PARABOLIC AND HYPERBOLIC SCREW MOTION SURFACES IN $\mathbb{H}^2 \times \mathbb{R}$

RICARDO SA EARP

Abstract. In this paper we find many families in the product space $\mathbb{H}^2 \times \mathbb{R}$ of complete embedded, simply connected, minimal and constant mean curvature surfaces less than or equal to $1/2$. We study complete surfaces invariant either by *parabolic* or by *hyperbolic* screw motions. We study the notion of isometric associate immersions. We exhibit an explicit formula for a Scherk type minimal surface. We give a one-parameter family of entire vertical graphs of mean curvature $1/2$. We prove a generalized Bour's lemma that can be applied to $\mathbb{H}^2 \times \mathbb{R}$, $\mathbb{S}^2 \times \mathbb{R}$, and to Heisenberg's space to produce a family of screw motion surfaces isometric to a given one.

1. Introduction

In a pioneer paper Harold Rosenberg studied minimal surfaces in $M^2 \times \mathbb{R}$, where M^2 is a round sphere, or a complete Riemannian surface with a metric of non-negative curvature, or $M^2 = \mathbb{H}^2$, the hyperbolic plane [17]. He has opened a quite interesting new branch of research in surface theory stimulating several works on the subject. The main scope in the present paper is to discover complete embedded minimal and constant mean curvature surfaces. Now, we briefly summarize our results, as follows:

We will study minimal and constant mean curvature surfaces in $\mathbb{H}^2 \times \mathbb{R}$, invariant by *parabolic* screw motions, i.e invariant by a one-parameter group of isometries such that each element is given by the composition of a parabolic translation with a vertical translation. We will find a two-parameter family of complete embedded, simply connected, minimal surfaces which contains surfaces invariant by parabolic translations. We also will obtain a one-parameter family of complete, embedded, simply connected, stable minimal surfaces which contains the hyperbolic plane $\mathbb{H}^2 \times \{0\}$. We will then construct a two-parameter family of complete H-surfaces with constant mean curvature H less

1991 *Mathematics Subject Classification.* 53C42.

Key words and phrases. The author likes to thanks CNPq and PRONEX of Brazil, for partial financial support.

than 1/2, simply connected and embedded, which contains a surface invariant by parabolic translations. We will find in this family, explicit non-parametric formulas for a one-parameter subfamily of complete, embedded, stable, H-surfaces ($4H^2 < 1$). Furthermore, we will exhibit a one-parameter family of minimal and constant mean curvature surfaces ($H = 1/2$), invariant by *hyperbolic* screw motions. We also obtain minimal and H-surfaces ($4H^2 < 1$) invariant by *hyperbolic* translations. Each such surface is complete, embedded, simply connected and stable. We remark that there are no stable H-surfaces, complete and non compact, for $H > 1/\sqrt{3}$. This is a result of Barbara Nelli and Harold Rosenberg [15].

Furthermore, we will derive a explicit simple non-parametric formula for a Scherk type minimal surface, invariant by *hyperbolic* translations, found independently by Uwe Abresch. Abresch and Rosenberg have applied it as a *barrier* studying the Dirichlet problem for the minimal surface equation in $\mathbb{H}^2 \times \mathbb{R}$. We will explain their geometric construction later on in the text. We observe that this Scherk type surface can be seen as a complete vertical graph over a domain in $\mathbb{H}^2$ taking $\pm\infty$ value boundary data on a geodesic and zero (or constant) asymptotic value boundary data. That is, zero on an arc of the circle at infinity. See the shadow domain in Figure 1.

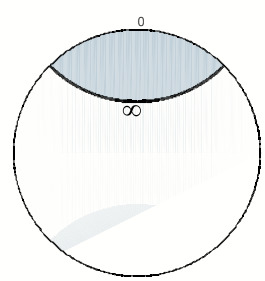

Figure 1: ball model for $\mathbb{H}^2 \times \{0\}$

In a paper with Eric Toubiana the author studied surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$ invariant by standard screw motions, that is a one-parameter group of isometries such that each element is the composition of a rotation around the vertical axis with a vertical translation. In that paper [19] the authors obtained for $\ell > \dfrac{1}{\sqrt{2}}$ a complete embedded

simply connected minimal screw motion surface in $\mathbb{H}^2 \times \mathbb{R}$ with pitch ℓ. If $\ell = 1$, each one has Gaussian curvature $K \equiv -1$.

The author with Eric Toubiana and Laurent Hauswirth established an uniqueness theorem in $\mathbb{H}^2 \times \mathbb{R}$, or $\mathbb{S}^2 \times \mathbb{R}$. They proved that the conformal metric and the related holomorphic Hopf function, arising from the theory of minimal immersions and conformal mappings, determine a minimal conformal immersion, up to an isometry of ambient space. They proved an existence theorem that produce the existence of the minimal associate family as a corollary. They also generalized a theorem by Krust (see [6], page 118) which state that an associate surface of a minimal vertical graph on a convex domain is a vertical graph. These theorem is true in $\mathbb{M} \times \mathbb{R}$ when the Gaussian curvature satisfies $K_{\mathbb{M}} \leq 0$. See [11].

Now let us make some more detailed comments about the present paper. Our idea is simple: We will consider $\mathbb{H}^2 = \{x + iy, y > 0\}$ the upper half-plane model of hyperbolic plane and we will consider the product space $\mathbb{H}^2 \times \mathbb{R}$, with coordinates (x, y, t), endowed with the metric $d\sigma^2 = \dfrac{dx^2}{y^2} + \dfrac{dy^2}{y^2} + dt^2$. We will search for surfaces invariant by *parabolic* screw motions- a parabolic translation is identified with a horizontal Euclidean translation in this model. That is, we will study immersions of the form

$$X(x, y) = (x, y, \lambda(y) + \ell x) \tag{1}$$

Thus we will search for minimal and for constant mean curvature surfaces generated by applying Euclidean translations to the vertical graph $t = \lambda(y)$ lying in the (y, t) vertical plane along the directions of the vector $(1, 0, \ell)$. We say ℓ is the pitch. When $\ell = 0$, we obtain a surface invariant by parabolic translations. Of course, in this model this is related with two notions of non-parametric graphs. We will introduce them now for motivation. First, let us consider prescribed mean curvature *horizontal H-graphs* given by $y = g(x, t)$, where $g(x, t)$ is a positive C^2 function. A computation shows that the *horizontal mean curvature equation* in $\mathbb{H}^2 \times \mathbb{R}$, is given by

$$\begin{aligned} &\frac{2H}{g^2}\left(g_t^2 + g^2(1 + g_x^2)\right)^{3/2} \\ &\qquad = g_{xx}(g^2 + g_t^2) + g_{tt}(1 + g_x^2) - 2g_x g_t g_{xt} + g(1 + g_x^2) \end{aligned} \tag{2}$$

An interesting question that arises is the Bernstein problem for these graphs: Are the only entire such H-graphs invariant under parabolic screw motions ?

Some classical constant mean curvature surfaces in $\mathbb{H}^2 \times \mathbb{R}$ that arise naturally from this non-parametric point of view are related to this question: Indeed, notice that when g is constant we obtain the vertical cylinder over a horocyle of mean curvature $2H = 1$. If $g_t \equiv 0$, we obtain the minimal vertical cylinder over a geodesic, the constant mean curvature vertical cylinder over an equidistant curve ($2H < 1$), and the constant mean curvature vertical cylinder over a circle ($2H > 1$) . If H is constant and $g_x \equiv 0$ we obtain the Abresch-Rosenberg [1] surface with mean curvature $2H < 1$ given by $y = e^{t/a}, a = \frac{-2H}{\sqrt{1-4H^2}}$. Notice that if $H = 0$ and $g_x \equiv 0$, we can obtain a very simple formula for a one-parameter family of minimal stable complete surfaces invariant by a parabolic motion $y = \frac{\sin t}{d}, d \neq 0$. This formula has already been established by Benoît Daniel [2] and this surface was known to Laurent Hauswirth [10]. We will construct a two-parameter family $y = g_{\mathcal{M}}(x, t, d, \ell), d \neq 0$ of horizontal minimal graphs, which produces complete, simply connected and embedded minimal surfaces. If $\ell = 0$, we rediscover Daniel-Hauswirth minimal surface. Fixing the parameter d varying ℓ we get a family of non-isometric deformations of Daniel-Hauswirth minimal surface, i.e two of them with different pitch ℓ are non-isometric.

Let us now turn attention to *vertical H-graphs* given by a C^2 function $t = u(x, y)$. The *vertical mean curvature equation* in $\mathbb{H}^2 \times \mathbb{R}$, is given by the following equation:

$$\operatorname{div}_{\mathbb{H}} \left(\frac{\nabla_{\mathbb{H}} u}{W_u} \right) = 2H \tag{3}$$

where $\operatorname{div}_{\mathbb{H}}$, $\nabla_{\mathbb{H}}$ are the hyperbolic divergence and gradient respectively and $W_u = \sqrt{1 + |\nabla_{\mathbb{H}} u|^2_{\mathbb{H}}}$, being $|\cdot|_{\mathbb{H}}$ the norm in $\mathbb{H}^2$.

Gradient interior estimates and infinite boundary value problems for H-vertical graphs were inferred by Spruck [21] and Hauswirth-Rosenberg-Spruck [12].

Consider the halfspace model for $\mathbb{H}^2$, with Euclidean coordinates x, y, $y > 0$. In such model, the above equation takes the following form

$$\begin{aligned} &\frac{2H}{y^2} \left(1 + y^2 u_x^2 + y^2 u_y^2)\right)^{3/2} \\ &\qquad = (1 + y^2 u_x^2) u_{yy} + (1 + y^2 u_y^2) u_{xx} - 2y^2 u_x u_y u_{xy} - y u_y (u_x^2 + u_y^2) \end{aligned} \tag{4}$$

Of course, on account of (3), the above equation is a second order quasilinear elliptic equation of divergence form, namely

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+y^2\left|\nabla u\right|^2}}\right)=\frac{2H}{y^2} \tag{5}$$

where ∇u stands for the Euclidean gradient. If $t = a\ln y$, we rediscover Abresch-Rosenberg surface given above. Notice that the *tilted Euclidean plane* $t = \ell x, y > 0$, $\ell \neq 0$ gives rise to a solution of (4). It turns out that, for $\ell = 1$ we will obtain a *conjugate* of a Scherk type minimal surface. If $\ell = 0$, we obtain the totally geodesic hyperbolic plane $\mathbb{H}^2\{0\}$. Notice that we obtain a one-parameter family of minimal, non-totally geodesic stable complete surface in $\mathbb{H}^2 \times \mathbb{R}$ foliated by geodesics of the ambient space (different from the helicoids) invariant by *parabolic* screw motions. Varying ℓ we have a family of non-isometric deformations of the hyperbolic plane, i.e two elements of this family with different pitch ℓ are non-isometric.

We will give an explicit formula of a one-parameter family of vertical H-graphs, $t = u_{\mathcal{H}}(x, y, \ell)$ in $\mathbb{H}^2 \times \mathbb{R}$ over the entire $\mathbb{H}^2$ with pitch ℓ and constant mean curvature $4H^2 < 1$. Varying ℓ we obtain a family of non-isometric simply connected, embedded and stable deformations of Abresch-Rosenberg surface ($\ell = 0$).

In fact, we will construct a two-parameter family of constant mean curvature $4H^2 < 1$, horizontal graphs $y = g_{\mathcal{H}}(x, t, d, \ell)$ over the whole xt plane that contains the above family; by setting $d = 0$ we get the family $t = u_{\mathcal{H}}(x, y, \ell)$ mentioned above. Again, fixing d and varying ℓ we obtain a family of non-isometric deformations.

Moreover, we will obtain vertical minimal and H-graphs ($4H^2 \leqslant 1$), over the entire hyperbolic plane invariant by *hyperbolic* screw motions (with pitch ℓ). Incidentally, we will give an explicit, simple, non-parametric formula, for a family of minimal entire vertical graphs invariant by *hyperbolic* screw motions.

We also exhibit a one-parameter family of entire vertical graphs of mean curvature 1/2. Harold Rosenberg asked about the question of the uniqueness of these vertical graphs in the class of all 1/2-entire graphs. This is the "Bernstein Problem" for 1/2-surfaces in $\mathbb{H}^2 \times \mathbb{R}$. We notice that very recently Isabel Fernández and Pablo Mira classified the entire minimal graphs in Heisenberg space Nil_3 [9]. This is related to the Bernstein problem cited before.

We would like to remark that the notions of *horizontal* and *vertical* graphs have appeared in the study of H-surfaces in hyperbolic space (see, for instance [18]).

Finally, let M^2 be a two dimensional Riemannian manifold. Assume the existence of a one-parameter group Γ of isometries acting on M^2. We will say that S is a Γ-screw motion surface, if it is invariant by successive compositions of an element of Γ with a vertical translation. If Γ consists of rotations about the vertical axis we say simply screw motion or standard screw motions, instead of Γ-screw motion. If $M^3 = \mathbb{H}^2 \times \mathbb{R}$, and Γ is the group of parabolic translations we say *parabolic* screw motions, if Γ is the group of hyperbolic translations we say *hyperbolic* screw motions. We will prove a generalized Bour's Lemma which is general enough to be applied to $\mathbb{H}^2 \times \mathbb{R}$, $\mathbb{S}^2 \times \mathbb{R}$, (and to Heisenberg space). Given a Γ-screw motion surface S we will obtain a two-parameter family of isometric Γ-screw motions surfaces to S, say $\mathcal{F}(\ell, m), m \neq 0$. In the case of *parabolic* screw motions in $\mathbb{H}^2 \times \mathbb{R}$ this family contains a parabolic translation. The same is true for *hyperbolic* screw motions in $\mathbb{H}^2 \times \mathbb{R}$. The case of standard screw motions is treated in [19]. More precisely, we find natural parameters (s, τ), so that the metric is of the form $\mathrm{d}\sigma^2 = \mathrm{d}s^2 + U^2(s)\,\mathrm{d}\tau^2$, and we are able to describe entirely any screw motion surface in terms of the parameters ℓ, m and the metric determined by U^2. This can be also applied to Heisenberg space. In this paper we will apply the Bour generalized Lemma to screw motion surfaces in $\mathbb{H}^2 \times \mathbb{R}$, to show that any two minimal isometric *parabolic* screw motion immersions are associate; that is the absolute value of their Hopf functions are equal. The author with Eric Toubiana proved that this result also holds for standard screw motion surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$. Besides, these authors in that paper proved that in $\mathbb{H}^2 \times \mathbb{R}$, a catenoid is conjugate to a helicoid of pitch $\ell < 1$. In this paper we deduce that if $\ell = 1$ the helicoid is conjugate to Daniel-Hauswirth minimal surface. Benoît Daniel has proved this result by another approach in [2]. It follows from his work that the helicoid of pitch $\ell > 1$ is conjugate to a surface invariant by hyperbolic isometries. We will deduce this fact by outlining an alternative proof. Furthermore, we show that each such helicoid is associate to a *parabolic screw motion surface*. On the other hand, there exist families of isometric *associate hyperbolic* screw motion immersions, but there exist also isometric non-associate *hyperbolic* screw motion immersions. Each *parabolic* screw motion surface is associate to an *hyperbolic* screw motion surface. For other relevant papers on this subject the reader is referred to William Meeks III and Harold Rosenberg [13], Barbara Nelli and Harold Rosenberg [14], Isabel Fernández and Pablo Mira [8] .

2. Complete embedded minimal and constant mean curvature surfaces in $\mathbb{H}^2 \times \mathbb{R}$ invariant by parabolic screw motions

Recall that we will consider the product $\mathbb{H}^2 \times \mathbb{R}$ (here $\mathbb{H}^2$ is the upper half plane model) with coordinates (x, y, t), endowed with the metric $d\sigma^2 = \dfrac{dx^2}{y^2} + \dfrac{dy^2}{y^2} + dt^2$, inner product denoted by $<,>$ and norm denoted by $\|,\|$. We turn now our attention to non-parametric *vertical graphs* $X : (x, y) \mapsto (x, y, u(x, y))$, given by C^2 functions $t = u(x, y)$ over $\mathbb{H}^2$. Let $\widetilde{\nabla}$ be the Riemannian connexion on $\mathbb{H}^2 \times \mathbb{R}$. Let X_x, X_y be the coordinate global frame field to the graph. It is straightforward to deduce that the upper unit normal N is given by $N = \dfrac{1}{\sqrt{1 + y^2(u_x^2 + u_y^2)}} \left(-u_x y^2, -u_y y^2, 1\right)$. We also easily deduce that the connexion is determined by the formulas $\widetilde{\nabla}_{X_x} X_x = (0, 1/y, u_{xx})$, $\widetilde{\nabla}_{X_y} X_y = (0, -1/y, u_{yy})$, $\widetilde{\nabla}_{X_y} X_x = (-1/y, 0, u_{xy})$. We have therefore that the coefficients of the second fundamental form $l :=< \widetilde{\nabla}_{X_x} X_x, N >, n :=< \widetilde{\nabla}_{X_y} X_y, N > m :=< \widetilde{\nabla}_{X_y} X_x, N >$ are given by $l = \frac{-u_y/y + u_{xx}}{\sqrt{1+y^2(u_x^2+u_y^2)}}, n = \frac{u_y/y + u_{yy}}{\sqrt{1+y^2(u_x^2+u_y^2)}}, m = \frac{u_x/y + u_{xy}}{\sqrt{1+y^2(u_x^2+u_y^2)}}$. Now the Gram-Schmidt orthogonalization process provide a tangent field $Y = X_y - < X_y, X_x > \frac{X_x}{\|X_x\|^2}$ orthogonal to X_x. Let $\overrightarrow{\boldsymbol{H}}$ the mean curvature vector. Now taking into account that the mean curvature H defined by $\overrightarrow{\boldsymbol{H}} = HN$ is given by $2H = \dfrac{1}{\|X_x\|^2} < \widetilde{\nabla}_{X_x} X_x, N > + \dfrac{1}{\|Y\|^2} < \widetilde{\nabla}_Y Y, N >$, we infer the formula for the *vertical mean curvature equation* in our model written in the introduction, see equation (4). We observe now that this equation is of *divergence form*, given by equation (5).

We observe that any vertical minimal or H-graph is stable. In fact, vertical translations provide a foliation of an open subset of ambient space given by H-surfaces, transverse to the Killing vertical vector field, see [15].

Particularly, we focus now on vertical graphs of the form $u(x, y) = \lambda(y) + \ell x$; we say $t = \lambda(y)$ is the *generating curve*. In view of (4), we have therefore that the generating curve $t = \lambda(y)$ of a *parabolic* screw motion surface satisfies the following equation:

Proposition 2.1 (Mean curvature equation). *The mean curvature equation is*

$$\frac{2H}{y^2}\left(1+y^2(\ell^2+\lambda'^2)\right)^{3/2}=(1+y^2\ell^2)\lambda''-y\lambda'(\ell^2+\lambda'^2) \tag{6}$$

Owing to the fact that the mean curvature equation (4) is of divergence form (5), we derive the following crucial formula for *parabolic* screw motion surfaces.

Lemma 2.1 (First integral).

$$\left(\frac{\lambda'}{\sqrt{1+y^2(\ell^2+\lambda'^2)}}\right)'=\frac{2H(y)}{y^2} \tag{7}$$

Particularly, if H constant, we obtain

$$\frac{\lambda'}{\sqrt{1+y^2(\ell^2+\lambda'^2)}}=\frac{yd-2H}{y}\quad\text{(first integral)} \tag{8}$$

We remark that equations (6) and (8) can be alternatively inferred using the techniques derived in [19].

At last, we observe that in our model the *horizontal mean curvature equation* (2), satisfied by a positive smooth function $y=g(x,t)$ can also be inferred in a similar way. For the readers benefit, we give now the principal quantities that arise in the derivation of this formula: The unit normal N pointing towards the half-space $\{y>0\}$ is given by $N=\dfrac{1}{\sqrt{g_t^2+g^2(g_x^2+1)}}\left(-g_xg^2,g^2,-g_t\right)$. The coefficients of the second fundamental form are given by (same notation as before) $l=\left(g_{xx}+1/g+g_x^2/g\right)/\sqrt{g_t^2+g^2(g_x^2+1)}$,
$n=\left(g_{tt}-g_t^2/g\right)/\sqrt{g_t^2+g^2(g_x^2+1)}, m=g_{xt}/\sqrt{g_t^2+g^2(g_x^2+1)}$.

We will see in the next section how these graphs arise in the construction of *parabolic* screw motion surfaces.

2.1. **Complete embedded minimal surfaces.** We will now state the following existence theorem for minimal surfaces in $\mathbb{H}^2\times\mathbb{R}$ obtained by the construction of complete horizontal graphs in our model.

Theorem 2.1 (Existence of embedded minimal surfaces).
A. *The tilted Euclidean half-plane*

$$t=\ell x, y>0 \tag{9}$$

gives rise to a one-parameter family of complete embedded simply connected minimal stable surfaces in $\mathbb{H}^2\times\mathbb{R}$ invariant by parabolic screw

motions. Two such surfaces with different pitch ℓ are non isometric. If $\ell = 0$ we get the hyperbolic plane.

B. *Let us assume $d \neq 0$. The generating curve is obtained by gluing together a convex vertical graph $t = \lambda(y), 0 \leqslant y \leqslant 1/d$, vertical at $y = 1/d$, and its vertical reflection. We obtain a horizontal graph family $y = g_{\mathcal{M}}(x, t, d, \ell), d \neq 0$, of complete embedded simply connected minimal surfaces invariant by parabolic screw motions. Fixing the parameter d varying ℓ we get a family of non-isometric deformations of Daniel-Hauswirth minimal surface, i.e two of them with different pitch ℓ are non-isometric.*

We will see in Section 5 that, making $\ell = 1$ in (9), we obtain a *conjugate Scherk type minimal surface.*

Proof. Observe now that it follows from (8), that $\lambda' \equiv 0$, produces a solution, hence up to vertical translation or symmetry about the xy plane, we may assume that $\lambda \equiv 0$. Now on account of (1), we deduce that the tilted Euclidean half-plane $t = \ell x, y > 0$ gives rise to a one-parameter family of complete embedded minimal stable surfaces in $\mathbb{H}^2 \times \mathbb{R}$ invariant by screw motions. On account of Corollary 4.1, equation (35) (making $H = 0$), we see that two such immersions with different pitch ℓ are non isometric. If $\ell = 0$ we get the hyperbolic plane. Not let us suppose $d \neq 0$. Owing to (8), up to vertical translation or symmetry about the xy plane, we have that the minimal parabolic screw motion vertical graph $t = \lambda(y) + \ell x$, generated by $t = \lambda(y) = \lambda(y, d, \ell), y \leqslant 1/d$ is given by

$$t = d \int_0^y \frac{\sqrt{1 + \xi^2 \ell^2}}{\sqrt{1 - \xi^2 d^2}} \, \mathrm{d}\xi, \qquad (d > 0) \tag{10}$$

Let EllipticE(k) be the complete elliptic integral of second kind. Notice now that we get an incomplete elliptic integral of second kind in (10), since $\lambda(y, d, \ell) = \text{EllipticE}(yd, \sqrt{(-1)\ell^2/d^2})$. Now it is a simple calculation to deduce that $t = \lambda(y)$, defined by (10) is increasing and convex in the interval $[0, 1/d)$, strictly convex in the interval$(0, 1/d)$, and it is vertical at the point $y = 1/d$. Another computation shows that the Euclidean curvature at $y = 1/d$ is finite and that $\lambda(1/d) = \text{EllipticE}(\sqrt{-\ell^2/d^2})$. Hence, by gluing together $t = \lambda(y)$ with its vertical reflection, say Schwarz reflection, given by $2\text{EllipticE}(\sqrt{-\ell^2/d^2}) - \lambda(y)$, we obtain an embedded curve which is complete in the ambient space and has " sinoidal shape" (from the Euclidean view point). Now, in view of (1), by applying successive screw motion to this curve, which is tilted Euclidean translation in the direction of the vector $(1, 0, \ell)$, we

get the horizontal minimal graph $y = g_{\mathcal{M}}(x, t, d, \ell), d \neq 0$. More precisely, this horizontal minimal graph is obtained by gluing together the vertical graph $t = \lambda(y, d, \ell) + \ell x, y \leqslant 1/d$, with its Schwarz reflection given by $2\text{EllipticE}(\sqrt{-\ell^2/d^2}) - \lambda(y, d, \ell) + \ell x$. Of course, this construction yields a complete embedded simply connected minimal surface invariant by parabolic screw motions. Now, by fixing the parameter d, and letting ℓ vary, we get from Corollary 4.1, equation (36)(making $H = 0$) a non-isometric deformation of Daniel-Hauswirth minimal surface ($\ell = 0$). This completes the proof of the Theorem, as desired. □

Remark 1. (1) *Let $t = \lambda(y) = \lambda(y, d, \ell), y \leqslant 1/d, d \neq 0$ a generating curve of an embedded minimal surface given by Theorem 2.1. Recall that by gluing together $t = \lambda(y)$ with its Schwarz reflection, given by $2EllipticE(\sqrt{-\ell^2/d^2}) - \lambda(y)$, we obtain an embedded curve generating an embedded minimal surface. Now, for $\zeta > 0$, let us define $f(y) := \lambda(\zeta y), y \leqslant 1/(\zeta d)$. It is easy to verify that $f(y)$ satisfies the minimal equation (8) ($H = 0$), with pitch $\zeta\ell$ and parameter ζd. Now we observe that the resulting minimal surface, can be obtained geometrically, by applying an horizontal translation (horizontal homothety) $(x, y, \lambda(y) + \ell x) \to (\zeta x, \zeta y, \lambda(y) + \ell x)$, to the original surface. Thus, making $d = \ell$, we obtain a foliation of an open set of the yt-vertical plane ($y > 0$) given by a one-parameter family of embedded complete curves generating a one- parameter family of embedded minimal surfaces. Some of these generating curves are drawn in Figure 2 ($d = \ell = 4, 2, 1, 2/3, 1/2$).*

(2) *Now let us fix ℓ, say $\ell = 1$, for simplicity. We will see that varying d in the interval $(0, \infty)$, we obtain a foliation of the half yt-vertical plane ($y > 0, t > 0$) by generating curves of embedded, minimal, parabolic screw motion surfaces, producing a foliation of an open set of ambient space. Consider positive numbers d_1, d_2 with $d_2 < d_1$. Notice that $\lambda(y, d_1, 1) > \lambda(y, d_2, 1)$ if $y \leqslant 1/d_1$. Now observe also that that $\lambda(1/d) = EllipticE(\sqrt{-1/d^2})$, is a strictly decreasing function in the variable d, satisfying $\lambda(1/d) \to \infty$, as $d \to 0$, and $\lambda(1/d) \to \pi/2$, as $d \to \infty$, Hence, we deduce that for $d_2 < d_1$ the generating curve obtained by $\lambda(y, d_2, 1)$ and its Schwarz reflection "involves" (in the Euclidean sense) entirely the curve determined by $\lambda(y, d_1, 1)$ and its Schwarz reflection, see Figure 3. Consequently, we infer the desired foliation of the half-vertical plane. Hence, we get a foliation of an open set of ambient space by minimal parabolic screw motion surfaces with pitch $\ell = 1$. In Figure 3 are drawn some generating curves for $\ell = 1$ and $d = 1/2, 2/3, 1, 3/2, 2, 10$.*

(3) *Finally, we see that $t = \lambda(y, 1, \ell), y \leqslant 1$ gives the generating curve of a family of non-isometric deformation of Daniel-Hauswirth*

minimal surface, obtained by fixing the parameter $d = 1$, *and varying the pitch* ℓ. *We obtain an embedded minimal surface by gluing together* $t = \lambda(y, 1, \ell) + \ell x, y \leqslant 1$ *and its Schwarz reflection, as we have explained before. Notice that this family of generating curves have self-intersections. We draw in Figure 4, some examples for* $\ell = 1/100, 1, 2, 3$. *We leave to the reader the description of their asymptotic boundaries.*

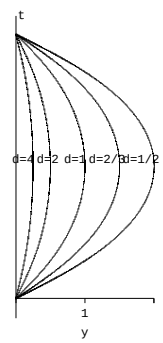

Figure 2

Generating curves of embedded minimal surfaces obtained by horizontal translations varying the pitch $\ell = d$

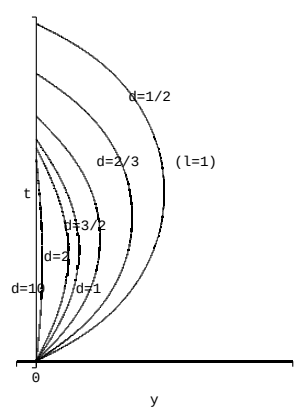

Figure 3

Foliation given by generating curves of embedded minimal surfaces obtained varying d *and fixing the pitch* $\ell = 1$

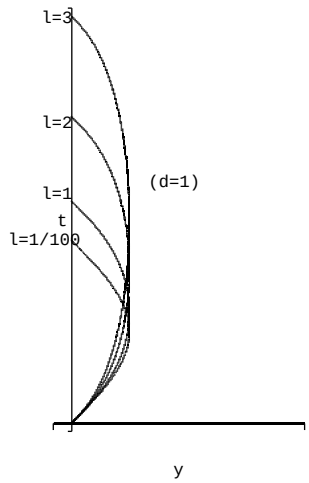

Figure 4

Generating curves of embedded minimal surfaces obtained varying the pitch ℓ and fixing the parameter $d = 1$

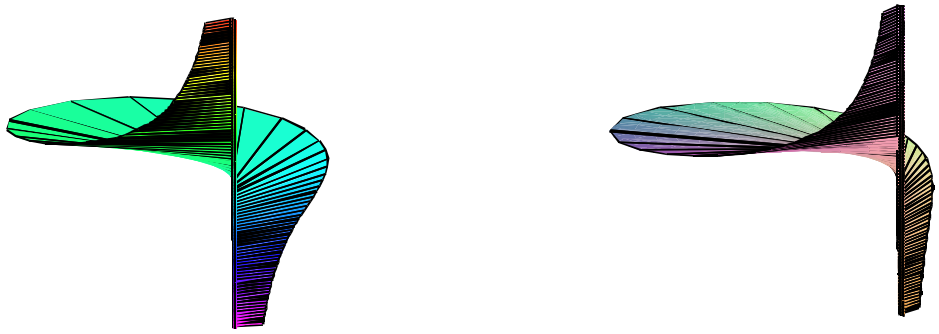

Figure 5: ball model for $\mathbb{H}^2 \times \{0\}$

Non-isometric deformation in $\mathbb{H}^2 \times \mathbb{R}$ of hyperbolic plane ($\ell = 1$)

Conjugate Scherk type minimal surface

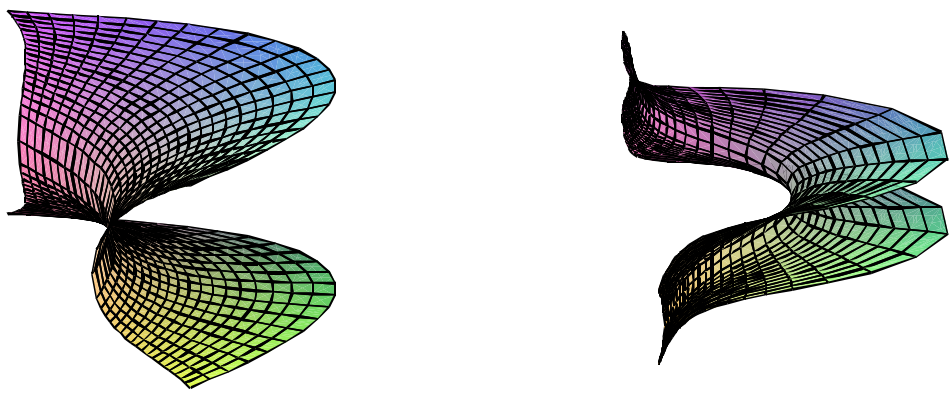

Figure 6: ball model for $\mathbb{H}^2 \times \{0\}$
Non-isometric deformation ($d = \ell = 1$) in $\mathbb{H}^2 \times \mathbb{R}$
of Daniel-Hauswirth minimal surface

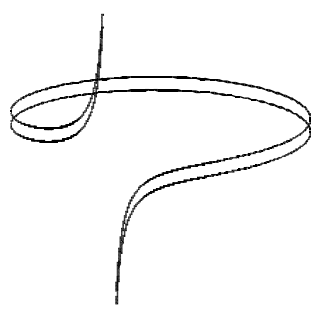

Figure 6b: ball model for $\mathbb{H}^2 \times \{0\}$
Asymptotic boundary of a non-isometric deformation
of Daniel-Hauswirth minimal surface ($d = \ell = 1$)

2.2. **Complete embedded H-surfaces.** We will prove our existence results for parabolic screw motion H-surfaces in $\mathbb{H}^2 \times \mathbb{R}$, if the mean curvature satisfies $|H| < 1/2$. We will construct complete H-surfaces given by vertical graphs over the entire hyperbolic plane; hence they are stable. It is an interesting question if there exists a non compact stable H-surface in $\mathbb{H}^2 \times \mathbb{R}$ with mean curvature bigger than $1/2$. This is related with results derived by Barbara Nelli and Harold Rosenberg. These authors proved that in $\mathbb{H}^2 \times \mathbb{R}$, there is no non compact stable H-surface with $H > 1/\sqrt{3}$, either with compact boundary or without boundary [15]. Does there exist a complete non compact stable H-surface with $\frac{1}{2} < H \leqslant \frac{1}{\sqrt{3}}$? For $H = 1/2$ surfaces of revolution provide examples of complete non compact stable H-surfaces, which have a simple explicit expression, see [1] or [15]. See also [19]. We will obtain in the next section examples of entire vertical graphs with mean curvature $H = 1/2$. We have the following:

Theorem 2.2 (Existence of embedded H-surfaces). *For any $H < 0$ satisfying $4H^2 < 1$, there exists a one-parameter explicit family of vertical H-graphs with pitch ℓ, each one is stable and embedded, given by $t = u_{\mathcal{H}}(x, y, \ell) := \lambda(y, \ell) + \ell x$ over the entire hyperbolic plane $\mathbb{H}^2 \times \{0\}$. Varying ℓ we obtain a family of non-isometric, simply connected and embedded, stable deformations of Abresch-Rosenberg surface ($\ell = 0$). More precisely, the generating curve $t = \lambda(y, \ell)$ is given by*

$$t = \lambda(y, \ell) = \frac{-2H}{\sqrt{1 - 4H^2}} \left[\ln \left(\frac{\sqrt{1 + \ell^2 y^2} - 1}{\sqrt{1 + \ell^2 y^2} + 1} \right)^{1/2} + \sqrt{1 + \ell^2 y^2} \right] \ (\ell \neq 0) \tag{11}$$

For any $H < 0$ satisfying $4H^2 < 1$, there exists a two-parameter explicit family of horizontal H-graphs given by $y = g_{\mathcal{H}}(x, t, d, \ell)$ over the entire xt plane, each one is embedded, that contains the above family : If $d = 0$ the generating curve is given by (11). These horizontal H-graphs are obtained by applying Schwarz reflection to a vertical parabolic screw motion graph generated by $t = \lambda(y, d, \ell), 0 < y < (1 - 2|H|)/d$, and its vertical reflection at the vertical point $y = (1 - 2|H|)/d$. Fixing d and letting ℓ vary, we obtain a family of non-isometric deformations. Thus, each such H-surface is a complete simply connected embedded surface, invariant by parabolic screw motions.

Proof. Owing to (8), up to vertical translation or symmetry about the xy plane, we have that a screw motion H-vertical graph
$t = \lambda(y, d, \ell) + \ell x$, with $4H^2 < 1$, generated by $t = \lambda(y, d, \ell)$, is given

by

$$t = \int_{*}^{y} \frac{(2|H| + \xi d)\sqrt{1+\xi^2\ell^2}}{\xi\sqrt{1-(2|H|+\xi d)^2}}\,\mathrm{d}\xi \qquad (d \geqslant 0) \tag{12}$$

Letting $d = 0$ in (12) we infer

$$t := \lambda(y,\ell) = \frac{2|H|}{\sqrt{1-4H^2}} \int_{*}^{y} \frac{\sqrt{1+\xi^2\ell^2}}{\xi}\,\mathrm{d}\xi \tag{13}$$

The behavior of $\lambda(y,\ell)$ can be analyzed as follows. Notice that on account of (8) we are assuming that $4H^2 < 1$, with $H < 0$. Now clearly $\lambda(y,\ell)$ is an increasing function for $y > 0$. A computation shows that $\lambda(y,\ell)$ is strictly concave. Next, notice that $\sqrt{1+y^2\ell^2} = 1 + \frac{\ell^2}{2}y^2 + o(y^2)$, near $y = 0$; hence $\lambda(y,\ell)$ has a log behavior as $y \to 0$. Clearly at infinity $\lambda(y,\ell)$ has a linear behavior. Thus, $u_{\mathcal{H}}(x,y,\ell) := \lambda(y,\ell) + \ell x$ is a vertical graph over the entire hyperbolic plane $\mathbb{H}^2 \times \{0\}$, that yields a complete horizontal H-graph over the (x,t) plane, as well. We conclude therefore there exists a one-parameter family of vertical H-graphs, and each one is stable and embedded. On the other hand, observe that equation (13) is easily solved by elementary integration techniques: we therefore obtain the explicit form (11) in the Statement, whose graph, for $H = -1/4, \ell = 1$, is drawn in Figure 7.

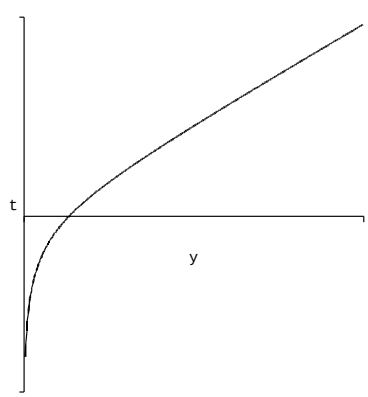

Figure 7

Generating curve of an embedded H-surface belonging to the family ($d = 0$) given by the elementary formula (11)

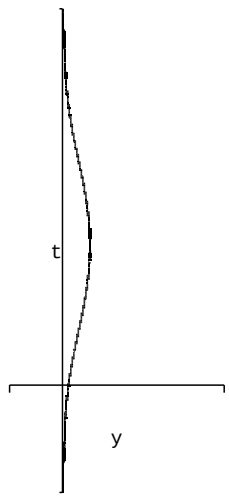

Figure 8

Generating curve of an embedded H-surface ($d = \ell = 1$)

Now, if $d \neq 0$, we define $\lambda(y\ d, \ell)$ by (12). Clearly, $\lambda(y, d, \ell)$ is an increasing function, vertical at $y = (1 - 2|H|)/d$. An analogous discussion as before ensures that the local behavior of $\lambda(y, d, \ell)$ at $y = 0$ is log type. Now after a computation we infer that at $y = (1 - 2|H|)/d$ the Euclidean curvature is finite, hence by vertical reflection at $y = (1-2|H|)/d$, we obtain a complete horizontal H-graph $y = g_{\mathcal{H}}(x, t, d, \ell)$ ($4H^2 < 1$) over the entire (x, t) plane, which is simply connected, embedded and invariant by parabolic screw motions. The generating curve for $H = -1/4, d = \ell = 1$, is drawn in Figure 8. The fact that fixing d varying ℓ the family $t = u_{\mathcal{H}}(x, y, \ell)$ ($d = 0$) determined by (11), and the family $y = g_{\mathcal{H}}(x, t, d, \ell)$ is formed by non-isometric H-surfaces follows from Corollary 4.1. This concludes the proof of the Theorem, as desired. $\square$

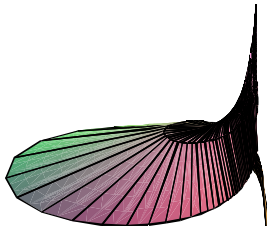

Figure 9: ball model for $\mathbb{H}^2 \times \{0\}$
Non-isometric, simply connected and embedded stable deformation ($d = 0, \ell = 1$) in $\mathbb{H}^2 \times \mathbb{R}$ of Abresch-Rosenberg surface, $H = -1/4$

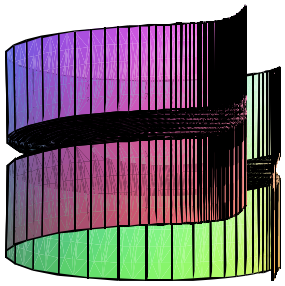

Figure 10: ball model for $\mathbb{H}^2 \times \{0\}$
Complete embedded H-surface ($d = 1, \ell = 1$) in $\mathbb{H}^2 \times \mathbb{R}$, invariant by parabolic screw motions, $H = -1/4$

3. A generalized Bour's Lemma

We will now prove a generalized Bour's Lemma that can be applied to parabolic and hyperbolic screw motion surfaces in $\mathbb{H}^2 \times \mathbb{R}$, and screw motion surfaces in Heisenberg's space. This generalizes a result in [19]. Javier Ordóñes in his Doctoral Thesis at PUC-Rio [16], following ideas of Manfredo Do Carmo and Marcos Dajczer [4], proved another generalization of Bour's Lemma applied to surfaces invariant by screw motions in a three dimensional space form. It is reasonable to expect a general Bour's Lemma that holds in all situations. But we will not pursue this task here.

We will now consider the three dimensional Riemannian manifold M^3 given by an open set of the Euclidean three space $\mathbb{R}^3$ equipped with coordinates $\{(\rho, \varphi, t), \rho > 0\}$ and metric $d\sigma^2$ given by

$$d\sigma^2 = \Psi^2(\rho)\,\mathrm{d}\rho^2 + \Phi^2(\rho)\,\mathrm{d}\varphi^2 - \Lambda^2(\rho)\,\mathrm{d}t\,\mathrm{d}\varphi + \mathrm{d}t^2 \tag{14}$$

where $\Psi^2, \Phi^2, \Lambda^2$, satisfy some further conditions, as we will see in the sequel. We will call Γ the one-parameter group of isometries acting on M^3 by translation on the φ variable.

Definition 3.1. *We say that a surface S immersed into M^3 is a Γ-screw motion surface (with vertical axis t), if it is invariant by successive compositions of a element of Γ with a vertical translation. More precisely, using coordinates (ρ, φ, t), S is given by $(\rho, \varphi) \mapsto (\rho, \varphi, \lambda(\rho) + \ell\varphi)$, where $\ell \geqslant 0$, is called the* pitch. *We say that the curve $(\rho, \lambda(\rho))$ lying on $\{\varphi = 0\}$ is the generating curve.*

We have studied in [19] the generating curve of standard screw motion surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$. In the previous section, see Section 2, we studied the generating curve of *parabolic* screw motion surfaces in $\mathbb{H}^2 \times \mathbb{R}$.

Next, let us give some motivating examples to see that the metric given by (14) appears naturally in this context:

Remark 2.

(1) The ambient space is $\mathbb{H}^2 \times \mathbb{R}$.

a) Standard *screw motion surfaces. Here we take the ball model $\mathbb{H}^2 = \{(x, y), x^2 + y^2 < 1\}$ as the hyperbolic plane equipped with the hyperbolic metric $\dfrac{4}{(1 - |z|^2)^2}|\,\mathrm{d}z|^2$. The metric in $\mathbb{H}^2 \times \mathbb{R}$ using cylindrical coordinates (ρ, φ, t), (here ρ is the hyperbolic distance measure from the origin of $\mathbb{H}^2$ i.e $R = \tanh \rho/2, R = \sqrt{x^2 + y^2}$ and t is the height), is given by $\mathrm{d}\sigma^2 = \mathrm{d}\rho^2 + \sinh^2 \rho\,\mathrm{d}\varphi^2 + \mathrm{d}t^2$. Thus we have*

$\Phi^2 = \sinh^2\rho, \Phi'^2 = \cosh^2\rho = F_1(\Phi^2)$, *where* $F_1(u) = 1+u$. $\Psi^2 \equiv 1$ *and* $\Lambda \equiv 0$.
b) Parabolic *screw motions surfaces. We take the upper half plane model* $\mathbb{H}^2 = \{(x,y), y>0\}$ *equipped with the hyperbolic metric* $\frac{\mathrm{d}x^2}{y^2} + \frac{\mathrm{d}y^2}{y^2}$. *We set* $y=\rho$ *and* $x=\varphi$. *The metric is given by* $\mathrm{d}\sigma^2 = \frac{\mathrm{d}x^2}{y^2} + \frac{\mathrm{d}y^2}{y^2} + \mathrm{d}t^2$. *We have* $\Phi^2 = 1/y^2, \Phi'^2 = 1/y^4 = F_1(\Phi^2)$, *where* $F_1(u) = u^2$. $\Psi^2 = G_1(\Phi^2)$, *where* $G_1(u) = u$ *and* $\Lambda \equiv 0$.
c) Hyperbolic *screw motion surfaces. We take again the upper half plane model and polar coordinates* $x = R\cos\theta, y = R\sin\theta, 0<\theta<\pi, R>0$. *In view of* (14), *we set* $\rho := \theta$ *and* $\mathrm{e}^{\varphi} := R$. *Thus we have* $\mathbb{H}^2 = \{(x = \mathrm{e}^{\varphi}\cos\rho, y = \mathrm{e}^{\varphi}\sin\rho), 0<\rho<\pi\}$, *equipped with the hyperbolic metric. The metric in* $\mathbb{H}^2\times\mathbb{R}$ *using coordinates* (ρ,φ,t) *is given by* $\mathrm{d}\sigma^2 = \frac{\mathrm{d}\rho^2}{\sin^2\rho} + \frac{\mathrm{d}\varphi^2}{\sin^2\rho} + \mathrm{d}t^2$. *We have* $\Phi^2 = 1/\sin^2\rho, \Phi'^2 = F_1(\Phi^2)$ *where* $F_1(u) = u^2 - u$. $\Psi^2 = G_1(\Phi^2)$, *where* $G_1(u) = u$, *and* $\Lambda \equiv 0$.
(2) Screw motion *surfaces in* $\mathbb{S}^2\times\mathbb{R}$. *Now let* $\mathbb{S}^2 = \mathbb{C}\cup\{\infty\}$ *be the sphere equipped with the spherical metric* $\frac{4}{(1+|z|^2)^2}|\,\mathrm{d}z|^2$. *The metric in* $\mathbb{S}^2\times\mathbb{R}$ *using cylindrical coordinates* (ρ,φ,t), *where* ρ *is the sphere distance measure from the origin of* $\mathbb{S}^2$ *i.e* $R = \tan\rho/2, R = \sqrt{x^2+y^2}$ *and* t *is the height, is given by* $\mathrm{d}s^2 = \mathrm{d}\rho^2 + \sin^2\rho\,\mathrm{d}\varphi^2 + \mathrm{d}t^2$. *We have* $\Phi^2 = \sin^2\rho, \Phi'^2 = F_1(\Phi^2)$, *where* $F_1(u) = 1-u$. $\Psi^2 \equiv 1$ *and* $\Lambda \equiv 0$.
(3) Screw motion *surfaces in Heisenberg's space. Consider* $\mathbb{R}^3$ *equipped with the metric* $\mathrm{d}\sigma^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \left[\mathrm{d}t + \frac{1}{2}(y\,\mathrm{d}x - x\,\mathrm{d}y)\right]^2$. *We write* $x = 2\sinh(\rho/2)\cos\varphi, y = 2\sinh(\rho/2)\sin\varphi$. *The metric using coordinates* (ρ,φ,t) *is given by*
$\mathrm{d}\sigma^2 = \cosh^2(\rho/2)\,\mathrm{d}\rho^2 + \sinh^2\rho\,\mathrm{d}\varphi^2 - 4\sinh^2(\rho/2)\,\mathrm{d}t\,\mathrm{d}\varphi + \mathrm{d}t^2$. *We have* $\Phi^2 = \sinh^2\rho, \Phi'^2 = F_1(\Phi^2)$, *where* $F_1(u) = 1+u$. $\Psi^2 = G_1(\Phi^2)$, *where* $G_1(u) = (1+\sqrt{1+u})/2$. $\Lambda^2 = G_2(\Phi^2)$, *where*
$G_2(u) = 2(\sqrt{1+u}-1), \Lambda'^2 = G_1(\Phi^2)$.

Next, we will prove a central result in this Section. It contains some apparently complicated formulas but, as we will see afterwards, when it is specialized to parabolic or hyperbolic screw motions surfaces, the formulas look much nicer.

Theorem 3.1 (Generalized Bour's Lemma). *Let* M^3 *be the Euclidean three space endowed with the metric given by* (14) *and one-parameter group of isometries* Γ. *Let us assume* $\Phi^2 + \ell^2 > \ell\Lambda^2$ *and* $\Phi^2 > \Lambda^4/4$.

Then, any surface invariant by Γ-screw motions can be parametrized locally by natural coordinates s, τ, such that the induced metric $\mathrm{d}\mu^2$ is given by

$$\mathrm{d}\mu^2 = \mathrm{d}s^2 + U^2(s)\,\mathrm{d}\tau^2 \tag{15}$$

Furthermore, assume that $\Phi^2 = f(U^2), \Phi'^2 = F_1(\Phi^2), \Psi^2 = G_1(\Phi^2), \Lambda^2 = G_2(\Phi^2)$ and $\Lambda'^2 = G_3(\Phi^2)$, where $f(u), F_1(u), G_1(u), G_2(u), G_3(u)$, are smooth real functions for $u \geqslant 0$. Let S be such a Γ-screw motion surface. Then, there exists a two-parameter family $\mathcal{F}(m, \ell), m \neq 0$ of Γ-screw motion surfaces isometric to S, given by

$$\begin{aligned}
m^2U^2 &= \ell^2 + \Phi^2 - \Lambda^2\ell \\
\rho'^2(s) &= \frac{\ell^2 + \Phi^2 - \Lambda^2\ell}{\Psi^2\left(\ell^2 + \Phi^2 - \Lambda^2\ell\right) + \lambda'^2(\rho)(\Phi^2 - \Lambda^4/4)} \\
\rho'^2(s) &= \frac{m^4U^2U'^2}{\left[\sqrt{f(U^2)}\sqrt{F_1(f(U^2))} - \ell\sqrt{G_2(f(U^2))G_3(f(U^2))}\right]^2}
\end{aligned}$$

$$\lambda\circ\rho(s) = \int \frac{mU}{\left[m^2U^2 - \left(\ell - \frac{G_2(f(U^2))}{2}\right)^2\right]^{1/2}} \cdot B\,\mathrm{d}s$$

where

$$B(s) = \left[1 - \frac{m^4U^2U'^2G_1(f(U^2))}{\left[\sqrt{f(U^2)F_1(f(U^2))} - \ell\sqrt{G_2(f(U^2))G_3(f(U^2))}\right]^2}\right]^{1/2}$$

$$\varphi(s,\tau) = \frac{\tau}{m} - \int \frac{(\lambda\circ\rho)'\left[\ell - \frac{G_2(f(U^2))}{2}\right]}{m^2U^2}\,\mathrm{d}s \tag{16}$$

Proof. The proof will proceed in the same sprit as in [19]. In view of (14) and $t = \lambda(\rho) + \ell\varphi$, we deduce that the induced metric $\mathrm{d}\mu^2$ of a given screw motion Γ-surface S immersed into M^3 is given by

$$\mathrm{d}\mu^2 = \left(\Psi^2 + \lambda'^2\right)\mathrm{d}\rho^2 + \left(\Phi^2 + \ell^2 - \Lambda^2\ell\right)\mathrm{d}\varphi^2 + \left(2\ell\lambda' - \Lambda^2\lambda'\right)\mathrm{d}\rho\,\mathrm{d}\varphi \tag{17}$$

Now re-write the above equation to obtain

$$\begin{aligned}\mathrm{d}\mu^2 = &\underbrace{\left[\Psi^2 + \frac{\lambda'^2\left(\Phi^2 - \Lambda^4/4\right)}{\ell^2 + \Phi^2 - \Lambda^2\ell}\right]\mathrm{d}\rho^2}_{\mathrm{d}s^2} \\ &+ \underbrace{\left[\ell^2 + \Phi^2 - \Lambda^2\ell\right]}_{U^2}\underbrace{\left[\mathrm{d}\varphi + \frac{(\ell - \Lambda^2/2)}{\ell^2 + \Phi^2 - \Lambda^2\ell}\,\mathrm{d}\lambda{\circ}\rho\right]^2}_{\mathrm{d}\tau^2}\end{aligned} \tag{18}$$

This equation leads to the following system

$$\mathrm{d}s = \sqrt{\Psi^2 + \frac{\lambda'^2\left(\Phi^2 - \Lambda^4/4\right)}{\ell^2 + \Phi^2 - \Lambda^2\ell^2}}\;\mathrm{d}\rho \tag{19}$$

$$\mathrm{d}\tau = \mathrm{d}\varphi + \frac{\lambda'\left(\ell - \Lambda^2/2\right)}{\ell^2 + \Phi^2 - \Lambda^2\ell}\,\mathrm{d}\rho \tag{20}$$

By the Implicit Function Theorem (due to the fact $\frac{\partial(s,\tau)}{\partial(\rho,\varphi)} \neq 0$), we define locally natural coordinates s, τ. Notice that ρ and λ do not depend on τ, hence $U = U(s)$ and we deduce (15). Next we search for an explicit parametrization of an arbitrary screw motion surface (with pitch denoted by ℓ for convenience also) isometric to S by natural coordinates s, τ, involving a simple expression in terms of U and parameters ℓ, m as in the statement. Notice that we may suppose $U > 0$, since we assume momentarily $\rho > 0$ (extension to $\rho = 0$ or negative requires some additional argument). In view of (19) we infer the *third equation* in the Statement, hence we have

$$(\lambda{\circ}\rho)'^2 = \left(\frac{\ell^2 + \Phi^2 - \Lambda^2\ell}{\Phi^2 - \Lambda^4/4}\right)\left(1 - \rho'^2\Psi^2\right) \tag{21}$$

Now on account of (18) we see that the expression of U^2 is given by

$$U\,\mathrm{d}\tau = \pm\sqrt{\ell^2 + \Phi^2 - \Lambda^2\ell}\,\left[\mathrm{d}\varphi + \frac{\lambda'\left(\ell - \Lambda^2/2\right)}{\ell^2 + \Phi^2 - \Lambda^2\ell}\,\mathrm{d}\rho\right]$$

Thus

$$\frac{\partial\varphi}{\partial\tau} = \pm\frac{U}{\sqrt{\ell^2 + \Phi^2 - \Lambda^2\ell}} \tag{22}$$

$$\frac{\partial\varphi}{\partial s} = \frac{-\lambda'\rho'\left(\ell - \Lambda^2/2\right)}{\ell^2 + \Phi^2 - \Lambda^2\ell} \tag{23}$$

In view of (22) and (23) we deduce that $\dfrac{\partial\varphi}{\partial\tau}$, does not depend on s; hence we obtain the crucial formula

$$\pm\frac{U}{\sqrt{\ell^2+\Phi^2-\Lambda^2\ell}}=\frac{1}{m},\qquad m\neq 0 \tag{24}$$

Then we derive the *second formula* in the Statement. Owing to (24) by performing some calculations we are able to express ρ'^2 in terms of U^2, U'^2, m^4 and $(\Phi\Phi'-\Lambda\Lambda'\ell)^2$. Again taking into account (24) by substituting $\Phi,\Phi',\Lambda,\Lambda'$ in terms of $\Phi^2=f(U^2)$, in view of(21), we derive the *fourth and fifth formula* in the Statement. The *last formula* in the Statement follows combining (22), (23), (24) with the fifth formula in the Statement. This concludes the proof of the second part of the Statement and concludes the proof of the Theorem.

□

Remark 3. *We remark that fromTheorem 3.1, we recover the formulas given in* [19]*, Theorem 19 and Theorem 20, for screw motions surfaces in* $\mathbb{H}^2\times\mathbb{R}$ *and* $\mathbb{S}^2\times\mathbb{R}$*, see Remark 2 example* (1)a), (2)*, above. In fact, when the ambient space is the product* $M^2\times\mathbb{R}$*, we have* $\Lambda\equiv 0$ *(*$G_2=G_3\equiv 0$*). Moreover, we have* $\Phi^2=f(U^2)$*, where* $f(u)=m^2u-\ell^2$*,* $m>0,\ell\geqslant 0$*. Hence, the second and the third formulas in* (16) *simplify. The forth and the last two formulas become simpler*

$$\begin{aligned}
\rho'^2 =&\frac{m^4U^2U'^2}{(m^2U^2-\ell^2)F_1(m^2U^2-\ell^2)}\\
\lambda\circ\rho(s)=&\int\frac{mU}{\left[m^2U^2-\ell^2\right]^{1/2}}\left[1-\frac{m^4U^2U'^2G_1(m^2U^2-\ell^2)}{(m^2U^2-\ell^2)F_1(m^2U^2-\ell^2)}\right]^{1/2}\mathrm{d}s\\
\varphi(s,\tau)=&\frac{\tau}{m}-\ell\int\frac{(\lambda\circ\rho)'}{m^2U^2}\,\mathrm{d}s
\end{aligned} \tag{25}$$

Now let us turn attention to Heisenberg's space: In view of Remark 2, example (3) *and Theorem 3.1, if* M^3 *is Heisenberg space,* $\Phi^2=f(U^2)$*, where*
$f(u)=\left(\ell+\sqrt{1-2\ell+m^2u}\right)^2-1, m>0, 0\leqslant\ell\leqslant 1$*, since* $\Phi^2=\sinh^2\rho$ *and* $m^2U^2=\ell^2+\sinh^2\rho-4\ell\sinh^2(\rho/2),\quad(0\leqslant\ell\leqslant 1)$*. Hence, we can establish the following formulas for screw motion surfaces in*

Heisenberg's space.

$$\lambda\circ\rho(s) = \int \frac{mU\left[\left(2(\ell-1)+2\sqrt{1-2\ell+m^2U^2}\right)(1-2\ell+m^2U^2)-m^4U^2U'^2\right]^{1/2}}{2\left(\sqrt{1-2\ell+m^2U^2}+\ell-1\right)\sqrt{1-2\ell+m^2U^2}}$$

$$\varphi(s,\tau) = \frac{\tau}{m} - \int \frac{(\lambda\circ\rho)'\left[1-\sqrt{1-2\ell+m^2U^2}\right]}{m^2U^2}\,\mathrm{d}s \tag{26}$$

We also observe that, in view again of the previous Remark 2, examples (1)c) and (3) and (25), explicit formulas may be written either for *parabolic* or *hyperbolic* screw motion surfaces in $\mathbb{H}^2\times\mathbb{R}$.

4. Complete embedded minimal and constant mean curvature surfaces in $\mathbb{H}^2\times\mathbb{R}$ invariant by hyperbolic screw motions

Notice that *hyperbolic* screw motion surfaces in $\mathbb{H}^2\times\mathbb{R}$, can be studied in the same way as *parabolic* screw motion surfaces. Recall the definition: $x = \mathrm{e}^{\varphi}\cos\rho, y = \mathrm{e}^{\varphi}\sin\rho, t = \lambda(\rho)+\ell\varphi, 0<\rho<\pi$. Here we take the upper half-plane model for $\mathbb{H}\times\{0\}$. As in [19], we derive that the mean curvature equation (with respect to the unit normal $N = (1/\sqrt{1+\sin^2\rho\,(\ell^2+\lambda'^2)}\,)(\sin^2\rho\;\mathrm{e}^{\varphi+i\rho}(\ell+i\lambda'),-1)$[complex notation]) is the following:

$$\begin{aligned}2H\left(1+\ell^2\sin^2\rho+\lambda'^2\sin^2\rho\right)^{3/2}\\ = -\lambda''\sin^2\rho\,(1+\ell^2\sin^2\rho)+\lambda'\cos\rho\sin^3\rho\,(\ell^2+\lambda'^2)\end{aligned}$$

If H is constant, we deduce:

$$\frac{\lambda'}{\sqrt{1+\ell^2\sin^2\rho+\lambda'^2\sin^2\rho}} = 2H\cot\rho+d \quad (\ H \text{ is constant}) \tag{27}$$

Notice that in view of (27), we deduce that the generating curve of an H-hyperbolic screw motion surface is given by (write $\theta=\rho$)

$$t=\lambda(\theta) = \int_*^{\theta} \frac{\sqrt{1+\ell^2\sin^2\rho}\;(d+2H\cot\rho)}{\sqrt{1-\sin^2\rho\;(d+2H\cot\rho)^2}}\,\mathrm{d}\rho \tag{28}$$

Now observe that any *hyperbolic* screw motion surface obtained by (28) is stable, since it is a vertical graph given by

$$t=\lambda(\operatorname{arc\,cot}(x/y))+\frac{\ell}{2}\ln\left(x^2+y^2\right)$$

Recall that vertical translations of an H-vertical graph are isometric deformations of ambient space producing a foliation that ensure

stability. Notice that if $\lambda \equiv 0$ (or $d = 0$), we obtain a complete, embedded, simply connected, stable minimal surface in $\mathbb{H}^2 \times \mathbb{R}$, invariant by hyperbolic screw motions (with pitch ℓ): Define $(R, \theta) \mapsto (R\cos\theta, R\sin\theta, \ell \ln R), R > 0, 0 < \theta < \pi$. In fact , in view of (27) and (28), just make $\theta = \rho, R = e^{\varphi}$. In fact, these one-parameter family of minimal surfaces, as vertical graphs over the entire hyperbolic plane, are given by the following explicit non-parametric formula:

$$t = \frac{\ell}{2} \ln(x^2 + y^2), \qquad y > 0 \tag{29}$$

Notice that we may check that (29) yields a minimal surface, with the aid of the non-parametric equation (4). Observe now that by making $2H = 1$ (downward pointing inner unit normal) and $d = 0$ in (27) or (28), we obtain a complete, embedded, simply connected, stable H-surface in $\mathbb{H}^2 \times \mathbb{R}$, invariant by hyperbolic screw motions (with pitch ℓ). This surface is a vertical H-graph with $H = -1/2$ (upward pointing inner unit normal), over the entire hyperbolic plane, given by the following explicit formula:

$$t = -\frac{\sqrt{x^2 + y^2 + \ell^2 y^2}}{y} + \ell \left[\ln \left(\sqrt{x^2 + y^2 + \ell^2 y^2} + \ell y \right) \right], \qquad y > 0 \tag{30}$$

Letting $\ell = 0$ in (30) and taking the the symmetric with respect the horizontal xy-plane, we derive a quite simple formula for a H-vertical graph, over the entire hyperbolic plane ($H = 1/2$), invariant by hyperbolic translations:

$$t = \frac{\sqrt{x^2 + y^2}}{y}, \qquad y > 0 \tag{31}$$

Notice that this entire 1/2-graph (31), invariant by hyperbolic translations, has the property that the level curve $\{t = 1\}$ is a geodesic and the level curves $\{t = c,\ c > 1\}$ are equidistant curves in $\mathbb{H}^2$.

Of course, using (4) we readily check that (30) or (31) are H-vertical graphs, indeed. Very recently Isabel Fernández and Pablo Mira [7] gave a characterization of (31).

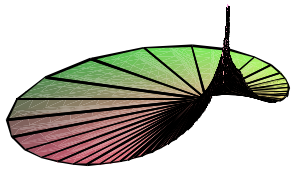

Figure 11: ball model for $\mathbb{H}^2 \times \{0\}$
Complete embedded stable minimal surface ($d = 0, \ell = 1$) in $\mathbb{H}^2 \times \mathbb{R}$, invariant by hyperbolic screw motions

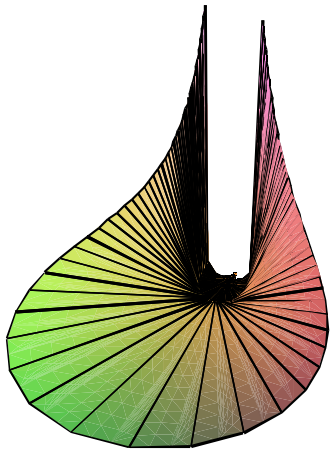

Figure 12: ball model for $\mathbb{H}^2 \times \{0\}$
Complete embedded stable H-surface ($d = 0, \ell = 0$) in $\mathbb{H}^2 \times \mathbb{R}$, invariant by hyperbolic translations, $H = 1/2$

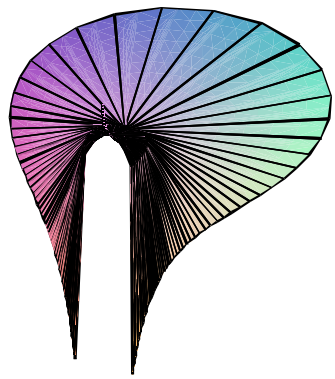

Figure 13: ball model for $\mathbb{H}^2 \times \{0\}$
Complete embedded stable H-surface ($d = 0, \ell = 1$) in $\mathbb{H}^2 \times \mathbb{R}$, invariant by hyperbolic screw motions, $H = -1/2$

Now letting $H = 0$ and $d = 1, \ell = 0$ in (28), we obtain a Scherk type minimal surface, invariant by hyperbolic translations, given by the following explicit formula

$$t = \ln\left(\frac{\sqrt{x^2+y^2}+y}{x}\right), \qquad y > 0, x > 0 \tag{32}$$

Notice that this function takes infinite boundary value data on the positive y axis and zero asymptotic value boundary data at the positive x axis. So by applying an (horizontal) isometry of ambient space, we have the situation that we have drawn schematically in Figure 1: Scherk can be seen taking $\pm\infty$ value boundary data on a geodesic and zero (or constant) asymptotic value boundary data.

We remark that formula (32) was used by Pascal Collin and Harold Rosenberg [5] in the construction of entire minimal graphs in $\mathbb{H}^2 \times \mathbb{R}$ that are conformally the complex plane $\mathbb{C}$, disproving a conjecture by R. Schoen.

Figure 14: ball model for $\mathbb{H}^2 \times \{0\}$

Part of Scherk type minimal surface ($\ell = 0, d = 1$) in $\mathbb{H}^2 \times \mathbb{R}$, invariant by hyperbolic translations

Again, in view of (4), it is easily verified that (32) yields a minimal surface. Letting $H = 0$ and $\ell = 0, d = 1/2$ in (28), we obtain a complete, embedded, stable minimal surface invariant by hyperbolic translations. Letting $H = 1/4, \ell = 0, d = 1/2$ in (28), yields a complete, embedded, stable H-surface invariant by hyperbolic translations. The facts that

these H-surfaces are complete and embedded, follows from the analysis of the generating curves. Other such surfaces may be constructed in the same way.

We pause now to say that Scherk type surface (32) (see Figure 14) can be applied to derive results for the Dirichlet problem for the minimal equation in $\mathbb{H}^2 \times \mathbb{R}$. Very recently, Pascal Collin and Harold Rosenberg have proved the existence of harmonic diffeomorfism from $\mathbb{C}$ to $\mathbb{H}^2$ by studying complete minimal graphs in $\mathbb{H}^2 \times \mathbb{R}$. They used formula (32), as we remarked before [5].

The following remark is due to Harold Rosenberg:

Remark 4. *Abresch and Rosenberg studied the asymptotic values of minimal graphs in $\mathbb{H}^2 \times \mathbb{R}$ over domains in $\mathbb{H}^2$. In particular, they observed that there is no minimal graph u in a domain W of $\mathbb{H}^2$, taking infinite asymptotic values on an arc l of the asymptotic boundary of W. See Figure 17.*

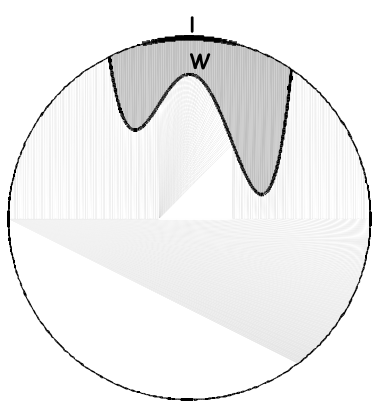

Figure 17: ball model for $\mathbb{H}^2 \times \{0\}$

This can be seen using the graph S given by (32)*, see also Figure 1, with value $-\infty$ on a geodesic arc g inside W, with boundary two points of l. Choose g so that the assumed solution u (plus infinite on l), is*

positive on the domain U bounded by g and arc of l, see Figure 18.

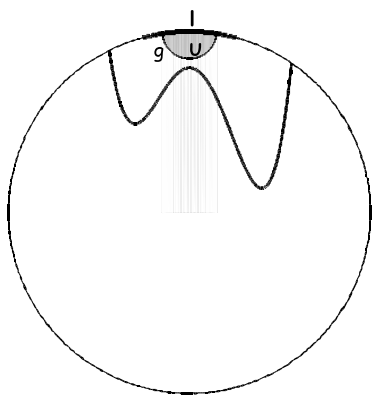

Figure 18: ball model for $\mathbb{H}^2 \times \{0\}$

Clearly the graph of Scherk is bellow u, where Scherk is defined. Now vertically translate up the graph of Scherk. Then there is a first point of contact of the graph with the graph of u which is impossible, by the maximum principle.

Rosenberg conjectured that there is no minimal graph over a domain in $\mathbb{H}^2$ with asymptotic values infinity on a set of positive measure of the circle at infinity.

We will now establish the following results for further reference.

THEOREM 4.1. *Any surface invariant by parabolic screw motions in $\mathbb{H}^2 \times \mathbb{R}$ can be parametrized locally by natural coordinates s, τ. Let S be such a parabolic screw motion surface. Then there exists a two parameters family $\mathcal{F}(m, \ell), m \neq 0$, containing a surface invariant by parabolic translations, such that each element of the family is a parabolic screw motion surface isometric to S given by*

$$
\begin{aligned}
m^2U^2(s) &= \frac{1}{y^2} + \ell^2 \\
\rho'^2(s) &= \frac{y^2(1+y^2\ell^2)}{1+y^2\ell^2+y^2(\lambda'(y))^2} \\
\rho'^2(s) &= \frac{m^4U^2U'^2}{(m^2U^2-\ell^2)^3} \\
\lambda\circ\rho(s) &= \int \frac{mU\sqrt{(m^2U^2-\ell^2)^2 - m^4U^2U'^2}}{\sqrt{(m^2U^2-\ell^2)^3}}\,\mathrm{d}s \\
\varphi(s,\tau) &= \frac{\tau}{m} - \ell\int \frac{\sqrt{(m^2U^2-\ell^2)^2 - m^4U^2U'^2}}{mU\sqrt{(m^2U^2-\ell^2)^3}}\,\mathrm{d}s
\end{aligned}
\tag{33}
$$

Proof. Owing to Remark 2 example (1)b), Theorem 3.1 and Remark 3, we deduce the formulas (33) in the Statement. To see that the family contains a surface invariant by parabolic translations, we argue as follows. Now looking at equations (33) we see that if these formulas hold for some pitch $\ell \geqslant 0$ then they hold also for a pitch $\widetilde{\ell}$ in the interval $[0,\ell]$ since $(m^2U^2-\ell^2) \geqslant 0$ and $(m^2U^2-\ell^2)^2 - m^4U^2U'^2 \geqslant 0$. We obtain thereby a family $\mathcal{F}(m,\ell), m \neq 0$ of isometric surfaces to S containing a parabolic translation surface ($\ell = 0$). This completes the proof of the Theorem. □

THEOREM 4.2. *Any surface invariant by hyperbolic screw motions in* $\mathbb{H}^2 \times \mathbb{R}$ *can be parametrized locally by natural coordinates* s, τ. *Let* S *be such a hyperbolic screw motion surface. Then there exists a two parameters family* $\mathcal{F}(m,\ell), m \neq 0$, *containing a surface invariant by hyperbolic translations, such that each element of the family is a hyperbolic screw motion surface isometric to* S *given by*

$$
\begin{aligned}
m^2U^2(s) &= \frac{1}{\sin^2\rho} + \ell^2 \\
\rho'^2(s) &= \frac{\sin^2\rho\,(1+\ell^2\sin^2\rho)}{1+\ell^2\sin^2\rho+\sin^2\rho(\lambda'(\rho))^2} \\
\rho'^2(s) &= \frac{m^4U^2U'^2}{\left(m^2U^2-\ell^2\right)^2\left(m^2U^2-\ell^2-1\right)} \\
\lambda\circ\rho(s) &= \int \frac{mU\sqrt{\left(m^2U^2-\ell^2\right)\left(m^2U^2-\ell^2-1\right)-m^4U^2U'^2}}{\left(m^2U^2-\ell^2\right)\sqrt{\left(m^2U^2-\ell^2-1\right)}}\,\mathrm{d}s \\
\varphi(s,\tau) &= \frac{\tau}{m} - \ell\int \frac{\sqrt{\left(m^2U^2-\ell^2\right)\left(m^2U^2-\ell^2-1\right)-m^4U^2U'^2}}{mU\left(m^2U^2-\ell^2\right)\sqrt{\left(m^2U^2-\ell^2-1\right)}}\,\mathrm{d}s
\end{aligned}
\tag{34}
$$

Proof. The proof is the same as in Theorem 4.1 □

We wish now to complete the geometric description of the families of minimal and constant mean curvature *parabolic* screw motions surfaces given in Section 2. Given a constant H satisfying $1-4H^2>0$, we need to determine explicitly the metric of all such isometric immersions with the same mean curvature H. This is established by the following result.

Corollary 4.1. *Let S be a parabolic screw motion minimal or H-surface with pitch ℓ immersed into $\mathbb{H}^2\times\mathbb{R}$ parametrized by natural coordinates s,τ. Let d be the parameter given by* (8)*. Let us assume $1-4H^2>0$.*

If $d=0$, then

$$
m^2U^2 = \mathrm{e}^{\pm 2\sqrt{1-4H^2}\,(s-s_0)} + \ell^2 \tag{35}
$$

If $d>0$, then

$$
\sqrt{m^2U^2-\ell^2} = \frac{2|H|d}{1-4H^2} + \frac{d}{1-4H^2}\cosh\left(\sqrt{1-4H^2}\,(s-s_0)\right) \tag{36}
$$

Proof. Assume $1-4H^2>0, H\leqslant 0$. We combine the three first equations in Theorem 4.1, with the integral formula. After some computations we obtain the following:

$$
m^4U^2U'^2-(1-4H^2)(m^2U^2-\ell^2)^2+4|H|d(m^2U^2-\ell^2)^{3/2}+d(m^2U^2-\ell^2)=0 \tag{37}
$$

Now by making the change of variables $Z = (m^2U^2 - \ell^2)^{1/2}$, we deduce the formula

$$Z'^2 = (1-4H^2)Z^2 - 4|H|dZ - d^2$$

Treating separately the cases $d = 0$ and $d \neq 0$, using elementary ordinary differential equations techniques, we deduce (35) and (36), as desired. This completes the proof of the Corollary. □

We remark that if $\ell = 0$, then the Gaussian curvature $K := -U''/U$, of a minimal parabolic screw motion immersion is $K \equiv -1$. Otherwise K satisfies the inequality $-1 < K < 0$.

Finally, we give a description of the isometric H-surfaces invariant by *hyperbolic* screw motions.

Corollary 4.2. *Let S be a hyperbolic screw motion minimal or H-surface with pitch ℓ immersed into $\mathbb{H}^2 \times \mathbb{R}$ parametrized by natural coordinates s, τ. Let d be the parameter given by* (27).

(1) *Assume* $1 - 4H^2 > 0$.
   (a) *If* $d^2 < 1 - 4H^2$ *then*

$$\sqrt{m^2U^2 - \ell^2 - 1} = \pm\frac{\sqrt{1-4H^2-d^2}}{1-4H^2}\sinh\left((\sqrt{1-4H^2}(s-s_0)\right) \pm \frac{2Hd}{1-4H^2} \tag{38}$$

   (b) *If* $d^2 > 1 - 4H^2$ *then*

$$\sqrt{m^2U^2 - \ell^2 - 1} = \frac{\sqrt{d^2-(1-4H^2)}}{1-4H^2}\cosh\left(\sqrt{1-4H^2}(s-s_0)\right) \pm \frac{2Hd}{1-4H^2} \tag{39}$$

   (c) *If* $d^2 = 1 - 4H^2$ *then*

$$\sqrt{m^2U^2 - \ell^2 - 1} = \mathrm{e}^{\pm\sqrt{1-4H^2}\,(s-s_0)} \pm \frac{2H}{\sqrt{1-4H^2}} \tag{40}$$

(2) *Assume* $1 - 4H^2 = 0$.
   (a) *If* $d = 0$ *then*

$$\sqrt{m^2U^2 - \ell^2 - 1} = \pm(s - s_0)$$

   (b) *If* $0 < d < 1$ *then*

$$\sqrt{m^2U^2 - \ell^2 - 1} = \pm\left[\frac{d}{2}(s-s_0)^2 - \frac{1-d^2}{d^2}\right]$$

(3) *Assume* $1 - 4H^2 < 0$. *Then*

$$\sqrt{m^2U^2 - \ell^2 - 1} = \pm\frac{\sqrt{d^2+4H^2-1}}{4H^2-1}\sin\left(\sqrt{4H^2-1}(s-s_0)\right) \pm \frac{2Hd}{4H^2-1}$$

Proof. As in Corollary 4.1, by applying the relations derived in Theorem 4.2 together with the first integral formula (27), we infer he following differential equation

$$Z'^2 = Z^2(1-4H^2) \pm 4HdZ + 1 - d^2$$

where $Z = \sqrt{m^2U^2 - \ell^2 - 1}$. Now by working with elementary ordinary differential equations, we deduce the formulas in the Statement, as desired. □

5. Associate and conjugate parabolic and hyperbolic screw motion immersions

Let M^2 be a two dimensional Riemannian manifold. Let (x, y, t) be local coordinates in $M^2 \times \mathbb{R}$, where $z = x+iy$ are conformal coordinates on M^2 and $t \in \mathbb{R}$. Let $\sigma^2 |\,\mathrm{d}z|^2$, be the conformal metric in M^2, hence $\mathrm{d}s^2 = \sigma^2 |\,\mathrm{d}z|^2 + \mathrm{d}t^2$ is the metric in the product space $M^2 \times \mathbb{R}$. Let $\Omega \subset \mathbb{C}$ be a planar domain, $w = u+iv \in \Omega$. We recall that if $X : \Omega \looparrowright M^2 \times \mathbb{R}, w \mapsto (h(w), f(w)), w \in \Omega$ is a conformal minimal immersion with induced metric $\mathrm{d}s^2 = \mu^2 |\,\mathrm{d}w|^2$, then $h : \Omega \subset \mathbb{C} \to (M^2, \sigma^2 |\,\mathrm{d}z|^2), w \mapsto h(w)$ is a harmonic map, that is, it satisfies

$$h_{w\overline{w}} + 2\frac{\sigma_z}{\sigma} h_w h_{\overline{w}} = 0 \tag{41}$$

see, for instance, [19]. We recall also that for any harmonic map $h : \Omega \subset \mathbb{C} \mapsto M^2$ there exists a related Hopf holomorphic function given by

$$\phi = (\sigma \circ h)^2\, h_w \overline{h}_w \tag{42}$$

see [20], [22]. Eric Toubiana and the author have introduced the notions of associate and conjugate immersions in [19], following a work in progress with Laurent Hauswirth [11]. Namely, two conformal isometric immersions $X, \widetilde{X} : \Omega \looparrowright \mathbb{H}^2 \times \mathbb{R}$ are said *associate* if the Hopf functions satisfy the relation $\widetilde{\phi} = \mathrm{e}^{i\theta}\, \phi$. If $\widetilde{\phi} = -\phi$, then the two immersions are said *conjugate*. Benoît Daniel gave an alternative and equivalent definition [2].

As we said in the introduction, Eric Toubiana and the author proved that any two minimal isometric screw motion immersions in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$ are associate. The same authors proved that in $\mathbb{H}^2 \times \mathbb{R}$, a catenoid is conjugate to a helicoid of pitch $\ell < 1$.

We will now prove that two minimal isometric parabolic screw motion immersions are associate.

Theorem 5.1. *Any two minimal isometric parabolic screw motion immersions into $\mathbb{H}^2 \times \mathbb{R}$ are associate. Furthermore, a helicoid of pitch*

$\ell = 1$ is conjugate to a surface invariant by parabolic translations, i.e it is conjugate to Daniel-Hauswirth minimal surface.

We remark that Benoît Daniel proved the second part of the above Statement by another approach in [2].

Proof. Let us now take natural coordinates s, τ so that the induced metric is given by $\mathrm{d}\mu^2 = \mathrm{d}s^2 + U^2\,\mathrm{d}\tau^2$, see Theorem 4.1. We have therefore natural conformal coordinates $\upsilon + i\tau$ where $\upsilon = \int 1/U\,\mathrm{d}s$. Of course, the induced metric becomes $\mathrm{d}\mu^2 = U^2(\mathrm{d}\upsilon^2 + \mathrm{d}\tau^2)$. We therefore may compute the Hopf function ϕ for parabolic screw motion minimal immersions. After, a somewhat long computation, working as in [19], we obtain the relations

$$\begin{aligned} 4\Re\phi &= \frac{1}{m^2}\left[\ell^2 - \frac{\lambda'^2}{1 + r^2(\ell^2 + \lambda'^2)}\right] \\ \Im\phi &= 0, \text{if} \qquad \ell = 0 \\ 4\Re\phi &= \frac{\ell^2}{m^2} - \frac{4m^2}{\ell^2}\left(\Im\phi\right)^2, \text{if} \qquad \ell \neq 0 \end{aligned} \tag{43}$$

Owing to (43) and (8) we deduce that ϕ is given by

$$m^2\phi = \frac{\ell^2}{4} - \frac{d^2}{4} + i\frac{\ell d}{2}, \qquad (d \geqslant 0)$$

Thus

$$16|\phi|^2 = \left(\frac{\ell^2}{m^2} + \frac{d^2}{m^2}\right)^2 \tag{44}$$

Now two isometric minimal parabolic screw motion immersions have the metric given by Corollary 4.1, making $H = 0$; that is either by

$$m^2U^2 = \mathrm{e}^{2(s-s_0)} + \ell^2, \text{if} \qquad d = 0$$

or

$$m^2U^2 = \frac{d^2}{2}\cosh\left(2(s - s_0)\right) + \ell^2 + \frac{d^2}{2}, \text{if} \qquad d \neq 0 \tag{45}$$

In view of (44) and (45), the absolute value of their Hopf functions are the same, hence they are associate as well. Now, on account of [19] the helicoid ($d = 0$) of pitch $\ell = 1$ has metric given by $U^2 = \frac{1}{m^2}\cosh^2(s - s_0)$, and Hopf function given by $\phi = \frac{1}{4m^2}$. On the other hand, in view of Corollary 4.1, Daniel-Hauswirth minimal surface ($\ell = 0, m = 1$) has metric given by $U^2 = d^2\cosh^2(s - s_0)$, and Hopf function given by $\phi = -\frac{d^2}{4}$. Thus we have that if $d^2m^2 = 1$, they are conjugate, as desired. This completes the proof of the Theorem.

□

Next, we will prove that the non-isometric deformation of the hyperbolic plane given by *tilted Euclidean plane* (9), invariant by parabolic screw motions with pitch $\ell = 1$, is conjugate to *Scherk type minimal surface*, invariant by hyperbolic translations given by (32). See Figures 5 and 14.

Let d be the parameter given in (27).

THEOREM 5.2. *The conjugate of the Scherk type minimal surface invariant by hyperbolic translations is the minimal surface generated by a horizontal line invariant by parabolic screw motions with pitch $\ell = 1$. If $d^2 < 1$, any two minimal isometric hyperbolic screw immersions in this family are associate. The same holds if either $d^2 = 1$ or $d^2 > 1$. To each minimal hyperbolic screw motion immersion in the family $d^2 < 1$, there exists a minimal isometric non associate hyperbolic screw motion immersion in the family $d^2 > 1$. Furthermore, each* parabolic *screw motion surface is associate to an* hyperbolic *screw motion surface. Any helicoid with pitch $\ell > 1$ is conjugate to a minimal surface invariant by hyperbolic translations ($\ell = 0$ and $d^2 > 1$)*

We remark that last part of the Statement (about the helicoid) is a result of Benoît Daniel [2]. We will give an alternative proof.

Proof. We first observe that the structure of the proof is the same as in Theorem 5.1. On account of Theorem 4.2, we can deduce the following relations

$$\begin{aligned} 4\Re\phi &= \frac{1}{m^2}\left[\ell^2 - \frac{\lambda'^2}{1+\sin^2\rho\,(\ell^2+\lambda'^2)}\right] \\ \Im\phi &= 0, \text{if} \qquad \ell = 0 \\ 4\Re\phi &= \frac{\ell^2}{m^2} - \frac{4m^2}{\ell^2}\left(\Im\phi\right)^2, \text{if} \qquad \ell \neq 0 \end{aligned} \tag{46}$$

Owing to (46) and (27) we deduce that ϕ is given by

$$m^2\phi = \frac{\ell^2}{4} - \frac{d^2}{4} + i\frac{\ell d}{2}, \qquad (d \geqslant 0)$$

Hence

$$16|\phi|^2 = \left(\frac{\ell^2}{m^2} + \frac{d^2}{m^2}\right)^2 \tag{47}$$

Now it follows from (40) and from (47) that the metric and the Hopf function of the Scherk type minimal surface (32) are given by $m^2U^2 = \mathrm{e}^{2(s-s_0)} + 1, \phi = -\frac{1}{4m^2}$, respectively. On the other hand, owing to (35) and (44) the metric and the Hopf function of the non-isometric minimal deformation of hyperbolic plane (9) with pitch $\ell = 1$, are given

by $m^2U^2 = \mathrm{e}^{2(s-s_0)}+1, \phi = \frac{1}{4m^2}$, respectively. Hence the immersions are conjugate. This proves the first part of the Statement.

Let us first assume that $d^2 < 1$. According to (38) the metric of two isometric minimal immersions are given by

$$\begin{aligned} m^2U^2 &= (1-d^2)\sinh^2(s-s_0) + \ell^2 + 1 \\ &= (1-d^2)\cosh^2(s-s_0) + d^2 + \ell^2 \end{aligned} \tag{48}$$

Thus on account of (47), any two such isometric immersions are associate. Secondly, if $d^2 = 1$, we infer from (40) and (47) the same result. Analogously, if $d^2 > 1$, since the metric is given by

$$\begin{aligned} m^2U^2 &= \frac{d^2-1}{2}\cosh\left(2(s-s_0)\right) + \ell^2 + \frac{d^2+1}{2} \\ &= (d^2-1)\cosh^2(s-s_0) + \ell^2 + 1 \end{aligned} \tag{49}$$

Now take a minimal hyperbolic screw motion immersion in the first family ($d^2 < 1$), setting $m = m_{\hbar_1}, d = d_{\hbar_1}, \ell = \ell_{\hbar_1}$ in (48). We see that there exists a minimal hyperbolic screw motion immersion (m, d, ℓ) in the third family ($d^2 > 1$), see (49), by setting

$$\frac{d^2-1}{m^2} = \frac{1-d_{\hbar_1}^2}{m_{\hbar_1}^2}, \qquad \frac{\ell^2+1}{m^2} = \frac{d_{\hbar_1}^2 + \ell_{\hbar_1}^2}{m_{\hbar_1}^2} \tag{50}$$

Let $\phi_{\hbar_1}$ and ϕ be their Hopf's functions, respectively. Notice that owing to (47) and (50), we deduce the sctrict inequality $|\phi_{\hbar_1}| < |\phi|$, hence these isometric immersions are non associate.

Notice that any minimal parabolic screw motion immersion $S = S(d_{\mathcal{P}}, m_{\mathcal{P}}, \ell_{\mathcal{P}})$, with $d_{\mathcal{P}}^2 \neq 0$, has metric given by (36), setting $H = 0$. In view of (49), (44) and (47), by setting

$$\frac{d^2-1}{m^2} = \frac{d_{\mathcal{P}}^2}{m_{\mathcal{P}}^2}, \qquad \frac{\ell^2+1}{m^2} = \frac{\ell_{\mathcal{P}}^2}{m_{\mathcal{P}}^2} \tag{51}$$

we obtain a minimal hyperbolic screw motion surface in the third family associate to S. In the same way, we obtain that any minimal parabolic screw motion immersion $(d_{\mathcal{P}}, m_{\mathcal{P}}, \ell_{\mathcal{P}})$, with $d_{\mathcal{P}}^2 = 0$, is associate to a minimal hyperbolic screw motion surface in the second family ($d^2 = 1$). Therefore, we conclude that each *parabolic* screw motion surface is associate to an *hyperbolic* screw motion surface.

Finally, consider a minimal surface invariant by hyperbolic translations with parameters $(d^2 > 1, m_\hbar, \ell = 0)$, metric given by (39) or (49) (with $m = m_\hbar, \ell = 0, d^2 > 1$), and Hopf function given by $\phi_\hbar = -\frac{d^2}{4m_\hbar^2}\cdot$ Owing to [19] the helicoid with pitch $\ell_{\mathcal{H}}$ has metric given by

$$m_{\mathcal{H}}^2U^2 = \frac{1}{2}\cosh\left(2(s-s_0)\right) + \ell_{\mathcal{H}}^2 - \frac{1}{2}$$

and Hopf function given by $\phi = \frac{\ell_{\mathcal{H}}^2}{4m_{\mathcal{H}}^2}$. Therefore, if $\frac{d^2-1}{2m_{\hbar}^2} = \frac{1}{2m_{\mathcal{H}}^2}$, and $\frac{d^2+1}{2m_{\hbar}^2} = \frac{2\ell_{\mathcal{H}}^2-1}{2m_{\mathcal{H}}^2}$, we deduce that any helicoid of pitch $\ell_{\mathcal{H}} > 1$ is conjugate to a minimal surface invariant by hyperbolic translations, as desired. This completes the proof of the Theorem. □

References

[1] U. Abresch and H. Rosenberg. *A Hopf differential for constant mean curvature surfaces in $S^2 \times R$ and $H^2 \times R$.* Acta Math. **193**, No 2, 141-174, 2004.

[2] B. Daniel. *Isometric immersions into $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ and applications to minimal surfaces.* Institut de Math. de Jussieu, Prépublications 375, Juin 2004.

[3] E. Bour. *Mémoire sur la déformation des surfaces* . J. Ecole Pol., **39**, 1-148, 1862.

[4] M. P. Do Carmo and M. Dajczer. *Helicoidal surfaces with constant mean curvature.* Tohoku Math. J., **34**, 425-435, 1982.

[5] P. Collin and H. Rosenberg. *Construction of harmonic diffeomorfisms and minimal graphs.* Preprint.

[6] U. Dierkes, S. Hildebrandt, A. Küster and O. Wohlrab.*Minimal surfaces I and II*; A series of comprehensive studies in mathematics. Springer Verlag.

[7] I. Fernández and P. Mira. *Harmonic maps and constant mean curvature surfaces in $\mathbb{H}^2 \times \mathbb{R}$.* To appear in American J. Math.

[8] I. Fernández and P. Mira. *A characterization of constant mean curvature surfaces in homogeneous 3-manifold.* To appear in Diff. Geom. Appl., 2007.

[9] I. Fernández and P. Mira. *Holomorphic quadratic differentials and the Bernstein problem in Heisenberg space.* Preprint.

[10] L. Hauswirth. *Generalized Riemann examples in three-dimensional manifolds.* Pacific J. Math. **2**24, No 1, 91-117, 2006.

[11] L. Haurwirth, R. Sa Earp and E. Toubiana. *Associate and conjugate minimal immersions in $M^2 \times R$.* Preprint.

[12] L. Hauswirth, H. Rosenberg and J. Spruck *Infinite boundary value problems for constant mean curvature graphs in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$.* http://www.math.jhu.edu/ js/jsfinal.pdf

[13] W. Meeks, III and H. Rosenberg. *The theory of minimal surfaces in $M \times R$.* Comment. Math. Helv. **8**0, No 4, 811-858, 2005.

[14] B. Nelli and H. Rosenberg. *Minimal surfaces in $H^2 \times R$.* Bull. Braz. Math. Soc. **33**, 263-292, 2002.

[15] B. Nelli and H. Rosenberg. *Global properties of conastant mean curvature surfaces in $H^2 \times R$.* Pacific Journ. Math. **226**, No 1, 137-152, 2006.

[16] J. Ordóñes. *Superfícies helicoidais com curvatura constante no espaço de formas tridimensionais.* Doctoral Thesis, PUC-Rio, 1995.

[17] H. Rosenberg. *Minimal surfaces in $M^2 \times R$.* Illinois J. Math. **46**, 1177-1195, 2002.

[18] R. Sa Earp and E. Toubiana *Existence and uniqueness of minimal graphs in hyperbolic space.* Asian J. Math. **4**, 669–694, 2000.

[19] R. Sa Earp and E. Toubiana. *Screw motion surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$.* Illinois J. Math. **49**, No. 3, 2005.

[20] R. Schoen and S.T. Yau. *Lectures on harmonic maps.* Conference Proceedings and Lecture Notes in Geometry and Topology, II. International Press, Cambridge, MA, 1997.

[21] J. Spruck. *Interior Gradient Estimates and Existence Theorems for Constant Mean Curvature Graphs in $M^n \times \mathbb{R}$.* http://www.math.jhu.edu/ js/grad2.pdf

[22] T. Y. H. Wan. *Constant mean curvature surface, harmonic maps, and universal Teichmüller space.* J. Diff. Geom. **35**, 643-657, 1992.

E-mail address: `earp@mat.puc-rio.br`

Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro, Rio de Janeiro, 22453-900 RJ, Brazil