Minimal Graphs in Nil_3 : existence and non-existence results

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Abstract

We study the minimal surface equation in the Heisenberg space, Nil_3. A geometric proof of non existence of minimal graphs over non convex, bounded and unbounded domains is achieved for some prescribed boundary data (our proof holds in the Euclidean space as well). We solve the Dirichlet problem for the minimal surface equation over bounded and unbounded convex domains, taking bounded, piecewise continuous boundary value. We are able to construct a Scherk type minimal surface and we use it as a barrier to construct non trivial minimal graphs over a wedge of angle $\theta \in [\frac{\pi}{2}, \pi[$, taking non negative continuous boundary data, having at least quadratic growth. In the case of an half-plane, we are also able to give solutions (with either linear or quadratic growth), provided some geometric hypothesis on the boundary data are satisfied. Finally, some open problems arising from our work, are posed.

Keywords: Maximum principle, minimal extension, Dirichlet problem, quadratic height growth.
MSC 2010: 53A10, 53C42, 35J25.

1 Introduction

In this paper we study the minimal surface equation in the Heisenberg space, Nil_3. We first consider non convex (non convex, smooth) domains $\Omega \subset \mathbb{R}^2$ and we provide a construction of a continuous (smooth) boundary data that does not admit a minimal extension over Ω (Theorems 3.1, 3.2). We point out that our proof is geometric in nature and holds for the minimal surface equation in the Euclidean three space as well. Notice that, for bounded non convex domains, the non solvability of the Euclidean minimal equation in two variables was established by Finn [12] for continuous boundary data, and by Jenkins and Serrin [15] for C^2 boundary data, with arbitrarily small absolute value.

Secondly, we prove that, for any convex domain Ω (bounded or unbounded) different from the half-plane, given a piecewise continuous or smooth boundary data φ over $\partial\Omega$, there exists a minimal extension u of φ over Ω. Moreover, we prove that the boundary of the graph of u is the union of the graph of φ with Euclidean vertical segments at the discontinuity points of φ (Theorem 4.3).
We would like to mention that, in [1, Theorem 1], Alias, Dajczer and Rosenberg prove an existence result for minimal graphs on bounded domains in Nil_3, whose boundary is C^3 and strictly convex, with continuous boundary data.
We also solve the Dirichlet problem in a half-plane with piecewise continuous or smooth bounded boundary data (Theorem 4.3). All these graphs over the half-plane have linear growth (see Definition 4.2).

Thirdly, we provide a geometric construction of a Scherk type surface over a triangle and we use it to prove that any non negative prescribed continuous boundary data φ on the boundary of a wedge of angle $\theta \in [\pi/2, \pi[$, has a minimal extension u in the wedge with at least quadratic growth. The existence of the Scherk type surface and of the graph over a wedge, are consequences of more general existence results (Theorems 4.2, 4.4). When the domain is a half-plane, we are able to construct bounded

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minimal graphs and minimal graphs with quadratic growth, provided some geometric hypothesis on the boundary data are satisfied (Corollary 4.4).
Our result in Nil_3 is in contrast with the classical results for the minimal equation in Euclidean space. In $\mathbb{R}^3$, if the boundary data on a wedge of angle less that π is zero (respectively bounded) then the minimal solution is zero (resp. bounded) [22].

Finally, here are some questions that arise from a glimpse of our developments together with the known results in the Euclidean space:

- The following maximum principle holds for the minimal surface equation in $\mathbb{R}^3$. Let Ω be a strip: if the boundary data φ on $\partial\Omega$ is bounded above by a constant A, then any minimal extension in Ω is also bounded above by the same constant A [22, Theorem 2.2, pg. 168].

  It is very interesting to check if this *maximum principle* holds in the Heisenberg space. The solutions given by Theorem 4.3(A) satisfy such property.

- Let Ω be a strip. In $\mathbb{R}^3$, the solution of the Dirichlet problem for the minimal surface equation in Ω is not unique in general. In fact, P. Collin [6], using the Jenkins-Serrin construction [14], showed that there exists a smooth unbounded boundary data φ such that φ admits an infinity of minimal extensions to Ω. Any such extension has at most linear growth.

  The non uniqueness (or the uniqueness) of the minimal extension on a strip in the Heisenberg space is an open problem.

  On the other hand, in [7, Theorem 3.3], Collin and Krust proved the following uniqueness result in $\mathbb{R}^3$: if Ω is a strip and if φ is a continuous bounded boundary data over $\partial\Omega$, then there exists a unique minimal extension u in $\mathbb{R}^3$ to Ω.

  It is very interesting to investigate if an analogous uniqueness result holds in the Heisenberg space.

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2 $Nil_3(\tau)$ and the minimal surface equation

2.1 The Setting

The three dimensional Heisenberg group $Nil_3(\tau)$ can be viewed as $\mathbb{R}^3$ with the following metric:

$$ds^2_\tau = dx_1^2 + dx_2^2 + (\tau(x_2dx_1 - x_1dx_2) + dx_3)^2\,,$$

where τ is a constant different from zero. If $\tau = 0$, we recover the Euclidean space. In fact, most of our results hold for $\tau = 0$. We will point out when our results are new in the Euclidean case, as well.
The isometries of $Nil_3(\tau)$ in this model are generated by the following maps (see [11] for further details).

$$\begin{aligned}
\varphi_1(x_1,x_2,x_3) &= (x_1 + c, x_2, x_3 + \tau c x_2)\\
\varphi_2(x_1,x_2,x_3) &= (x_1, x_2 + c, x_3 - \tau c x_1)\\
\varphi_3(x_1,x_2,x_3) &= (x_1, x_2, x_3 + c)\\
\varphi_4(x_1,x_2,x_3) &= (x_1\cos\theta - x_2\sin\theta, x_1\sin\theta + x_2\cos\theta, x_3)\\
\varphi_5(x_1,x_2,x_3) &= (x_1, -x_2, -x_3).
\end{aligned}$$

Notice that φ_3, φ_4 and φ_5 are also Euclidean isometries.

We can express the isometries of $Nil_3(\tau)$ in a complex form, that will be useful in the following. Let $z := x_1 + ix_2$ on $\mathbb{R}^2$, then any isometry of $Nil_3(\tau)$ is of one of the following forms

$$\Psi_1(z, x_3) := \big(\psi_1(z),\ x_3 + \tau \mathrm{Im}(\bar{z}_0 e^{i\theta} z)\big), \quad \Psi_2(z, x_3) := \big(\psi_2(z),\ -x_3 + \tau \mathrm{Im}(\bar{z}_0 e^{i\theta} \bar{z})\big) \tag{1}$$

where $\psi_1(z) = e^{i\theta} z + z_0$ and $\psi_2(z) = e^{i\theta} \bar{z} + z_0$, for some $\theta \in \mathbb{R}$ and some $z_0 \in \mathbb{C}$. Notice that both ψ_1 and ψ_2 are isometries of $\mathbb{R}^2$.

In this work we always assume that the open subsets $\Omega \subset \mathbb{R}^2$ considered have properly embedded boundary. More precisely for any $p \in \partial\Omega$, there exists an open ball $B \subset \mathbb{R}^2$ centered at p such that $B \cap \partial\Omega$ is the graph of a C^k function defined on some open interval, $k \geqslant 0$.

Let $u : \Omega \to \mathbb{R}$ be a C^2 function. The graph of the function $x_3 = u(x_1, x_2)$ is minimal in $Nil_3(\tau)$ if and only if u satisfies the (*vertical*) *minimal surface equation*

$$\mathcal{D}_\tau(u) := \big(1 + (u_2 - \tau x_1)^2\big)\, u_{11} - 2(u_1 + \tau x_2)(u_2 - \tau x_1) u_{12} + \big(1 + (u_1 + \tau x_2)^2\big)\, u_{22} = 0 \tag{2}$$

The previous equation can be written in divergence form, that is:

$$\mathrm{div}_{\mathbb{R}^2}\left(\frac{\tau x_2 + u_1}{W_u}, \frac{-\tau x_1 + u_2}{W_u}\right) = 0$$

where

$$W_u = \sqrt{1 + (\tau x_2 + u_1)^2 + (-\tau x_1 + u_2)^2}. \tag{3}$$

Notice that, as u is C^2 on Ω then, by Morrey's regularity result [17], u is analytic on Ω. Moreover the standard interior and boundary maximum principle hold for solutions of equation (2). Let us recall the geometric formulations of the maximum principle, that will be largely applied in this article. One can find a proof of the maximum principle in [13].

Maximum Principle.

1. *Let M_1 and M_2 be two connected minimal surfaces immersed in $Nil_3(\tau)$ and assume M_2 is complete. Let p be an interior point of both of them. Assume that M_1 lies on one side of M_2 in a neighborhood of p, then, M_1 coincides with M_2 in a neighborhood of p and, by analytic continuation, $M_1 \subset M_2$.*

2. *Let M_1 and M_2 be two compact, connected minimal surfaces in $Nil_3(\tau)$, both with boundary. Assume that $M_1 \cap \partial M_2 = \emptyset$ and that $M_2 \cap \partial M_1 = \emptyset$.*

   Let p be an interior point of both of them. Then it cannot occur that M_1 lies on one side of M_2 in a neighborhood of p.

Definition 2.1. *Let Ω be an open subset of $\mathbb{R}^2$ and let φ be a function continuous on $\partial\Omega$ except, possibly, at a discrete and closed set $Z \subset \partial\Omega$ of points where φ has left and right limit. We say that u is a* minimal extension *of φ over $\overline{\Omega}$ if*

1. *$u : \overline{\Omega} \setminus Z \longrightarrow \mathbb{R}$, is continuous, smooth in Ω, and satisfies the minimal surface equation (2).*

2. *$u_{|\,(\partial\Omega \setminus Z)} = \varphi$.*

Remark 2.1. (A) *Let us describe the effect of the isometries of $Nil_3(\tau)$ on a curve Γ in the x_1-x_2 plane. Let φ be any isometry of $Nil_3(\tau)$. The curve $\varphi(\Gamma)$ is not contained in the x_1-x_2 plane in general. The projection $\pi(\varphi(\Gamma))$ of such curve on the x_1-x_2 plane is obtained from the curve Γ by an isometry of the Euclidean x_1-x_2 plane, because the trace of any isometry of $Nil_3(\tau)$ on the x_1-x_2 plane is an isometry of $\mathbb{R}^2$.*

Then, for example, the notion of convexity of a curve Γ in the x_1-x_2 plane is somewhat intrinsic, for the following reason. Assume that Γ is convex (in Euclidean sense), then $\pi(\varphi(\Gamma))$ is convex, for any isometry φ of $Nil_3(\tau)$.

(B) *Let Ω be a domain in $\mathbb{R}^2$ and let $v : \Omega \to \mathbb{R}$ be any function defined in Ω with boundary value $v_{|\partial\Omega} = \varphi$. Apply the isometry Ψ_1 defined in (1) to the graph of u :*

$$\begin{aligned}\Psi_1\left(\{(z, v(z)),\ z \in \Omega\}\right) &= \left\{\big(\psi_1(z),\ v(z) + \tau Im(\overline{z}_0 e^{i\theta} z)\big),\ z \in \Omega\right\} \\ &= \left\{\big(\widetilde{z},\ \widetilde{v}(\widetilde{z})\big),\ \widetilde{z} \in \psi_1(\Omega)\right\},\end{aligned}$$

where $\widetilde{z} := \psi_1(z)$ and $\widetilde{v}(\widetilde{z}) := (v \circ \psi_1^{-1})(\widetilde{z}) + \tau Im(\overline{z}_0 e^{i\theta} \psi_1^{-1}(\widetilde{z}))$ for any $\widetilde{z} \in \psi_1(\Omega)$. Consequently the image of the graph of v by the isometry Ψ_1 is again a graph with boundary value $\widetilde{v}$.

2.2 Examples of Minimal Graphs

1. The graph of a linear entire function $u(x_1, x_2) = ax_1 + bx_2 + c$, $a, b, c \in \mathbb{R}$, is a minimal surface, that is usually called *plane*. Notice that a vertical Euclidean plane is also a minimal surface and it is flat.

2. Let $\phi(t, s) = (r(t)\cos s, r(t)\sin s, h(t))$ a parametrization of a rotationally invariant surface ($\tau = \frac{1}{2}$). Looking for minimal solutions one gets either planes ($h(t) = const$) or a 1-parameter family of surfaces, *vertical catenoids*, depending on a parameter $r_0 > 0$, given by:

$$h(r) = \pm \int_{r_0}^{r} \frac{r_0\sqrt{s^2 + 4}}{2\sqrt{s^2 - r_0^2}} ds, \qquad r \geqslant r_0.$$

   This means that the half of the catenoid is a graph over the exterior domain $r \geqslant r_0$ with zero boundary value [16]. One finds a detailed study of the vertical catenoids in [4].

3. C.B. Figueroa, F. Mercuri and R. H. Pedrosa [11], [10] classified all the minimal graphs invariant by a one parameter group of left invariant isometries. Such surfaces are the graphs of functions of the following form:

$$u_a(x_1, x_2) = \tau x_1 x_2 + a\left[2\tau x_2\sqrt{1 + 4\tau^2 x_2^2} + \sinh^{-1}(2\tau x_2)\right], \quad a \in \mathbb{R} \tag{4}$$

   For these examples, see also M. Bekkar and T. Sari [2, 3].

4. B. Daniel [8, Examples 8.4 and 8.5] constructed entire minimal graphs of the form $f(x_1, x_2) = x_1 g(x_2)$ for some real function g with linear growth.

5. M. Bekkar and T. Sari [2, 3] described some examples of minimal graphs ruled by Euclidean straight lines (that are not geodesic in Nil_3) and classified minimal surfaces in Nil_3 ruled by geodesics.

2.3 Horizontal Catenoids in $Nil_3(\tau)$

Let us describe *Horizontal Catenoids* in $Nil_3(\tau)$. In [9], B. Daniel and L. Hauswirth have constructed a family of horizontal catenoids $\mathcal{C}_\alpha$, $\alpha \in]0, +\infty[$, in $Nil_3(\frac{1}{2})$.
The catenoids $\mathcal{C}_\alpha$ have the following description [9, Theorem 5.6].

- The intersection of $\mathcal{C}_\alpha$ with any vertical plane $x_2 = c$, $c \in \mathbb{R}$, is a nonempty, embedded, closed curve, convex with respect to the Euclidean metric.

- The surface $\mathcal{C}_\alpha$ is properly embedded.
- $\mathcal{C}_\alpha$ is conformally equivalent to $\mathbb{C} \setminus \{0\}$.
- The projection of $\mathcal{C}_\alpha$ in the x_1-x_2 plane is the following subset of $\mathbb{R}^2$

$$\pi(\mathcal{C}_\alpha) = \left\{(x_1, x_2) \in \mathbb{R}^2,\ |x_1| \leqslant \alpha \cosh\left(\frac{x_2}{\alpha}\right)\right\}$$

- The surface $\mathcal{C}_\alpha$ is invariant by rotation of angle π around the x_1, x_2 and x_3 axis.

Remark 2.2. *We extend the construction of the family $\mathcal{C}_\alpha$ in $Nil_3(\tau)$, for any value of $\tau > 0$. Consider a new copy of $\mathbb{R}^3$ with coordinates $y = (y_1, y_2, y_3)$ and, for any real number $\lambda > 0$, let $f_\lambda : (\mathbb{R}^3, y) \to (\mathbb{R}^3, x)$ be the map defined by $x = f_\lambda(y) = \lambda y$.*
Then, the pullback metric of $(\mathbb{R}^3, x, ds^2_{1/2})$ on $(\mathbb{R}^3, y)$ induced by f_λ is:

$$\begin{aligned} g_\lambda &= \lambda^2 \left[dy_1^2 + dy_2^2 + \left(\frac{\lambda}{2}(y_2 dy_1 - y_1 dy_2) + dy_3\right)^2\right] \\ &= \lambda^2 ds^2_{\lambda/2}. \end{aligned}$$

Let $X : \Sigma \to (\mathbb{R}^3, x, ds^2_{1/2})$ be a conformal and minimal immersion, where Σ is a Riemann surface. We deduce that $Y := f_{2\tau}^{-1} \circ X : \Sigma \to (\mathbb{R}^3, y, ds^2_\tau)$ is also a conformal and minimal immersion.
Consequently, for any $\tau > 0$, and for any $\alpha > 0$, the surface $\mathcal{C}^\tau_\alpha := \frac{1}{2\tau}\mathcal{C}_\alpha$ is a horizontal catenoid in $(\mathbb{R}^3, ds^2_\tau)$, that is in $Nil_3(\tau)$.
We deduce from the construction that, for any $\tau > 0$, the family of horizontal catenoids $\mathcal{C}^\tau_\alpha$ of $Nil_3(\tau)$ has a geometric description analogous to that of the case $\tau = 1/2$. In particular the family of the projections $\{\pi(\mathcal{C}^\tau_\alpha),\ \alpha > 0\}$ is the same for any $\tau > 0$ since, for any $\alpha, \beta > 0$, we can obtain $\pi(\mathcal{C}_\alpha)$ from $\pi(\mathcal{C}_\beta)$ by a suitable homothety.

3 Non existence Results in $Nil_3(\tau)$

As, in the following, we will deal with convex and non convex curves, let us recall some properties of them. Let Γ be a Jordan curve in the x_1-x_2 plane and let Ω be the open bounded subset of the x_1-x_2 plane such that $\partial\Omega = \Gamma$. As the projection of $Nil_3(\tau)$ on the first two coordinates is a Riemannian submersion on $\mathbb{R}^2$, it makes sense to assume that Γ is convex as an Euclidean curve. Recall that Γ is convex if and only if, for any point $p \in \Gamma$, there exists a straight line l_p passing through p such that $\Omega \subset \mathbb{R}^2 \setminus l_p$. Recall, moreover, that this definition is equivalent to say that $\overline{\Omega}$ is convex, that is: for any two points $p, q \in \overline{\Omega}$, the segment between p and q is contained in $\overline{\Omega}$. We observe that, if Γ is not convex, then either

(*) There exists a point $p \in \Gamma$, a straight line L passing through p and a neighborhood V of p in $\mathbb{R}^2$, such that $(L \cap V) \setminus p \subset \Omega$ (see Figure 1(a)),

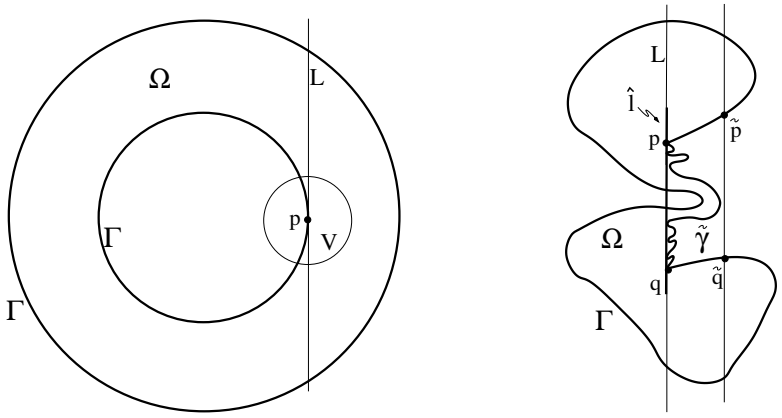
or

(**) There exists a closed arc $\widetilde{\gamma} \subset \Gamma$ with endpoints $\widetilde{p}, \widetilde{q}$, and there exists a segment $\hat{l}$ such that:

- the straight line L containing $\hat{l}$ is parallel (and distinct) to the straight line passing by $\widetilde{p}$ and $\widetilde{q}$, (therefore $\widetilde{p}, \widetilde{q} \notin L$),
- $\widetilde{\gamma} \cap L$ is infinite,
- $\widetilde{\gamma}$ remains in one closed side of L,

– Let $p \in \hat{l}$ (resp. $q \in \hat{l}$) be the first intersection point of $\widetilde{\gamma}$ with $\hat{l}$, coming from $\widetilde{p}$ (resp. $\widetilde{q}$). Denote by l the segment $[p, q]$.

  Then p and q are interior points of $\hat{l}$ and $\hat{l} \setminus l \subset \Omega$ (see Figure 1(b)).



(a) Non convex (∗) (b) Non convex (∗∗)

Figure 1: Non Convexity

Our first result is a non existence theorem on bounded domains.

Theorem 3.1. *Let Ω be a bounded domain such that $\Gamma = \partial\Omega$ is a non convex C^k curve, $k \geqslant 0$. Then there exists a C^k function φ on Γ that does not admit a minimal extension over $\overline{\Omega}$ in $Nil_3(\tau)$.*

Proof. We do the proof for $\tau = \frac{1}{2}$. It will be clear that the proof is analogous for any $\tau > 0$ (by Remark 2.2). By Remark 2.1 (B), it is enough to prove the result for $\psi(\Gamma)$, where ψ is any Euclidean isometry of $\mathbb{R}^2$ and for some function $\widetilde{\varphi}$ over $\psi(\Gamma)$. The desired function φ will be obtained by modify $\widetilde{\varphi}$ accordingly.

We assume that Γ satisfies (**) and we use the same notations as there. The proof in the case of Γ satisfying (*) follows easily.

Without loss of generality, we can assume that l is parallel to the x_2 axis and that the x_1 axis intersects l at its middle point p_0 of coordinates $(d, 0)$. Moreover we assume that each point of $l \cap \widetilde{\gamma}$ is a local minimum of $\widetilde{\gamma}$ for the coordinate x_1, since the case where each point of $l \cap \widetilde{\gamma}$ is a local maximum for the coordinate x_1 can be handed analogously (see Figure 2).

We will consider the continuous 1-parameter family of catenoids $\mathcal{C}_\alpha$, $\alpha > 0$, described in Section 2.3. Recall that the projection of $\mathcal{C}_\alpha$ in the x_1-x_2 plane is the following subset of $\mathbb{R}^2$:

$$\left\{(x_1, x_2) \in \mathbb{R}^2,\ |x_1| \leqslant \alpha \cosh\left(\frac{x_2}{\alpha}\right)\right\}, \alpha > 0.$$

Notice that the projection $\pi(\mathcal{C}_\alpha)$ is equal to the projection of an Euclidean catenoid with axis x_2 on the x_1-x_2 plane. Therefore, the projections are obtained one from the other by an homothety and, next to the waist, they become flatter as α increases.

Let $\varepsilon > 0$ and let $p_1 \in \widetilde{\gamma}$ (resp. $q_1 \in \widetilde{\gamma}$), be the first point on $\widetilde{\gamma}$ with x_1 coordinate equals to $d + \varepsilon$, coming from p (resp. q) and going to $\widetilde{p}$ (resp. $\widetilde{q}$). Such points exist if ε is small enough. We denote by γ_1 the sub-arc of $\widetilde{\gamma}$ with endpoints p_1 and q_1.

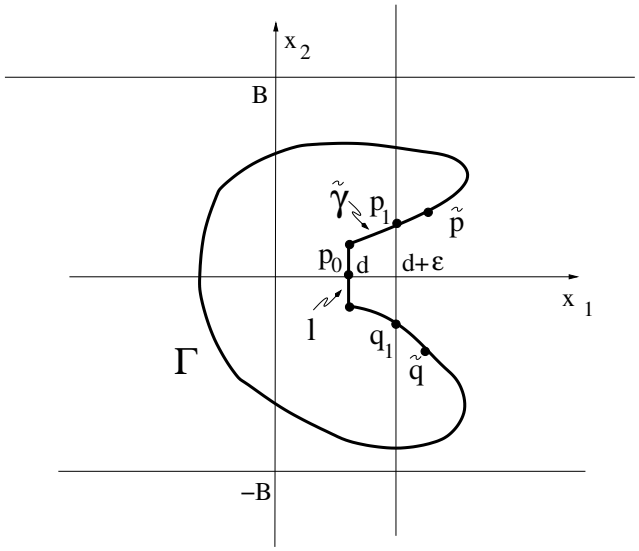


Figure 2: The curve Γ

As (**) holds, the points of $\gamma_1 \setminus l$ have x_1 coordinate strictly greater that d.
Let B a positive constant such that $\Gamma \subset \{(x_1, x_2) \in \mathbb{R}^2,\ |x_2| \leqslant \frac{B}{2}\}$.
Consider the unique catenoid $\mathcal{C}_\mu$ such that

$$\mu + \frac{\varepsilon}{4} = \mu \cosh \frac{B}{\mu}. \tag{5}$$

This means that the portion of $\pi(\mathcal{C}_\mu)$ defined by:

$$\pi(\mathcal{C}_\mu) \cap \{(x_1, x_2) \in \mathbb{R}^2,\ x_1 \geqslant 0,\ 0 \leqslant x_2 \leqslant B\},$$

is contained in the following vertical strip of $\mathbb{R}^2$ of width $\varepsilon/4$:

$$\{(x_1, x_2) \in \mathbb{R}^2,\ \mu \leqslant x_1 \leqslant \mu + \frac{\varepsilon}{4}\}.$$

Up to translate Γ along the x_1 axis, we can assume that the x_1 coordinate of p_0 is $d = \mu + \frac{\varepsilon}{4}$. This choice guarantees that $\pi(\mathcal{C}_\mu)$ does not intersect γ_1.
Then, deform $\pi(\mathcal{C}_\alpha)$ by an homothety from the origin, α going from μ to $\mu + \varepsilon/4$, in order to find a first contact point of $\pi(\mathcal{C}_\alpha)$ with the curve γ_1. Let $\mu' \in (\mu, \mu + \varepsilon/4)$ such that $\pi(C_{\mu'})$ is the desired homothetic image (see Figure 3).

By our choice of μ and d and because of the geometry of the curves $\pi(\mathcal{C}_\alpha)$, we can assume that the contact points between $\pi(\mathcal{C}_{\mu'})$ and γ_1 are interior points of γ_1. Denote by q_0 one of the contact points and notice that q_0 does not belong to the interior of l. Denote by γ_0 the sub-arc of γ_1 whose endpoints are q_0 and one of the endpoints of l, and such that $\gamma_0 \cap l$ is infinite.
Let $D > 0$ be such that, for any $\alpha \in [\mu, \mu']$ and any $p \in \mathcal{C}_\alpha \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3,\ |x_2| \leqslant B\}$, we have $|x_3(p)| \leqslant D$.
Now consider a non negative C^k function $\widetilde{\varphi}$ on Γ such that

$$\widetilde{\varphi} = \begin{cases} 0 & \text{on } \Gamma \setminus \widetilde{\gamma}, \\ 3D & \text{on } \gamma_1. \end{cases}$$

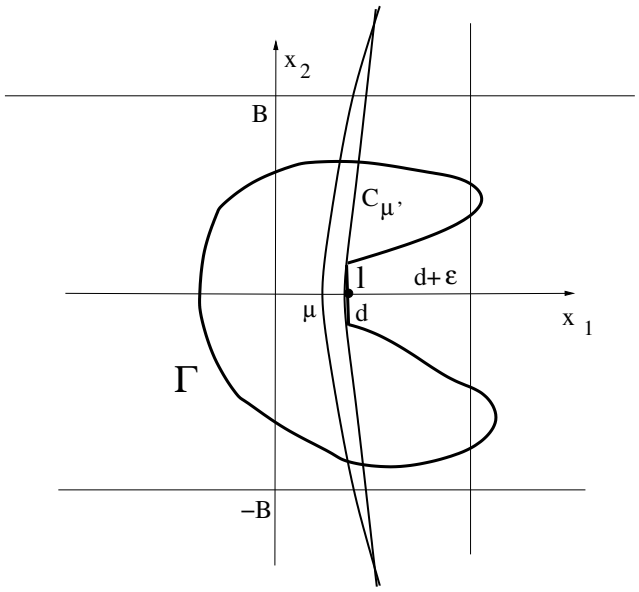


Figure 3: The deformation of $\mathcal{C}_\mu$

Assume that there exists a minimal extension u of $\widetilde{\varphi}$ over $\overline{\Omega}$. As the function u is continuous in $\overline{\Omega}$, for any $\eta_1 > 0$ there exists $\eta_2 > 0$ such that

$$|u(\widehat{p}) - 3D| \leqslant \eta_1, \quad \text{for any } \widehat{p} = (x_1, x_2) \in \overline{\Omega},\ x_1(q_0) - \eta_2 \leqslant x_1 \leqslant x_1(q_0),\ x_2 = x_2(q_0) \tag{6}$$

where $x_i(q_0)$ is the x_i coordinate of q_0, $i = 1, 2$.
This means that u has a small variation on a small segment, δ, parallel to the x_1 axis, δ starting at q_0 and having x_1 coordinates less than $x_1(q_0)$.
Consider a homothety $\pi(\mathcal{C}_{\mu''})$ of $\pi(\mathcal{C}_{\mu'})$ such that $\mu'' \in (\mu, \mu')$ is very close to μ'. As $\mathcal{C}_{\mu'}$ is the first catenoid of the family, such that its projection touches γ_1, then $\pi(\mathcal{C}_{\mu''}) \cap \gamma_1 = \emptyset$ and $\pi(\mathcal{C}_{\mu''}) \cap \delta \neq \emptyset$.
Now, translate vertically (along the x_3 axis) upward $\mathcal{C}_{\mu''}$ such that the translation of $\mathcal{C}_{\mu''}$ does not touch the graph of u. Then translate vertically downward till the first contact point between the translation of $\mathcal{C}_{\mu''}$ and the graph of u occurs. We observe that our construction yields that the first contact point between the translation of $\mathcal{C}_{\mu''}$ and the graph of u

- occurs before that $\mathcal{C}_{\mu''}$ touches $x_3 = 0$,
- does not occur at a boundary point of the graph of u.

Then, the first contact point between the translation of $\mathcal{C}_{\mu''}$ and the graph of u is an interior point of both surfaces. This is a contradiction by the maximum principle. □

In the next theorem, we extend the result of Theorem 3.1 to the case of unbounded domains.

Theorem 3.2. *Let Ω be an unbounded and non convex domain. Assume that the boundary $\Gamma = \partial\Omega$ is composed by a finite number of connected components, each one being a properly embedded (possibly compact) C^k curve, $k \geqslant 0$. Then there exists a C^k function φ on Γ that does not admit a minimal extension over $\overline{\Omega}$.*

Proof. As Ω is non convex, there exist points $p, q \in \Gamma$ such that the segment I joining p and q is not entirely contained in $\overline{\Omega}$. Let J be a connected component of the complement of $I \cap \overline{\Omega}$ in I. Since Ω is

connected and J is an open segment contained in the complement of $\overline{\Omega}$, the endpoints of J belong to the same connected component of Γ. From now on, the proof is analogous to the proof of Theorem 3.1, taking as $\widetilde{p}$, $\widetilde{q}$, the endpoints of the segment J. □

Remark 3.1. *We notice that our proofs of Theorem 3.1 and 3.2 hold in $\mathbb{R}^3$ ($\tau = 0$), as well. In the case of bounded domain, an almost analogous result is stated by H. Jenkins and J. Serrin in [15], page 185. The result in $\mathbb{R}^3$ for unbounded domain was unknown, to the best of our knowledge.*

4 Existence Results in $Nil_3(\tau)$

4.1 Compactness Theorem

In the proof of existence results either on unbounded domain or with infinite boundary data, we will use strongly the *Compactness Theorem for minimal graphs in $Nil_3(\tau)$*. Despite of the fact that it is a classical result for the minimal surface equation in several ambient spaces, we clarify which are the main ingredients of the proof in $Nil_3(\tau)$.
The Compactness Theorem yields that any $C^{2,\alpha}$ bounded sequence (u_n) of solutions of the minimal surface equation on a domain Ω in $\mathbb{R}^2$ admits a subsequence that converges uniformly in the C^2 topology, on any compact subset of Ω, to a solution of the minimal surface equation. The proof of this result relies on Ascoli-Arzelá Theorem, so one needs to prove that the sequence (u_n) is uniformly bounded in the $C^{2,\alpha}$ topology. By Schauder theory, $C^{2,\alpha}$ *a-priori* estimates follow from $C^{1,\beta}$ *a-priori* estimates. These last estimates follow, by Ladyzhenskaya-Ural'ceva theory, from C^1 *a-priori* estimates. By [23, Theorem 3.6], such estimates follow from uniform height estimates. As [23, Theorem 3.6] is stated in a more general situation, we state it on our case.

Theorem. (see [23, Theorem 3.6]) *Let $\Omega \subset \mathbb{R}^2$ be a relatively compact domain and $u : \Omega \longrightarrow \mathbb{R}$ be a C^2 function on Ω satisfying* (2). *Assume that C_1 is a positive constant such that $|u| < C_1$ on Ω. Then, for any positive constant C_2, there exists a constant $\alpha = \alpha(C_1, C_2, \Omega)$ such that for any $p \in \Omega$ with $d(p, \partial\Omega) \geqslant C_2$, we have*

$$W_u(p) < \alpha.$$

where W_u is defined in (3)

The uniform bound on W_u clearly gives C^1 a-priori estimates on any compact subset of Ω.

Remark 4.1. *Let (u_n) be a sequence of C^2 functions satisfying the minimal surface equation on a domain Ω. We deduce from above that, once we have uniform height estimates for (u_n), there exists a subsequence of (u_n) that converges C^2 on any compact subset of Ω, to a solution of the minimal surface equation.*

4.2 Construction of Barriers

In the proof of existence results, in order to prove that the solutions takes the given boundary value, it is important to get barriers at a convex point of the boundary, where the boundary data is continuous. Our construction of barriers is strongly inspired by the analogous construction in $\mathbb{H}^2 \times \mathbb{R}$, by the second and the third author in [25]. Let us first define a convex point of a domain Ω.

Definition 4.1. *Let $\Omega \subset \mathbb{R}^2$ be a domain. We say that a boundary point $p \in \partial\Omega$ is a* convex point of Ω *if there exist an open neighborhood V of p in $\mathbb{R}^2$ and a straight line $L \subset \mathbb{R}^2$ passing through p such that:*

- *$V \cap \Omega$ stays in one side of L,*

- $(V \cap L) \cap \Omega = \emptyset$.

Let Ω be a (not necessarily bounded) domain and let $\varphi : \partial\Omega \longrightarrow \mathbb{R}$ be a function. We recall what is a barrier at a point $p_0 \in \Gamma := \partial\Omega$ with respect to the function φ.
Let p_0 be a point of Γ. One says that p_0 admits an *upper* (*lower*) *barrier* with respect to φ if the following holds. For any positive M and any $k \in \mathbb{N}$, there exists an open set V_k containing p_0 in its boundary and a function ω_k^+ (resp. ω_k^-) in $C^2(V_k \cap \Omega) \cap C^0(\overline{V_k \cap \Omega})$ such that

1. $\omega_k^+(q)_{|\partial\Omega\cap V_k} \geqslant \varphi(q)$, $\omega_k^+(q)_{|\Omega\cap\partial V_k} \geqslant M$ (resp. $\omega_k^-(q)_{|\partial\Omega\cap V_k} \leqslant \varphi(q)$, $\omega_k^-(q)_{|\Omega\cap\partial V_k} \leqslant -M$).
2. $\mathcal{D}(\omega_k^+) \leqslant 0$ (resp. $\mathcal{D}(\omega_k^-) \geqslant 0$) where $\mathcal{D}$ is defined in (2) .
3. $\omega_k^+(p_0) = \varphi(p_0) + \frac{1}{k}$ (resp. $\omega_k^-(p_0) = \varphi(p_0) - \frac{1}{k}$).

Now, let $p_0 \in \Gamma$ be a convex point of Ω and assume that φ is continuous at p_0. Let M be any positive real number. We show how to construct an upper barrier at the point p_0 (see Figure 4).

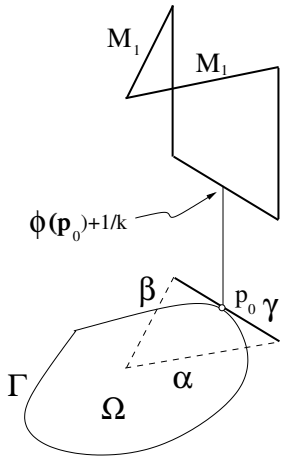


Figure 4: The upper barrier

Consider a triangle T in $\mathbb{R}^2$ with sides α, β, γ. Let A, B, C be the vertices of T labeled such that A, B, C are opposite to α, β, γ respectively. Let M_1 be a positive constant to be chosen later and let $\widetilde{A}$, ($\widetilde{B}$, $\widetilde{C}$) be the endpoints of the vertical segment above A (B, C respectively) of Euclidean lenght M_1. Let $\widetilde{\alpha}$ ($\widetilde{\beta}$ respectively) be the segment projecting one to one on α (β respectively) with endpoints $\widetilde{B}$, $\widetilde{C}$ ($\widetilde{A}$, $\widetilde{C}$ respectively) and with third coordinate equal to M_1. Denote by L_B (L_A) the vertical segment between B and $\widetilde{B}$ (A and $\widetilde{A}$ respectively). We solve the Plateau problem with boundary γ, L_B, L_A, $\widetilde{\alpha}$, $\widetilde{\beta}$ (see [18]). The solution of the Plateau problem is a graph in the interior of T of a function v having value zero on the interior of γ and value M_1 on $\alpha \cup \beta$ (see the proof of [1, Theorem 1]).
As φ is continuous, there exists $\varepsilon > 0$ such that, for any $q \in \partial\Omega$ such that $d(p_0, q) < \varepsilon$, one has $\varphi(q) < \varphi(p_0) + \frac{1}{k}$.
In the definition of barrier, let V_k be the triangle T defined above, such that γ touches Γ at p_0, $\gamma \cap \Omega = \emptyset$ and $T \cap \Omega \neq \emptyset$. Moreover, choose T such that for any $q \in T \cap \partial\Omega$, one has $d(q, p_0) < \varepsilon$.
We choose M_1 such that $M_1 > \max(M, \varphi(p_0) + \frac{1}{k})$ and we set $\omega_k^+ = v + \varphi(p) + \frac{1}{k}$, where v is the function on T described above.
Hence the function ω_k^+ satisfies the required properties.

The construction of the lower barrier is analogous.

4.3 Existence on Bounded Domains

We state an existence result and we prove it by using classical tools of minimal surfaces theory. See [25, Corollary 4.1, page 325] for analogous results in $\mathbb{H}^2 \times \mathbb{R}$. In [1, Theorem 1] and [21, Theorem 1.1] one finds existence results in Nil_3, with different regularity assumptions.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain. Let φ be a continuous function on $\Gamma = \partial\Omega$ except, possibly, at a finite number of points $\{p_1, \dots, p_n\}$, where φ has finite left and right limits. Then, there exists a unique minimal extension u of φ over $\overline{\Omega}$. Moreover, the boundary of the graph of u in $Nil_3(\tau)$ is the Jordan curve γ given by the graph of φ over $\Gamma \setminus \{p_1, \dots, p_n\}$ and the vertical segments between the left and right limit of φ at any $p_1, \dots, p_n$.*

Proof. We first prove the existence part. Let S be a solution of the Plateau problem for the Jordan curve γ defined in the statement (see [18]). As Γ is convex, we can compare S with the vertical planes and by the maximum principle, we get that the surface S is contained in the Euclidean cylinder over $\overline{\Omega}$. Then, $S \cap (\Omega \times \mathbb{R})$ is a graph, as it is proved in [1].

The uniqueness, in the case of continuous boundary data, follows by a standard up and down argument and the maximum principle. When φ has a finite number of discontinuity points, one uses the classical argument of H. Jenkins and J. Serrin [14] adapted to the metric of Nil_3, by A. L. Pinheiro [21, Theorem 2.1]. □

Remark 4.2. *The result analogous to Theorem 4.1 in $\mathbb{H}^2 \times \mathbb{R}$ and $\widetilde{PSL_2(\mathbb{R})}$ are consequences of the results in [19], [20], [25] and [27] respectively. The proof of Theorem 4.1 holds in such spaces as well.*

4.4 Scherk's Type Surface

Theorem 4.2. *Let Ω be a bounded, convex domain such that $\partial\Omega = C \cup \gamma$, where C is a convex curve and γ is a segment. Let $\varphi : C \to \mathbb{R}$ be a continuous function. Then, φ has a unique minimal extension over Ω assuming the value $+\infty$ on the interior of γ.*
More precisely, there exists a unique continuous function $u : \overline{\Omega} \setminus \gamma \to \mathbb{R}$ such that:

- *u is C^2 on Ω and verifies the minimal surface equation (2),*
- *$u(p_n) \to +\infty$ for any sequence (p_n) of Ω converging to an interior point of γ,*
- *$u(p) = \varphi(p)$ for any $p \in C \setminus \partial C$.*

As a particular case of Theorem 4.2 we get the following existence result. Let $T \subset \mathbb{R}^2$ be a triangle with sides α, β, γ. Let A, B, C the vertices of T labeled such that they are opposite to α, β, γ respectively.

Corollary 4.1. *Let $\varphi : \alpha \cup \beta \to \mathbb{R}$ be a continuous function. Then, φ has a unique minimal extension over T assuming the value $+\infty$ on the interior of γ.*
More precisely, there exists a unique continuous function $u : T \setminus \gamma \to \mathbb{R}$ such that:

- *u is C^2 on the interior of T and verifies the minimal surface equation (2),*
- *$u(p_n) \to +\infty$ for any sequence (p_n) of $\mathrm{int}(T)$ converging to an interior point of γ,*
- *$u(p) = \varphi(p)$ for any $p \in \alpha \cup \beta$, $p \neq A, B$.*

Proof of Theorem 4.2. We do the proof for $\tau = \frac{1}{2}$. It will be clear that the proof is analogous for any $\tau > 0$ (by Remark 2.2).
By Remark 2.1(B), it is enough to prove the result for $\psi(\Omega)$ where ψ is any Euclidean isometry of $\mathbb{R}^2$ and for any continuous function on $\psi(C)$.

It follows that we can assume that the segment γ is orthogonal to the x_1 axis, with: $0 < x_1(C) < x_1(\gamma)$.
Let $n \in \mathbb{N}$ and let φ_n be the piecewise continuous function on $\partial\Omega$ with value φ on C and n on $\mathrm{int}(\gamma)$. Theorem 4.1 insures the existence and uniqueness of a minimal extension u_n of φ_n to Ω. Namely, for any $n \in \mathbb{N}$:

- u_n is continuous on $\overline{\Omega} \setminus \partial C$,
- u_n is C^2 on Ω and satisfies the minimal surface equation (2),
- $u_n(p) = \varphi_n(p)$ for any $p \in C \setminus \partial C$.

Claim 1. *There is a subsequence of (u_n) converging to a C^2 function u on Ω, satisfying the minimal surface equation.*
By Remark 4.1, it is enough to prove that for any compact subset K of Ω, there exist uniform height estimates for the sequence (u_n).
We show uniform height estimates for (u_n) from above on K. A similar reasoning shows that uniform height estimates from below hold as well.

Let $d_K > 0$ be the Euclidean distance between K and γ, and let B be a positive constant such that $\Omega \subset \{(x_1, x_2) \in \mathbb{R}^2,\ |x_2| < B/2\}$.
As in the proof of Theorem 3.1, we consider the family of horizontal catenoids $\mathcal{C}_t$ of Nil_3, $t \in \mathbb{R}_+$. Recall that the projection of $\mathcal{C}_t$, $t > 0$, on the x_1-x_2 plane is given by:

$$\pi(\mathcal{C}_t) = \left\{(x_1, x_2) \in \mathbb{R}^2,\ |x_1| \leqslant t\cosh\left(\frac{x_2}{t}\right)\right\}.$$

We set $L_B^+(t) = \partial\pi(\mathcal{C}_t) \cap \{(x_1, x_2) \in \mathbb{R}^2,\ x_1 > 0\,, |x_2| \leqslant B\}$. Then:

$$L_B^+(t) \cap \{(x_1, B),\ x_1 \in \mathbb{R}\} = \{(t\cosh\left(\frac{B}{t}\right), B)\}$$
$$L_B^+(t) \cap \{(x_1, 0),\ x_1 \in \mathbb{R}\} = \{(t, 0)\}.$$

Then we have

$$L_B^+(t) \subset \{(x_1, x_2) \in \mathbb{R}^2,\ t \leqslant x_1 \leqslant t\cosh\left(\frac{B}{t}\right),\ |x_2| \leqslant B\}.$$

Let $t > 0$ be large enough to have

$$t\cosh\left(\frac{B}{t}\right) - t < d_K/2. \tag{7}$$

Now, let us translate the domain Ω along the x_1 axis so that $x_1(\gamma) = t\cosh\left(\frac{B}{t}\right)$, recall that the side γ is orthogonal to the x_1 axis. Because of inequality (7) we have that $K \subset \mathrm{int}\big(\pi(\mathcal{C}_t)\big)$ (see Figure 5). Translate vertically the catenoid $\mathcal{C}_t$ so that it stays above the graph of φ on $\overline{\Omega} \cap \pi(\mathcal{C}_t)$. Then, as the boundary of the graph of any of the functions u_n over $\overline{\Omega} \cap \pi(\mathcal{C}_t)$ stays below the translated catenoid, by the maximum principle the graph of any of the functions u_n on $\overline{\Omega} \cap \pi(\mathcal{C}_t)$ remains below this catenoid. This gives uniform upper estimates for the sequence (u_n) on K.

Claim 2. *The sequence (u_n) is strictly increasing on Ω. Consequently, $u(p_n) \to +\infty$ for any sequence (p_n) of Ω converging to an interior point of γ.*
The first assertion is a consequence of the *general maximum principle* of H. Jenkins and J. Serrin [14], adapted to the metric of Nil_3 by A. L. Pinheiro [21, Theorem 2.1], because the boundary data of the sequence (u_n) are not decreasing in n. The second assertion of the Claim follows easily.

One can use the barrier constructed in Section 4.2 in the same way as in [24, Theorem 3.4], to prove that the function u extends continuously up to $\partial\Omega \setminus \gamma$ taking value $u = \varphi$ on $C \setminus \partial C$.

Finally, uniqueness follows by the generalization of Jenkins-Serrin result [14, Section 6] to Nil_3 by A. L. Pinheiro [21, Theorem 4.2]. □

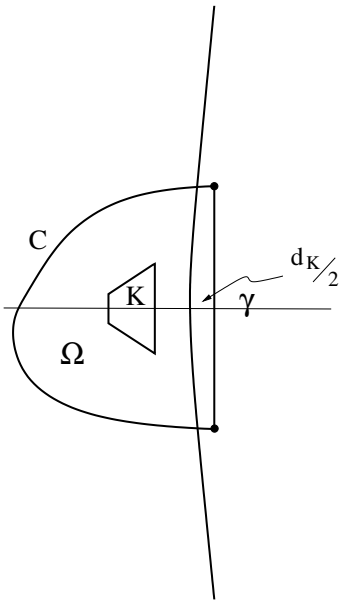


Figure 5: The subset K

Remark 4.3. *Theorem 4.2 also holds for functions $\varphi : \alpha \cup \beta \to \mathbb{R}$ continuous except at a finite number of points, where φ has finite left and right limits.*

4.5 Existence on Unbounded Domains

In next theorem we prove that we can solve the Dirichlet problem for the minimal surface equation on any convex unbounded domain, different from the half-plane, with arbitrary piecewise continuous boundary data, and on a half-plane for bounded piecewise continuous boundary data.

Theorem 4.3.

(A) *Let $\Omega \subset \mathbb{R}^2$ be an unbounded convex domain different from a half-plane. Let φ be a function on $\Gamma = \partial\Omega$ continuous except at a discrete and closed set $Z \subset \partial\Omega$ of points where φ has finite left and right limits.*

Then there exists a minimal extension of φ over $\overline{\Omega}$.

(B) *Let Ω be a half-plane and let Γ the straight line that is the boundary of Ω. Let φ be a bounded function on Γ continuous except at a discrete and closed set $Z \subset \Gamma$ of points where φ has left and right limits. Then there exists a 1-parameter family of minimal extension of φ over $\overline{\Omega}$.*

In both cases, the boundary of the minimal extension is the union of the graph of φ over $\partial\Omega \setminus Z$ with the vertical segment between the left and the right limit of φ at the discontinuity points.

Proof. (A) As Ω is convex and is not a half-plane, it is either contained in a wedge or it is a strip, (in this case it has two boundary components.)

Case 1. *Ω is contained in a wedge.*
Assume that the wedge is contained in the half plane $x_1 > 0$. Let B_n be the open ball of radius n, centered at the origin in the x_1-x_2 plane and let $\Omega_n = B_n \cap \Omega$. Denote by Γ_n the boundary of Ω_n contained in ∂B_n. Let r_n, s_n be the intersection points between Γ_n and $\partial\Omega$. Since Z is discrete, we can

assume that φ is continuous at r_n and s_n. On the boundary of Ω_n, we consider a piecewise continuous function φ_n, continuous on Γ_n, with value between $\varphi(r_n)$ and $\varphi(s_n)$ on Γ_n such that

$$\varphi_n(q) = \begin{cases} \varphi(q) \text{ if } q \in \partial\Omega_n \setminus \Gamma_n \\ \varphi(r_n) \text{ if } q = r_n \\ \varphi(s_n) \text{ if } q = s_n \end{cases}$$

As Ω_n is bounded and convex and φ_n is piecewise continuous, Theorem 4.1 guarantees the existence of a minimal extension u_n of φ_n on Ω_n. We recall that the boundary of the graph of each u_n contains the vertical segment above any discontinuity point with endpoints the left and the right limit of φ at the point. Moreover, there are no other points of the closure of the graph of u_n on the vertical geodesic passing through the discontinuity points.

We want to prove that there is a subsequence of (u_n) converging to a minimal solution u on every compact subset of Ω and such that

- u extends continuously up to $\overline{\Omega} \setminus Z$,
- the boundary of the graph of u consists in the graph of φ over $\partial\Omega \setminus Z$ and the vertical segments between the right limit and the left limit of φ at any point $p \in Z$.

By Remark 4.1, in order to prove the convergence, we only need to prove that uniform height estimates hold for the sequence (u_n) on every compact subset of Ω.
Let K be a compact subset of Ω and let n_0 be such that $K \subset \Omega_{n_0}$. Consider a horizontal catenoid $\mathcal{C}_{\alpha(n_0)}$ such that $\Omega_{n_0} \subset \pi(\mathcal{C}_{\alpha(n_0)})$. Moreover let $\widetilde{\mathcal{C}}_{\alpha(n_0)} = \mathcal{C}_{\alpha(n_0)} \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3,\ \ |x_2| \leqslant B\}$, where B is chosen such that $\pi(\widetilde{\mathcal{C}}_{\alpha(n_0)})$ strictly contains K. Now, let $N_0 > n_0$ (depending only on n_0) such that $\pi(\widetilde{\mathcal{C}}_{\alpha(n_0)}) \cap \Gamma_{N_0} = \emptyset$ (see figure 6).
As, $\overline{\Omega}_{N_0}$ is compact, φ_{N_0} is bounded there, so that it is possible to translate vertically upward $\widetilde{\mathcal{C}}_{\alpha(n_0)}$ such that it is above $\varphi_{N_0}(\partial\Omega_{N_0} \cap \partial\Omega)$. Notice that, for any $n \geqslant N_0$, the graph of $u_{n|\partial\Omega_{N_0}}$ is below the translation of $\widetilde{\mathcal{C}}_{\alpha(n_0)}$. This means that, for any $n \geqslant N_0$, the boundary of the graph of $u_{n|\Omega_{N_0}}$ stays below the translation of $\widetilde{\mathcal{C}}_{\alpha(n_0)}$ and hence, by the maximum principle, the graph of u_n stays below it. This gives the uniform height estimates from above on K for every u_n, $n \geqslant N_0$.

It is clear that analogously, one finds height estimates from below on K, for every u_n, $n \geqslant N_0$.

Now we prove that the function u, previously defined as the limit of a subsequence of (u_n), takes the desired boundary values.
At any point of $\Gamma \setminus Z$, that is where φ is continuous, one can use the barrier that we constructed before in the same way as in [24, Theorem 3.4], to prove that u extends continuously up to $\overline{\Omega} \setminus Z$, by setting $u = \varphi$ on $\Gamma \setminus Z$.

Now, we show what happens at a discontinuity point. The proof is analogous to the one in [25, Corollary 4.1, page 325]. Let p be a discontinuity point of φ. Let $\varphi_1(p)$ and $\varphi_2(p)$ be the left and the right limit of φ at p. We assume that $\varphi_1(p) < \varphi_2(p)$. We first prove that the vertical segment between $(p, \varphi_1(p))$ and $(p, \varphi_2(p))$ is contained in the boundary of the graph of the function u. Let l be such that $\varphi_1(p) < l < \varphi_2(p)$.
Let p_n, q_n, distinct points of Γ at distance $\frac{1}{n}$ from p, such that p_n and q_n are not in the same component of $\Gamma \setminus \{p\}$. One has that, for n large enough:

$$u(p_n) = u_n(p_n) = \varphi(p_n) < l < \varphi(q_n) = u_n(q_n) = u(q_n).$$

Let δ_n be a small arc contained in the intersection of Ω with an Euclidean disk centered at p of radius $\frac{2}{n}$, with endpoints p_n and q_n. By the previous inequality and by the continuity of u, there exists a point $r_n \in \delta_n$ such that $u(r_n) = l$. This means that, for n great enough, any point (r_n, l) belongs to the graph of u. Now, when $n \longrightarrow \infty$, one has that (p, l) belongs to the closure of the graph of u.

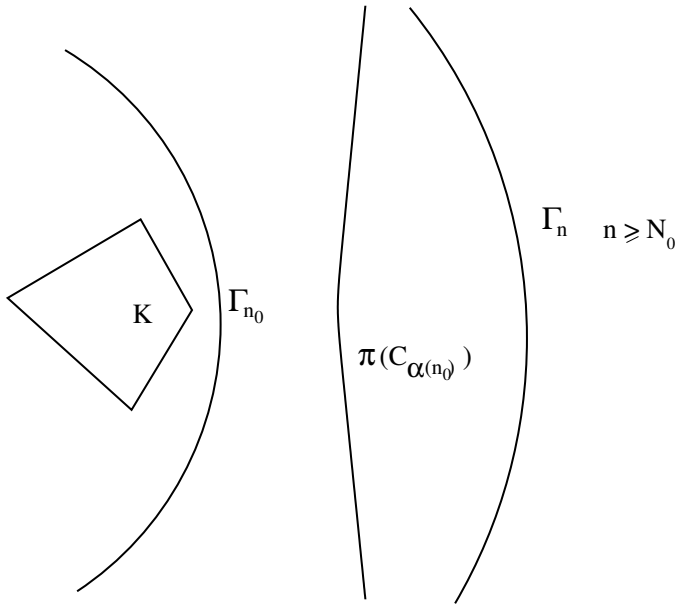


Figure 6: Uniform bounds on K : the graph of φ_{N_0} over $\partial\Omega_{N_0} \cap \partial\Omega$ is below the vertical translation of $\mathcal{C}_{\alpha(n_0)}$.

Finally, we have to prove that there are no other points of the closure of the graph of u on the vertical geodesic above p.
Let h be a real number such that $h > \max\{\varphi_1(p), \varphi_2(p)\}$. Then, for any $\varepsilon > 0$ such that $h - \varepsilon > \max\{\varphi_1(p), \varphi_2(p)\}$, we construct a standard upper barrier on a triangle T with sides α, β, γ. Since Ω is convex, we can choose the triangle such that the side γ touches Γ at the point p. We choose the value $h - \varepsilon$ on γ and the value M on α and β, such that $M > \sup_{q \in T \cap \Omega, n \in \mathbb{N}} u_n(q)$ (observe that by construction, the functions u_n are uniformly bounded on any compact subset of $\overline{\Omega}$). We denote by v the function on T given by this upper barrier.
Then, by the maximum principle, we get that $u_n(q) \leqslant v(q)$ for any $q \in T \cap \overline{\Omega}$, $q \neq p$. Consequently we have $u(q) \leqslant v(q)$ for any $q \in T \cap \overline{\Omega}$, $q \neq p$. Since $v_{|\gamma} = h - \varepsilon$, we obtain that the point (p, h) is not in the closure of the graph of u.

We can show in the same way that any point (p, h) with $h < \min\{\varphi_1(p), \varphi_2(p)\}$ is not in the closure of the graph of u.

Case 2. Ω *is a strip.*
The proof is analogous to the proof of the Case 1. We only describe what are Ω_n and φ_n in this case. We assume the strip is $\Omega = \{(x_1, x_2) \in \mathbb{R}^2,\ 0 < x_2 < d\}$. For each $n \in \mathbb{N}^*$, let Ω_n be the following rectangle:

$$\Omega_n = \{(x_1, x_2) \in \mathbb{R}^2,\ |x_1| < n,\ 0 < x_2 < d\}. \tag{8}$$

We choose $\varphi_n : \partial\Omega_n \longrightarrow \mathbb{R}$ to be a piecewise continuous function such that $\varphi_n(q) = \varphi(q)$ if $q \in \partial\Omega \cap \partial\Omega_n$ and such that it is monotone on each vertical side of $\partial\Omega_n$.

(B) We can assume that the half-plane is $\Omega = \{(x_1, x_2) \in \mathbb{R}^2,\ x_2 > d\}$, $d > 0$. Let us consider a plane $x_3 = ax_2 + b$, where $a > 0$, $b > \sup_{\{x_2 = d\}} \varphi$. For any $n \in \mathbb{N}$, we consider the strip $\Omega_n = \{(x_1, x_2) \in$

$\mathbb{R}^2,\ d < x_2 < n\}$ and the piecewise continuous function $\varphi_n : \partial\Omega_n \longrightarrow \mathbb{R}$ defined as follows

$$\varphi_n(p) = \begin{cases} \varphi(p) & \text{if } p = (x_1, d) \\ an + b & \text{if } p = (x_1, n) \end{cases}$$

We solve the Dirichlet problem on Ω_n with boundary values equal to φ_n as in (A) and denote by u_n the solution. Recall that each u_n is obtained as uniform limit on compact subsets of the strip, of functions $(u_{n,k})_{k\in\mathbb{N}}$ defined on rectangles $\mathcal{R}_{n,k}$ exhausting the strip Ω_n. Moreover, by the maximum principle, comparing the graphs of $u_{n,k}$ with the vertical translations of the plane $x_3 = ax_2 + b$, one has that $u_{n,k}(x_1, x_2) \leqslant ax_2 + b$, for all $(x_1, x_2) \in \mathcal{R}_{n,k}$, for all n, k. Therefore the graph of u_n is below the plane $x_3 = ax_2 + b$, for every n. This gives the desired estimate from above for the sequence (u_n). We do the same with planes staying below the boundary values and we find uniform estimates from below, as well.
As the solution that we find is contained between two planes, the existence of one parameter family of solutions is easily achieved by changing the slope of the initial plane that one uses as supersolution. □

Remark 4.4.

(A) *The existence of solutions on a half-plane can be proved in a more general case. Assume that the half-plane is $x_2 > 0$ and let the boundary value φ be piecewise continuous and such that $\varphi(x_1, 0) = cx_1$ for $|x_1| > n$. The proof is analogous to the proof of Theorem 4.3(B), using suitable tilted planes that do not contain the x_1-direction.*

(B) *In the proof of Theorem 4.3-(A), Case 1, we can use, as barriers, the surfaces constructed in Theorem 4.1.*

(C) *The proof of Theorem 4.3-(A) yields that, when the boundary value φ is bounded above (respectively below) by a constant A, then the solution given by our proof is also bounded above (respectively below) by the same constant A.*

(D) *The proof of Theorem 4.3-(B) yields that, when the boundary value φ is non negative, then the solution given by our proof is non negative as well. Moreover, such solution has linear growth (see Definition 4.2).*

On the contrary, it will be clear from the examples below that there are many unbounded solutions on a wedge, with zero boundary value. The existence of such surfaces means that, on a wedge, the boundedness of the boundary value does not imply the boundedness of the extension. Two questions arises about solutions on a strip.

- *(a) Is the minimal solution with zero boundary value on a strip, unique?*
- *(b) Let u be any minimal solution on a strip with boundary value φ such that $|\varphi| \leqslant M$ for some $M > 0$. Is $|u| \leqslant M$?*

Note that, by [16, Theorem 7], any non trivial solution of the minimal surface equation, with zero boundary value on a strip has at least linear growth (see Definition 4.2). In the same article Manzano and the first author prove that the growth of an entire minimal graph in Nil_3 has order at most three (Theorem 6).

Let us recall the definition of growth of a graph.

Definition 4.2. *Let Ω be an unbounded subset of $\mathbb{R}^2$ and let Ω_R be the intersection of Ω with the ball of radius R centered at the origin. Let $f : \mathbb{R} \longrightarrow \mathbb{R}_+$ be a continuous non decreasing function. The graph of a continuous function $u : \Omega \longrightarrow \mathbb{R}$ has growth at least $f(R)$ (respectively at most $f(R)$) if*

$$\liminf_{R\longrightarrow\infty} \frac{\sup_{\partial\Omega_R} |u|}{f(R)} > 0, \quad (\text{resp. } \limsup_{\mathrm{R}\longrightarrow\infty} \frac{\sup_{\partial\Omega_\mathrm{R}} |\mathrm{u}|}{\mathrm{f(R)}} < \infty)$$

In particular we say that the growth is $\alpha > 0$ if there exists a constant $c > 0$ such that

$$\lim_{R \longrightarrow \infty} \frac{\sup_{\partial\Omega_R} |u|}{R^\alpha} = c.$$

(*linear if* $\alpha = 1$, *and quadratic if* $\alpha = 2$)

In the following we describe some examples of minimal graphs on unbounded domain with linear and quadratic growth.

1. Recall the examples by Figueroa, Mercuri and Pedrosa described in Section 2.2. The graph of the function $u_0 : \mathbb{R}^2 \longrightarrow \mathbb{R}$, defined by $u_0(x_1, x_2) = \tau x_1 x_2$, is a complete minimal surface in $Nil_3(\tau)$ that is invariant by the isometry $\varphi(x_1, x_2, x_3) = (x_1 + c, x_2, x_3 + c\tau x_3)$, $c \in \mathbb{R}$. The function u_0 over the wedge $\{(x_1, x_2) \in \mathbb{R}^2,\ x_1 \geqslant 0,\ x_2 \geqslant 0\}$ has zero boundary value and quadratic growth.

2. In [5, Corollary 3.8], S. Cartier proved that there exist non-zero minimal graphs on any wedge with angle less than π, with zero boundary value. The growth of such graphs is linear.

In the following theorem we show a new example of minimal graph on a convex unbounded domain, containing a wedge of angle $\theta = \frac{\pi}{2}$, with non negative boundary value and at least quadratic growth.

Theorem 4.4. *Let Ω be a convex, unbounded domain, different from the half-plane, containing a wedge W_θ with vertex at the origin and angle $\theta \in [\frac{\pi}{2}, \pi[$, and let φ be a non-negative, continuous function on $\partial\Omega$. Then, there exists a non-negative minimal extension of φ to Ω in $Nil_3(\tau)$, $\tau \neq 0$, with at least quadratic growth.*

Proof. Up to an isometry, we can assume that $x_2 \geqslant 0$ on the wedge W_θ contained in Ω and that W_θ is symmetric with respect to the x_2 axis. Consider the graph Σ of the function $v(x_1, x_2) = \tau x_1 x_2$ over the wedge $\{(x_1, x_2) \in \mathbb{R}^2,\ x_1 \geqslant 0,\ x_2 \geqslant 0\}$. We rotate Σ in order to obtain a graph $\widetilde{\Sigma}$ over the wedge $\{(x_1, x_2) \in \mathbb{R}^2,\ x_2 \geqslant |x_1|\}$ with zero boundary value. Moreover $x_3(p) > 0$ for any $p \in \widetilde{\Sigma}$.
For any $n \in \mathbb{N}^*$ we denote by A_n and B_n the intersection points of the boundary of Ω with the horizontal line $\{x_2 = n\}$.
Let $\Omega_n = \Omega \cap \{(x_1, x_2),\ x_2 \leqslant n\}$. The set Ω is convex and bounded. Let $C_n = \partial\Omega \cap \partial\Omega_n$ and $\gamma_n = \partial\Omega_n \cap \{x_2 = n\}$. Observe that the sequence (Ω_n) gives an exhaustion of the domain Ω.
Let $\varphi_n : \partial\Omega_n \to \mathbb{R}$ be a continuous function such that:

- φ_n is positive and stays above $\widetilde{\Sigma}$ along the open segment γ_n.
- $\varphi_{n_{|C}} = \varphi$.

Let u_n be the minimal extension of φ_n to Ω_n given by Theorem 4.1. Let h_n be the function whose graph is the Scherk-type surface with boundary value zero on C_n and with value $+\infty$ on γ_n, given by Theorem 4.2.

Finally let $K \subset \Omega$ be a compact subset and let $n(K) \in \mathbb{N}^*$ such that $K \subset \Omega_{n(K)-1}$. Let $L_{n(K)-1} = \max_{C_{n(K)-1}} \varphi$. By the maximum principle, for any $n \geqslant n(K)$ we have $u_n \leqslant h_{n(K)} + L_{n(K)-1}$ on K. This gives uniform estimates from above for the sequence (u_n) on K. Moreover the sequence (u_n) is bounded below by 0.

We deduce from Remark 4.1 that a subsequence of (u_n) converges to a non-negative function u on Ω, satisfying equation (2). Furthermore, the function u extends continuously to $\partial\Omega$ with prescribed value φ. This can be seeing by comparing with the barriers described in Section 4.2.

Observe that, for any n, by the maximum principle, the graph of u_n stays above $\widetilde{\Sigma}$, so the graph of u also stays above $\widetilde{\Sigma}$. Since the function whose graph is $\widetilde{\Sigma}$ has quadratic growth, we have that u has at least quadratic growth. □

As a consequence of Theorem 4.4, we have the following Corollaries.

Corollary 4.2. *Let W_θ be a wedge with vertex at the origin and angle $\theta \in [\frac{\pi}{2}, \pi[$, and let φ be a non-negative, continuous function on ∂W_θ. Then, there exists a non-negative minimal extension of φ to W_θ in $Nil_3(\tau)$, $\tau \neq 0$, with at least quadratic growth.*

The proof of Corollary 4.2 is an immediate consequence of Theorem 4.4, taking $\Omega = W_\theta$. Notice that, when the boundary value is zero, the fact that the growth is at least quadratic yields that the examples of Corollary 4.2 are different from those in [5].
Next Corollary show that, we are able to construct graphs with at least quadratic growth on any wedge of angle $\theta \in [\pi, 2\pi[$, provided the boundary data satisfies suitable conditions. Notice that, when $\theta = \pi$, the wedge is an half-plane.

Corollary 4.3. *Let W_θ be the wedge with vertex at the origin and angle $\theta \in [\pi, 2\pi[$, bounded by the following two half-lines*

$$L_1 = \left\{ x_2 = x_1 \tan\left(\frac{\pi-\theta}{2}\right), \quad x_1 \geqslant 0 \right\} \qquad L_2 = \left\{ x_2 = x_1 \tan\left(\frac{\pi+\theta}{2}\right), \quad x_1 \leqslant 0 \right\}.$$

Let $\varphi : L_1 \cup L_2 \longrightarrow \mathbb{R}$ be a continuous function such that $\varphi(x_1, x_2) = -\varphi(-x_1, x_2)$ and $\varphi \geqslant 0$ on L_1. Then, there exists a minimal extension of φ to W_θ in $Nil_3(\tau)$ with at least quadratic growth.

Proof. Let $Q = W_\theta \cap \{(x_1, x_2) \in \mathbb{R}^2,\ x_1 \geqslant 0\}$. We define the function $\psi : \partial Q \longrightarrow \mathbb{R}$ by $\psi = \varphi$ on $\partial Q \cap \{x_2 = x_1 \tan\left(\frac{\pi-\theta}{2}\right)\}$, $\psi \equiv 0$ on $\partial Q \cap \{x_1 = 0\}$.
Let v be the minimal extension of ψ over Q given by Corollary 4.2. Then, extend v to W_θ by the Reflection Principle (as in [26, Lemma 3.6]) along $\partial Q \cap \{x_1 = 0\}$. As $\varphi(x_1, x_2) = -\varphi(-x_1, x_2)$, the extended function gives the desired solution with at least quadratic growth. □

In the case of the half-plane, we have a further result.

Corollary 4.4. *Let Π be the half-plane $\{(x_1, x_2) \in \mathbb{R}^2,\ x_2 \geqslant 0\}$, and let $\varphi : \partial\Pi \longrightarrow \mathbb{R}$ an odd function. Assume that φ is continuous except as a discrete and closed set $Z \subset \partial\Pi$, $\varphi(x_1) \geqslant 0$ for $x_1 \geqslant 0$ and there exists a constant M such that $|\varphi| \leqslant M$. Then, there exists a minimal extension of φ to Π in $Nil_3(\tau)$ ($\tau \geqslant 0$, including the Euclidean 3-space), that is bounded in Π by the same constant M. The boundary of the minimal extension is the union of the graph of φ over $\partial\Pi \setminus Z$ with the vertical segment between the left and the right limit of φ at the discontinuity points.*

Proof. Let Q be the subset of Π given by $\{(x_1, x_2) \in \mathbb{R}^2,\ x_1, x_2 \geqslant 0\}$. We define the function $\psi : \partial Q \longrightarrow \mathbb{R}$ by $\psi = \varphi$ on $\partial Q \cap \{x_2 = 0\}$, $\psi \equiv 0$ on $\partial Q \cap \{x_1 = 0\}$.
Let v be the minimal extension of ψ over Q given by Theorem 4.3(A). Notice that, by Remark 4.4, the function v is bounded by the constant M. Then, extend v to Π by the Reflection Principle along $\partial Q \cap \{x_1 = 0\}$. The extended function gives the desired solution. □

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