Halfspace Theorems

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Halfspace theorem of Hoffman and Meeks

D. Hoffman and W. Meeks proved a beautiful theorem on minimal surfaces, the so-called Halfspace Theorem:

Theorem

There is no non planar, complete, minimal surface properly immersed in a halfspace of $\mathbb{R}^3$.

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Sketch of the proof:

Special Weingarten surfaces (joint work with Eric Toubiana)

We shall consider surfaces M immersed in $\mathbb{R}^3$, which are oriented by a unit normal vector field N and whose mean curvature $H = H(N)$ and Gaussian curvature K, satisfies a Weingarten relation of the form

$$H = f(H^2 - K)$$

where f is a C^1 function defined in the interval $[0, +\infty]$. We assume that f is elliptic i.e

$$4t(f'(t))^2 < 1$$

We call such a surface M special Weingarten surface, or simply special Weingarten surface, if H, K satisfy the above condition for f elliptic.

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We remark that if there is a plane in the class of M, i.e. $f(0) = 0$, then the theory is "minimal type". Otherwise, if there is a sphere in the class of M, i.e. $f(0) \neq 0$, then the theory is constant (non zero) mean curvature type.

In both cases, ellipticity ensures that M satisfies a maximum principle.

Basic Statements for a Special Minimal Type Surface ($f(0) = 0$):

- The Gaussian curvature K of M is non positive;
- The zeros of K are isolated;
- M is contained inside the convex hull of the boundary;
- If M is complete with zero Gaussian curvature, i.e. $K \equiv 0$, then M is a plane.

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Basic Statements for a Special Constant (non zero) Mean Curvature Type Surface ($f(0) \neq 0$):

- Alexandrov's theorem hold: If M is closed (compact without boundary) and embedded then M is a round sphere;
- Hopf's theorem hold: If M is closed immersed with genus zero then M is a round sphere (here, ellipticity is not required);
- If M is complete with zero Gaussian curvature then M is a right cylinder;

Existence of Special Catenoids

Theorem 1:

Let f be an elliptic function with $f(0) = 0$ and let $\tau > 0$ be a real number that verifies

$$\frac{1}{\tau} < \lim_{t \to +\infty} (t - f(t^2))$$

Then there is a unique complete rotational symmetric special surface M_τ such that its generating curve is a graph of a C^2 convex, strictly positive function $y = y(x)$, which attains a minimum at 0 and is symmetric respect to the y axis, i.e., y verifies:

$$y > 0, y(0) = \tau, y'(0) = 0, y'' > 0 \text{ and } y(x) = y(-x).$$

The geometry of special catenoids

Theorem 2:

Let f be a C^2 elliptic function near 0 with $f(0) = 0$. Let M be a rotational symmetric surface given by Theorem 1.

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We remark that if f is non negative with $f(0) = 0$ and $\lim_{t\to+\infty}(t - f(t^2)) = +\infty$ then there is a 1-parameter family $M_\tau, \tau > 0$, of special catenoids converging to a vertical plane as $\tau \to 0$. This yields the following generalization of the well-known "Halfspace Theorem" for minimal type special surfaces.

Halfspace Theorem for special surfaces

Theorem 3:

Let f be a elliptic non negative ($f \geqslant 0$) function. Assume f is C^2 in a neighborhood of the origin. Suppose f verifies

$$f(0) = 0, \lim_{t \to +\infty} (t - f(t^2)) = +\infty$$

If M is a complete connected properly immersed special surface (respect to f) contained inside a halfspace of $\mathbb{R}^3$, then M is a plane.

Rotational surfaces with mean curvature 1/2 in $\mathbb{H}^2 \times \mathbb{R}$

For any $\alpha \in \mathbb{R}_+$, there exists a rotational surface $\mathcal{H}_\alpha$ of constant mean curvature $H = \frac{1}{2}$.
For $\alpha \neq 1$, the surface $\mathcal{H}_\alpha$ has two vertical ends (where a vertical end is a topological annulus, with no asymptotic point at finite height) that are vertical graphs over the exterior of a disk D_α (see Figure 1).

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Up to vertical translation, one can assume that $\mathcal{H}_\alpha$ is symmetric with respect to the horizontal plane $t = 0$.
For $\alpha = 1$, the surface $\mathcal{H}_1$ has only one end, it is a graph over $\mathbb{H}^2$ and it is denoted by S.
When $\alpha > 1$ the surface $\mathcal{H}_\alpha$ is not embedded (first picture in Figure 1). The self intersection set is a horizontal circle on the plane $t = 0$. Denote by ρ_α the radius of the intersection circle. For $\alpha < 1$ the surface $\mathcal{H}_\alpha$ is embedded (second picture in Figure 1).

For any $\alpha \in \mathbb{R}_+$, let $u_\alpha : \mathbb{H}^2 \times \{0\} \setminus D_\alpha \longrightarrow \mathbb{R}$ be the function such that the end of the surface $\mathcal{H}_\alpha$ is the vertical graph of u_α.

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The asymptotic behavior of u_α has the following form: $u_\alpha(\rho) \simeq \frac{1}{\sqrt{\alpha}} e^{\frac{\rho}{2}}$, $\rho \longrightarrow \infty$, where ρ is the hyperbolic distance from the origin. The positive number $\frac{1}{\sqrt{\alpha}} \in \mathbb{R}_+$ is called the growth of the end.

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The function u_α is vertical along the boundary of D_α. Furthermore the radius r_α is always greater or equal to zero, it is zero if and only if $\alpha = 1$ and tends to infinity as $\alpha \longrightarrow 0$ or $\alpha \longrightarrow \infty$.

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Notice that, any end of an immersed rotational surface $(\alpha > 1)$ has growth smaller than the growth of S, while any end of an embedded rotational surface $(\alpha < 1)$ has growth greater than the growth of S. This means that the intersection between any $\mathcal{H}_\alpha$ and S is a compact set.

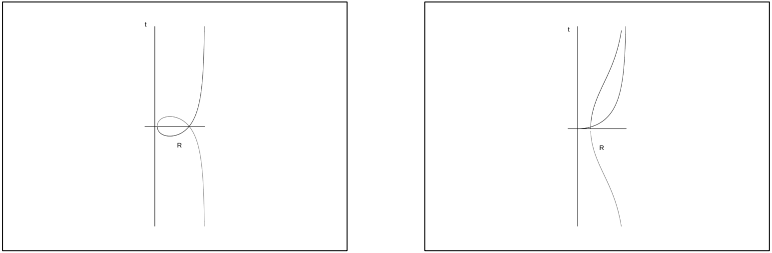

Figure: $H = \frac{1}{2}$: the profile curve in the immersed and embedded case ($R = \tanh \rho/2$).

R. Sa Earp and E. Toubiana find explicit integral formulas for rotational surfaces of constant mean curvature $H \in (0, \frac{1}{2}]$. A careful description of the geometry of these surfaces is a work by B. Nelli, R. Sa Earp, W. Santos and E. Toubiana. Generalizations in $\mathbb{H}^n \times \mathbb{R}$ can be found in the work by P. Bérard and R. Sa Earp.

Vertical halfspace Theorem (joint work with Barbara Nelli)

Theorem:

Let S be a simply connected rotational surface with constant mean curvature $H = \frac{1}{2}$. Let Σ be a complete surface with constant mean curvature $H = \frac{1}{2}$, different from a rotational simply connected one. Then, Σ cannot be properly immersed in the mean convex side of S.

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We remark that L. Hauswirth, H. Rosenberg and J. Spruck prove a halfspace type theorem for $H = \frac{1}{2}$ surfaces on one side of a horocylinder. Their result is different in nature from our result the "halfspace" is one side of a horocylinder, while for us, the "halfspace" is the mean convex side of a rotational simply connected surface.