^1[0, \tau)$ satisfying*

$$|h(0)| \leq \mathcal{H}_\Gamma(y) \tag{10}$$

and

$$(n-1)\left(|h'(t)| - (h(t))^2\right) \leq \operatorname{Ricc}_{\gamma_y(t)}(\gamma_y'(t)) \ \forall\ t \in [0, \tau), \tag{11}$$

where $\gamma_y(t) = \exp_y(tN_y) \in \Gamma_t$. Then

$$|h(t)| \leq \mathcal{H}_{\Gamma_t}(\gamma_y(t)) \ \forall\ 0 \leq t < \tau, \tag{12}$$

where $\mathcal{H}_{\Gamma_t}$ is computed with respect to $\gamma_y'(t)$. Furthemore, $\mathcal{H}_{\Gamma_t}(\gamma_y(t))$ is increasing as a function of t.

Proof. Let $\mathcal{H}(t) := \mathcal{H}_{\Gamma_t}(\gamma_y(t))$. It is known that (see [2, Cor. B.4 p. 66])

$$\mathcal{H}'(t) \geq \frac{\operatorname{Ricc}_{\gamma_y(t)}(\gamma_y'(t))}{n-1} + (\mathcal{H}(t))^2 .$$

Since we are assuming (11) it follows

$$\mathcal{H}'(t) \geq |h'(t)| - (h(t))^2 + (\mathcal{H}(t))^2 . \tag{13}$$

Then,

$$(\mathcal{H}(t) - h(t))' \geq (\mathcal{H}(t) + h(t))\,(\mathcal{H}(t) - h(t)) \tag{14}$$

and

$$(\mathcal{H}(t) + h(t))' \geq (\mathcal{H}(t) - h(t))\,(\mathcal{H}(t) + h(t)) . \tag{15}$$

Let us define $v(t) = \mathcal{H}(t) - h(t)$ and $g(t) = \mathcal{H}(t) + h(t)$. From (14) we have

$$\left(\frac{v(t)}{e^{\int_0^t g(s)ds}}\right)' \geq 0,$$

so $v(t) \geq v(0)e^{\int_0^t g(s)ds}$ for each $t \in [0, \tau)$. As a consequence of (10) we obtain

$$\mathcal{H}(t) \geq h(t) \ \forall t \in [0, \tau).$$

Using (15) we obtain in a similar way that

$$\mathcal{H}(t) \geq -h(t) \ \forall t \in [0, \tau).$$

Therefore,

$$\mathcal{H}(t) \geq |h(t)| \ \ \forall t \in [0, \tau). \tag{16}$$

Substituting (16) in (13) we also obtain $\mathcal{H}'(t) \geq 0$. □

Roughly speaking, lemma 7 says that, under condition (11), the parallel hypersurfaces inherit the initial condition on Γ throughout the orthogonal geodesics. Moreover, the mean curvature of the parallel hypersurfaces in Ω increases along the inner normal geodesics.

2.1 A priori height estimate

We point out that in theorem 1 of Serrin the combination of condition (3) with the Serrin condition (4) provides height estimate for the Dirichlet problem (P) in the Euclidean case. Analogously for theorem 3 of Spruck. We generalize these geometric ideas in the next theorem.

Theorem 8. *Let $\Omega \in M$ be a bounded domain with $\partial\Omega$ of class $\mathscr{C}^2$ and $\varphi \in \mathscr{C}^0(\partial\Omega)$. Let $H \in \mathscr{C}^1(\overline{\Omega} \times \mathbb{R})$ satisfying $\partial_z H \geq 0$,*

$$\mathrm{Ricc}_x \geq n \sup_{z\in\mathbb{R}} \|\nabla_x H(x,z)\| - \frac{n^2}{n-1} \inf_{z\in\mathbb{R}} \left(H(x,z)\right)^2 \ \ \forall\ x \in \Omega, \tag{17}$$

and

$$(n-1)\mathcal{H}_{\partial\Omega}(y) \geq n\,|H(y,\varphi(y))| \ \ \forall\ y \in \partial\Omega. \tag{18}$$

If $u \in \mathscr{C}^2(\Omega) \cap \mathscr{C}^0(\overline{\Omega})$ is a solution of problem (P)*, then*

$$\sup_{\Omega} |u| \leq \sup_{\partial\Omega} |\varphi| + \frac{e^{\mu\delta} - 1}{\mu},$$

where $\mu > n \sup \left\{ |H(x,z)|\,, (x,z) \in \overline{\Omega} \times \left[-\sup_{\partial\Omega} |\varphi|\,, \sup_{\partial\Omega} |\varphi| \right] \right\}$ *and* $\delta = \mathrm{diam}(\Omega)$.

Proof. For $x \in \Omega$ let us define the distance function $d(x) = \mathrm{dist}(x, \partial\Omega)$. Let Ω_0 be the biggest open subset of Ω having the unique nearest point property; that is, for every $x \in \Omega_0$ there exists a unique $y \in \partial\Omega$ such that $d(x) = \mathrm{dist}(x,y)$. Then $d \in \mathscr{C}^2(\Omega_0)$ (see [19, Prop. 4.1 p. 794], [14]).

We now define $w = \phi \circ d + \sup_{\partial\Omega} |\varphi|$ over Ω, where

$$\phi(t) = \frac{e^{\mu\delta}}{\mu} \left(1 - e^{-\mu t}\right).$$

If we prove that $u \leq w$ in $\overline{\Omega}$ we obtain the desired estimate. By the sake of contradiction we suppose that the function $v = u - w$ attains a maximum $m > 0$ at $x_0 \in \Omega$.

Let $y_0 \in \partial\Omega$ be such that $d(x_0) = \mathrm{dist}(x_0, y_0) = t_0$ and γ the minimizing geodesic orthogonal to $\partial\Omega$ joining x_0 to y_0. Restricting u and w to γ we see that $v'(t_0) = 0$. Hence, $u'(t_0) = w'(t_0) = \phi'(t_0) > 0$ which implies that $\nabla u(x_0) \neq 0$. Therefore, $\Gamma_0 = \{x \in \Omega; u(x) = u(x_0)\}$ is of class $\mathscr{C}^2$ near x_0. Then, there exists a geodesic ball $B_\epsilon(z_0)$ tangent to Γ_0 in x_0 such that

$$u > u(x_0) \text{ in } \overline{B_\epsilon(z_0)} \setminus \{x_0\}. \tag{19}$$

We note that

$$\mathrm{dist}(z_0, y_0) \leq \mathrm{dist}(z_0, x_0) + \mathrm{dist}(x_0, y_0) = \epsilon + d(x_0).$$

Hence, for $\tilde{z}$ lying in the intersection of $\partial B_\epsilon(z_0)$ with a minimizing geodesic joining z_0 to y_0, we have

$$d(\tilde{z}) \leq \mathrm{dist}(\tilde{z}, y_0) = \mathrm{dist}(z_0, y_0) - \epsilon \leq d(x_0) + \epsilon - \epsilon = d(x_0).$$

Thus, $w(\tilde{z}) \leq w(x_0)$ since ϕ is increasing. Consequently,

$$u(\tilde{z}) - w(x_0) \leq u(\tilde{z}) - w(\tilde{z}) \leq u(x_0) - w(x_0)$$

and $u(\tilde{z}) \leq u(x_0)$. By (19) one has that $\tilde{z} = x_0$, so $z_0 = \gamma(t_0 + \epsilon)$. This ensures that $x_0 \in \Omega_0$ because if there exists $y_1 \neq y_0$ satisfying $d(x_0) = \mathrm{dist}(x_0, y_1)$, then

$$\mathrm{dist}(z_0, y_1) < \mathrm{dist}(z_0, x_0) + \mathrm{dist}(x_0, y_1) = \mathrm{dist}(z_0, x_0) + \mathrm{dist}(x_0, y_0) = d(z_0),$$

which is a contradiction.

However, let's show that this is also impossible. After some computations we have

$$\mathcal{M}w = \phi'(1 + \phi'^2)\Delta d + \phi'' \quad \text{in } \Omega_0. \tag{20}$$

For $x \in \Omega_0$, let $y = y(x)$ in $\partial\Omega$ be the nearest point to x and $\gamma_y(t)$ the orthogonal geodesic to $\partial\Omega$ from y to x. Let us define

$$h(t) = \frac{n}{n-1} H\left(\gamma_y(t), \varphi(y)\right).$$

Note that y is now fixed. From the Serrin condition (18) it follows that

$$|h(0)| = \frac{n}{n-1} \left| H\left(y, \varphi(y)\right) \right| \leq \mathcal{H}_{\partial\Omega}(y) = \mathcal{H}(0).$$

Besides,

$$h'(t) = \frac{n}{n-1} \big\langle \nabla_x H(\gamma_y(t), \varphi(y)), \gamma_y'(t) \big\rangle.$$

Taking into account the additional hypothesis (17) we see that

$$(n-1)\left(|h'(t)| - (h(t))^2\right) \leq \mathrm{Ricc}_{\gamma_y(t)}(\gamma_y'(t)).$$

Then we can apply lemma 7 to the function $h(t)$ to obtain

$$n \left| H(\gamma_y(t), \varphi(y)) \right| \leq (n-1)\mathcal{H}_{\Gamma_t}(\gamma_y(t)),$$

where Γ_t is parallel to some portion of $\partial\Omega$. Therefore

$$\Delta d(x) \leq -n \left| H\left(x, \varphi(y(x))\right) \right| \;\; \forall\, x \in \Omega_0.$$

Using this estimate in (20) we obtain

$$\mathcal{M}w \leq -n \left| H\left(x, \varphi(y(x))\right) \right| \phi'(1 + \phi'^2) + \phi''.$$

Also

$$\phi''(t) = -\mu e^{\mu(\delta - t)} = -\mu\phi'(t) < -n \left| H(x, \varphi(y(x))) \right| \phi'(t)$$

and $\phi' \geq 1$, so

$$\mathcal{M}w \leq -n \left|H\left(x, \varphi(y(x))\right)\right| \phi'(2+\phi'^2) < -n \left|H\left(x, \varphi(y(x))\right)\right| \left(1+\phi'^2\right)^{3/2}. \quad (21)$$

On the other hand, the hypothesis $\partial_z H \geq 0$ implies that

$$\mp H(x, \pm w) \leq \mp H\left(x, \varphi(y(x))\right) \leq \left|H\left(x, \varphi(y(x))\right)\right|. \quad (22)$$

From this fact and (21) we conclude that

$$\pm\mathfrak{Q}(\pm w) = \mathcal{M}w \mp nH\left(x, \pm w\right)\left(1+\phi'^2\right)^{3/2} \leq 0.$$

Therefore

$$\mathfrak{Q}(w+m) = \mathcal{M}(w+m) - nH(x, w+m)\left(1+\phi'^2\right)^{3/2} \leq \mathfrak{Q}w \leq \mathfrak{Q}u.$$

Moreover $u \leq w+m$ and $u(x_0) = w(x_0)+m$. By the maximum principle $u \equiv w+m$ in Ω_0 which is a contradiction since $u < w+m$ in $\partial\Omega$. This proves that $u \leq w$ in $\overline{\Omega}$.

Similarly we prove that $u \geq -w$ in Ω. $\square$

Remark 9. Instead of condition (17), the proof shows that it is suffice to assume that

$$\operatorname{Ricc}_x \geq n \left\|\nabla_x H(x, \varphi(y))\right\| - \frac{n^2}{n-1}\left(H(x, \varphi(y))\right)^2 \ \forall\ x \in \Omega_0,$$

where Ω_0 is the biggest open subset of Ω having the unique nearest point property, and $y \in \partial\Omega$ is the nearest point to x.

2.2 A priori boundary gradient estimates

In this section we use the classical idea to find upper and a lower barriers for u on $\partial\Omega$ to get a control for ∇u along $\partial\Omega$.

Theorem 10. *Let $\Omega \in M$ be a bounded domain with $\partial\Omega$ of class $\mathscr{C}^2$ and $\varphi \in \mathscr{C}^2(\overline{\Omega})$. Let $H \in \mathscr{C}^1\left(\overline{\Omega}\times\mathbb{R}\right)$ satisfying $\partial_z H \geq 0$,*

$$\operatorname{Ricc}_x \geq n \sup_{z\in\mathbb{R}} \left\|\nabla_x H(x,z)\right\| - \frac{n^2}{n-1} \inf_{z\in\mathbb{R}} \left(H(x,z)\right)^2 \ \forall\ x \in \Omega, \quad (23)$$

and

$$(n-1)\mathcal{H}_{\partial\Omega}(y) \geq n\left|H(y, \varphi(y))\right| \ \forall\ y \in \partial\Omega. \quad (24)$$

If $u \in \mathscr{C}^2(\Omega)\cap\mathscr{C}^1(\overline{\Omega})$ is a solution of (P)*, then*

$$\sup_{\partial\Omega}\left\|\nabla u\right\| \leq \left\|\varphi\right\|_1 + e^{C\left(1+\|H\|_1+\|\varphi\|_2\right)\left(1+\|\varphi\|_1\right)^3\left(\|u\|_0+\|\varphi\|_0\right)} \quad (25)$$

for some $C = C(n, \Omega)$.

Proof. Again, for $x \in \Omega$, we set $d(x) = \operatorname{dist}(x, \partial\Omega)$. Let $\tau > 0$ be such that d is of class $\mathscr{C}^2$ over the set of points in Ω for which $d(x) \leq \tau$. Let $\psi \in \mathscr{C}^2([0,\tau])$ be a non-negative function satisfying

P1. $\psi'(t) \geq 1$, P2. $\psi''(t) \leq 0$, P3. $t\psi'(t) \leq 1$.

For $a < \tau$ to be fixed latter on we consider the set

$$\Omega_a = \{x \in M; d(x) < a\}.$$

We now define $w^{\pm} = \pm\psi \circ d + \varphi$. Firstly, let's estimate $\pm\mathcal{M}w^{\pm}$ in Ω_a. A straightforward computation yields

$$\begin{aligned}\pm\mathcal{M}w^{\pm} =& \psi' W_{\pm}^2 \Delta d + \psi'' W_{\pm}^2 - \psi'' \langle \nabla d, \pm\psi'\nabla d + \nabla\varphi\rangle^2 \\ &- \psi'\, \nabla^2\, d(\nabla\varphi, \nabla\varphi) \mp \nabla^2\, \varphi(\pm\psi'\nabla d + \nabla\varphi, \pm\psi'\nabla d + \nabla\varphi),\end{aligned} \tag{26}$$

where

$$W_{\pm} = \sqrt{1 + \|\nabla w^{\pm}\|^2} = \sqrt{1 + \|\pm\psi'\nabla d + \nabla\varphi\|^2}.$$

Since $\psi'' < 0$ and $\langle \nabla d, \pm\psi'\nabla d + \nabla\varphi\rangle^2 \leq \|\pm\psi'\nabla d + \nabla\varphi\|^2$, then

$$\psi'' W_{\pm}^2 - \psi''\langle \nabla d, \pm\psi'\nabla d + \nabla\varphi\rangle^2 \leq \psi''. \tag{27}$$

Once $\nabla^2\, d(x)$ is a continuous bilinear form and $\psi' \geq 1$ we have

$$\psi' \left|\nabla^2\, d(\nabla\varphi, \nabla\varphi)\right| \leq \psi'^2 \|d\|_2 \|\varphi\|_1^2. \tag{28}$$

Note also that

$$\|\pm\psi'\nabla d + \nabla\varphi\|^2 = \left(\psi'^2 + 2\psi'\langle \pm\nabla d, \nabla\varphi\rangle + \|\nabla\varphi\|^2\right) \leq (1 + \|\varphi\|_1)^2\, \psi'^2, \tag{29}$$

hence

$$\left|\nabla^2\, \varphi(\pm\psi'\nabla d + \nabla\varphi, \pm\psi'\nabla d + \nabla\varphi)\right| \leq \|\varphi\|_2\, (1 + \|\varphi\|_1)^2\, \psi'^2. \tag{30}$$

Substituting (27), (28), (30) in (26) it follows

$$\pm\mathcal{M}w^{\pm} \leq \psi' W_{\pm}^2 \Delta d + \psi'' + c\psi'^2, \tag{31}$$

where

$$c = \|d\|_2 \|\varphi\|_1^2 + \|\varphi\|_2\, (1 + \|\varphi\|_1)^2. \tag{32}$$

Observe now that

$$\pm\mathfrak{Q}w^{\pm} = \pm\mathcal{M}w^{\pm} \mp nH(x, w^{\pm})W_{\pm}^3.$$

Moreover

$$\mp H(x, w^{\pm}(x)) = \mp H(x, \pm\psi(d(x)) + \varphi(x)) \leq \mp H(x, \varphi(x))$$

since we are assuming that $\partial_z H \geq 0$, so

$$\pm\mathfrak{Q}w^{\pm} \leq \pm\mathcal{M}w^{\pm} \mp nH(x, \varphi(x))W_{\pm}^3 \leq \pm\mathcal{M}w^{\pm} + n\, |H(x, \varphi(x))|\, W_{\pm}^3.$$

Using the estimate in (31) we obtain

$$\pm\mathfrak{Q}w^{\pm} \leq \psi' W_{\pm}^2 \Delta d + \psi'' + c\psi'^2 + n\left|H(x,\varphi(x))\right| W_{\pm}^3. \tag{33}$$

Let now $y \in \partial\Omega$ be fixed and $\gamma_y(t) = \exp_y(tN_y)$ for $0 \leq t \leq a$, where N is the inner normal field to $\partial\Omega$. Applying again lemma 7 to $h(t) = \frac{n}{n-1}H(\gamma_y(t), \varphi(y))$, we see that $\mathcal{H}'(t) \geq 0$, for $0 \leq t \leq \tau$. Then, $\mathcal{H}_{\Gamma_t}(\gamma_y(t)) \geq \mathcal{H}_{\partial\Omega}(y)$ for $0 \leq t \leq a$, where Γ_t is parallel to $\partial\Omega$. Therefore,

$$\Delta d(x) \leq \Delta d(y) \leq -n\left|H(y,\varphi(y))\right| \ \forall\, x \in \Omega_a, \tag{34}$$

where we denote by $y = y(x) \in \partial\Omega$ the nearest point to x. Substituting (34) in (33) we obtain

$$\begin{aligned}\pm\mathfrak{Q}w^{\pm} \leq & n\psi' W_{\pm}^2(\left|H(x,\varphi(x))\right| - \left|H(y,\varphi(y))\right|) \\ & + n\left|H(x,\varphi(x))\right| W_{\pm}^2\left(W_{\pm} - \psi'\right) + \psi'' + c\psi'^2.\end{aligned} \tag{35}$$

It follows directly from (29) that

$$W_{\pm}^2 \leq 1 + \left(1 + \|\varphi\|_1\right)^2 \psi'^2 \leq 2\left(1 + \|\varphi\|_1\right)^2 \psi'^2. \tag{36}$$

In addition

$$\left|H(x,\varphi(x))\right| - \left|H(y,\varphi(y))\right| \leq h_1(1 + \|\varphi\|_1)d(x),$$

where

$$h_1 = \sup_{\Omega\times\left[-\sup\limits_{\Omega}|\varphi|,\sup\limits_{\Omega}|\varphi|\right]} \left\|\nabla_{M\times\mathbb{R}}H(x,z)\right\|.$$

Then,

$$n\psi' W_{\pm}^2(\left|H(x,\varphi(x))\right| - \left|H(y,\varphi(y))\right|) \leq 2nh_1\left(1 + \|\varphi\|_1\right)^3 d(x)(\psi'(d(x)))^3.$$

Using the assumption P3 it follows

$$n\psi' W_{\pm}^2(\left|H(x,\varphi(x))\right| - \left|H(y,\varphi(y))\right|) \leq 2nh_1\left(1 + \|\varphi\|_1\right)^3 \psi'^2. \tag{37}$$

On the other hand,

$$W_{\pm} - \psi' \leq 1 + \left\|\pm\psi'\nabla d + \nabla\varphi\right\| - \psi' \leq 1 + \|\varphi\|_1. \tag{38}$$

From (36) and (38) we obtain

$$n\left|H(x,\varphi(x))\right|\left(W_{\pm} - \psi'\right) W_{\pm}^2 \leq 2nh_0\left(1 + \|\varphi\|_1\right)^3 \psi'^2, \tag{39}$$

where

$$h_0 = \sup_{\Omega\times\left[-\sup\limits_{\Omega}|\varphi|,\sup\limits_{\Omega}|\varphi|\right]} \left|H(x,z)\right|.$$

Using (37) and (39) in (35) we get

$$\pm\mathfrak{Q}w^{\pm} \leq \left(c + 2n\|H\|_1\left(1 + \|\varphi\|_1\right)^3\right)\psi'^2 + \psi'',$$

where we are using the notation $\|H\|_1 = h_0 + h_1$.

Remembering the expression for c given in (32) and making some algebraic computation we infer that

$$c + 2n \|H\|_1 (1 + \|\varphi\|_1)^3 < C (1 + \|\varphi\|_2 + \|H\|_1) (1 + \|\varphi\|_1)^3 ,$$

where

$$C = 2n (1 + \|d\|_2 + 1/\tau) . \tag{40}$$

Choosing

$$\nu = C (1 + \|H\|_1 + \|\varphi\|_2) (1 + \|\varphi\|_1)^3 \tag{41}$$

we define ψ by

$$\psi(t) = \frac{1}{\nu} \log(1 + kt).$$

So,

$$\psi'(t) = \frac{k}{\nu(1 + kt)} \tag{42}$$

and

$$\psi''(t) = -\frac{k^2}{\nu(1 + kt)^2}, \tag{43}$$

hence

$$\pm\mathfrak{Q}w^{\pm} < \nu\psi'^2 + \psi'' = 0, \quad \text{in} \quad \Omega_a.$$

Besides

$$t\psi'(t) = \frac{kt}{\nu(1 + kt)} \leq \frac{1}{\nu} < 1,$$

which is property P3. From (43) we see that property P2 is also satisfied. This implies that $\psi'(t) > \psi'(a)$ for all $t \in [0, a]$ as well, thus property P1 is ensured provided that

$$\psi'(a) = \frac{k}{\nu(1 + ka)} = 1. \tag{44}$$

Furthermore, if we choose

$$\psi(a) = \frac{1}{\nu} \log(1 + ka) = \|u\|_0 + \|\varphi\|_0 , \tag{45}$$

we would have

$$\pm w^{\pm}(x) = \psi(a) \pm \varphi(x) = \|u\|_0 + \|\varphi\|_0 \pm \varphi(x) \geq \pm u(x) \;\forall\; x \in \partial\Omega_a \setminus \partial\Omega.$$

By combining (44) and (45) we see that

$$k = \nu e^{\nu(\|u\|_0 + \|\varphi\|_0)} \tag{46}$$

and, therefore,

$$a = \frac{e^{\nu(\|u\|_0 + \|\varphi\|_0)} - 1}{\nu e^{\nu(\|u\|_0 + \|\varphi\|_0)}}.$$

Note also that $a < \frac{1}{\nu} < \tau$ as required.

Finally, if $x \in \partial\Omega$, then $w^{\pm}(x) = \pm\psi(0) + \varphi(x) = u(x)$. By the maximum principle we can conclude that $w^- \le u \le w^+$ in Ω_a, thus

$$-\psi \circ d \le u - \varphi \le \psi \circ d \text{ in } \Omega_a.$$

Recall that

$$-\psi \circ d = u - \varphi = \psi \circ d = 0 \text{ in } \partial\Omega.$$

Consequently, for $y \in \partial\Omega$ and $0 \le t \le a$, we have that

$$-\psi(t) + \psi(0) \le (u - \varphi)(\gamma_y(t)) - (u - \varphi)(\gamma_y(0)) \le \psi(t) - \psi(0).$$

Dividing by $t > 0$ and passing to the limit as t goes to zero we infer that

$$|\langle \nabla u(y), N\rangle| \le |\langle \nabla\varphi(y), N\rangle| + \psi'(0). \tag{47}$$

As $u = \varphi$ on $\partial\Omega$, using (47) we derive

$$\|\nabla u(y)\| \le \|\nabla\varphi(y)\| + \psi'(0).$$

which yields the desired estimate. □

Remark 11. It is suffice to assume in the statement of theorem 10 that

$$\mathrm{Ricc}_x \ge n\,\|\nabla_x H(x, \varphi(y))\| - \frac{n^2}{n-1}\,(H(x, \varphi(y)))^2 \ \ \forall\, x \in \Omega_0,$$

where Ω_0 is the biggest open subset of Ω having the unique nearest point property, and $y \in \partial\Omega$ is the nearest point to x.

Now, we observe that the combination of assumption (23) with the Serrin condition (24) ensures that the mean curvature of the parallel hypersurfaces Γ_t in Ω increases along the inner normal geodesics.

On the other hand, this behavior of $\mathcal{H}_{\Gamma_t}$ is guaranteed indeed by the geometric condition

$$\mathrm{Ricc}_{\gamma_y(t)}(\gamma_y'(t)) \ge -(n-1)\,(\mathcal{H}_{\partial\Omega}(y))^2 \ \ \forall\, y \in \partial\Omega. \tag{48}$$

This can be seen applying lemma 7 to the constant function $h(t) = \mathcal{H}_{\partial\Omega}(y)$ (see also [9, Th. 1 p. 232]).

Therefore, if (48) holds we do not need the assumption (23) in the statement of theorem 10. So, we are able to establish the following result for later reference.

Theorem 12. *Suppose that for* $\mathrm{Ricc}_x \ge -(n-1)c^2$ *for each* $x \in M$*, where* $c > 0$*. Let* $\Omega \in M$ *be a bounded domain with* $\partial\Omega$ *of class* $\mathscr{C}^2$ *such that* $\mathcal{H}_{\partial\Omega} \ge c$ *and* $\varphi \in \mathscr{C}^2(\overline{\Omega})$*. Let* $H \in \mathscr{C}^1\left(\overline{\Omega} \times \mathbb{R}\right)$ *satisfying* $\partial_z H \ge 0$ *and*

$$(n-1)\mathcal{H}_{\partial\Omega}(y) \ge n\,|H(y, \varphi(y))| \ \ \forall\, y \in \partial\Omega. \tag{49}$$

If $u \in \mathscr{C}^2(\Omega) \cap \mathscr{C}^1(\overline{\Omega})$ *is a solution of* (P)*, then*

$$\sup_{\partial\Omega} \|\nabla u\| \le \|\varphi\|_1 + e^{C\left(1+\|H\|_1+\|\varphi\|_2\right)\left(1+\|\varphi\|_1\right)^3\left(\|u\|_0+\|\varphi\|_0\right)} \tag{50}$$

for some $C = C(n, \Omega)$*.*

Proof. By the previous discussion we see that

$$\Delta d(x) \leq \Delta d(y) \ \forall \ x \in \Omega_a,$$

where $y \in \partial\Omega$ is the nearest point to x. The rest of the proof is the same as before. $\square$

Now we consider a mean convex domain Ω in the hyperbolic space $\mathbb{H}^n$ and let $y \in \partial\Omega$. If $\lambda_i(t)$ represents the ith principal curvature of Γ_t in $\gamma_y(t)$, then (see [1, p. 17])

$$\lambda_i(t) = \frac{-\tanh t + \lambda_i(0)}{1 - \lambda_i(0)\tanh t}, \tag{51}$$

hence

$$\lambda_i'(t) = \frac{\operatorname{sech}^2(t)\left((\lambda_i(0))^2 - 1\right)}{(1 - \lambda_i(0)\tanh t)^2}. \tag{52}$$

Thus, $\mathcal{H}_{\Gamma_t}(\gamma_y(t))$ decrease if $|\lambda_i| < 1$ for all $1 \leq i \leq n$. In any case we can choose τ small enough such that

$$\left|\mathcal{H}_{\partial\Omega}(y) - \mathcal{H}_{d(x)}(x)\right| \leq \kappa d(x)$$

for some $\kappa > 0$ depending on Ω. Using this fact we are able to deduce the following result.

Theorem 13. *Let $\Omega \in \mathbb{H}^n$ be a bounded domain with $\partial\Omega$ of class $\mathscr{C}^2$ and $\varphi \in \mathscr{C}^2(\overline{\Omega})$. Let $H \in \mathscr{C}^1\left(\overline{\Omega} \times \mathbb{R}\right)$ satisfying $\partial_z H \geq 0$, and*

$$(n-1)\mathcal{H}_{\partial\Omega}(y) \geq n\,|H(y, \varphi(y))| \ \ \forall \ y \in \partial\Omega.$$

If $u \in \mathscr{C}^2(\Omega) \cap \mathscr{C}^1(\overline{\Omega})$ is a solution of (P)*, then*

$$\sup_{\partial\Omega} \|\nabla u\| \leq \|\varphi\|_1 + e^{C\left(1+\|H\|_1+\|\varphi\|_2\right)\left(1+\|\varphi\|_1\right)^3\left(\|u\|_0+\|\varphi\|_0\right)} \tag{53}$$

for some $C = C(n, \Omega)$.

Proof. The proof follows the steps of the proof of theorem 10 with the difference that we need to replace relation (34) by

$$\Delta d(x) \leq \Delta d(y) + (n-1)\kappa d(x) \leq -n\,|H(y, \varphi(y))| + n\kappa d(x).$$

In this case $C = 2n\left(1 + \kappa + \|d\|_2 + 1/\tau\right)$ instead of (40). $\square$

2.3 A priori global gradient estimate

In order to obtain a priori global gradient estimate we use techniques introduced by Caffarelli-Nirenberg-Spruck [8, p. 51] in the Euclidean context. See other applications in the works of Nelli-Sa Earp [15, Lemma 3.1 p. 4] and Barbosa-Sa Earp [4, Lemma 5.2 p. 62] in the hyperbolic setting.

Theorem 14. *Let $\Omega \in M$ be a bounded domain with $\partial\Omega$ of class $\mathscr{C}^2$. Let $u \in \mathscr{C}^3(\Omega) \cap \mathscr{C}^1(\overline{\Omega})$ be a solution of (1), where $H \in \mathscr{C}^1\left(\Omega \times \left[-\sup_{\overline{\Omega}} |u|, \sup_{\overline{\Omega}} |u|\right]\right)$ satisfies $\partial_z H \geq 0$. Then*

$$\sup_{\Omega} \|\nabla u(x)\| \leq \left(\sqrt{3} + \sup_{\partial\Omega} \|\nabla u\|\right) \exp\left(2 \sup_{\Omega} |u| \left(1 + 8n \left(\|H\|_1 + R\right)\right)\right),$$

where $R \geq 0$ is such that $\mathrm{Ricc}_x \geq -R$ for each $x \in \Omega$.

Proof. Let $w(x) = \|\nabla u(x)\| e^{Au(x)}$ where $A \geq 1$. Suppose w attains a maximum at $x_0 \in \overline{\Omega}$. If $x_0 \in \partial\Omega$, then

$$w(x) \leq w(x_0) = \|\nabla u(x_0)\| e^{Au(x_0)}.$$

So,

$$\sup_{\Omega} \|\nabla u(x)\| \leq \sup_{\partial\Omega} \|\nabla u\| e^{2A \sup_{\Omega} |u|}. \tag{54}$$

Suppose now that $x_0 \in \Omega$ and that $\nabla u(x_0) \neq 0$. Let us define normal coordinates at x_0 in such a way that $\frac{\partial}{\partial x_1}\Big|_{x_0} = \frac{\nabla u(x_0)}{\|\nabla u(x_0)\|}$. Then,

$$\partial_k u(x_0) = \left\langle \frac{\partial}{\partial x_k}\Big|_{x_0}, \nabla u(x_0) \right\rangle = \|\nabla u(x_0)\| \delta_{k1}. \tag{55}$$

Denoting by σ the metric in this coordinates system we recall that

$$\sigma_{ij}(x_0) = \sigma^{ij}(x_0) = \delta_{ij}, \tag{56}$$

$$\partial_k \sigma_{ij}(x_0) = \partial_k \sigma^{ij}(x_0) = 0, \tag{57}$$

$$\Gamma_{ij}^k(x_0) = 0. \tag{58}$$

Also $\nabla u(x) = \sum_i u^i \frac{\partial}{\partial x_i}$, where

$$u^i = \sum_{j=1}^n \sigma^{ij} \partial_j u. \tag{59}$$

Thus,

$$\|\nabla u(x)\|^2 = \sum_{i,j=1}^n \sigma^{ij} \partial_i u \partial_j u. \tag{60}$$

Observe now that the function $\tilde{w}(x) = \ln w(x) = Au(x) + \ln \|\nabla u(x)\|$ also attains a maximum at x_0. Therefore, for each $0 \le k \le n$, we have the relations $\partial_k \tilde{w}(x_0) = 0$ and $\partial_{kk} \tilde{w}(x_0) \le 0$. Thus

$$\partial_k \tilde{w}(x) = A\partial_k u(x) + \frac{\partial_k \left(\|\nabla u\|^2 \right)(x)}{2\|\nabla u(x)\|^2},$$

$$\partial_{kk} \tilde{w}(x) = A\partial_{kk} u(x) + \frac{1}{2}\partial_k \left(\|\nabla u\|^{-2} \right)(x)\partial_k \left(\|\nabla u\|^2 \right)(x) + \frac{\partial_{kk} \left(\|\nabla u\|^2 \right)(x)}{2\|\nabla u(x)\|^2}.$$

Since

$$\partial_k \left(\|\nabla u\|^{-2} \right) = \partial_k \left(\|\nabla u\|^2 \right)^{-1} = -\left(\|\nabla u\|^2 \right)^{-2} \partial_k \left(\|\nabla u\|^2 \right),$$

then

$$\partial_{kk} \tilde{w}(x) = A\partial_{kk} u(x) - \frac{\left(\partial_k \left(\|\nabla u\|^2 \right)(x) \right)^2}{2\|\nabla u(x)\|^4} + \frac{\partial_{kk} \left(\|\nabla u\|^2 \right)(x)}{2\|\nabla u(x)\|^2}.$$

Hence,

$$A\partial_k u(x_0) + \frac{\partial_k \left(\|\nabla u\|^2 \right)(x_0)}{2\|\nabla u(x_0)\|^2} = 0, \tag{61}$$

and

$$A\partial_{kk} u(x_0) - \frac{\left(\partial_k \left(\|\nabla u\|^2 \right)(x_0) \right)^2}{2\|\nabla u(x_0)\|^4} + \frac{\partial_{kk} \left(\|\nabla u\|^2 \right)(x_0)}{2\|\nabla u(x_0)\|^2} \le 0. \tag{62}$$

From (60) it follows

$$\partial_k \left(\|\nabla u\|^2 \right) = \sum_{i,j=1}^{n} \left(\left(\partial_k \sigma^{ij} \right) \partial_i u \partial_j u + 2\sigma^{ij} \partial_{ki} u \partial_j u \right) \tag{63}$$

From (55), (56) and (57) we obtain

$$\partial_k \left(\|\nabla u\|^2 \right)(x_0) = 2 \sum_{i,j=1}^{n} \delta_{ij} \partial_{ki} u(\|\nabla u(x_0)\| \delta_{j1}),$$

so

$$\partial_k \left(\|\nabla u\|^2 \right)(x_0) = 2\|\nabla u(x_0)\| \partial_{1k} u(x_0). \tag{64}$$

Substituting (55) and (64) in (61) we derive

$$A\|\nabla u(x_0)\| \delta_{k1} + \frac{2\|\nabla u(x_0)\| \partial_{1k} u(x_0)}{2\|\nabla u(x_0)\|^2} = 0,$$

thus,

$$\partial_{1k}u(x_0) = -A\,\|\nabla u(x_0)\|^2\,\delta_{k1}. \tag{65}$$

Substituting also (65) in (64) we obtain

$$\partial_k\left(\|\nabla u\|^2\right)(x_0) = -2A\,\|\nabla u(x_0)\|^3\,\delta_{k1}. \tag{66}$$

On the other hand, taking into account the expression (63) it follows

$$\begin{aligned}\partial_{kk}\left(\|\nabla u\|^2\right)(x) = \sum_{i,j=1}^{n} &\left(\left(\partial_{kk}\sigma^{ij}\right)\partial_i u\partial_j u + \left(\partial_k\sigma^{ij}\right)\partial_k\left(\partial_i u\partial_j u\right)\right.\\ &\left.+2\left(\left(\partial_k\sigma^{ij}\right)\partial_{ki}u\partial_j u + \sigma^{ij}\partial_{kki}u\partial_j u + \sigma^{ij}\partial_{ki}u\partial_{kj}u\right)\right).\end{aligned}$$

From (55), (56) and (57) we have

$$\begin{aligned}\partial_{kk}\left(\|\nabla u\|^2\right)(x_0) = \|\nabla u(x_0)\|^2\left(\partial_{kk}\sigma^{11}\right) + 2\,\|\nabla u(x_0)\|\,\partial_{kk1}u&\\ +2\sum_{i=1}^{n}(\partial_{ki}u(x_0))^2.&\end{aligned} \tag{67}$$

Differentiating two times with respect to x_k the equation $\sigma \circ \sigma^{-1} = Id$ and evaluating in x_0 we see that $\partial_{kk}\sigma^{-1}(x_0) = -\partial_{kk}\sigma(x_0)$. Besides,

$$\begin{aligned}\partial_{kk}\sigma_{11} =& \tfrac{\partial}{\partial x_k}\tfrac{\partial}{\partial x_k}\left\langle \tfrac{\partial}{\partial x_1}, \tfrac{\partial}{\partial x_1}\right\rangle = 2\tfrac{\partial}{\partial x_k}\left\langle \nabla_{\frac{\partial}{\partial x_k}}\tfrac{\partial}{\partial x_1}, \tfrac{\partial}{\partial x_1}\right\rangle\\ =&2\left(\left\langle \nabla_{\frac{\partial}{\partial x_k}}\nabla_{\frac{\partial}{\partial x_k}}\tfrac{\partial}{\partial x_1}, \tfrac{\partial}{\partial x_1}\right\rangle + \left\|\nabla_{\frac{\partial}{\partial x_k}}\tfrac{\partial}{\partial x_1}\right\|^2\right).\end{aligned}$$

Recalling (58) we then have

$$\partial_{kk}\sigma^{11}(x_0) = -\partial_{kk}\sigma_{11}(x_0) = -2\left\langle \nabla_{\frac{\partial}{\partial x_k}}\nabla_{\frac{\partial}{\partial x_k}}\tfrac{\partial}{\partial x_1}, \tfrac{\partial}{\partial x_1}\right\rangle. \tag{68}$$

Substituting (68) in (67) we can conclude that

$$\begin{aligned}\partial_{kk}\left(\|\nabla u\|^2\right)(x_0) = 2\Big(-\|\nabla u(x_0)\|^2\left\langle \nabla_{\frac{\partial}{\partial x_k}}\nabla_{\frac{\partial}{\partial x_k}}\tfrac{\partial}{\partial x_1}, \tfrac{\partial}{\partial x_1}\right\rangle&\\ + \|\nabla u(x_0)\|\,\partial_{kk1}u(x_0) + \sum_{i=1}^{n}\left(\partial_{ki}u(x_0)\right)^2\Big).&\end{aligned} \tag{69}$$

Using expressions (66) and (69) in (62) we verify that

$$\begin{aligned}A\partial_{kk}u(x_0) - 2A^2\,\|\nabla u(x_0)\|^2\,\delta_{k1} + \frac{\partial_{kk1}u(x_0)}{\|\nabla u(x_0)\|} - \left\langle \nabla_{\frac{\partial}{\partial x_k}}\nabla_{\frac{\partial}{\partial x_k}}\tfrac{\partial}{\partial x_1}, \tfrac{\partial}{\partial x_1}\right\rangle&\\ +\frac{\sum_{i=1}^{n}\left(\partial_{ki}u(x_0)\right)^2}{\|\nabla u(x_0)\|^2} \leq 0.&\end{aligned}$$

From (65) we have for $k=1$

$$-A^2\left\|\nabla u(x_0)\right\|^2-2A^2\left\|\nabla u(x_0)\right\|^2+\frac{\partial_{111}u(x_0)}{\|\nabla u(x_0)\|}-\left\langle \nabla_{\frac{\partial}{\partial x_1}}\nabla_{\frac{\partial}{\partial x_1}}\tfrac{\partial}{\partial x_1},\tfrac{\partial}{\partial x_1}\right\rangle$$
$$+\frac{\sum_{i=1}^{n}\left(-A\left\|\nabla u(x_0)\right\|^2\right)^2\delta_{i1}}{\left\|\nabla u(x_0)\right\|^2}\leq 0,$$

then,

$$\partial_{111}u(x_0)\leq 2A^2\left\|\nabla u(x_0)\right\|^3+\|\nabla u(x_0)\|\left\langle \nabla_{\frac{\partial}{\partial x_1}}\nabla_{\frac{\partial}{\partial x_1}}\tfrac{\partial}{\partial x_1},\tfrac{\partial}{\partial x_1}\right\rangle. \tag{70}$$

If $k>1$, then

$$A\partial_{kk}u(x_0)+\frac{\partial_{kk1}u(x_0)}{\|\nabla u(x_0)\|}-\left\langle \nabla_{\frac{\partial}{\partial x_k}}\nabla_{\frac{\partial}{\partial x_k}}\tfrac{\partial}{\partial x_1},\tfrac{\partial}{\partial x_1}\right\rangle\leq-\frac{\sum_{i=1}^{n}\left(\partial_{ki}u(x_0)\right)^2}{\left\|\nabla u(x_0)\right\|^2}\leq 0,$$

so,

$$\partial_{kk1}u(x_0)\leq -A\partial_{kk}u(x_0)\,\|\nabla u(x_0)\|+\|\nabla u(x_0)\|\left\langle \nabla_{\frac{\partial}{\partial x_k}}\nabla_{\frac{\partial}{\partial x_k}}\tfrac{\partial}{\partial x_1},\tfrac{\partial}{\partial x_1}\right\rangle. \tag{71}$$

In the sequel we evaluate at x_0 the mean equation (2). First we recall that

$$\nabla^2_{ij}u(x)=\nabla^2 u(x)\left(\tfrac{\partial}{\partial x_i},\tfrac{\partial}{\partial x_j}\right)=\partial_{ij}u-\sum_{k=1}^{n}\Gamma^k_{ij}\partial_k u, \tag{72}$$

$$\Delta u(x)=\operatorname{tr}\left(X\longrightarrow\nabla_X\nabla u\right)=\sum_{ij}\sigma^{ij}\nabla^2_{ij}u(x). \tag{73}$$

From (59), (72) and (73) we have

$$u^i(x_0)=\partial_i u(x_0)=\|\nabla u(x_0)\|\,\delta_{i1}, \tag{74}$$

$$\nabla^2_{ij}u(x_0)=\partial_{ij}u(x_0), \tag{75}$$

$$\Delta u(x_0)=\sum_{i=1}^{n}\partial_{ii}u(x_0). \tag{76}$$

Substituting these expressions in (2), using (65), we see that

$$\begin{aligned} nH_0W_0^3 =& W_0^2\Delta u(x_0) - \sum_{i,j=1}^{n} \left(\|\nabla u(x_0)\|\,\delta_{i1}\right)\left(\|\nabla u(x_0)\|\,\delta_{j1}\right)\partial_{ij}u \\ =& W_0^2\Delta u(x_0) - \|\nabla u(x_0)\|^2\,\partial_{11}u(x_0) \\ =& W_0^2\sum_{i>1}\partial_{ii}u(x_0) + \partial_{11}u(x_0) \\ =& W_0^2\sum_{i>1}\partial_{ii}u(x_0) - A\,\|\nabla u(x_0)\|^2, \end{aligned}$$

where $H_0 = H(x_0, u(x_0))$ and $W_0 = \sqrt{1 + \|\nabla u(x_0)\|^2}$. Therefore,

$$\sum_{i>1}\partial_{ii}u(x_0) = nH_0W_0 + \frac{A\,\|\nabla u(x_0)\|^2}{W_0^2}. \tag{77}$$

Finally let us differentiate (2) with respect to x_1. We have

$$\begin{aligned} \partial_1\left(W^2\right)\Delta u + W^2(\partial_1\Delta u) - 2\sum_{i,j=1}^{n} u^i\left(\partial_1 u^j\right)\nabla^2_{ij}u - \sum_{i,j=1}^{n} u^iu^j\partial_1(\nabla^2_{ij}u) \\ = n(\partial_1 H + \partial_z H\partial_1 u)W^3 + nH\partial_1\left(W^3\right). \end{aligned} \tag{78}$$

Let us calculate the derivative involved in this equation and evaluate at x_0. Since (66) holds we deduce

$$\partial_1\left(W^2\right)(x_0) = \partial_1\left(\|\nabla u\|^2\right)(x_0) = -2A\,\|\nabla u(x_0)\|^3, \tag{79}$$

$$\partial_1\left(W^3\right)(x_0) = \frac{3}{2}W_0\partial_1\left(W^2\right)(x_0) = -3AW_0\,\|\nabla u(x_0)\|^3. \tag{80}$$

Using (59), we have

$$\partial_1 u^i = \partial_1\sum_{j=1}^{n}\sigma^{ij}\partial_j u = \sum_{j=1}^{n}\left(\left(\partial_1\sigma^{ij}\right)\partial_j u + \sigma^{ij}\partial_{1j}u\right).$$

Using now (65) we obtain

$$\partial_1 u^i(x_0) = \partial_{1i}u(x_0) = -A\,\|\nabla u(x_0)\|^2\,\delta_{i1}. \tag{81}$$

On the other hand, from (72) we deduce

$$\partial_1(\nabla^2_{ij}u)(x) = \partial_{1ij}u(x) - \left\langle \nabla_{\frac{\partial}{\partial x_1}}\nabla u, \nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j}\right\rangle - \left\langle \nabla u(x), \nabla_{\frac{\partial}{\partial x_1}}\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j}\right\rangle.$$

Hence,

$$\partial_1(\nabla^2_{ij}u)(x_0) = \partial_{1ij}u(x_0) - \left\langle \nabla_{\frac{\partial}{\partial x_1}} \nabla_{\frac{\partial}{\partial x_i}} \tfrac{\partial}{\partial x_j}, \nabla u(x_0) \right\rangle. \tag{82}$$

Finally, it follows from (73),

$$\partial_1 \Delta u(x) = \sum_{ij} \left(\left(\partial_1 \sigma^{ij}\right) \nabla^2_{ij} u(x) + \sigma^{ij} \partial_1 \left(\nabla^2_{ij} u(x)\right)\right).$$

From (82) we also have

$$\partial_1(\Delta u)(x_0) = \sum_{i=1}^{n} \left(\partial_{1ii} u(x_0) - \left\langle \nabla_{\frac{\partial}{\partial x_1}} \nabla_{\frac{\partial}{\partial x_i}} \tfrac{\partial}{\partial x_i}, \nabla u(x_0) \right\rangle \right). \tag{83}$$

Substituting (74), (75), (79), (80), (81), (82) and (83) in (78) we obtain

$$
\begin{aligned}
& n\partial_1 H(x_0) W_0^3 + n\partial_z H(x_0) \left\|\nabla u(x_0)\right\| W_0^3 - 3nAH_0 W_0 \left\|\nabla u(x_0)\right\|^3 \\
= & -2A \left\|\nabla u(x_0)\right\|^3 \Delta u(x_0) + W_0^2 \sum_{i=1}^{n} \left(\partial_{1ii} u(x_0) - \left\langle \nabla_{\frac{\partial}{\partial x_1}} \nabla_{\frac{\partial}{\partial x_i}} \tfrac{\partial}{\partial x_i}, \nabla u(x_0) \right\rangle \right) \\
& + 2A \left\|\nabla u(x_0)\right\|^3 \partial_{11} u(x_0) \\
& - \left\|\nabla u(x_0)\right\|^2 \left(\partial_{111} u(x_0) - \left\langle \nabla_{\frac{\partial}{\partial x_1}} \nabla_{\frac{\partial}{\partial x_1}} \tfrac{\partial}{\partial x_1}, \nabla u(x_0) \right\rangle \right) \\
= & -2A \left\|\nabla u(x_0)\right\|^3 \left(\Delta u(x_0) - \partial_{11} u(x_0)\right) \\
& + W_0^2 \sum_{i>1} \left(\partial_{1ii} u(x_0) - \left\langle \nabla_{\frac{\partial}{\partial x_1}} \nabla_{\frac{\partial}{\partial x_i}} \tfrac{\partial}{\partial x_i}, \nabla u(x_0) \right\rangle \right) \\
& + W_0^2 \left(\partial_{111} u(x_0) - \left\|\nabla u(x_0)\right\| \left\langle \nabla_{\frac{\partial}{\partial x_1}} \nabla_{\frac{\partial}{\partial x_1}} \tfrac{\partial}{\partial x_1}, \tfrac{\partial}{\partial x_1} \right\rangle \right) \\
& - \left\|\nabla u(x_0)\right\|^2 \left(\partial_{111} u(x_0) - \left\langle \nabla_{\frac{\partial}{\partial x_1}} \nabla_{\frac{\partial}{\partial x_1}} \tfrac{\partial}{\partial x_1}, \nabla u(x_0) \right\rangle \right) \\
= & -2A \left\|\nabla u(x_0)\right\|^3 \sum_{i>1} \partial_{ii} u(x_0) \\
& + W_0^2 \sum_{i>1} \left(\partial_{1ii} u(x_0) - \left\langle \nabla_{\frac{\partial}{\partial x_1}} \nabla_{\frac{\partial}{\partial x_i}} \tfrac{\partial}{\partial x_i}, \nabla u(x_0) \right\rangle \right) \\
& + \partial_{111} u(x_0) - \left\|\nabla u(x_0)\right\| \left\langle \nabla_{\frac{\partial}{\partial x_1}} \nabla_{\frac{\partial}{\partial x_1}} \tfrac{\partial}{\partial x_1}, \tfrac{\partial}{\partial x_1} \right\rangle.
\end{aligned}
$$

Using (70), (71), (77) we obtain

$$
\begin{aligned}
& n\partial_1 H(x_0)W_0^3 + n\partial_z H(x_0)\left\|\nabla u(x_0)\right\| W_0^3 - 3nAH_0W_0\left\|\nabla u(x_0)\right\|^3 \\
\leq & -2A\left\|\nabla u(x_0)\right\|^3 \sum_{i>1}\partial_{ii}u(x_0) \\
& + W_0^2\sum_{i>1}\Bigg(-A\partial_{ii}u(x_0)\left\|\nabla u(x_0)\right\| + \left\|\nabla u(x_0)\right\|\left\langle \nabla_{\frac{\partial}{\partial x_i}}\nabla_{\frac{\partial}{\partial x_i}}\tfrac{\partial}{\partial x_1}, \tfrac{\partial}{\partial x_1}\right\rangle \\
& \qquad\qquad - \left\|\nabla u(x_0)\right\|\left\langle \nabla_{\frac{\partial}{\partial x_1}}\nabla_{\frac{\partial}{\partial x_i}}\tfrac{\partial}{\partial x_i}, \tfrac{\partial}{\partial x_1}\right\rangle\Bigg) \\
& + 2A^2\left\|\nabla u(x_0)\right\|^3 + \left\|\nabla u(x_0)\right\|\left\langle \nabla_{\frac{\partial}{\partial x_1}}\nabla_{\frac{\partial}{\partial x_1}}\tfrac{\partial}{\partial x_1}, \tfrac{\partial}{\partial x_1}\right\rangle \\
& \qquad\qquad - \left\|\nabla u(x_0)\right\|\left\langle \nabla_{\frac{\partial}{\partial x_1}}\nabla_{\frac{\partial}{\partial x_1}}\tfrac{\partial}{\partial x_1}, \tfrac{\partial}{\partial x_1}\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
= & -2A\left\|\nabla u(x_0)\right\|^3 \sum_{i>1}\partial_{ii}u(x_0) \\
& - A\left\|\nabla u(x_0)\right\| W_0^2\sum_{i>1}\partial_{ii}u(x_0) + 2A^2\left\|\nabla u(x_0)\right\|^3 \\
& + \left\|\nabla u(x_0)\right\| W_0^2\sum_{i>1}\left(\left\langle \nabla_{\frac{\partial}{\partial x_i}}\nabla_{\frac{\partial}{\partial x_1}}\tfrac{\partial}{\partial x_i}, \tfrac{\partial}{\partial x_1}\right\rangle - \left\langle \nabla_{\frac{\partial}{\partial x_1}}\nabla_{\frac{\partial}{\partial x_i}}\tfrac{\partial}{\partial x_i}, \tfrac{\partial}{\partial x_1}\right\rangle\right)
\end{aligned}
$$

$$
\begin{aligned}
= & \left(-2A\left\|\nabla u(x_0)\right\|^3 - A\left\|\nabla u(x_0)\right\| W_0^2\right)\sum_{i>1}\partial_{ii}u(x_0) \\
& + 2A^2\left\|\nabla u(x_0)\right\|^3 + \left\|\nabla u(x_0)\right\| W_0^2\sum_{i>1}\left\langle R\left(\tfrac{\partial}{\partial x_i}, \tfrac{\partial}{\partial x_1}\right)\tfrac{\partial}{\partial x_i}, \tfrac{\partial}{\partial x_1}\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
\leq & -A\left\|\nabla u(x_0)\right\|\left(1+3\left\|\nabla u(x_0)\right\|^2\right)\left(nH_0W_0 + \frac{A\left\|\nabla u(x_0)\right\|^2}{W_0^2}\right) \\
& + 2A^2\left\|\nabla u(x_0)\right\|^3 - \left\|\nabla u(x_0)\right\| W_0^2\,\mathrm{Ricc}_{x_0}\left(\tfrac{\partial}{\partial x_1}\right)
\end{aligned}
$$

$$
\begin{aligned}
= & -A\left\|\nabla u(x_0)\right\| nH_0W_0\left(1+3\left\|\nabla u(x_0)\right\|^2\right) - \frac{A^2\left\|\nabla u(x_0)\right\|^3}{W_0^2}\left(1+3\left\|\nabla u(x_0)\right\|^2\right) \\
& + 2A^2\left\|\nabla u(x_0)\right\|^3 - \left\|\nabla u(x_0)\right\| W_0^2\,\mathrm{Ricc}_{x_0}\left(\tfrac{\partial}{\partial x_1}\right).
\end{aligned}
$$

Since $\partial_z H \geq 0$ we have

$$
\begin{aligned}
& n\partial_1 H W_0^3 \\
\leq & AnH_0 W_0 \|\nabla u(x_0)\| \left(3\|\nabla u(x_0)\|^2 - 1 - 3\|\nabla u(x_0)\|^2\right) \\
& + \frac{A^2\|\nabla u(x_0)\|^3}{W_0^2}\left(2W_0^2 - 1 - 3\|\nabla u(x_0)\|^2\right) - \|\nabla u(x_0)\| W_0^2 \operatorname{Ricc}_{x_0}\left(\tfrac{\partial}{\partial x_1}\right) \\
= & -AnH_0 W_0 \|\nabla u(x_0)\| + \frac{A^2\|\nabla u(x_0)\|^3}{W_0^2}\left(1 - \|\nabla u(x_0)\|^2\right) \\
& - \|\nabla u(x_0)\| W_0^2 \operatorname{Ricc}_{x_0}\left(\tfrac{\partial}{\partial x_1}\right).
\end{aligned}
$$

Let

$$
\begin{aligned}
h_0 &= \sup_{\Omega\times\left[-\sup_\Omega|u|,\sup_\Omega|u|\right]} |H| \\
h_1 &= \sup_{\Omega\times\left[-\sup_\Omega|u|,\sup_\Omega|u|\right]} \left(\|\nabla_x H\| + \partial_z H\right).
\end{aligned}
$$

and $R \geq 0$ such that $-\operatorname{Ricc} \leq R$ in Ω. Then

$$
\frac{A^2\|\nabla u(x_0)\|^3}{W_0^2}\left(\|\nabla u(x_0)\|^2 - 1\right) \leq Anh_0 W_0\|\nabla u(x_0)\| + \|\nabla u(x_0)\| W_0^2 R + nh_1 W_0^3.
$$

Dividing by W_0^3 it follows

$$
\begin{aligned}
\frac{A^2\|\nabla u(x_0)\|^3}{W_0^5}\left(\|\nabla u(x_0)\|^2 - 1\right) &\leq Anh_0\frac{\|\nabla u(x_0)\|}{W_0^2} + nh_1 + \frac{\|\nabla u(x_0)\|}{W_0}R, \\
&\leq Anh_0 + nh_1 + R, \qquad (*) \\
&\leq An\left(h_0 + h_1 + R\right) \qquad (**)
\end{aligned}
$$

where for $(*)$ we used the fact that $W_0^2 > W_0 > \|\nabla u(x_0)\|$, and for $(**)$ that $A, n > 1$. Denoting by $H_1 = h_0 + h_1$ and dividing by A^2 we obtain

$$
\frac{\|\nabla u(x_0)\|^3}{W_0^5}\left(\|\nabla u(x_0)\|^2 - 1\right) < \frac{n}{A}\left(H_1 + R\right).
$$

We can suppose that $\|\nabla u(x_0)\| > 1$. Since

$$
W_0^3 = \left(1 + \|\nabla u(x_0)\|^2\right)^{3/2} < \left(2\|\nabla u(x_0)\|^2\right)^{3/2} < 4\|\nabla u(x_0)\|^3,
$$

we see that

$$\frac{\|\nabla u\|^3}{W_0^3} > \frac{1}{4}.$$

Then,

$$\frac{1}{4}\frac{\|\nabla u(x_0)\|^2 - 1}{W_0^2} < \frac{\|\nabla u(x_0)\|^3}{W_0^3}\frac{\|\nabla u(x_0)\|^2 - 1}{W_0^2} < \frac{n}{A}\left(H_1 + R\right),$$

that is,

$$\frac{\|\nabla u(x_0)\|^2 - 1}{\|\nabla u(x_0)\|^2 + 1} < \frac{4n}{A}\left(H_1 + R\right),$$

Choosing $A > 8n\left(H_1 + R\right)$ it follows

$$\frac{\|\nabla u(x_0)\|^2 - 1}{\|\nabla u(x_0)\|^2 + 1} < \frac{1}{2},$$

so,

$$\|\nabla u(x_0)\| < \sqrt{3}.$$

As a consequence,

$$w(x) \leq w(x_0) = \|\nabla u(x_0)\|\, e^{Au(x_0)} \leq \sqrt{3}\, e^{Au(x_0)},$$

thus

$$\sup_{\Omega}\|\nabla u(x)\| \leq \sqrt{3}\, e^{2A\sup\limits_{\Omega}|u|}. \tag{84}$$

Joining (54) and (84) we obtain

$$\sup_{\Omega}\|\nabla u(x)\| \leq \sqrt{3}\, e^{2A\sup\limits_{\Omega}|u|} + \sup_{\partial\Omega}\|\nabla u\|\, e^{2A\sup\limits_{\Omega}|u|},$$

Choosing

$$A = 1 + 8n\left(\|H\|_1 + R\right).$$

we obtain the desire estimate. ☐

Remark 15. A related global gradient estimate was obtained independently in [7, Prop. 2.2 p. 5].

3 Proof of the theorems

Proof of the main theorem (theorem 4). Let $\Omega \subset M$ with $\partial\Omega$ of class $\mathscr{C}^{2,\alpha}$ for some $\alpha \in (0,1)$ and $\varphi \in \mathscr{C}^{2,\alpha}(\overline{\Omega})$. Elliptic theory assures that the solvability of problem (P) strongly depends on $\mathscr{C}^1$ a priori estimates for the family of related problems

$$\begin{cases} \operatorname{div}\left(\dfrac{\nabla u}{W}\right) = \tau n H(x,u) \ \text{ in } \ \Omega, \\ \qquad\qquad u = \tau\varphi \ \text{ in } \ \partial\Omega, \end{cases} \tag{P_τ}$$

not depending on τ or u.

Let u be a solution of problem (P_τ) for arbitrary $\tau \in [0,1]$. Let $w = \phi \circ d + \sup\limits_{\partial\Omega} |\varphi|$ as in the proof of theorem 8. Then

$$u \leq \sup_{\partial\Omega} |\tau\varphi| \leq \sup_{\partial\Omega} |\varphi| = w \ \text{ on } \ \partial\Omega.$$

As before, let Ω_0 be the biggest open subset of Ω having the unique nearest point property. Let $x \in \Omega_0$ and $y = y(x) \in \partial\Omega$ the nearest point to x. Once (22) holds and $\tau \in [0,1]$ we have that

$$\mp n\tau H(x, \pm w) \leq n\tau \left|H(x, \varphi(y))\right| \leq n \left|H(x, \varphi(y))\right|.$$

From (21) we have

$$\pm\mathfrak{Q}_\tau(\pm w) = \mathcal{M}w \mp n\tau H(x, \pm w)(1+\phi'^2)^{3/2} \leq 0.$$

Proceeding as in the proof of theorem 8, we get that w and $-w$ are supersolution and subsolution in Ω_0, respectively, for the problem (P_τ). This provides a priori height estimate for any solution of the problems (P_τ) independently of τ.

On account of assumptions (8) and (9), we can apply theorem 10 to obtain a priori boundary gradient estimate for the solutions of the problems (P_τ).

Elliptic regularity guarantees that any solution u of the related problems (P_τ) belongs to $\mathscr{C}^3(\Omega)$. We conclude therefore, by applying theorem 14, the desired a priori global gradient estimate independently of τ and u.

Classical elliptic theory (see [12, Th. 11.4 p. 281]), ensures the existence of a solution $u \in \mathscr{C}^{2,\alpha}(\overline{\Omega})$ for our problem (P). Uniqueness follows from the maximum principle. $\square$

Proof of theorem 5. We first recall that in $\mathbb{H}^n \times \mathbb{R}$ there exists an entire vertical graph of constant mean curvature $\frac{n-1}{n}$. Explicit formulas were given by Bérard-Sa Earp [5, Th. 2.1 p. 22]. The a priori height estimate for the solutions of the related problems (P_τ) follows directly from the convex hull lemma [5, Prop. 3.1 p. 41].

Now, the a priori boundary gradient estimate and the a priori global gradient estimate follows from theorem 13 and theorem 14, respectively. The rest of the proof is the same as before. $\square$

Proof of theorem 6. Under the hypothesis on M and Ω, Galvez-Lozano [11, Th. 6 p. 12] proved the existence of a vertical graph over Ω with constant mean curvature $\frac{n-1}{n}$ and zero boundary data. As a matter of fact, such a graph constitutes a barrier for the solutions of the related problems (P_τ).

On the other hand, the strong Serrin condition trivially holds since for $y \in \partial\Omega$ we have

$$(n-1)\mathcal{H}_{\partial\Omega}(y) > (n-1)c > n-1 \geq n \sup_{\Omega\times\mathbb{R}} |H(x,z)| \, .$$

Also,

$$\mathrm{Ricc}_x \geq -(n-1)c^2 > -(n-1)\mathcal{H}_{\partial\Omega}(y).$$

Thus, the boundary gradient estimate follows from our theorem 12. The rest of the proof is the same as before. □

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