

A Halfspace Theorem for Mean Curvature $H = \frac{1}{2}$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$

Barbara Nelli - Ricardo Sa Earp

Abstract

We prove a vertical halfspace theorem for surfaces with constant mean curvature $H = \frac{1}{2}$, properly immersed in the product space $\mathbb{H}^2 \times \mathbb{R}$, where \mathbb{H}^2 is the hyperbolic plane and \mathbb{R} is the set of real numbers. The proof is a geometric application of the classical maximum principle for second order elliptic PDE, using the family of non compact rotational $H = \frac{1}{2}$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$.

1 Introduction

D. Hoffman and W. Meeks proved a beautiful theorem on minimal surfaces, the so-called “Halfspace Theorem” in [4]: there is no non planar, complete, minimal surface properly immersed in a halfspace of \mathbb{R}^3 . In this paper, we focus on complete surfaces with constant mean curvature $H = \frac{1}{2}$ in the product space $\mathbb{H}^2 \times \mathbb{R}$, where \mathbb{H}^2 is the hyperbolic plane and \mathbb{R} is the set of real numbers. In the context of $H = \frac{1}{2}$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$, it is natural to investigate halfspace type results.

Before stating our result we would like to emphasize that, in last years there has been work on constant mean curvature surfaces in homogeneous 3-manifolds, in particular in the product space $\mathbb{H}^2 \times \mathbb{R}$: new examples were produced and many theoretical results as well.

A halfspace theorem for minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ is false, in fact there are many vertically bounded complete minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ [12]. On the contrary, we are able to prove the following result for $H = \frac{1}{2}$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$.

Theorem 1. *Let S be a simply connected rotational surface with constant mean curvature $H = \frac{1}{2}$. Let Σ be a complete surface with constant mean curvature $H = \frac{1}{2}$, different from a rotational simply connected one. Then, Σ cannot be properly immersed in the mean convex side of S .*

In [5], L. Hauswirth, H. Rosenberg and J. Spruck prove a halfspace type theorem for $H = \frac{1}{2}$ surfaces on one side of a horocylinder.

The result in [5] is different in nature from our result because in [5], the “halfspace” is one side of a horocylinder, while for us, the “halfspace” is the mean convex side of a rotational simply connected surface.

The proof of our result is a geometric application of the classical maximum principle to surfaces with mean curvature $H = \frac{1}{2}$ in $\mathbb{H}^2 \times \mathbb{R}$.

Maximum Principle. *Let S_1 and S_2 be two surfaces with constant mean curvature $H = \frac{1}{2}$ that are tangent at a point $p \in \text{int}(S_1) \cap \text{int}(S_2)$. Assume that the mean curvature vectors of S_1 and S_2 at p coincide and that, around p , S_1 lies on one side of S_2 . Then $S_1 \equiv S_2$. When the intersection point p belongs to the boundary of the surfaces, the result holds as well, provided further that the two boundaries are tangent and both are local graphs over a common neighborhood in $T_p S_1 = T_p S_2$.*

The proof of the Maximum Principle is based on the fact that a constant mean curvature surface in $\mathbb{H}^2 \times \mathbb{R}$ locally satisfies a second order elliptic PDE (cf. [1] [3], [2], [6] for the classical proof of the Maximum Principle is in \mathbb{R}^n , the proof generalizes to space forms and to $\mathbb{H}^2 \times \mathbb{R}$ as well).

We notice that our surfaces are not compact, while the classical maximum principle applies at a finite point. It will be clear in the proof of Theorem 1 that we are able to reduce the analysis to finite tangent points, because of the geometry of rotational surfaces of mean curvature $H = \frac{1}{2}$.

Our halfspace Theorem leads to the following conjecture (strong halfspace theorem).

Conjecture. *Let Σ_1, Σ_2 be two complete properly embedded surfaces with constant mean curvature $H = \frac{1}{2}$, different from the rotational simply connected surface S of Theorem 1. Then Σ_i cannot lie in the mean convex side of Σ_j , $i \neq j$.*

For $H > \frac{1}{\sqrt{2}}$ the conjecture is true and it is known as the maximum principle at infinity (cf. [8]).

2 Vertical Halfspace Theorem

We recall some properties of rotational surfaces of mean curvature $H = \frac{1}{2}$ that will be crucial in the proof of Theorem 1.

R. Sa Earp and E. Toubiana find explicit integral formulas for rotational surfaces of constant mean curvature $H \in (0, \frac{1}{2}]$ in [11]. A careful description of the geometry of these surfaces is contained in Lemma 5.2 and Proposition 5.2 in the Appendix of [9]. For any $\alpha \in \mathbb{R}_+$, there exists a rotational surface \mathcal{H}_α of constant mean curvature $H = \frac{1}{2}$. For $\alpha \neq 1$, the surface \mathcal{H}_α has two vertical ends (where a vertical end is a topological annulus, with no asymptotic point at finite height) that are vertical graphs over the

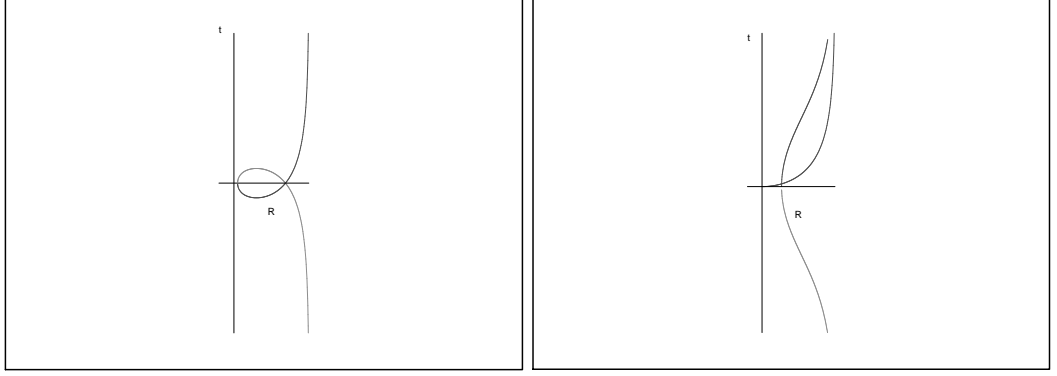


Figure 1: $H = \frac{1}{2}$: the profile curve in the immersed and embedded case ($R = \tanh \rho/2$).

exterior of a disk D_α (see Figure 1).

By vertical graph we mean the following: the vertical graph of a function u defined on a subset Ω of \mathbb{H}^2 is $\{(x, y, t) \in \Omega \times \mathbb{R} \mid t = u(x, y)\}$. When the graph has constant mean curvature H , u satisfies the following second order elliptic PDE

$$\operatorname{div}_{\mathbb{H}} \left(\frac{\nabla_{\mathbb{H}} u}{W_u} \right) = 2H \quad (1)$$

where $\operatorname{div}_{\mathbb{H}}$, $\nabla_{\mathbb{H}}$ are the hyperbolic divergence and gradient respectively and $W_u = \sqrt{1 + |\nabla_{\mathbb{H}} u|_{\mathbb{H}}^2}$, being $|\cdot|_{\mathbb{H}}$ the norm in $\mathbb{H}^2 \times \{0\}$.

Up to vertical translation, one can assume that \mathcal{H}_α is symmetric with respect to the horizontal plane $t = 0$.

For $\alpha = 1$, the surface \mathcal{H}_1 has only one end, it is a graph over \mathbb{H}^2 and it is denoted by S .

When $\alpha > 1$ the surface \mathcal{H}_α is not embedded (first picture in Figure 1). The self intersection set is a horizontal circle on the plane $t = 0$. Denote by ρ_α the radius of the intersection circle. For $\alpha < 1$ the surface \mathcal{H}_α is embedded (second picture in Figure 1). For any $\alpha \in \mathbb{R}_+$, let $u_\alpha : \mathbb{H}^2 \times \{0\} \setminus D_\alpha \rightarrow \mathbb{R}$ be the function such that the end of the surface \mathcal{H}_α is the vertical graph of u_α . The asymptotic behavior of u_α has the following form: $u_\alpha(\rho) \simeq \frac{1}{\sqrt{\alpha}} e^{\frac{\rho}{2}}$, $\rho \rightarrow \infty$, where ρ is the hyperbolic distance from the origin. The positive number $\frac{1}{\sqrt{\alpha}} \in \mathbb{R}_+$ is called the *growth* of the end.

The function u_α is vertical along the boundary of D_α . Furthermore the radius r_α is always greater or equal to zero, it is zero if and only if $\alpha = 1$ and tends to infinity as $\alpha \rightarrow 0$ or $\alpha \rightarrow \infty$. As we pointed out before, the function $u_1 = 2 \cosh\left(\frac{\rho}{2}\right)$ is entire and its graph corresponds to the unique simply connected rotational example S .

Notice that, any end of an immersed rotational surface ($\alpha > 1$) has growth smaller than the growth of S , while any end of an embedded rotational surface ($\alpha < 1$) has growth greater than the growth of S . This means that the intersection between any \mathcal{H}_α and S is a compact set.

Theorem 1 is called “vertical” because the end of the surface Σ is vertical, as it is contained in the mean convex side of S .

Proof of Theorem 1. One can assume that the surface S is tangent to the slice $t = 0$ at the origin and it is contained in $\{t \geq 0\}$. Suppose, by contradiction, that Σ is contained in the mean convex side of S . Lift vertically S . If there is an interior contact point between Σ and the translation of S , one has a contradiction by the maximum principle. As Σ is properly immersed, Σ is asymptotic at infinity to a vertical translation of S . One can assume that the surface Σ is asymptotic to the S tangent to the slice $t = 0$ at the origin and contained in $\{t \geq 0\}$.

Let h be the height of one lowest point of Σ , i.e. $h = \min\{t \mid (x, y, t) \in \Sigma\}$. Denote by $S(h)$ the vertical lifting of S of length h . One has one of the following facts.

- $S(h)$ and Σ have a first finite contact point p : this means that $S(h - \varepsilon)$ does not meet Σ at a finite point, for $\varepsilon > 0$ and then $S(h)$ and Σ are tangent at p with mean curvature vector pointing in the same direction. In this case, by the maximum principle $S(h)$ and Σ should coincide. Contradiction.
- $S(h)$ and Σ meet at a point p , but p is not a first contact point. Then, for ε small enough, $S(h - \varepsilon)$ intersects Σ transversally.

Denote by W the non compact subset of $\mathbb{H}^2 \times \mathbb{R}$ above S and below $S(h - \varepsilon)$.

It follows from the maximum principle that there are no compact components of Σ contained in W . Denote by Σ_1 a non compact connected component of Σ contained in W . By definition of Σ_1 , the boundary $\partial\Sigma_1$ is contained in $\partial W \setminus S = S(h - \varepsilon)$. Consider the family of rotational non embedded surfaces \mathcal{H}_α , $\alpha > 1$. Translate each \mathcal{H}_α vertically in order to have the waist on the plane $t = h - \varepsilon$. By abuse of notation, we continue to call the translation, \mathcal{H}_α . Denote by \mathcal{H}_α^+ , the part of the surface outside the vertical cylinder of radius ρ_α . Notice that \mathcal{H}_α^+ is embedded and it is a vertical graph. When $\alpha \rightarrow \infty$, then $\rho_\alpha \rightarrow \infty$ as well. Furthermore the growth of the end of \mathcal{H}_α^+ is smaller than the growth of S . Hence when α is great enough, say α_0 , $\mathcal{H}_{\alpha_0}^+$ is outside the mean convex side of S . Then, $\mathcal{H}_{\alpha_0}^+$ does not intersect Σ . Furthermore, when $\alpha \rightarrow 1$, \mathcal{H}_α^+ converge to $S(h - \varepsilon)$. Now, start to decrease α from α_0 to one. Before reaching $\alpha = 1$, the surface \mathcal{H}_α^+ first meets S and then touches Σ_1 tangentially at an interior finite point, with Σ_1 above \mathcal{H}_α^+ . This depends on the following two facts.

- The boundary of Σ_1 lies on $S(h - \varepsilon)$ and the boundary of any of the \mathcal{H}_α^+ lies on the horizontal plane $t = h - \varepsilon$.
- The growth of any of the \mathcal{H}_α^+ is strictly smaller than the growth of S . Thus the end of \mathcal{H}_α^+ is outside the mean convex side of S .

The existence of an such interior tangency point is a contradiction by the maximum principle. \square

Remark 1. Our result is sharp in the following sense. There are examples of complete, $H = \frac{1}{2}$ surfaces properly immersed in the non mean convex side of the simply connected rotational surface S of Theorem 1. Consider any \mathcal{H}_α with $\alpha > 1$. As the growth of \mathcal{H}_α is less than the growth of S , $\mathcal{H}_\alpha \cap S$ is a compact set. Then, a suitable vertical downward translation of \mathcal{H}_α is contained in the non mean convex side S .

It is worth noticing that there are many examples of entire graphs of mean curvature $H = \frac{1}{2}$ with a non vertical end, i.e., with points of the asymptotic boundary at finite height; see, for instance, [10]. A significant example is given by the following graph (halfplane model for \mathbb{H}^2):

$$t = \frac{\sqrt{x^2 + y^2}}{y}, \quad y > 0$$

It has mean curvature $H = \frac{1}{2}$, and its asymptotic boundary contains two vertical half straight lines (see [10, Equation (31), Figure (12)]).

Let us now discuss some consequences of Theorem 1.

First we need to recall the following notion. For a given circle C in $\mathbb{H}^2 \times \{0\}$, denote by Z the vertical cylinder over C , that is $Z = \{(x, y, t) \mid (x, y) \in C, t \in \mathbb{R}\}$. An end E is *cylindrically bounded* if there exists a vertical cylinder Z such that, up to a reflection about the slice $\{t = 0\}$, E is contained in the mean convex side of $Z \cap \{t \geq 0\}$.

Corollary 1. *Let Σ be a complete properly immersed surface with mean curvature $H = \frac{1}{2}$ with cylindrically bounded ends. Then Σ must have more than one end.*

Proof. Assume by contradiction that Σ has only one cylindrically bounded end E . Then there exists a vertical cylinder Z such that E lies in the mean convex side of $Z \cap \{t > 0\}$. In particular, one can choose the cylinder Z such that the whole surface Σ is contained in $Z \cap \{t > 0\}$. It is clear that $Z \cap \{t > 0\}$ is contained in the mean convex side of a suitable vertical translation of the simply connected surface S . Hence Σ is contained in the mean convex side of some vertical translation of S as well. Now Theorem 1 yields $\Sigma = S$ (in fact we only need the first part of the proof of Theorem 1), which is a contradiction, since S is not cylindrically bounded. □

As we remarked before, there many entire graphs of mean curvature $H = \frac{1}{2}$. The following consequence of Theorem 1, gives some information about their geometry.

Corollary 2. *Let Σ be a entire graph of mean curvature $H = \frac{1}{2}$. If Σ is not rotational, then it intersects the interior of the complement of the mean convex side of S .*

□

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BARBARA NELLI
Università di L'Aquila
nelli@univaq.it
 RICARDO SA EARP
PUC, Rio de Janeiro
earp@mat.puc-rio.br