A REFLECTION PRINCIPLE FOR MINIMAL SURFACES IN SMOOTH THREE MANIFOLDS

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ABSTRACT. We prove a reflection principle for minimal surfaces in smooth (non necessarily analytic) three manifolds and we give an explicit application when the ambient space is just a smooth manifold.

1. INTRODUCTION

In this paper we prove a reflection principle for minimal surfaces in a smooth (non necessarily analytic) setting. The precise statement is given in the Theorem 1.1 below. One of the motivations to seek such a reflection principle is the fact that there are many three dimensional smooth (non analytic) Riemannian manifolds endowed with natural reflections about a geodesic line (see, exemple 2.3-(3)). We derive an application when the ambient is just a smooth manifold, using our main theorem combined with some classical theorems about area minimizing solution of the Plateau problem, see Proposition 3.4.

A form of the reflection principle says that if a minimal surface in $\mathbb{R}^3$ contains a segment of straight line L, then the minimal surface is invariant by reflection with respect to L, see [12]. Such a form was generalized by Leung [15, Thm. 1] for analytic manifolds.

We are interested in the knowledge of a reflection principle across a geodesic line in the boundary of a minimal surface. We assume the surface is contained in a smooth non necessarily analytic manifold. We assume also that the geodesic line admits a reflection in the ambient space (Definition 2.1).

First, we emphasize that the "reflection principle" in the Euclidean space with only the usual hypothesis that the minimal surface contains a segment L of a straight line in its (topological) boundary, is not established. In fact, even with the strongest assumption that the surface is an embedded disk up to the boundary segment L, as far as we know, there is no proof of the "reflection principle" in the Euclidean space.

Of course, when we impose some additional conditions a proof can be done. For example, when we know that the minimal immersion is conformal in the interior and

continuous up to the boundary segment L or when the surface with its boundary L is a minimal graph, continuous up to L.

In fact, the reflection principle for conformal minimal immersions in $\mathbb{R}^3$ is a generalization of the well-known Schwarz reflection principle for harmonic functions. The proof uses the fact that the coordinates of a conformal minimal immersion in Euclidean space are harmonic, then the Schwarz principle for harmonic functions is applied. See an elegant deduction in [4, Thm. 1, Sec. 4.8] or in [21, Lemma 7.3].

When the ambient space is the sphere $\mathbb{S}^3$, Lawson produced a proof following the same idea of the proof in the Euclidean case (which, in fact, holds in the three-dimensional hyperbolic space). The precise statement is as follows: When the conformal minimal immersion in the sphere $\mathbb{S}^3$ contains an arc of geodesic L of the ambient space on its boundary and has C^2 regularity up to this arc, then the surface can be extended analytically by reflection in L [14, Prop. 3.1]. The proof makes use of a Lichtenstein theorem, see [11, Thm. 1].

The reflection principle has been used by several authors (including the present authors) in the theory of minimal surfaces in homogeneous three-dimensional spaces, see for example Rosenberg [22], Abresh-Rosenberg [1], Menezes [18], Mazet-Rodríguez-Rosenberg [17].

On the other hand, the authors have established the reflection principle for minimal vertical graphs, when the ambient space is the product space $\mathbb{H}^n \times \mathbb{R}$, where $\mathbb{H}^n$ is the n-dimensional hyperbolic space, see [24, Lemma 3.6]. The proof also works in $\mathbb{R}^n \times \mathbb{R}$. Furthermore the authors use the reflection principle to construct Scherk type minimal hypersurface in $\mathbb{H}^n \times \mathbb{R}$ [24, Theorem 5.10].

In the statement of our Main Theorem below we use the notion of a reflection I_γ about a geodesic γ in a C^∞ Riemannian manifold (M, g). We denote by $U_\gamma \subset M$ the domain of definition of I_γ, see Definition 2.1.

Theorem 1.1 (Main Theorem). *Let (M, g) be a C^∞ Riemannian three manifold. Let $\gamma \subset M$ be an open geodesic arc which admits a reflection I_γ.*
Let $S \subset U_\gamma$ be an embedded minimal surface. We assume that $S \cup \gamma$ is a C^1 surface with boundary.
Then the reflection of S about γ gives rise to a C^∞ continuation of S across γ. That is, $S \cup \gamma \cup I_\gamma(S)$ is a smooth immersed minimal surface which is embedded near γ.

Theorem 1.2 (Main Theorem bis). *Let M be a C^∞ Riemannian three manifold and let $\gamma \subset M$ be an open geodesic arc which admits a reflection I_γ.*
Let $S \subset U_\gamma$ be an embedded minimal surface such that $S \cup \gamma$ is a C^0 surface with boundary. We assume that $S \cup \gamma$ is the graph of a C^0 function $x_3 = f(x_1, x_2)$ for some local coordinates (x_1, x_2, x_3) of M.

We assume also that f restricted to the projection of S is a C^2 function with bounded gradient.
Then the reflection of S about γ gives rise to a C^∞ continuation of S across γ. That is, $S \cup \gamma \cup I_\gamma(S)$ is a smooth immersed minimal surface which is a graph near γ.

We point out that a crucial tool in the proof of the above theorems is a Hölder gradient regularity up to the boundary for solutions of Dirichlet problems for quasilinear elliptic equations, see Theorems 2.4, 2.7 and Remark 2.8.

2. Analytic and geometric background

2.1. **Geodesic arc of reflection.**
Throughout this paper, M is a connected C^∞ manifold of dimension three, equipped with a C^∞ metric g.

Definition 2.1. Let (M, g) be a complete C^∞ Riemannian three manifold.
We say that an open geodesic arc $\gamma \subset M$ admits a *reflection* if there exist an oriented open subset $U_\gamma \subset M$ containing γ, and a non trivial isometry $I_\gamma : (U_\gamma, g) \longrightarrow (U_\gamma, g)$ such that

- I_γ is orientation preserving,
- $I_\gamma(p) = p$ for any $p \in \gamma$,
- $I_\gamma \circ I_\gamma = \mathrm{Id}_{U_\gamma}$.

Next we describe some relevant examples.

Example 2.2. (1) We denote by $\mathbb{H}^2$ the hyperbolic plane and $g_\mathbb{H}$ its hyperbolic metric.

Let $M = \mathbb{H}^2 \times \mathbb{R}$ endowed with the product metric $g := g_\mathbb{H} + dt^2$.

The natural reflections about vertical or horizontal geodesic lines in $\mathbb{H}^2 \times \mathbb{R}$ satisfy the conditions of Definition 2.1. We observe that there is no other geodesic lines of reflection.

(2) For any $\kappa \leqslant 0$ we denote by $\mathbb{M}^2(\kappa)$ the complete and simply connected surface with constant intrinsic curvature κ.

We set $M = \mathbb{E}(\kappa, \tau)$, $\kappa \leqslant 0$, $\tau > 0$, the simply connected and complete homogeneous three manifold, see for example [3]. More precisely, $\mathbb{E}(\kappa, \tau)$ has a four dimensional isometry group, it is a fibration over $\mathbb{M}^2(\kappa)$, the canonical projection $\mathbb{E}(\kappa, \tau) \longrightarrow \mathbb{M}^2(\kappa)$ is a Riemannian submersion and the bundle curvature is τ.

We treat separately the cases $\kappa < 0$ and $\kappa = 0$.

(a) We suppose $\kappa < 0$, (in particular for $\kappa = -1$ we have $\mathbb{E}(-1,\tau) = \widetilde{\mathrm{PSL}}_2(\mathbb{R},\tau)$).
We choose the disk model for $\mathbb{M}^2(\kappa)$, that is the open disk of radius $2/\sqrt{-\kappa}$.
We denote : $\mathbb{M}^2(\kappa) := \mathbb{D}(2/\sqrt{-\kappa})$ provided with the metric

$$ds^2 = \lambda^2(x,y)(dx^2 + dy^2), \quad \text{where} \quad \lambda(x,y) = \frac{1}{1 + \kappa\frac{x^2+y^2}{4}}.$$

Then we have $\mathbb{E}(\kappa,\tau) = \mathbb{D}(2/\sqrt{-\kappa}) \times \mathbb{R}$ provided with the metric

$$g = \lambda^2(dx^2 + dy^2) + \left(-\frac{2\tau}{\kappa}\frac{\lambda_y}{\lambda}dx + \frac{2\tau}{\kappa}\frac{\lambda_x}{\lambda}dy + dt\right)^2$$

$$= \lambda^2(dx^2 + dy^2) + \left(\lambda\tau\big(ydx - xdy\big) + dt\right)^2.$$

The isometries of $(\mathbb{E}(\kappa,\tau), g)$ are given by (setting $z = x + iy$)

$$(1) \qquad\qquad F(z,t) = \left(f(z), t + \frac{2\tau}{\kappa}\arg\frac{\partial f}{\partial z}(z) + c\right)$$

where f is a positive isometry of $(\mathbb{D}(2/\sqrt{-\kappa}), ds^2)$, and

$$(2) \qquad\qquad G(z,t) = \left(g(z), -t - \frac{2\tau}{\kappa}\arg\frac{\partial\bar{g}}{\partial z}(z) + c\right),$$

where g is a negative isometry of $(\mathbb{D}(2/\sqrt{-\kappa}), ds^2)$, and $c \in \mathbb{R}$. Observe
that any isometry is orientation preserving.
In particular $F(z,t) = (-z,t)$ is an isometry. Set $L_0 := \{(0,t), \ t \in \mathbb{R}\}$.
Observe that each point of L_0 is a fixed point for F. Therefore L_0 is a
geodesic line. Since $F \circ F = \mathrm{Id}$, F is a reflection about L_0.
Now let $z_0 \in \mathbb{D}(2/\sqrt{-\kappa})$ be any point. Let h be a positive isometry of
$\mathbb{D}(2/\sqrt{-\kappa})$ such that $h(z_0) = 0$. We set $H(z,t) = (h(z), t + \frac{2\tau}{\kappa}\arg\frac{\partial h}{\partial z}(z))$
and $L_{z_0} := \{(z_0,t), \ t \in \mathbb{R}\}$. Note that for any $p \in L_{z_0}$ we have $H(p) \in L_0$.
Therefore setting $I_{z_0} := H^{-1} \circ F \circ H$ we get for any $p \in L_{z_0}$

$$I_{z_0}(p) = H^{-1}\Big(F\big(H(p)\big)\Big) = H^{-1}\big(H(p)\big) = p.$$

We conclude as before that L_{z_0} is a geodesic line and that I_{z_0} is a reflection
about L_{z_0}.
Now we set $D_y : \{(0,y,0), \ -2/\sqrt{-\kappa} < y < 2/\sqrt{-\kappa}\} \subset \mathbb{D}(2/\sqrt{-\kappa}) \times \{0\}$.
We consider the isometry $G(z) = (-\bar{z}, -t)$. Observe that $G(p) = p$ for any
$p \in D_y$. Therefore D_y is a geodesic line and, since $G \circ G = \mathrm{Id}$, we get that
G is a reflection about D_y.
For any $\theta \in [0,\pi)$ we set $D_\theta := \{(se^{i\theta}, 0), \ -2/\sqrt{-\kappa} < s < 2/\sqrt{-\kappa}\}$, thus
$D_y = D_{\pi/2}$. Observe that for any $\alpha \in \mathbb{R}$ the map $R_\alpha(z,t) := (e^{i\alpha}z, t)$ is
an isometry. Since $R_{\pi/2-\theta}(D_\theta) = D_y$, we get that the map $G_\theta := R_{\pi/2-\theta}^{-1} \circ$
$G \circ R_{\pi/2-\theta}$ fixes any point of D_θ. Thus D_θ is a geodesic line and G_θ is a
reflection about D_θ.

We can prove in the same way that any horizontal geodesic admits a reflection.

(b) We suppose now $\kappa = 0$, therefore we have that $\mathbb{E}(0, \tau) = \text{Nil}_3(\tau)$ is the Heisenberg group and it can be viewed as $\mathbb{R}^3$ with the metric

$$g_\tau = dx^2 + dy^2 + (\tau(ydx - xdy) + dt)^2.$$

The isometries of $(\mathbb{R}^3, g_\tau)$ are (setting $z = x + iy$)

(3) $$F(z, t) = (e^{i\theta}z + a + ib, t + \tau\text{Im}(a - ib)e^{i\theta}z + c)$$

and

(4) $$G(z, t) = (e^{i\theta}\overline{z} + a + ib, -t - \tau\text{Im}(a + ib)e^{-i\theta}z + c)$$

where $a, b, c, \theta \in \mathbb{R}$ are any real numbers. Observe again that any isometry is orientation preserving.

Arguing as in the case $\kappa < 0$, it can be shown that the Euclidean lines $L_{z_0} := \{(z_0, t), \ t \in \mathbb{R}\}$ and $D_\theta := \{(se^{i\theta}, 0), \ s \in \mathbb{R}\}$ are geodesic lines which admit a reflection, for any $z_0 = (x_0, y_0) \in \mathbb{R}^2$ and any $\theta \in [0, \pi)$.
Any horizontal geodesic admits a reflection.

(3) At last, it is not difficult to construct smooth and non analytic three manifolds having geodesic lines of reflection. For example let M^2 be a smooth Riemannian surface.

(a) Let $\gamma \subset M^2$ be a geodesic arc. Assume that there exist an open oriented neighborhood V of γ in M^2 and a non trivial isometry $i_\gamma : V \longrightarrow V$ such that
  - i_γ reverses the orientation,
  - $i_\gamma(p) = p$ for any $p \in \gamma$,
  - $i_\gamma \circ i_\gamma = Id_V$.

Then setting $U = V \times \mathbb{R} \subset M^2 \times \mathbb{R} := M^3$ and $I_\gamma(x, t) = (i_\gamma(x), -t)$, we see that γ admits also a reflection in the Riemannian product $M^2 \times \mathbb{R}$.

(b) Now assume that there exist a point $p_0 \in M^2$, an open oriented neighborhood $V \subset M^2$ of p_0 and a non trivial isometry $\varphi : V \longrightarrow V$ such that
  - φ is orientation preserving,
  - $\varphi(p_0) = p_0$,
  - $\varphi \circ \varphi = Id_V$.

Then setting $U = V \times \mathbb{R} \subset M^2 \times \mathbb{R} := M^3$ and $I_\gamma(x, t) = (\varphi(x), t)$, we see that $\gamma := \{p_0\} \times \mathbb{R}$ admits also a reflection in the Riemannian product $M^2 \times \mathbb{R}$.

2.2. **Boundary regularity.**

Definition 2.3. Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, be a domain and let Q be a second order quasilinear operator of the following form

$$Q(u) := \sum_{i,j=1}^{n} a^{ij}(x, u, Du) D_{ij} u + b(x, u, Du),$$

where $x \in \Omega$, and the functions a^{ij}, b are defined and C^1 on $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$.
We assume that $(a^{ij}(x, z, p))_{1 \leqslant i,j \leqslant n}$ is a symmetric matrix for any $(x, z, p) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$.
We denote by $\lambda(x, z, p)$, respectively $\Lambda(x, z, p)$, the minimum eigenvalue, respectively the maximum eigenvalue, of the symmetric matrix $(a^{ij}(x, z, p))$.
We say that Q is an *elliptic operator*, if $0 < \lambda(x, z, p)$ for any $(x, z, p) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$.
Assume now that Ω is a bounded domain. Then, by continuity, for any $K > 0$, there exist constant numbers $0 < \lambda_K \leqslant \Lambda_K$ and $\mu_K > 0$ such that

$$0 < \lambda_K \leqslant \lambda(x, z, p) \leqslant \Lambda(x, z, p) \leqslant \Lambda_K,$$

$$\left(|a^{ij}| + |D_{p_k} a^{ij}| + |D_z a^{ij}| + |D_{x_k} a^{ij}| + |b| \right)(x, z, p) \leqslant \mu_K,$$

for any $x \in \overline{\Omega}$ and $(z, p) \in \mathbb{R} \times \mathbb{R}^n$ satisfying $|z| + |p| \leqslant K$.

A crucial ingredient in the proof of the Main Theorem is a uniform global Hölder estimates for the gradient for a solution of a general second order elliptic quasilinear equation. This can be seen as an extension of the well-know Ladyzhenskaya and Ural'tseva fundamental global a priori Hölder estimates [13, Chapter IV, Theorem 6.3].

Theorem 2.4. *Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, be a bounded domain with C^2 boundary, and let Q be a quasilinear operator as above with $a^{ij}, b \in C^1(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$.*
Let $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ be a function satisfying

$$\begin{cases} Q(u) = 0 & on \quad \Omega, \\ u = \varphi & on \quad \partial\Omega, \end{cases}$$

where $\varphi \in C^2(\overline{\Omega})$.
Assume there is $K > 0$ such that $|u| + |Du| \leqslant K$ on Ω.
Then, there exists a constant $\tau \in (0, 1)$, such that $u \in C^{1,\tau}(\overline{\Omega})$.
More precisely, $u \in C^1(\overline{\Omega})$ and there exist positive numbers $C = C(n, K, \lambda_{\widehat{K}}, \mu_{\widehat{K}}, |\varphi|_{C^2(\overline{\Omega})}, \Omega)$ and $\tau = \tau(n, K, \lambda_{\widehat{K}}, \mu_{\widehat{K}}, |\varphi|_{C^2(\Omega)}) \in (0, 1)$ such that for any $x_1, x_2 \in \overline{\Omega}$ we have

$$(5) \qquad\qquad |Du(x_1) - Du(x_2)| \leqslant C \, |x_1 - x_2|^{\tau},$$

where $\widehat{K} := K + |\varphi|_{C^1(\Omega)}$.

Remark 2.5. We observe that the assumption that u has bounded gradient in Theorem 2.4 is crucial. Indeed, consider in $\mathbb{R}^3$ a vertical catenoid C. Assume that the *neck* of C stays in the horizontal plane $\{x_3 = 0\}$. Thus the part of C staying between the planes

$\{x_3 = 0\}$ and $\{x_3 = 1\}$ is the graph of a function u defined on an annulus in the plane $\{x_3 = 0\}$. The function u satisfies the minimal equation on this annulus of $\mathbb{R}^2$ and the gradient of u is not bounded near the inner circle of the annulus. Therefore u has not extension $C^{1,\tau}$ up to the boundary.

Proof. We present Trudinger's proof, see [27, Theorem 4], omitting the derivation of certain assertions, but giving further details for the sake of clarity.

We set $\widehat{u} := u - \varphi$. Thus $|\widehat{u}| + |D\widehat{u}| \leqslant \widehat{K}$ in Ω, $\widehat{u} = 0$ on $\partial\Omega$ and $\widehat{u}$ satisfies

$$\sum_{i,j=1}^{n} \widehat{a}^{ij}(x, \widehat{u}, \widehat{D}u) D_{ij}\widehat{u} + \widehat{b}(x, \widehat{u}, D\widehat{u}) = 0$$

with

$$\widehat{a}^{ij}(x, \widehat{u}, \widehat{D}u) := a^{ij}(x, \widehat{u} + \varphi, D\widehat{u} + D\varphi)$$

$$\widehat{b}(x, \widehat{u}, D\widehat{u}) := b(x, \widehat{u} + \varphi, D\widehat{u} + D\varphi) + \sum_{i,j=1}^{n} \widehat{a}^{ij}(x, \widehat{u}, \widehat{D}u) D_{ij}\varphi.$$

We consider the linear operator on Ω

$$L(\omega) = \sum_{i,j} \alpha^{ij}(x) D_{ij}\omega$$

for $\omega \in C^2(\Omega)$, where $\alpha^{ij}(x) := \widehat{a}^{ij}(x, \widehat{u}(x), D\widehat{u}(x))$. Observe that $(\alpha^{ij}(x))$ is a bounded and continuous symmetric matrix on Ω. Furthermore we have $0 < \lambda_{\widehat{K}} \leqslant \rho(x) \leqslant \Lambda_{\widehat{K}}$ for any $x \in \Omega$, where $\rho(x)$ is any eigenvalue of the matrix $(\alpha^{ij}(x))$.

Observe also that $\widehat{u}$ satisfies $L(\widehat{u}) = f(x)$, where $f(x) := -\widehat{b}(x, \widehat{u}(x), D\widehat{u}(x))$.

Next we state without proofs some structure facts that we use in the sequel.

For any vector $d = (d_1, \ldots, d_k) \in \mathbb{R}^k$, $k \in \mathbb{N}^*$, we set $|d| = \sqrt{d_1^2 + \cdots + d_k^2}$.

Let $p \in \partial\Omega$ be any boundary point. Since $\partial\Omega$ is a compact embedded hypersurface of $\mathbb{R}^n$ with C^2 regularity, we deduce first that there exists a constant $A > 0$ such that for any $p \in \partial\Omega$ and for any normal curvature $k_n(p)$ of $\partial\Omega$ at p, we have $|k_n(p)| \leqslant A$.

We deduce also that there exists a positive constant $R < 1$, depending only on the geometry of $\partial\Omega$, and not on p, such that

- $\Omega \cap B_R(p)$ is connected, where $B_R(p)$ is the ball centered at p with radius R.
- We choose orthonormal coordinates $(y_1, \cdots, y_n)$ in $\mathbb{R}^n$ such that $p = 0$ in those coordinates and $(\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_{n-1}})$ is a basis of the tangent space of $\partial\Omega$ at p. Then a neighborhood of p in $\partial\Omega$ is the graph of a C^2 function h defined in a neighborhood of 0 in $\{y_n = 0\}$ containing the *disk* $\{|y| < R, \ y_n = 0\}$. Moreover h satisfies
    - $h(0) = 0$ and $Dh(0) = 0$,

   − $|Dh| < 1/4$ and $|\frac{\partial^2 h}{\partial y_i \partial y_j}| \leqslant 8A$ on the whole domain where h is defined.
- The map $F : B_R(p) \longrightarrow \mathbb{R}^n$ defined by $F(y) := y - (0, \ldots, 0, h(y_1, \ldots, y_{n-1}))$ is
  a C^2 diffeomorphism onto its image, satisfying $F(p) = 0$ and
    − $F(\Omega \cap B_R(p)) \subset \{y_n > 0\}$,
    − $F(\partial\Omega \cap B_R(p)) \subset \{y_n = 0\}$,
    − $|DF_i| \leqslant 2$, where $F = (F_1, \ldots, F_n)$,
    − $|D_{qs}F_i| \leqslant 8A$, $q, s = 1, \ldots, n$,
    − for any positive $r \leqslant R$, setting $B_r := \{y \in \mathbb{R}^n, \ |y| < r\}$ and $B_r^+ = B_r \cap \{y_n > 0\}$, we have $B_{r/2}^+ \subset F(\Omega \cap B_r(p))$ and $B_{r/4}(p) \cap \Omega \subset F^{-1}(B_{r/2}^+)$.

Thus, the function $w := \widehat{u} \circ F^{-1}$ is defined on $B_{R/2}^+$ and satisfies the linear elliptic equation

(6) $$\widetilde{L}(w) := \sum_{i,j} \widetilde{\alpha}^{ij}(y) D_{ij} w = \widetilde{f}(y) \quad \text{in} \quad B_{R/2}^+,$$

$$w = 0 \quad \text{on} \quad \partial B_{R/2}^+ \cap \{y_n = 0\},$$

where

- $\widetilde{\alpha}^{ij}(y) := \sum_{q,s} \alpha^{qs}(F^{-1}(y)) \, D_s F_j \, D_q F_i$,
- $\widetilde{f}(y) := -\sum_i \left(\sum_{q,s} \alpha^{qs}(F^{-1}(y)) D_{qs} F_i \right) D_i w + f(F^{-1}(y))$.

Observe that we have $(\widetilde{\alpha}^{ij}) = DF\,(\alpha^{ij})\,^t DF$. Taking into account the definition of F, a straightforward computation shows that we have $\dfrac{\lambda_{\widehat{K}}}{2} \leqslant \widetilde{\rho}(y) \leqslant 4\Lambda_{\widehat{K}}$ for any eigenvalue $\widetilde{\rho}(y)$ of the matrix $(\widetilde{\alpha}^{ij}(y))$ and that $|w| + |Dw| \leqslant (1 + 2\sqrt{n})\widehat{K}$.
Since u has bounded gradient we observe that the function $\widetilde{f}$ is bounded on $B_{R/2}^+$.

We consider the function $v(y) := \dfrac{w(y)}{y_n}$ on $B_{R/2}^+$ and we set $\delta := \frac{\lambda_{\widehat{K}}}{48n\Lambda_{\widehat{K}}}$.
Since u has bounded gradient we deduce from the proof of [9, Theorem 1.2.16] that there exist real numbers $C_1 = C_1(n, K, |\varphi|_{C^1(\Omega)}, \lambda_{\widehat{K}}, \Lambda_{\widehat{K}}) > 0$ and $\alpha = \alpha(n, K, |\varphi|_{C^1(\Omega)}, \lambda_{\widehat{K}}, \Lambda_{\widehat{K}}) \in (0, 1)$, such that for any positive $r \leqslant \frac{\delta}{4}\frac{R}{2}$ and for any $x, y \in B_r^+$ we have

(7) $$|v(x) - v(y)| \leqslant \frac{C_1}{R^\alpha} r^\alpha \Big(\sup_{B_{R/2}^+} |Dw| + R \sup_{B_{R/2}^+} |\widetilde{f}| \Big).$$

Observe that in [9, Theorem 1.2.16] it is assumed that $w \in C^1(B_{R/2}^+ \cup \Sigma_{R/2})$ where $\Sigma_{R/2} := B_{R/2} \cap \{y_n = 0\}$. Nevertheless, what is required in the proof is that $|Dw|$ is bounded on $B_{R/2}^+$.
It follows from Proposition 4.1 in the Appendix that the function v extends to a continuous function on $B_{\delta R/16}^+ \cup \Sigma_{\delta R/16}$. Moreover we obtain the boundary Hölder

estimates of Krylov: for any $x', y' \in \Sigma_{\delta R/64}$ we have

$$(8) \qquad |v(x') - v(y')| \leqslant \frac{C_1}{R^\alpha} \big(\sup_{B_{R/2}^+} |Dw| + R \sup_{B_{R/2}^+} |\widetilde{f}| \big) |x' - y'|^\alpha.$$

Thus the function w admits normal derivative along $\Sigma_{\delta R/32}$. Using Formula (8) it can be shown that v is Hölder continuous on $B_{\delta R/256}^+ \cup \Sigma_{\delta R/256}$, see Proposition 4.2 in the Appendix.

More precisely there exist positive numbers $C_2 = C_2(n, K, |\varphi|_{C^1(\Omega)}, \lambda_{\widehat{K}}, \Lambda_{\widehat{K}}, R)$ and $\beta = \beta(n, K, |\varphi|_{C^1(\Omega)}, \lambda_{\widehat{K}}, \Lambda_{\widehat{K}}) \leqslant \alpha < 1$, such that for any $x, y \in B_{\delta R/256}^+ \cup \Sigma_{\delta R/256}$ we have

$$(9) \qquad |v(x) - v(y)| \leqslant \frac{C_2}{R^\beta} \big(\sup_{B_{R/2}^+} |Dw| + R \sup_{B_{R/2}^+} |\widetilde{f}| \big) |x - y|^\beta.$$

We know from [13, Chapter IV, Theorem 6.1] and [6, Theorem 13.6] that we have the Ladyzhenskaya and Ural'tseva a priori interior estimates for the function $\widehat{u}$ on Ω. Namely, there exist positive constants $C_3, \eta < 1$ such that for any subdomain $\Omega' \subset \overline{\Omega}' \subset \Omega$, we have for any $x_1, x_2 \in \Omega'$

$$(10) \qquad |D\widehat{u}(x_1) - D\widehat{u}(x_2)| \leqslant C_3 |x_1 - x_2|^\eta \operatorname{dist}(\Omega', \partial\Omega)^{-\eta}$$

with $C_3 = C_3(n, K, \mu_{\widehat{K}}, \lambda_{\widehat{K}}, |\varphi|_{C^2(\Omega)}, \operatorname{diam}(\Omega)$ and $\eta = \eta(n, K, \mu_{\widehat{K}}, \lambda_{\widehat{K}}, |\varphi|_{C^2(\Omega)})$. Note that the dependence of $|\varphi|_{C^2(\Omega)}$ arises from the definition of $\widehat{b}$.

Now we extend the function w by odd reflection to the whole ball $B_{R/2}$ setting for any $y = (y', y_n) \in B_{R/2}$

$$\overline{w}(y) = \begin{cases} w(y) & \text{if } y_n \geqslant 0 \\ -w(y', -y_n) & \text{if } y_n < 0. \end{cases}$$

Observe that $\overline{w}$ is a continuous function and that $\overline{w} \in C^2(B_{R/2} \backslash \Sigma_{R/2})$. Consequently, v extends also to a continuous function $\widetilde{v}$ on the whole ball $B_{\delta R/16}$ by setting $\widetilde{v}(y', y_n) := \frac{\overline{w}(y', y_n)}{y_n}$ for any $y \in B_{\delta R/16} \setminus \Sigma_{\delta R/16}$ and $\widetilde{v}(y', 0) := v(y', 0)$ for any $(y', 0) \in \Sigma_{\delta R/16}$.

From (9) we get that for any $x, y \in B_{\delta R/256}$ we have

$$(11) \qquad |\widetilde{v}(x) - \widetilde{v}(y)| \leqslant 2\frac{C_2}{R^\beta} \big(\sup_{B_{R/2}^+} |Dw| + R \sup_{B_{R/2}^+} |\widetilde{f}| \big) |x - y|^\beta.$$

We set $R_1 := \delta R/256$. Using the a priori interior Hölder estimates (10) for the gradient of $\widehat{u}$ and the Hölder estimates (11) for $\widetilde{v}$, Trudinger derived in the proof of [27, Theorem 4] that for any $x, y \in B_{R_1/4}$ we have

$$|\overline{w}(x + y) + \overline{w}(x - y) - 2\overline{w}(x)| \leqslant C_4 |y|^{1+\gamma},$$

where $\gamma := \beta\eta/(1 + \eta)$ and $C_4 = C_4(n, K, \mu_{\widehat{K}}, \lambda_{\widehat{K}}, |\varphi|_{C^2(\Omega)}, \Omega)$.

Now consider a C^∞ function ψ on the whole $\mathbb{R}^n$ such that

$$\begin{cases} \psi(y) = 1 & \text{if } |x| \leqslant R_1/32 \\ 0 < \psi(x) < 1 & \text{if } R_1/32 < |x| < 3R_1/64 \\ \psi(y) = 0 & \text{if } |x| \geqslant 3R_1/64. \end{cases}$$

Then $\psi\overline{w}$ is a continuous function defined on $\mathbb{R}^n$. It can be shown that there exists a constant $C_5 = C_5(n, K, \mu_{\widehat{K}}, \lambda_{\widehat{K}}, |\varphi|_{C^2(\Omega)}, \Omega)$ such that for any $x, y \in \mathbb{R}^n$ we have

$$|\psi\overline{w}(x+y) + \psi\overline{w}(x-y) - 2\psi\overline{w}(x)| \leqslant C_5 |y|^{1+\gamma}.$$

We deduce from [26, Chapter V, section 4, Propositions 8 and 9] that $\psi\overline{w} \in C^{1,\gamma}(\mathbb{R}^n)$. More precisely, $\psi\overline{w} \in C^1(\mathbb{R}^n)$ and there exists a universal constant $\Upsilon > 0$ such that for any $x, y \in \mathbb{R}^n$ we have

$$|D\psi\overline{w}(x) - D\psi\overline{w}(y)| \leqslant \Upsilon C_5 |x - y|^\gamma.$$

Therefore, $w \in C^{1,\gamma}(B^+_{R_1/32} \cup \Sigma_{R_1/32})$ and for any $x, y \in B^+_{R_1/32} \cup \Sigma_{R_1/32}$ we have

$$|Dw(x) - Dw(y)| \leqslant \Upsilon C_5 |x - y|^\gamma.$$

Recall that $\widehat{u} = w \circ F$ on $\Omega \cap B_{R_1/64}(p)$ and $F(\Omega \cap B_{R_1/64}(p)) \subset B^+_{R_1/32}$. Therefore for any $p \in \partial\Omega$, the restriction of $\widehat{u}$ at $\Omega \cap B_{R_1/64}$ belongs to $C^{1,\gamma}(\overline{\Omega} \cap B_{R_1/64}(p))$.
More precisely there exist positive constants $C_6 = C_6(n, K, \lambda_{\widehat{K}}, \mu_{\widehat{K}}, |\varphi|_{C^2(\Omega)}, \Omega)$, and $\gamma = \tau(n, K, \lambda_{\widehat{K}}, \mu_{\widehat{K}}, |\varphi|_{C^2(\Omega)}) < 1$, but which do not depend on $p \in \partial\Omega$, such that for any $x_1, x_2 \in \overline{\Omega} \cap B_{R_1/64}(p)$ we have

(12) $$|D\widehat{u}(x_1) - D\widehat{u}(x_2)| \leqslant C_6 |x_1 - x_2|^\gamma.$$

Thus $\widehat{u} \in C^1(\overline{\Omega})$.
Finally, since $\partial\Omega$ is compact, there exist a finite number of points $p_1, \ldots, p_k \in \partial\Omega$ such that $\partial\Omega \subset \bigcup_{i=1}^k \Omega \cap B_{R_1/128}(p_i)$.
We set $\Omega_0 := \Omega \setminus \bigcup_{i=1}^k \Omega \cap B_{R_1/128}(p_i)$, thus $\Omega = \Omega_0 \cup \left(\bigcup_{i=1}^k \Omega \cap B_{R_1/64}(p_i) \right)$.
Considering the interior estimates (10) for $\Omega' = \Omega_0$, the boundary Hölder estimates (12) at each subset $\Omega \cap B_{R_1/64}(p_i)$, $i = 1, \ldots, k$, and a ball chain argument, we conclude that $D\widehat{u} \in C^\tau(\overline{\Omega})$ where $\tau := \min(\gamma, \eta)$. More precisely $\widehat{u} \in C^1(\overline{\Omega})$ and there exist positive constants $C_7(n, K, \lambda_{\widehat{K}}, \mu_{\widehat{K}}, |\varphi|_{C^2(\Omega)}, \Omega)$, and $\tau(n, K, \lambda_{\widehat{K}}, \mu_{\widehat{K}}, |\varphi|_{C^2(\Omega)}) < 1$, such that for any $x_1, x_2 \in \overline{\Omega}$ we have

$$|D\widehat{u}(x_1) - D\widehat{u}(x_2)| \leqslant C_7 |x_1 - x_2|^\tau.$$

Since $\varphi \in C^2(\overline{\Omega})$, there exists a positive constant $C_8 = C_8(|\varphi|_{C^2(\Omega)}, \Omega, \tau)$ such that

$$|D\varphi(x_1) - D\varphi(x_2)| \leqslant C_8 |x_1 - x_2|^\tau,$$

for any $x_1, x_2 \in \overline{\Omega}$.
Finally, since $u = \widehat{u} + \varphi$, setting $C := C_7 + C_8$, we have for any $x_1, x_2 \in \overline{\Omega}$

$$|Du(x_1) - Du(x_2)| \leqslant C |x_1 - x_2|^\tau,$$

where $C = C(n, K, \lambda_{\widehat{K}}, \mu_{\widehat{K}}, |\varphi|_{C^2(\Omega)}, \Omega)$. Thus we obtain that $u \in C^{1,\tau}(\overline{\Omega})$ as desired.
□

We infer from the proof of Theorem 2.4 the following local version.

Theorem 2.6. *Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, be a bounded domain with C^2 boundary, and let Q be a quasilinear operator as in Definition 2.3, with $a^{ij}, b \in C^1(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$.*
Let $T \subset \partial\Omega$ be a nontrivial C^2 domain of the boundary of Ω. Let $\Omega_0 \subset \Omega$ be a subdomain with C^2 boundary such that $\overline{\Omega}_0 \cap \partial\Omega \subset T$.
Let $u \in C^0(\Omega \cup T) \cap C^2(\Omega)$ be a function satisfying

$$\begin{cases} Q(u) = 0 & on \quad \Omega, \\ u = \varphi & on \quad T, \end{cases}$$

where $\varphi \in C^2(\Omega \cup T)$.
Assume there is $K > 0$ such that $|u| + |Du| \leqslant K$ on Ω.
Then, there exists a constant $\tau \in (0, 1)$, such that $u \in C^{1,\tau}(\overline{\Omega}_0)$
More precisely, $u \in C^1(\overline{\Omega}_0)$ and there exist positive numbers $C = C(n, K, \lambda_{\widehat{K}}, \Lambda_{\widehat{K}}, |\varphi|_{C^2(\overline{\Omega}_0)}, \Omega_0)$ and $\tau = \tau(n, K, \lambda_{\widehat{K}}, \Lambda_{\widehat{K}}, |\varphi|_{C^2(\overline{\Omega}_0)})$ such that for any $x_1, x_2 \in \overline{\Omega}_0$ we have

$$|Du(x_1) - Du(x_2)| \leqslant C|x_1 - x_2|^\tau,$$

where $\widehat{K} := K + |\varphi|_{C^1(\overline{\Omega}_0)}$.

Elliptic regularity leads to the following.

Theorem 2.7. *Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, be a bounded domain with C^{k+2} boundary, $k \geqslant 0$, and let Q be a quasilinear operator as above with $a^{ij}, b \in C^{k+1}(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$.*
Let $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ be a function satisfying

$$\begin{cases} Q(u) = 0 & on \quad \Omega, \\ u = \varphi & on \quad \partial\Omega, \end{cases}$$

where $\varphi \in C^{k+2}(\overline{\Omega})$.
Assume there exists a constant $K > 0$ such that $|u| + |Du| \leqslant K$ on Ω.
Then, there exists a constant $\tau \in (0, 1)$, such that $u \in C^{k+1,\tau}(\overline{\Omega})$, where $\tau = \tau(n, K, \lambda_{\widehat{K}}, \mu_{\widehat{K}}, |\varphi|_{C^2(\overline{\Omega})})$ and $\widehat{K} := K + |\varphi|_{C^1(\overline{\Omega})}$.

Proof. The proof proceeds by induction on $k \geqslant 0$.
For $k = 0$ this is Theorem 2.4.
The rest of the proof is a straightforward consequence of Schauder theory, see [6, Theorem 6.19].
□

Remark 2.8. There is a local version of Theorem 2.7. Namely let $T \subset \partial\Omega$ be a nontrivial domain of the boundary of Ω. Let us assume that $\varphi \in C^{k+2}(\Omega \cup T)$. Let $\Omega_0 \subset \Omega$ be a subdomain with C^{k+2} boundary such that $\overline{\Omega}_0 \cap \partial\Omega \subset T$. Let $u \in C^2(\Omega) \cap C^0(\Omega \cup T)$ satisfying $Q(u) = 0$ on Ω and $u = \varphi$ on T. Then we have $u \in C^{k+1,\beta}(\overline{\Omega}_0)$, where $\beta = \beta(n, K, \lambda_{\widehat{K}}, \mu_{\widehat{K}}, |\varphi|_{C^2(\overline{\Omega}_0)})$ and $\widehat{K} := K + |\varphi|_{C^1(\overline{\Omega}_0)}$.

3. Proof of the Main Theorem

3.1. Minimal equation.

We first give the minimal equation for a graph $x_3 = u(x_1, x_2)$ in some arbitrary local coordinates (x_1, x_2, x_3) of M, following Colding-Minicozzi [2, Equation (7.21)] and Gulliver [7, Section 8].

Let u be a C^∞ function defined on a domain Ω contained in the x_1, x_2 plane of coordinates. Let $S \subset M$ be the graph of u.

We use the usual convention for the partial derivative of a C^2 function u: $u_i = \frac{\partial u}{\partial x_i}$, $u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$, $i, j = 1, 2$. We denote $g_{ij} \in C^\infty$, $1 \leqslant i, j \leqslant 3$, the coefficients of the Riemannian metric g in the local coordinates (x_1, x_2, x_3) and we call G the 3×3 matrix (g_{ij}). Up to restricting the local coordinates, we can assume that the matrix G is bounded.

Let $\Gamma_{ij}^m \in C^\infty$ be the Christoffel symbols of the Riemannian metric g, $1 \leqslant i, j, m \leqslant 3$. We set $\partial_i := \frac{\partial}{\partial x_i}$, $i = 1, 2, 3$. Then, $X_i := \partial_i + u_i \partial_3$, $i = 1, 2$, is the adapted frame field generating the tangent plane of S.

Let h_{ij} be the coefficients of the metric induced on S, that is $h_{ij} = g(X_i; X_j)$, $1 \leqslant i, j \leqslant 2$.

Let $N := \sum_i N_i \partial_i$ be the unit normal field on S with $N_3 > 0$. We set $W := 1/g(N; \partial_3)$. We have

$$g(N; \partial_i) = -\frac{u_i}{W}, \quad i = 1, 2.$$

Note that the coordinates of N are given by

$$\begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} = \frac{1}{W} G^{-1} \begin{pmatrix} -u_1 \\ -u_2 \\ 1 \end{pmatrix}.$$

Since $g(N, N) = 1$ we obtain

$$W^2 = g^{33} - 2\sum_{i=1}^{2} u_i g^{i3} + \sum_{i,j=1}^{2} u_i u_j g^{ij},$$

where $G^{-1} = (g^{ij})$ is the inverse matrix of (g_{ij}).

The mean curvature H of S is given by

$$2H = \sum_{i,j=1}^{2} h^{ij} g(N; \nabla_{X_i} X_j),$$

where (h^{ij}) is the inverse matrix of (h_{ij}) and ∇ is the covariant derivative on (M, g). We define

$$F(x, u, u_1, u_2, u_{11}, u_{12}, u_{22}) := \sum_{i,j=1}^{2} \left[h^{ij} \left(u_{ij} + \Gamma_{ij}^3 + u_i \Gamma_{3j}^3 + u_j \Gamma_{3i}^3 + u_i u_j \Gamma_{33}^3 \right) \right.$$
$$\left. - \sum_{m=1}^{2} u_m h^{ij} \left(\Gamma_{ij}^m + u_i \Gamma_{3j}^m + u_j \Gamma_{i3}^m + u_i u_j \Gamma_{33}^m \right) \right].$$

Then by a computation it follows that the minimal equation $(H = 0)$ reads as

(13) $$F(x, u, u_1, u_2, u_{11}, u_{12}, u_{22}) = 0.$$

Since $h_{ij} = g(X_i; X_j)$, (h_{ij}) is a symmetric and positive matrix. This implies that (h^{ij}) is also a symmetric and positive matrix and, therefore, the equation (13) is an elliptic PDE. Furthermore, if u has bounded gradient then the equation (13) is uniformly elliptic. This means that there exist two positive constants $\lambda \leqslant \Lambda$ such that for any $x \in \Omega$ and for any eigenvalue $\rho(x)$ of the matrix $(h^{ij}(x))$, we have $0 < \lambda \leqslant \rho(x) \leqslant \Lambda$.

Remark 3.1. Let M be an analytic three manifold, and let $S \subset M$ be a minimal surface with an analytic open arc γ on its boundary. We assume that $S \cup \gamma$ is a C^1 surface with boundary.
Then for any $p \in \gamma$ there exists a neighborhood $U \subset M$ of p in M such that $U \cap S$ is a graph $x_3 = u(x_1, x_2)$ in some local coordinates (x_1, x_2, x_3) at p, of an analytic function u defined on a domain $\Omega \subset \{x_3 = 0\}$, containing an analytic arc γ_0 on its boundary. Furthermore γ_0 is the projection of γ and $u \in C^1(\Omega \cup \gamma_0)$.
We infer from Theorem 2.7 that $u \in C^k(\Omega \cup \gamma_0)$ for any $k \in \mathbb{N}$, and then $u \in C^{2,\mu}(\Omega \cup \gamma_0)$ for any $\mu \in (0, 1)$. We conclude from [19, Theorem 5.8.6'] that u is analytic on γ_0 and extends analytically across γ_0. Therefore, S can be extended analytically across γ as a minimal surface.

3.2. **Proof of Theorem 1.1.**
Let p be any point on the geodesic arc γ. We are going to construct convenient local coordinates (x_1, x_2, x_3) of M near $p \in \gamma$.
First we choose a parametrization by arc lenght, $x_1 \in (-\varepsilon, \varepsilon)$, of a open subarc of γ: $\widehat{\gamma} : (-\varepsilon, \varepsilon) \longrightarrow \gamma$, such that $\widehat{\gamma}(0) = p$.
Recall that by assumption, $S \cup \gamma$ is an embedded C^1 surface with boundary.

Let ν_p be the unit inner tangent vector of $S \cup \gamma$ at p, orthogonal to γ. We denote by ν the parallel vector field along γ such that $\nu(p) = \nu_p$. By abuse of notation we denote $\nu(x_1) = \nu(\widehat{\gamma}(x_1))$.

We set

$$\Sigma := \big\{ F(x_1, x_2) := \exp_{\widehat{\gamma}(x_1)} x_2 \nu(x_1), \ x_1, x_2 \in (-\varepsilon, \varepsilon) \big\}.$$

Clearly, if $\varepsilon > 0$ is small enough, then $\Sigma \subset M$ is a properly embedded C^∞-surface. Furthermore, Σ and $S \cup \gamma$ share the same tangent plane at p.

Let η be a C^∞ unit normal vector field along Σ. Thus if $\varepsilon > 0$ is small enough the map

$$(-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \longrightarrow M$$

$$G(x_1, x_2, x_3) := \exp_{F(x_1, x_2)} x_3 \eta,$$

is a C^∞ proper embedding. Therefore G provides local coordinates of M near p, and we have $G(0, 0, 0) = p$. We set

$$U_\varepsilon := \big\{ (x_1, x_2, x_3), \ x_1, x_2, x_3 \in (-\varepsilon, \varepsilon) \big\} \subset \mathbb{R}^3.$$

We define

$$V_\varepsilon := G(U_\varepsilon),$$

thus V_ε is an open neighborhood of p in M.

Observe that by construction we have

$$I_\gamma\big(G(x_1, x_2, 0)\big) = G(x_1, -x_2, 0) \quad \text{and} \quad I_\gamma\big(G(x_1, 0, x_3)\big) = G(x_1, 0, -x_3)$$

for any $x_1, x_2, x_3 \in (-\varepsilon, \varepsilon)$. Therefore by a continuity argument we get that

$$I_\gamma\big(G(x_1, x_2, x_3)\big) = G(x_1, -x_2, -x_3)$$

for any $x_1, x_2, x_3 \in (-\varepsilon, \varepsilon)$.

Now we establish that S can be locally extended near p by reflection across γ as a minimal surface.

By abuse of notations we identify V_ε with U_ε and a point $G(x_1, x_2, x_3)$ with (x_1, x_2, x_3). Therefore Σ is identified with $(-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \times \{0\}$. Observe that the reflection I_γ reads as

$$I_\gamma(x_1, x_2, x_3) = (x_1, -x_2, -x_3),$$

for any $(x_1, x_2, x_3) \in U_\varepsilon$.

Note that by construction we have $T_p \Sigma = T_p(S \cup \gamma)$. Therefore if $\varepsilon > 0$ is small enough then $G^{-1}\big((S \cup \gamma) \cap V_\varepsilon\big)$ is the graph $x_3 = u(x_1, x_2)$ of a C^1 function u defined on $\Sigma^+ := \{(x_1, x_2), \ x_1 \in (-\varepsilon, \varepsilon), x_2 \in [0, \varepsilon)\}$, using the identification $V_\varepsilon \equiv U_\varepsilon$.

Observe that by assumption u is C^2 on $(-\varepsilon, \varepsilon) \times (0, \varepsilon)$ and satisfies the minimal equation (13).

Since by assumption u is C^1 on Σ^+, we get that u has bounded gradient. Hence from Remark 2.8 we obtain that u is C^k on Σ^+ for any $k \in \mathbb{N}$.

We define a function v on $\Sigma^- := \{(x_1, x_2), \ x_1 \in (-\varepsilon, \varepsilon), x_2 \in (-\varepsilon, 0]\}$ setting

$$v(x_1, x_2) := -u(x_1, -x_2).$$

By construction $u(x_1, 0) = v(x_1, 0) = 0$ for any x_1 and $u_i(0, 0) = 0 = v_i(0, 0)$, $i = 1, 2$. Then we define a function w on Σ, identified with $\{(x_1, x_2), \ x_1, x_2 \in (-\varepsilon, \varepsilon)\}$, setting

$$w(x_1, x_2) := \begin{cases} u(x_1, x_2) \text{ if } (x_1, x_2) \in \Sigma^+ \\ v(x_1, x_2) \text{ if } (x_1, x_2) \in \Sigma^- \end{cases}$$

We have that v is C^2 on Σ^- and w is C^1 on Σ.

In order to check that w is C^2 on Σ it is enough to prove that the partial derivatives up to the second order of u and v agree along the arc $\Sigma^+ \cap \Sigma^- = \{(x_1, 0), \ x_1 \in (-\varepsilon, \varepsilon)\}$. For any $(x_1, x_2) \in \Sigma^-$ we have

$$v_1(x_1, x_2) = -u_1(x_1, -x_2), \quad v_2(x_1, x_2) = u_2(x_1, -x_2)$$

and

$$v_{11}(x_1, x_2) = -u_{11}(x_1, -x_2), \quad v_{12}(x_1, x_2) = u_{12}(x_1, -x_2), \quad v_{22}(x_1, x_2) = -u_{22}(x_1, -x_2).$$

Note also that $u_{ij}(0, 0) = 0$, $i, j = 1, 2$.

Therefore we deduce from the previous identities that along the geodesic line $\{x_2 = 0\}$ we have

$$F(x, u, u_1, u_2, u_{11}, u_{12}, u_{22}) - F(x, v, v_1, v_2, v_{11}, v_{12}, v_{22}) = h^{22}(u_{22}(x_1, 0) - v_{22}(x_1, 0)).$$

Since u and v satisfy the minimal equation (13) we infer

$$u_{22}(x_1, 0) = v_{22}(x_1, 0) = 0$$

for any $x_1 \in (-\varepsilon, \varepsilon)$. Therefore $u_{ij} = v_{ij}$ along the arc $\Sigma^+ \cap \Sigma^-$.

Thus we deduce that the function w is C^2 on the whole domain Σ, satisfying the minimal equation (13). Thereby, the graph of w, denoted by $\widetilde{S}$, is a minimal surface. Of course, observe that, by construction, $\widetilde{S}$ is invariant by reflection across γ.

We obtain therefore a minimal continuation, $\widetilde{S}$, of S across γ, embedded near γ. This accomplishes the proof of the theorem. $\qquad\qquad\square$

Proof of the Main Theorem bis. By assumption, f is defined on a domain Ω in the coordinates plane $\{x_3 = 0\}$. Moreover the boundary $\partial\Omega$ contains a C^∞ open arc γ_0 such that γ is the graph of f over γ_0.

Since, by assumption, f has bounded gradient, the proof follows readily using the regularity Theorem 2.4 as in the proof of the Main Theorem 1.1. $\qquad\qquad\square$

At last, we discuss several general remarks about the geometry of minimal surfaces.

Remark 3.2. We use the notations of the Main Theorem 1.2 bis.

Assume that the coordinates system (x_1, x_2, x_3) containing the minimal surface S has the following property: For certain (small enough) domain Ω contained in the coordinates plane $\{x_3 = 0\}$, we can uniquely solve the Dirichlet Problem for the minimal equation given any (small enough) continuous data on $\partial\Omega$.

Then we can drop the assumption that f has bounded gradient in the Main Theorem bis 1.2. For example this property occurs in $\mathbb{R}^3$, $\mathbb{H}^2 \times \mathbb{R}$ (see [24, Lemma 3.6]), and Nil_3 (see [20, Theorem 4.1 and Corollary 4.3]).

Remark 3.3. Let us consider the particular case where the ambient space M is analytic. Let $S \subset M$ be an embedded minimal surface such that $S \cup \gamma$ is a C^1 surface with boundary, where γ is an open geodesic arc of M which admits a reflection.
Since γ is an analytic arc, $S \cup \gamma$ is analytic and can be extended as an analytic surface $\widehat{S}$ across γ, see Remark 3.1
Now we can apply either Theorem 1 in [15] or Theorem 1.1 to infer that in a neighborhood of any point of γ, the extended surface $\widehat{S}$ is invariant by reflection across γ.

The following result have been applied in homogeneous three spaces by many authors to construct complete minimal surfaces, see for example [22], [1], [18], [17].

Proposition 3.4. *Let (M, g) be a C^∞ Riemannian three manifold and let Γ be a Jordan curve. Assume that Γ contains an open geodesic arc γ which admits a reflection. Let S be an area minimizing solution of the Plateau problem, if any.*
Then, as in Theorem 1.1, the reflection of S about γ gives rise to a C^∞ continuation of S across γ. That is, $S \cup \gamma \cup I_\gamma(S)$ is a smooth immersed minimal surface which is embedded near γ.

Proof. We set

$$B := \{(x, y) \in \mathbb{R}^2, \ x^2 + y^2 < 1\} \quad \text{and} \quad \overline{B} := \{(x, y) \in \mathbb{R}^2, \ x^2 + y^2 \leqslant 1\}.$$

By assumption, there exists a map $X \in C^0(\overline{B}, M) \cap C^2(B, M)$ such that

- $X(\overline{B}) = S$, and X maps monotonically ∂B onto Γ.
- $g(\partial_x X; \partial_x X) - g(\partial_y X; \partial_y X) = g(\partial_x X; \partial_y X) = 0$,
- $\partial_{xx} x_j + \partial_{yy} x_j + \sum_{k,l=1}^{3} \Gamma_{k\,l}^{j} \big(\partial_x x_k \partial_x x_l + \partial_y x_k \partial_y x_l \big) = 0$, $j = 1, 2, 3$,
- X minimize the functional area among all maps verifying the above properties,

where $\Gamma_{k\,l}^{j}$ stands for the Christoffel symbols, $\partial_x = \frac{\partial}{\partial x}$ and so on.
Let $\gamma_0 \subset \partial B$ be an open arc such that $X(\gamma_0) = \gamma$.
Since γ is of class C^3, at least, we get from [5, Theorem 4-(ii), Section 2.3] that $X \in C^2(B \cup \gamma_0, M)$. Since X is area minimizing we know from [7, Theorem 8.1] that X has no *true branch point* on B. We get from [10, Conclusion (ii) of Theorem] that X has, possibly, isolated boundary branch points on γ_0 and has no branch point (neither true nor false) in a neighborhood of γ_0 in B. We are going to prove that X has no boundary branch point on γ_0.
Let $p \in \gamma$ and $p_0 \in \gamma_0$ be such that $p = X(p_0)$. Up to a conformal transformation we can assume that $D^+ := \{(u, v) \in \mathbb{R}^2, \ u^2 + v^2 < 1, \text{ and } v \geqslant 0\}$ is a neighborhood of p_0 in $B \cup \gamma_0$, with p_0 corresponding to 0 and γ_0 to $\delta_0 := \{(u, 0), |u| < 1\} \subset \partial D^+$.
By an abuse of notation we assume also that X is defined on D^+. Furthermore, up to reducing the neighborhood of p_0, we can assume that X has no boundary branch point

on δ_0 except, possibly, at 0 and has no branch point on $D^+\backslash\delta_0$. Thus $X : D^+\backslash\delta_0 \longrightarrow M$ is a C^2 minimal immersion, $X(\delta_0) = \gamma$, $X(0) = p$ and X is of class C^2 up to δ_0.

We deduce from [10, Conclusion (ii) of Theorem] that the surface S has a well-defined tangent plane at any point of γ, even at boundary branch point.

Let ν_p be unit tangent vector of S at $p \in \gamma$. We consider the coordinates $(x_1, x_2, x_3) \in (-\varepsilon, \varepsilon)^3$ on a neighborhood of p in M constructed in the proof of Theorem 1.1. Thus p corresponds to $(0, 0, 0)$, γ corresponds to $x_2 = x_3 = 0$, and the reflection about γ (which is assumed to exist) reads as $I_\gamma(x_1, x_2, x_3) = (x_1, -x_2, -x_3)$.

We set

$$D^- := \{(u, v) \in \mathbb{R}^2, \, u^2+v^2 < 1, \, v \leqslant 0\} \text{ and } D = D^+\cup D^- = \{(u, v) \in \mathbb{R}^2, \, u^2+v^2 < 1\}.$$

We are going to show that X can be extended to a C^2 map $Z : D \longrightarrow M$ verifying

- $Z = X$ on D^+,
- $g(\partial_u Z; \partial_u Z) - g(\partial_v Z; \partial_v Z) = g(\partial_u Z; \partial_v Z) = 0$,
- $\partial_{uu} z_j + \partial_{vv} z_j + \sum_{k,l=1}^{3} \Gamma_{kl}^j \big(\partial_u z_k \partial_u z_l + \partial_v z_k \partial_v z_l\big) = 0$, $j = 1, 2, 3$ on D.

Then using [8, Theorem] we obtain that 0 is not a branch point of Z and consequently, 0 is not a boundary branch point of X. Using Theorem 1.1 we can deduce that $S \cup \gamma \cup I_\gamma(S)$ is a smooth immersed minimal surface which is embedded near γ as desired.

Recall that we can assume that X has no branch point on δ_0 except, possibly, at 0. We define the map $Y : D^- \longrightarrow M$ setting for any $(u, v) \in D^-$: $Y(u, v) := I_\gamma(X(u, -v)) = (x_1, -x_2, -x_3)(u, -v)$.

Then we define the continuous map $Z : D \longrightarrow M$ setting

$$Z(u, v) = \begin{cases} X(u, v), & \text{if } v \geqslant 0, \\ Y(u, v), & \text{if } v \leqslant 0. \end{cases}$$

We have $y_1(u, v) = x_1(u, -v)$, $y_2(u, v) = -x_2(u, -v)$ and $y_3(u, v) = -x_3(u, -v)$ on D^-. We want to show that partial derivatives of X and Y are equal along δ_0 up to second order, this will prove that Z is C^2 on D.

Since $X(\delta_0) = \gamma$ we have $x_2(u, 0) = x_3(u, 0) = 0$ along δ_0. So that $\partial_u x_k(u, 0) = \partial_u y_k(u, 0) = 0$ along δ_0, $k = 2, 3$. Since $\partial_v y_k(u, v) = \partial_v x_k(u, -v)$ on D^-, $k = 2, 3$, we deduce that z_2 and z_3 are C^1 on D.

By construction we have $\partial_u x_1(u, 0) = \partial_u y_1(u, 0)$ along δ_0.

Since X is a conformal map up to δ_0, we have $\partial_u x_1 \cdot \partial_v x_1 = 0$ along δ_0. Using the fact that $\partial_u X$ does not vanish along δ_0, except possibly at 0, we get that $\partial_v x_1 = 0$ along δ_0. Since $\partial_v y_1(u, v) = -\partial_v x_1(u, -v)$ on D^-, we get that $\partial_v x_1(u, 0) = \partial_v y_1(u, 0) = 0$ along δ_0 and therefore Z is a C^1 map on D.

By construction for $k = 2, 3$ we have $\partial_{uv} y_k(u, v) = \partial_{uv} x_k(u, -v)$ on D^-, so that $\partial_{uv} z_2$ and $\partial_{uv} z_3$ are well defined and continuous on D.

Since $\partial_v x_1 = 0$ along δ_0, we have $\partial_{uv} x_1(u, 0) = 0$ too for any u. With the relation $\partial_{uv} y_1(u, v) = -\partial_{uv} x_1(u, -v)$ on D^- we get that $\partial_{uv} z_1$ is well defined and continuous on D.

Furthermore we get by construction that $\partial_{uu} z_2$, $\partial_{uu} z_3$, $\partial_{uu} z_1$ and $\partial_{vv} z_1$ are continuous on D, so that z_1 is a C^2 function.

It remains to show that that $\partial_{vv} z_2$ and $\partial_{vv} z_3$ are well defined and continuous on D. Since X is a minimal and conformal immersion of $D^+ \setminus \delta_0$ and X has C^2 regularity up to δ_0, we have on D^+, for $j = 2, 3$:

$$\partial_{uu} x_j + \partial_{vv} x_j + \sum_{k,l=1}^{3} \Gamma_{kl}^{j} \big(\partial_u x_k \partial_u x_l + \partial_v x_k \partial_v x_l \big) = 0,$$

where the Christoffel symbols are evaluated at $(x_1, x_2, x_3)(u, v)$, $(u, v) \in D^+$.
Using the previous considerations, we get for $j = 2$ and for any $(u, 0) \in \delta_0$:

$$\partial_{vv} x_2(u, 0) + \Gamma_{11}^{2} \partial_u x_1(u, 0)^2 + \Gamma_{22}^{2} \partial_v x_2(u, 0)^2 + \Gamma_{33}^{2} \partial_v x_3(u, 0)^2 + 2\Gamma_{23}^{2} \partial_v x_2 \partial_v x_3(u, 0) = 0,$$

now the Christoffel symbols are evaluated at $\big(x_1(u, 0), 0, 0 \big)$.
By the way the coordinates x_1, x_2 and x_3 are chosen, we have $\Gamma_{kk}^{2}(x_1, 0, 0) = 0$ for any x_1, $k = 1, 2, 3$.
Moreover a computation shows that $\Gamma_{23}^{2} = \frac{1}{2g_{22}} \partial_{x_3} g_{22}$ on γ. Since I_γ is an isometry for the metric g we have $g_{22}(x_1, -x_2, -x_3) = g_{22}(x_1, x_2, x_3)$ so that $\partial_{x_3} g_{22}(x_1, 0, 0) = 0$ for any x_1.
Thus $\partial_{vv} x_2(u, 0) = 0$ along δ_0, we deduce that $\partial_{vv} y_2(u, 0) = 0$ also and then $\partial_{vv} z_2$ is a well defined and continuous map on D. This shows that z_2 is a C^2 map on D.
We prove in the same way that z_3 is a C^2 map on D. This achieves the prove.   □

We write now a typical example in $\mathbb{H}^2 \times \mathbb{R}$, see [23, Corollary 4.1].

Example 3.5. Let $T \subset \mathbb{H}^2$ be a geodesic triangle with sides A, B and C. We assign constant value a, b, c respectively on interior(A), interior(B), interior(C).
We solve the corresponding Dirichlet problem for the vertical minimal equation as in [23, Corollary 4.1]. We call f the solution and S the graph of f. Thus, S is a minimal surface of $\mathbb{H}^2 \times \mathbb{R}$.
It is a matter of fact that the boundary Γ of S is constituted of the union of three horizontal segments and three vertical segments. It turns out that S is the unique minimal surface having Γ as boundary. Therefore S is the solution of the Plateau problem for the boundary data Γ.
Henceforth, by Proposition 3.4, we can extend S as a minimal surface by reflection across any horizontal or vertical lines of Γ.

We refer to [23, Example 4.4] for a simple construction of a complete minimal surface of $\mathbb{H}^2 \times \mathbb{R}$, by solving a certain Dirichlet problem and using reflections about horizontal geodesics. The readers are also referred to [25].

4. Appendix

We recall some notations.

For any vector $d = (d_1, \ldots, d_k) \in \mathbb{R}^k$, $k \in \mathbb{N}^*$, we set $|d| = \sqrt{d_1^2 + \cdots + d_k^2}$.

We identify $\mathbb{R}^{n-1}$ with $\{x \in \mathbb{R}^n, \ x_n = 0\}$, that is with $\mathbb{R}^{n-1} \times \{0\}$. Therefore we identify any $y' \in \mathbb{R}^{n-1}$ with $(y', 0) \in \mathbb{R}^n$.

We note also for any $x \in \mathbb{R}^n : x = (x', x_n)$ where $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$.

For any $R > 0$ and for any $y' \in \mathbb{R}^{n-1}$, we set

$$B_R^+ := B_R \cap \{x_n > 0\} = \{x \in \mathbb{R}^n, \ |x| < R, \ x_n > 0\}$$

$$\Sigma_R := B_R \cap \{x_n = 0\} = \{x \in \mathbb{R}^n, \ |x| < R, \ x_n = 0\}$$

$$B_R^+(y') := \{x = (x', x_n) \in \mathbb{R}^n, \ |x - y'| < R, \ x_n > 0\}$$

$$\Sigma_R(y') := B_R^+(y') \cap \{x_n = 0\}.$$

We recall that $w \in C^2(B_{R/2}^+) \cap C^0(B_{R/2}^+ \cup \Sigma_{R/2})$ satisfies the linear elliptic equation (6). Furthermore $w \equiv 0$ on $\Sigma_{R/2}$ and the function $v(y) := \dfrac{w(y)}{y_n}$ defined on $B_{R/2}^+$ satisfies the estimate (7). We set $\delta := \frac{\lambda_{\widehat{K}}}{48n\Lambda_{\widehat{K}}}$, see the proof of Theorem 2.4.

Proposition 4.1. *The function v can be extended to a continuous function on $B_{\delta R/16}^+ \cup \Sigma_{\delta R/16}$. Moreover, for any $x', y' \in \Sigma_{\delta R/64}$ we have*

$$|v(x') - v(y')| \leqslant \frac{C_1}{R^\alpha} \big(\sup_{B_{R/2}^+} |Dw| + R \sup_{B_{R/2}^+} |\widetilde{f}| \big) |x' - y'|^\alpha,$$

where $C_1 > 0$ and $\alpha \in (0, 1)$ are the constants given in the inequality (7), that is $C_1 := C_1(n, K, |\varphi|_{C^1(\Omega)}, \lambda_{\widehat{K}}, \Lambda_{\widehat{K}})$ and $\alpha = \alpha(n, K, |\varphi|_{C^1(\Omega)}, \lambda_{\widehat{K}}, \Lambda_{\widehat{K}})$.

Proof. First observe that for any $y = (y', y_n) \in B_{\delta R/16}^+$ we have

$$B_{y_n}^+(y') \subset B_{\frac{\delta}{4}\frac{R}{4}}^+(y') \quad \text{and} \quad B_{\frac{R}{4}}^+(y') \subset B_{R/2}^+.$$

Consequently, from the inequality (7) applied to the function w on the half-balls $B_{y_n}^+(y')$ and $B_{R/4}(y')$, we deduce that for any $x \in B_{y_n}^+(y')$ we have

$$(14) \qquad |v(x) - v(y)| \leqslant \frac{C_1}{R^\alpha} y_n^\alpha \big(\sup_{B_{R/2}^+} |Dw| + R \sup_{B_{R/2}^+} |\widetilde{f}| \big).$$

Let (t_k) be a non increasing sequence of positive real numbers converging to 0. For any $p \leqslant q \in \mathbb{N}$ large enough we deduce from (14) (applied to the half-ball $B_{t_q}^+(y')$), that

$$|v(y', t_p) - v(y', t_q)| \leqslant \frac{C_1}{R^\alpha} t_q^\alpha \big(\sup_{B_{R/2}^+} |Dw| + R \sup_{B_{R/2}^+} |\widetilde{f}| \big).$$

Therefore $(v(y', t_k))$ is a Cauchy sequence. Consequently the sequence $(v(y', t_k))$ converges to some real number, momentarily denoted by $h(y')$. Moreover, the above

inequality shows also that the limit $h(y')$ does not depend on the positive sequence (t_k) converging to 0.

Now let $(x_k) = \big((x'_k, x_{k,n})\big)$ be a sequence in $B^+_{\delta R/16}$ converging to $(y', 0)$. We set $\delta_k := |x_k - (y', 0)|$. Since $x_k \in B^+_{2\delta_k}(y')$, we deduce from (14) that for k large enough we have

$$|v(x_k) - h(y')| \leqslant |v(x_k) - v(y', 2\delta_k)| + |v(y', 2\delta_k) - h(y')|$$

$$\leqslant \frac{C_1}{R^\alpha} (2\delta_k)^\alpha \big(\sup_{B^+_{R/2}} |Dw| + R \sup_{B^+_{R/2}} |\widetilde{f}| \big) + |v(y', 2\delta_k) - h(y')|.$$

Therefore we have that $v(x_k) \to h(y')$. Thus we can extend v to a continuous function on $B^+_{\delta R/16} \cup \Sigma_{\delta R/16}$.

Consider now $x', y' \in \Sigma_{\delta R/64}$. Observe that

$$B^+_{|x'-y'|}(y') \subset B^+_{\frac{\delta}{4}\frac{R}{4}}(y') \quad \text{and} \quad B^+_{R/4}(y') \subset B^+_{R/2}.$$

Therefore, we get from the inequality (7) that for any $z, z' \in B^+_{|x'-y'|}(y')$ we have

$$|v(z) - v(z')| \leqslant \frac{C_1}{R^\alpha} |x' - y'|^\alpha \big(\sup_{B^+_{R/2}} |Dw| + R \sup_{B^+_{R/2}} |\widetilde{f}| \big).$$

Now let (x'_k) be any sequence in $\Sigma_{|x'-y'|}(y')$ converging to x' and let (t_k) be a sequence of positive real numbers converging to 0 such that $(x'_k, t_k), (y', t_k) \in B^+_{|x'-y'|}(y')$ for any k. Therefore, we deduce from the previous inequality that for any k we have

$$|v(x'_k, t_k) - v(y', t_k)| \leqslant \frac{C_1}{R^\alpha} |x' - y'|^\alpha \big(\sup_{B^+_{R/2}} |Dw| + R \sup_{B^+_{R/2}} |\widetilde{f}| \big).$$

Letting k go to $+\infty$ we get

$$|v(x') - v(y')| \leqslant \frac{C_1}{R^\alpha} |x' - y'|^\alpha \big(\sup_{B^+_{R/2}} |Dw| + R \sup_{B^+_{R/2}} |\widetilde{f}| \big),$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

For the next result we recall that φ, K and $\widehat{K}$ are defined in Theorem 2.4 and that $\lambda_{\widehat{K}}$ and $\Lambda_{\widehat{K}}$ are defined in Definition 2.3.

We note also that, since w satisfies the linear elliptic equation (6), the function $v(y) = \dfrac{w(y)}{y_n}$ satisfies on $B^+_{R/2}$ the linear elliptic equation

$$(15) \qquad\qquad \widetilde{L}_0(v) := y_n \sum_{i,j=1}^n \widetilde{\alpha}^{ij}(y) D_{ij}v + 2 \sum_{i=1}^n \widetilde{\alpha}^{in}(y) D_i v = \widetilde{f}(y)$$

Proposition 4.2. *There exist positive numbers $C_2 = C_2(n, K, |\varphi|_{C^1(\Omega)}, \lambda_{\widehat{K}}, \Lambda_{\widehat{K}}, R)$ and $\beta = \beta(n, K, |\varphi|_{C^1(\Omega)}, \lambda_{\widehat{K}}, \Lambda_{\widehat{K}}) \leqslant \alpha < 1$, such that for any $x, y \in B^+_{\delta R/256} \cup \Sigma_{\delta R/256}$ we have*

$$(16) \qquad |v(x) - v(y)| \leqslant \frac{C_2}{R^\beta} \big(\sup_{B^+_{R/2}} |Dw| + R \sup_{B^+_{R/2}} |\widetilde{f}| \big) |x - y|^\beta.$$

Proof. We are going to consider successively the cases $x, y \in \Sigma_{\delta R/256}$, $x \in \Sigma_{\delta R/256}$, $y \in B^+_{\delta R/256}$ and $x, y \in B^+_{\delta R/256}$.

Case $x, y \in \Sigma_{\delta R/256}$.
We identify $x = (x', 0)$ with x' and $y = (x', 0)$ with y'.
Thanks to Proposition 4.1 we have

$$|v(x') - v(y')| \leqslant \frac{C_1}{R^\alpha} \big(\sup_{B^+_{R/2}} |Dw| + R \sup_{B^+_{R/2}} |\widetilde{f}| \big) |x' - y'|^\alpha,$$

Case $x \in \Sigma_{\delta R/256}$ and $y \in B^+_{\delta R/256}$.
We identify $x = (x', 0)$ with x'. Using inequality (14) and Proposition 4.1 we have

$$|v(x') - v(y)| \leqslant |v(x') - v(y')| + |v(y') - v(y)|$$

$$\leqslant \frac{C_1}{R^\alpha} \big(\sup_{B^+_{R/2}} |Dw| + R \sup_{B^+_{R/2}} |\widetilde{f}| \big) |x' - y'|^\alpha + \frac{C_1}{R^\alpha} y_n^\alpha \big(\sup_{B^+_{R/2}} |Dw| + R \sup_{B^+_{R/2}} |\widetilde{f}| \big)$$

$$\leqslant 2 \frac{C_1}{R^\alpha} \big(\sup_{B^+_{R/2}} |Dw| + R \sup_{B^+_{R/2}} |\widetilde{f}| \big) |x' - y|^\alpha.$$

Case $x, y \in B^+_{\delta R/256}$.
We can assume that $x_n \leqslant y_n$.
We are going to consider separately the cases $|x - y| < y_n/4$ and $|x - y| \geqslant y_n/4$.

Assume first that $|x - y| < y_n/4$.
Thus we have

$$B_{|x-y|}(y) \subset B_{y_n/2}(y) \subset B_{3y_n/5}(y) \subset B^+_{R/2}.$$

Recall that v satisfies the linear elliptic equation (15).
We are going to apply the extension of the Krylov-Safonov Hölder estimate done by Gilbarg and Trudinger, Corollary 9.24 of [6], to the function v with Ω, B_{R_0} and B_R (of Corollary 9.24) substituted respectively by $B_{3y_n/5}(y), B_{y_n/2}(y)$ and $B_{|x-y|}(y)$ (that is $R_0 = y_n/2$ and $R = |x - y|$).
Following the notations of [6, Section 9.7], the principal part of $\widetilde{L}_0$ is given by the symmetric matrix $\big(z_n \widetilde{\alpha}^{ij}(z) \big)$, $z \in \Omega = B_{3y_n/5}(y)$. The functions $b_i, i = 1, \ldots, n$, are given by $b_i = 2\widetilde{\alpha}^{in}$ (therefore $|b_i| \leqslant 8\Lambda_{\widehat{K}}$, see Equation (6)), and $c = 0$. For any $z \in \Omega = B_{3y_n/5}(y)$ we set $\lambda_0(z) = z_n \frac{\lambda_{\widehat{K}}}{2}$ and $\Lambda_0(z) = 4z_n \Lambda_{\widehat{K}}$. Thus, for any eigenvalue

$\rho_0(z)$ of the symmetric matrix $\big(z_n\widetilde{\alpha}^{ij}(z)\big)$, we have $\lambda_0(z) \leqslant \rho_0(z) \leqslant \Lambda_0(z)$, see the discussion after Equation (6). Therefore we can choose $\gamma = 8\frac{\Lambda_{\widehat{K}}}{\lambda_{\widehat{K}}}$ to achieve

$$\frac{\Lambda_0}{\lambda_0} \leqslant \gamma.$$

For any $z \in \Omega = B_{3y_n/5}(y)$ we have

$$\left(\frac{|b(z)|}{\lambda_0(z)}\right)^2 \leqslant n\left(16\frac{\Lambda_{\widehat{K}}}{z_n\lambda_{\widehat{K}}}\right)^2 \leqslant n\left(\frac{\Lambda_{\widehat{K}}}{\lambda_{\widehat{K}}}\right)^2 \frac{40^2}{y_n^2}.$$

Therefore we can choose $\nu := n\left(\frac{\Lambda_{\widehat{K}}}{\lambda_{\widehat{K}}}\right)^2 \frac{40^2}{y_n^2}$ to achieve

$$\left(\frac{|b|}{\lambda_0}\right)^2 \leqslant \nu$$

on $B_{3y_n/5}(y)$.

Thus, in Corollary 9.24 of [6] we have $\nu R_0^2 = 400n\left(\frac{\Lambda_{\widehat{K}}}{\lambda_{\widehat{K}}}\right)^2$, since $R_0 = y_n/2$. Therefore, using Corollary 9.24 of [6] we obtain

$$(17) \qquad |v(x) - v(y)| \leqslant C_1'\frac{|x-y|^\eta}{y_n^\eta}\left(\mathrm{osc}_{B_{y_n/2}(y)}v + \omega_n y_n^2 \sup_{B_{y_n/2}(y)}|\widetilde{f}|\right)$$

where ω_n is a constant depending only on n, and $C_1' > 0$ and $\eta \in (0,1)$ are constant real numbers depending only on n, γ and $\nu(y_n/2)^2 = 400n\left(\frac{\Lambda_{\widehat{K}}}{\lambda_{\widehat{K}}}\right)^2$, that is depending only on n and $\frac{\Lambda_{\widehat{K}}}{\lambda_{\widehat{K}}}$.
On the other hand we have

$$\begin{aligned}
\mathrm{osc}_{B_{y_n/2}(y)}v &= \mathrm{osc}_{B_{y_n/2}(y)}\big(v - v(y')\big) \\
&\leqslant 2\sup_{z\in B_{y_n/2}(y)}|v(z) - v(y')| \\
&\leqslant 2\sup_{z\in B_{y_n/2}(y)}|v(z) - v(z')| + 2\sup_{z\in B_{y_n/2}(y)}|v(z') - v(y')|,
\end{aligned}$$

where $z = (z', z_n)$.
Observe that since $y \in B_{\delta R/256}^+$ we have

$$\Sigma_{y_n/2}(y') \subset \Sigma_{\delta R/64}.$$

Consequently, for any $z \in B_{y_n/2}(y)$ we deduce from Proposition 4.1

$$|v(z') - v(y')| \leqslant \frac{C_1}{R^\alpha} \Big(\sup_{B_{R/2}^+} |Dw| + R \sup_{B_{R/2}^+} |\widetilde{f}| \Big) |z' - y'|^\alpha$$

$$\leqslant \frac{C_1}{R^\alpha} y_n^\alpha \Big(\sup_{B_{R/2}^+} |Dw| + R \sup_{B_{R/2}^+} |\widetilde{f}| \Big).$$

Note that for any $z \in B_{y_n/2}(y)$ we have

$$B_{z_n}^+(z') \subset B_{\frac{\delta}{4}\frac{R}{4}}^+(z') \quad \text{and} \quad B_{\frac{R}{4}}^+(z') \subset B_{R/2}^+.$$

Moreover, by Proposition 4.1 v extends continuously to z'. We deduce from the inequality (7) that for any $z \in B_{y_n/2}(y)$ we have

$$|v(z) - v(z')| \leqslant \frac{C_1}{R^\alpha} z_n^\alpha \Big(\sup_{B_{R/2}^+} |Dw| + R \sup_{B_{R/2}^+} |\widetilde{f}| \Big)$$

$$\leqslant \Big(\frac{3}{2} \Big)^\alpha \frac{C_1}{R^\alpha} y_n^\alpha \Big(\sup_{B_{R/2}^+} |Dw| + R \sup_{B_{R/2}^+} |\widetilde{f}| \Big).$$

Therefore we obtain

(18)
$$\operatorname{osc}_{B_{y_n/2}(y)} v \leqslant 5 \frac{C_1}{R^\alpha} y_n^\alpha \Big(\sup_{B_{R/2}^+} |Dw| + R \sup_{B_{R/2}^+} |\widetilde{f}| \Big).$$

Now we set $\beta = \min(\alpha, \eta)$. Since $|x - y| \leqslant y_n/4$ we have

$$\frac{|x - y|^\eta}{y_n^\eta} \leqslant \frac{|x - y|^\beta}{y_n^\beta} \quad \text{and} \quad \frac{y_n^\alpha}{R^\alpha} \leqslant \frac{y_n^\beta}{R^\beta}.$$

Therefore we deduce from (17) and (18) that

$$|v(x) - v(y)| \leqslant C_1' \frac{|x - y|^\beta}{y_n^\beta} \left(5 \frac{C_1}{R^\beta} y_n^\beta \Big(\sup_{B_{R/2}^+} |Dw| + R \sup_{B_{R/2}^+} |\widetilde{f}| \Big) + \omega_n y_n^2 \sup_{B_{y_n/2}(y)} |\widetilde{f}| \right)$$

$$\leqslant C_1' |x - y|^\beta \left(5 \frac{C_1}{R^\beta} \Big(\sup_{B_{R/2}^+} |Dw| + R \sup_{B_{R/2}^+} |\widetilde{f}| \Big) + \omega_n y_n^{2-\beta} \sup_{B_{y_n/2}(y)} |\widetilde{f}| \right)$$

$$\leqslant C_1' |x - y|^\beta \left(5 \frac{C_1}{R^\beta} \Big(\sup_{B_{R/2}^+} |Dw| + R \sup_{B_{R/2}^+} |\widetilde{f}| \Big) + \omega_n R^{2-\beta} \sup_{B_{y_n/2}(y)} |\widetilde{f}| \right).$$

Consequently, setting $C_1'' = C_1'\big(5C_1 + \omega_n R\big)$, we obtain

$$|v(x) - v(y)| \leqslant \frac{C_1''}{R^\beta}\Big(\sup_{B_{R/2}^+} |Dw| + R \sup_{B_{R/2}^+} |\widetilde{f}| \Big)|x - y|^\beta$$

where $C_1'' = C_1''(n, K, |\varphi|_{C^1(\Omega)}, \lambda_{\widehat{K}}, \Lambda_{\widehat{K}}, R) > 0$ and $\beta = \beta(n, K, |\varphi|_{C^1(\Omega)}, \lambda_{\widehat{K}}, \Lambda_{\widehat{K}}) \in (0,1)$, $\beta \leqslant \alpha$.

Assume now that $|x - y| \geqslant y_n/4$.
Recall that we are also assuming that $x_n \leqslant y_n$.
Observe that

$$B_{x_n}^+(x') \subset B_{\frac{\delta}{4}\frac{R}{8}}^+(x') \subset B_{\delta R/16}^+ \qquad \text{and} \qquad B_{R/8}^+(x') \subset B_{R/2}^+,$$

$$B_{y_n}^+(y') \subset B_{\frac{\delta}{4}\frac{R}{8}}^+(y') \subset B_{\delta R/16}^+ \qquad \text{and} \qquad B_{R/8}^+(y') \subset B_{R/2}^+.$$

Since v can be extended to a continuous function on $B_{\delta R/16}^+ \cup \Sigma_{\delta R/16}$ we get from the inequality (7) applied successively on the half balls $B_{x_n}^+(x')$ and $B_{y_n}^+(y')$

$$|v(x) - v(x')| \leqslant \frac{C_1}{R^\alpha}x_n^\alpha\Big(\sup_{B_{R/2}^+} |Dw| + R \sup_{B_{R/2}^+} |\widetilde{f}| \Big)$$

$$|v(y) - v(y')| \leqslant \frac{C_1}{R^\alpha}y_n^\alpha\Big(\sup_{B_{R/2}^+} |Dw| + R \sup_{B_{R/2}^+} |\widetilde{f}| \Big).$$

Moreover, since $x', y' \in \Sigma_{\delta R/64}$, Proposition 4.1 gives

$$|v(x') - v(y')| \leqslant \frac{C_1}{R^\alpha}\Big(\sup_{B_{R/2}^+} |Dw| + R \sup_{B_{R/2}^+} |\widetilde{f}| \Big)|x' - y'|^\alpha.$$

Therefore, using the above inequalities, $x_n \leqslant y_n$, $y_n \leqslant 4|x - y|$ and $|x' - y'| \leqslant |x - y|$, we get

$$|v(x) - v(y)| \leqslant |v(x) - v(x')| + |v(x') - v(y')| + |v(y') - v(y)|$$

$$\leqslant \frac{C_1}{R^\alpha}\big(2y_n^\alpha + |x' - y'|^\alpha\big)\Big(\sup_{B_{R/2}^+} |Dw| + R \sup_{B_{R/2}^+} |\widetilde{f}| \Big)$$

$$\leqslant 9\frac{C_1}{R^\alpha}\Big(\sup_{B_{R/2}^+} |Dw| + R \sup_{B_{R/2}^+} |\widetilde{f}| \Big)|x - y|^\alpha$$

$$\leqslant 9\frac{C_1}{R^\beta}\Big(\sup_{B_{R/2}^+} |Dw| + R \sup_{B_{R/2}^+} |\widetilde{f}| \Big)|x - y|^\beta,$$

since $\frac{|x-y|^\alpha}{R^\alpha} \leqslant \frac{|x-y|^\beta}{R^\beta}$.

Setting $C_2 := \max\big(9C_1, C_1''\big)$ and considering each case, we conclude that for any $x, y \in B_{\delta R/256}^+ \cup \Sigma_{\delta R/256}$ we have

$$|v(x) - v(y)| \leqslant \frac{C_2}{R^\beta}\big(\sup_{B_{R/2}^+}|Dw| + R\sup_{B_{R/2}^+}|\widetilde{f}|\big)|x - y|^\beta.$$

By construction, C_2 depends on $n, K, |\varphi|_{C^1(\Omega)}, \lambda_{\widehat{K}}, \Lambda_{\widehat{K}}$ and R, and β depends on n, K, $|\varphi|_{C^1(\Omega)}, \lambda_{\widehat{K}}$ and $\Lambda_{\widehat{K}}$, as desired. $\qquad\square$

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