A MINIMAL STABLE VERTICAL PLANAR END IN $\mathbb{H}^2 \times \mathbb{R}$ HAS FINITE TOTAL CURVATURE

RICARDO SA EARP AND ERIC TOUBIANA

Abstract. We prove that a minimal oriented stable annular end in $\mathbb{H}^2 \times \mathbb{R}$ whose asymptotic boundary is contained in two vertical lines has finite total curvature and converges to a vertical plane. Furthermore, if the end is embedded then it is a horizontal graph.

1. Introduction

Since the last decade there has been an increasing interest among geometers in studying minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with a certain prescribed asymptotic boundary, where $\mathbb{H}^2$ stands for the hyperbolic plane.

In a joint work with B. Nelli [11], the authors characterized the catenoids in $\mathbb{H}^2 \times \mathbb{R}$ among minimal surfaces with the same asymptotic boundary. Note that the catenoids are the unique minimal surfaces of revolution in $\mathbb{H}^2 \times \mathbb{R}$ and each catenoid M has infinite total curvature, that is $\int_M |K|\, dA = \infty$, where K is the Gaussian curvature of M.

In this paper, we prove that a minimal oriented stable annular end in $\mathbb{H}^2 \times \mathbb{R}$ whose asymptotic boundary is contained in two vertical lines has finite total curvature and converges to a vertical plane (Theorem 2.1). Furthermore, if the end is embedded then it is a horizontal graph with respect to a geodesic in $\mathbb{H}^2 \times \{0\}$ (Definition 2.4).

We point out that in Euclidean space a famous result of D. Fisher-Colbrie [3] states that a complete oriented minimal surface has finite index if and only if it has finite total curvature. Observe that in $\mathbb{H}^2 \times \mathbb{R}$, finite total curvature of a complete oriented minimal surface implies finite index [1], but the converse does not hold: there are many examples of oriented complete stable minimal surfaces with infinite total curvature.

Indeed, there are families of oriented complete stable minimal surfaces invariant by a nontrivial group of screw motions [15]. A particular example is a connected, complete and stable minimal surface whose asymptotic boundary is the union of an arc in $\partial_\infty \mathbb{H}^2 \times \{0\}$ with the two upper half vertical lines starting at the two boundary points $\{p_\infty, q_\infty\}$ of the arc [17, Proposition 2.1-(2)]. Furthermore, this surface has strictly negative

Date: October 11, 2015.

Key words and phrases. minimal surface, asymptotic boundary, stable minimal end, finite total curvature, curvature estimates, horizontal graph.

Mathematics subject classification: 53A10, 53C42, 49Q05.

The authors were partially supported by CNPq and FAPERJ of Brasil.

Gaussian curvature and is invariant by any translation along the geodesic line whose asymptotic boundary is $\{p_\infty, q_\infty\}$, and so does not have finite total curvature.

A second example is given by a one-parameter family of entire horizontal graphs with respect to a geodesic γ of $\mathbb{H}^2 \times \{0\}$, that is, each one is a horizontal graph (see Definition 2.4) over an entire vertical plane orthogonal to γ. In fact, each graph is stable and is invariant by hyperbolic screw motions, hence it has infinite total curvature. Moreover, it is also an entire vertical graph, given by a simple explicit formula taking the half-plane model for $\mathbb{H}^2$ (see [15, equation (4), section 4] or [17, example 2, section 3]). For each surface, the intersection with any horizontal slice is a geodesic of $\mathbb{H}^2$, which depends on the slice. Therefore, the asymptotic boundary of each surface comprises two analytic entire symmetric curves, each curve is a "exp type" graph over a vertical line (see Figure 1). We deduce that the asymptotic boundary is not contained in two vertical lines. Each surface is a minimally embedded plane in $\mathbb{H}^2 \times \mathbb{R}$, stable and has infinite total curvature. This shows that the hypothesis about the asymptotic boundary in Theorem 2.1 cannot be removed.

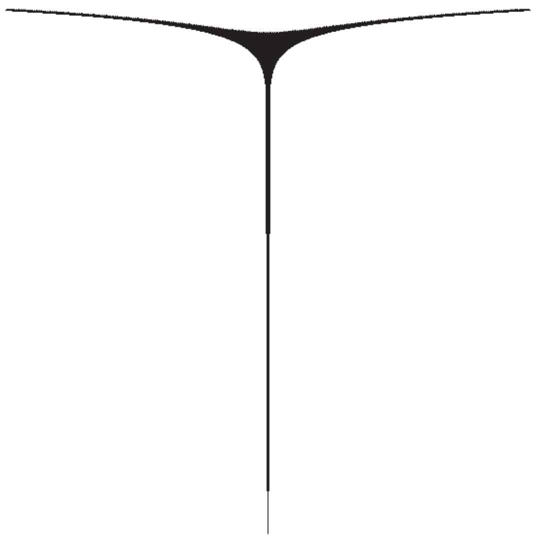


FIGURE 1. *Entire minimal horizontal graph with infinite total curvature (view from the outside of* $\mathbb{H}^2 \times \mathbb{R}$)

Another example is given by an end of a catenoid. Of course, a slice $\mathbb{H}^2 \times \{t\}$ is a trivial example.

We observe that there is another notion of horizontal graph in $\mathbb{H}^2 \times \mathbb{R}$ that appears in the literature [15].

We point out that as an immediate consequence of Theorem 2.1 and the main theorem in [5], we get an extension (Corollary 2.1) of the Schoen-type theorem proved by L. Hauswirth, B. Nelli and the authors [5].

Acknowledgements. The first author wishes to thank *Laboratoire Géométrie et Dynamique de l'Institut de Mathématiques de Jussieu* for their kind hospitality. The second author wishes to thank *Departamento de Matemática da PUC-Rio* for their kind hospitality. We are grateful to the referee for the helpful suggestions.

2. Main Theorem

First, we need to fix some definitions and terminology.
We choose the Poincaré disk model for the hyperbolic plane $\mathbb{H}^2$.
We identify $\mathbb{H}^2$ with $\mathbb{H}^2 \times \{0\}$. We define a *vertical plane* in $\mathbb{H}^2 \times \mathbb{R}$ to be a product $\gamma \times \mathbb{R}$, where $\gamma \subset \mathbb{H}^2$ is a complete geodesic line.
Next we define the notion of "asymptotic boundary" that we use in the text. We remark that it is different from the usual definition [2].

Definition 2.1 ((Asymptotic boundary))**.** We define the *asymptotic boundary* of $\mathbb{H}^2 \times \mathbb{R}$ setting:

$$\partial_\infty(\mathbb{H}^2 \times \mathbb{R}) := \big(\partial_\infty \mathbb{H}^2 \times \mathbb{R}\big) \cup \big(\mathbb{H}^2 \times \{-\infty, +\infty\}\big) \cup \big(\partial_\infty \mathbb{H}^2 \times \{-\infty, +\infty\}\big).$$

This decomposition means that for a divergent sequence (p_n) of $\mathbb{H}^2 \times \mathbb{R}$ there are three possibilities for *converging to infinity* (up to a subsequence) . That is, setting $p_n = (x_n, t_n) \in \mathbb{H}^2 \times \mathbb{R}$, we have the following cases:

- $x_n \to x_\infty \in \partial_\infty \mathbb{H}^2$ and $t_n \to t_0 \in \mathbb{R}$. We say that $p_\infty := (x_\infty, t_0) \in \partial_\infty \mathbb{H}^2 \times \mathbb{R}$ is an asymptotic point *at finite height.*
- $x_n \to x_0 \in \mathbb{H}^2$ and $t_n \to \pm\infty$. That is (p_n) converges to $p_\infty := (x_0, \pm\infty) \in \mathbb{H}^2 \times \{-\infty, +\infty\}$.
- $x_n \to x_\infty \in \partial_\infty \mathbb{H}^2$ and $t_n \to \pm\infty$. That is (p_n) converges to $p_\infty := (x_\infty, \pm\infty) \in \partial_\infty \mathbb{H}^2 \times \{-\infty, +\infty\}$.

Let $\Omega \subset \mathbb{H}^2 \times \mathbb{R}$ be a nonempty subset. We say that a point $p_\infty \in \partial_\infty(\mathbb{H}^2 \times \mathbb{R})$ is an *asymptotic point* of Ω if there is a sequence (p_n) of Ω converging to p_∞.
The set of asymptotic points of Ω, called the *asymptotic boundary of* Ω, is denoted by $\partial_\infty \Omega$.

Definition 2.2 ((Vertical planar end))**.** We say that $L \subset \partial_\infty(\mathbb{H}^2 \times \mathbb{R})$ is a *vertical line* if $L = \{p_\infty\} \times \mathbb{R}$ for some $p_\infty \in \partial_\infty \mathbb{H}^2$.
We say that a complete minimal surface E immersed in $\mathbb{H}^2 \times \mathbb{R}$ with compact boundary is *a vertical planar end*, if the surface is an oriented properly immersed annulus whose asymptotic boundary is contained in two distinct vertical lines L_1 and L_2. Precisely, there is a vertical plane $P \subset \mathbb{H}^2 \times \mathbb{R}$ such that $\partial_\infty E \cap (\partial_\infty \mathbb{H}^2 \times \mathbb{R}) \subset \partial_\infty P$.

Definition 2.3. Let $S \subset \mathbb{H}^2 \times \mathbb{R}$ be a surface and let $P = \gamma \times \mathbb{R}$ be a vertical plane. For any positive real number ρ, we denote by ${L_\rho}^+, {L_\rho}^- \subset \mathbb{H}^2$ the two equidistant lines of γ at distance ρ. Let Z_ρ be the component of $(\mathbb{H}^2 \times \mathbb{R}) \setminus ({L_\rho}^+ \cup {L_\rho}^-) \times \mathbb{R}$ containing P.
We say that S *converges to the vertical plane* P if the following two properties hold:

(1) For any $\rho > 0$ there is a compact part K_ρ of S such that $S \setminus K_\rho \subset Z_\rho$.
(2) $\partial_\infty S = \partial_\infty P$.

We observe that this definition and the definition of *asymptotic to the vertical plane* P given in [5, Definition 3.1] are different. But, under certain geometric assumptions they lead to the same conclusion: cf. Corollary 2.1 below with [5, Theorem 3.1].

We recall now the definition of *horizontal graph with respect to a geodesic* given in [5, Definition 3.2].

Definition 2.4. Let $\gamma \subset \mathbb{H}^2$ be a geodesic. We say that a nonempty set $S \subset \mathbb{H}^2 \times \mathbb{R}$ is a *horizontal graph with respect to the geodesic* γ, or simply a *horizontal graph*, if for any equidistant line $\widetilde{\gamma}$ of γ and for any $t \in \mathbb{R}$, the curve $\widetilde{\gamma} \times \{t\}$ intersects S at most in one point.

Now we state precisely our main theorem:

Theorem 2.1. *Let E be a minimal stable vertical planar end in $\mathbb{H}^2 \times \mathbb{R}$ and let P be the vertical plane such that $\partial_\infty E \cap (\partial_\infty \mathbb{H}^2 \times \mathbb{R}) \subset \partial_\infty P$.*
Then, E has finite total curvature and converges to the vertical plane P. Furthermore if E is embedded then, up to a compact part, E is a horizontal graph.

There are many examples of complete, possibly with compact boundary, minimal surfaces whose asymptotic boundary is contained in the union of vertical lines or copies of the asymptotic boundary of $\mathbb{H}^2$. For instance, we refer the reader to the first paper on this subject written by B. Nelli and H. Rosenberg [10]. We remark that the first paper about minimal ends of finite total curvature in $\mathbb{H}^2 \times \mathbb{R}$ was written by L. Hauswirth and H. Rosenberg [7].
We recall now that F. Morabito and M. Rodriguez [9] and J. Pyo [13] have constructed, independently, a family of minimal embedded annuli with finite total curvature. Each end of such annuli is asymptotic to a vertical geodesic plane. In [5] we called each such surface a *two ends model surface.*
The following corollary is an immediate consequence of Theorem 2.1 and the main theorem in [5].

Corollary 2.1. *An oriented complete and connected minimal surface immersed in $\mathbb{H}^2 \times \mathbb{R}$ with two distinct embedded (nonflat) annular ends, each one being stable and the asymptotic boundary of each end being contained in the asymptotic boundary of a vertical geodesic plane, is a two ends model surface.*

Proof of Theorem 2.1.
We have $P = \gamma \times \mathbb{R}$, where γ is a geodesic of $\mathbb{H}^2$. We set $\partial_\infty \gamma = \{p_\infty, q_\infty\}$.
Given any isometry T of $\mathbb{H}^2$ we denote also by T the isometry of $\mathbb{H}^2 \times \mathbb{R}$ induced by T: $(x, t) \mapsto (T(x), t)$.
For any geodesic $\alpha \subset \mathbb{H}^2$ we set $P_\alpha := \alpha \times \mathbb{R}$, that is P_α is the vertical plane containing α.
We will proceed with the proof of Theorem 2.1 in several steps.

Step 1. Let $\alpha \subset \mathbb{H}^2$ be any geodesic such that $\alpha \cap \gamma = \emptyset$. If a component of $(\mathbb{H}^2 \times \mathbb{R}) \setminus P_\alpha$, say $\mathcal{A}_\alpha$, contains $P \cup \partial E$ then $E \subset \mathcal{A}_\alpha$.

This is a consequence of the maximum principle established by the authors and B. Nelli [11, Theorem 3.1].

Step 2. For any $\rho > 0$ there is a compact part K_ρ of E such that $E \setminus K_\rho \subset Z_\rho$. Thus, $\partial_\infty E \subset \partial_\infty P$.

Proof. Let ρ be a positive real number. Arguing by contradiction, let us assume that $E \setminus Z_\rho$ is not compact, then there is an unbounded sequence $p_n = (x_n, t_n) \in E \setminus Z_\rho$. We can assume that the sequence (p_n) belongs to the component of $(\mathbb{H}^2 \times \mathbb{R}) \setminus ({L_\rho}^+ \times \mathbb{R})$ which does not contain P.

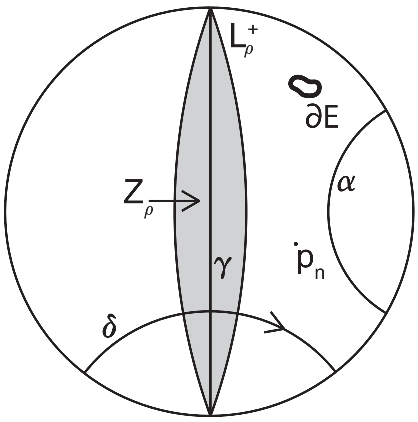


FIGURE 2

Let $\alpha \subset \mathbb{H}^2$ be any geodesic such that $\alpha \cap \gamma = \emptyset$ and such that ∂E and P belong to the same component of $(\mathbb{H}^2 \times \mathbb{R}) \setminus P_\alpha$ (Figure 2). We denote by ${P_\alpha}^-$ the other component of $(\mathbb{H}^2 \times \mathbb{R}) \setminus P_\alpha$. We deduce from Step 1 that ${P_\alpha}^- \cap E = \emptyset$. This implies that no subsequence of (x_n) converges either to p_∞ or to q_∞. Indeed, if a subsequence of (x_n) converges to p_∞, we could choose the geodesic α such that $p_\infty \in \partial_\infty \alpha$, contradicting ${P_\alpha}^- \cap E = \emptyset$. The same argument shows that there is no subsequence of (x_n) that converges to other points of $\partial_\infty \mathbb{H}^2$.
Then, we deduce from above and from the assumptions about the asymptotic boundary of E that (x_n) is a bounded sequence in $\mathbb{H}^2$. Since (p_n) is an unbounded sequence, we obtain that (t_n) is an unbounded sequence of real numbers. Thus, up to extracting a subsequence of (p_n) and up to a reflection with respect to $\mathbb{H}^2 \times \{0\}$, we may assume that $t_n \to +\infty$.
Now we consider the family of complete minimal surfaces M_d, $d > 1$, described in [17, Proposition 2.1-(1)] and in the proof of [11, Theorem 3.1]. Recall that M_d contains the equidistant line ${L_\rho}^+$ staying at the distance $\rho = \cosh^{-1}(d)$ from γ and that M_d is

contained in the closure of the nonmean convex component of $(\mathbb{H}^2 \times \mathbb{R}) \setminus ({L_\rho}^+ \times \mathbb{R})$. By abuse of notation, this surface is denoted M_ρ. The proof of the assertion follows from the maximum principle again [11, Theorem 3.1], using the surfaces M_ρ. We now give a short proof for the readers convenience.
Let V_ρ be the closure of the component of $(\mathbb{H}^2 \times \mathbb{R}) \setminus M_\rho$ which does not contain P. Observe that the height function is bounded on V_ρ. From the considerations above, there exists $t > 0$ such that the following conditions hold:

- $\partial E \cap \big(V_\rho + (0,0,t)\big) = \emptyset$,
- $E \cap \big(V_\rho + (0,0,t)\big) \neq \emptyset$.

Let $\delta \subset \mathbb{H}^2$ be a geodesic orthogonal to γ. Using that E is properly immersed, moving $M_\rho + (0,0,t)$ horizontally along δ and considering horizontal translated copies of $M_\rho + (0,0,t)$, we must find a last contact at an interior point of E and a copy of $M_\rho + (0,0,t)$. This yields a contradiction with the maximum principle. □

For any $r > 0$ we denote by B_r the open geodesic ball in $\mathbb{H}^2 \times \mathbb{R}$ centered at the origin $(0,0) \in \mathbb{H}^2 \times \mathbb{R}$, with radius r.

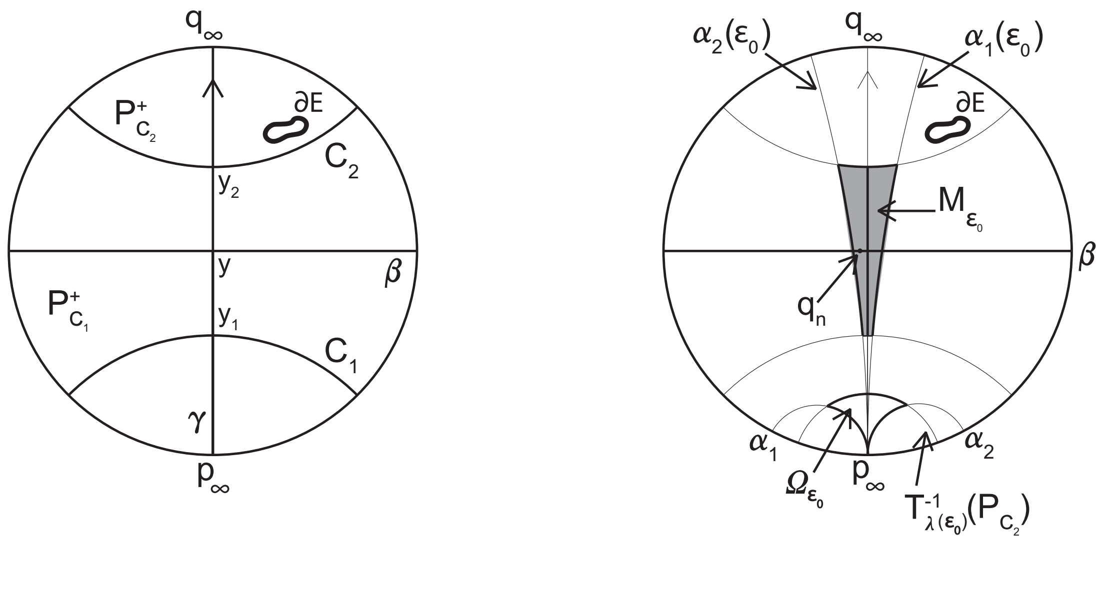
Step 3. Let n_3 be the third coordinate of the unit normal field on E with respect to the product metric on $\mathbb{H}^2 \times \mathbb{R}$. We have that $n_3(p) \to 0$ uniformly when $p \to \partial_\infty E$.
More precisely, for any $\varepsilon > 0$, there exists $\rho > 0$ such that for any $p \in (E \cap Z_\rho) \setminus B_{1/\rho}$, we have $|n_3(p)| < \varepsilon$.

Proof. Assume by contradiction that the assertion does not hold. Therefore there exist $\varepsilon > 0$ and a sequence of points $p_n := (x_n, t_n) \in E \cap Z_{1/n}$, $n \in \mathbb{N}^*$, such that $|n_3(p_n)| > \varepsilon$ and the sequence (p_n) is not bounded in $\mathbb{H}^2 \times \mathbb{R}$.
Up to extracting a subsequence, we can assume that the sequence (p_n) converges in $(\mathbb{H}^2 \times \mathbb{R}) \cup \partial_\infty(\mathbb{H}^2 \times \mathbb{R})$.

First case: Suppose $\lim x_n \in \partial_\infty \mathbb{H}^2$, thus we have $\lim x_n \in \partial_\infty \gamma$. Without loss of generality, we can assume that $x_n \to p_\infty$.
On the geodesic γ we consider the orientation given by $p_\infty \to q_\infty$. We choose two points $y_1, y_2 \in \gamma$ such that y_1 is between p_∞ and y_2. Let $C_i \subset \mathbb{H}^2$ be the geodesic orthogonal to γ through y_i, $i = 1, 2$, (Figure 3(a)).
Let ${P_{C_i}}^+$ be the connected component of $(\mathbb{H}^2 \times \mathbb{R}) \setminus P_{C_i}$ containing q_∞ in its asymptotic boundary, we denote by ${P_{C_i}}^-$ the other component. Since E has compact boundary, we can choose y_1 and y_2 so that $\partial E \subset {P_{C_2}}^+$. Let $y \in \gamma$ be the midpoint of the geodesic segment $[y_1, y_2]$ of γ and let $\beta \subset \mathbb{H}^2$ be the geodesic orthogonal to γ through y. Observe that the vertical planes P_{C_1} and P_{C_2} are symmetric with respect to P_β.

Let $\alpha_1 \subset \mathbb{H}^2$ be a geodesic so that $p_\infty \in \partial_\infty \alpha_1$, $\partial_\infty \alpha_1 \cap \partial_\infty C_1 = \emptyset$ and $\alpha_1 \cap C_1 = \emptyset$, thus $\alpha_1 \subset {P_{C_1}}^-$, (Figure 3(b)).
We denote by $\alpha_2 \subset \mathbb{H}^2$ the symmetric of α_1 with respect to γ. Using step 1, we can deduce that the component of $(\mathbb{H}^2 \times \mathbb{R}) \setminus (P_{\alpha_1} \cup P_{\alpha_2})$ containing P also contains $E \cup \partial E$.

Let $\varepsilon_0 > 0$. For any $\lambda > 0$ we denote by T_λ the hyperbolic translation of length λ along γ, with the orientation $p_\infty \to q_\infty$. There exists $\lambda(\varepsilon_0) > 0$ so that the geodesics



(a) (b)

Figure 3

$T_{\lambda(\varepsilon_0)}(\alpha_1)$ and $T_{\lambda(\varepsilon_0)}(\alpha_2)$ belong to an ε_0-neighborhood of γ in $\mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2$, in the Euclidean meaning. We set $T_{\lambda(\varepsilon_0)}(\alpha_i) := \alpha_i(\varepsilon_0)$, $i = 1, 2$. We remark that the vertical planes $P_{\alpha_i(\varepsilon_0)}$ belong to a ε_0-neighborhood of P_γ (in the Euclidean sense).

Let U_{ε_0} be the component of $(\mathbb{H}^2 \times \mathbb{R}) \setminus (P_{\alpha_1(\varepsilon_0)} \cup P_{\alpha_2(\varepsilon_0)})$ containing P. We denote by M_{ε_0} the component of $(\mathbb{H}^2 \times \mathbb{R}) \setminus (P_{\alpha_1(\varepsilon_0)} \cup P_{\alpha_2(\varepsilon_0)} \cup P_{C_1} \cup P_{C_2}) \times \mathbb{R}$ containing $\{y\} \times \mathbb{R}$. We have therefore

$$M_{\varepsilon_0} = U_{\varepsilon_0} \cap P_{C_1}{}^{+} \cap P_{C_2}{}^{-}.$$

Let Ω_{ε_0} be the component of $(\mathbb{H}^2 \times \mathbb{R}) \setminus (T_{\lambda(\varepsilon_0)}{}^{-1}(P_{C_2}) \cup P_{\alpha_1} \cup P_{\alpha_2})$ such that $\Omega_{\varepsilon_0} \cap P \neq \emptyset$. Then, by construction, for any $\lambda \geqslant \lambda(\varepsilon_0)$ and for any $p \in \Omega_{\varepsilon_0}$, we have $T_\lambda(p) \in U_{\varepsilon_0}$.

We may assume that $x_n \in \Omega_{\varepsilon_0}$ for any n.

For any $n \in \mathbb{N}^*$, there exists a unique $\lambda_n > 0$ such that $T_{\lambda_n}(x_n) \in \beta$. Therefore, setting $q_n := T_{\lambda_n}(p_n)$, we have $q_n \in P_\beta$.
For n large enough, say $n > n_0 > 0$, we have $\lambda_n > \lambda(\varepsilon_0)$ (since $x_n \to p_\infty$), therefore $q_n \in P_\beta \cap U_{\varepsilon_0}$, in particular $q_n \in M_{\varepsilon_0}$.

For any $n > n_0$, we denote by $E_n(\varepsilon_0)$ the connected component of $T_{\lambda_n}(E) \cap M_{\varepsilon_0}$ containing q_n. By construction, $E_n(\varepsilon_0)$ is the component of $T_{\lambda_n}(E \cap \Omega_{\varepsilon_0}) \cap M_{\varepsilon_0}$ containing q_n. Consequently, the boundary of $E_n(\varepsilon_0)$ belongs to $P_{C_1} \cup P_{C_2}$ and has no intersection with $P_{\alpha_1(\varepsilon_0)} \cup P_{\alpha_2(\varepsilon_0)}$, namely:

$$\partial E_n(\varepsilon_0) \subset P_{C_1} \cup P_{C_2} \quad \text{and} \quad \partial E_n(\varepsilon_0) \cap (P_{\alpha_1(\varepsilon_0)} \cup P_{\alpha_2(\varepsilon_0)}) = \emptyset. \tag{1}$$

Let $\overline{d_n}(\cdot,\cdot)$ be the intrinsic distance on $T_{\lambda_n}(E)$. By construction, for n large enough, say $n > n_1 > n_0$, we have $\overline{d_n}\big(p, \partial T_{\lambda_n}(E)\big) > \pi/2$ for any $p \in E_n(\varepsilon_0)$. Let $|\overline{A}_n|$ be the norm of the second fundamental form of $T_{\lambda_n}(E)$.
Recall that the sectional curvature of $\mathbb{H}^2 \times \mathbb{R}$ is bounded (in absolute value it is bounded by 1).
Since the end E is stable, the translated copy $T_{\lambda_n}(E)$ is also stable for any $n > n_1$. Using the fact that the distances between $E_n(\varepsilon_0)$ and the boundary of $T_{\lambda_n}(E)$ are uniformly bounded from below, we deduce from [14, Main Theorem] that there exists a constant $C' > 0$, which does not depend on $n > n_1$ or on $\varepsilon_0 > 0$, such that

$$|\overline{A}_n(p)| < C' \tag{2}$$

for any $p \in E_n(\varepsilon_0)$.
Furthermore, since the boundary of $E_n(\varepsilon_0)$ belongs to $P_{C_1} \cup P_{C_2}$, there exists a constant $C'' > 0$, which does not depend on n and or on ε_0, such that

$$\overline{d_n}(q_n, \partial E_n(\varepsilon_0)) > C''.$$

Now, we consider $\mathbb{H}^2 \times \mathbb{R}$ as an open set of Euclidean space $\mathbb{R}^3$, as well. We deduce from [14, Proposition 2.3] and from Proposition 3.1 in the Appendix, that there exists a real number $\delta > 0$, which does not depend on n or on ε_0, such that for any $n > n_1$, a part F_n of $E_n(\varepsilon_0)$ is the Euclidean graph of a function defined on the disk centered at point q_n with Euclidean radius δ in the tangent plane of E_n at q_n. Furthermore, the norm of the Euclidean gradient of this function is bounded by 1.

Let ν be the unitary normal along E in the Euclidean metric. We denote by ν_3 the vertical component of ν. Recall that $|n_3(p_n)| > \varepsilon$, hence $|n_3(q_n)| > \varepsilon$ for any n. Comparing the product metric of $\mathbb{H}^2 \times \mathbb{R}$ with the Euclidean metric, it can be shown that there exists $\varepsilon' > 0$, which does not depend on n, such that $|\nu_3(q_n)| > \varepsilon'$ for any $n > n_1$, (see the formula of the unit normal vector field of a vertical graph in the proof of [18, Proposition 3.2]). This implies that the tangent planes of E_n at points q_n have a slope bounded below uniformly (with respect to $n > n_1$).
Since the radius δ does not depend on ε_0, if we choose ε_0 small enough, the Euclidean graph F_n will have nonempty intersection with $P_{\alpha_1(\varepsilon_0)}$ and $P_{\alpha_2(\varepsilon_0)}$, which is not possible.

Second case: The sequence (x_n) has a finite limit in $\mathbb{H}^2$.
Therefore, since the sequence (p_n) is not bounded, up to considering a subsequence, and up to a vertical reflection, we can assume that $t_n \to +\infty$. We saw in step 2 that for any $\rho > 0$ there exists $t_\rho > 0$ such that $E \cap \{|t| > t_\rho\} \subset Z_\rho$. Then we can argue as in the first case replacing the vertical planes $P_{\alpha_1(\varepsilon_0)}$ and $P_{\alpha_2(\varepsilon_0)}$ by the surfaces ${L_\rho}^+ \times \mathbb{R}$ and ${L_\rho}^- \times \mathbb{R}$, recalling that ${L_\rho}^+ \cup {L_\rho}^- = \partial Z_\rho$.

Since we have proved both cases, then we have that $n_3(p) \to 0$, uniformly when $p \to \partial_\infty E$. $\square$

Notation. The end is conformally parametrized by $U_R := \{z \in \mathbb{C} \mid 1 \leqslant |z| < R\}$, for some $R > 1$. The conformal, complete and proper immersion $X : U_R \to \mathbb{H}^2 \times \mathbb{R}$

is given by $X = (F, h)$ where $F : U_R \to \mathbb{H}^2$ is a harmonic map and $h : U_R \to \mathbb{R}$ is a harmonic function. Let σ be the conformal factor of the hyperbolic metric on $\mathbb{H}^2$, we set $\phi := (\sigma \circ F)^2 F_z \overline{F}_z$. Since the immersion X is conformal, we have $\phi = -{h_z}^2$, [16, Proposition 1] and therefore, ϕ is holomorphic.
As above, we denote by n_3 the third coordinate of the unit normal field on E, with respect to the product metric. We define a function ω on E, or U_R, setting $n_3 = \tanh \omega$. The induced metric on U_R is, [6, Equation 14]:

$$ds^2 = 4\cosh^2(\omega)\,|\phi|\,|dz|^2. \tag{3}$$

Step 4. We have $R = \infty$, that is the end is conformally equivalent to a punctured disk. Moreover ϕ extends meromorphically up to the end and has the following expression on $U := \{z \in \mathbb{C} \mid 1 \leqslant |z|\}$:

$$\phi(z) = (\sum_{k\geqslant 1} \frac{a_{-k}}{z^k} + P(z))^2, \tag{4}$$

where P is a polynomial function.

Proof. From the expression of the metric ds^2, we deduce that if $z_0 \in U_R$ is a zero of ϕ, then ω must have a pole at z_0 and, therefore, $n_3(z_0) = \pm 1$. On the other hand, by properness we infer from step 3 that $n_3(z) \to 0$ uniformly when $|z| \to R$. Therefore $\omega(z) \to 0$ when $|z| \to R$. Consequently, we may assume that ϕ does not vanish on U_R. Since the metric ds^2 is complete and $\omega(z) \to 0$ uniformly when $|z| \to R$, the new metric given by $|\phi|\,|dz|^2$ is complete too.
Since ϕ is holomorphic, a result of Osserman shows that $R = \infty$ (see [12, Lemma 9.3]). Thus, $|\phi|\,|dz|^2$ is a complete metric on U. Furthermore, using the fact that ϕ does not vanish, another result of Osserman shows that ϕ has at most a pole at infinity, [12, Lemma 9.6].
Finally, recall that ϕ is the square of a holomorphic function : $\phi(z) = -(h_z(z))^2$ (see[16, Proposition 1]). This shows that ϕ has the required form. $\square$

From now on we assume that ϕ has no zero on U.

Step 5. The end E has finite total curvature.

We first show the following result.

Lemma 2.1. *The polynomial function P is not identically zero.*

of Lemma 2.1. We proceed as in the proof of [5, Lemma 2.1]. Assume by contradiction that $P \equiv 0$.
If $a_{-1} = 0$ we have

$$\int_U |\phi(z)|\,dA < \infty.$$

where dA is the Lebesgue measure on $\mathbb{R}^2$. Since $\omega(z) \to 0$ when $|z| \to \infty$, this would imply that the end E has finite area, which is absurd, see [4, Appendix: Theorem 3 and Remark 4].

If $a_{-1} \neq 0$ the argument is the same as in the proof of [5, Lemma 2.1]. $\square$

of Step 5. We set $\sqrt{\phi(z)} = \sum_{k\geqslant 1} \frac{a_{-k}}{z^k} + P(z)$ and $m = \deg(P)$. We define on U the, eventually, multivalued function:

$$W(z) := \int \big(\sum_{k\geqslant 1} \frac{a_{-k}}{z^k} + P(z)\big)\, dz = \int \sqrt{\phi(z)}\, dz.$$

If $a_{-1} \neq 0$, then the function $W(z)$ may be multivalued, but $\mathrm{Im}W(z)$ is single-valued, since $h(z) = 2\mathrm{Im}W(z)$. Noting that $W' = \sqrt{\phi}$, the holomorphic (possibly multivalued) function W has no critical point.

In [5, Section 2] it is shown that there exist connected and simply connected domains $\Omega_0, \dots, \Omega_{2m+1}$ in $\mathbb{C}$ such that:

- $\{z \in \mathbb{C} \mid |z| > R\} \subset \cup_{k=0}^{2m+1} \Omega_k$ for some $R > 1$.
- the restricted map $W_k := W_{|\Omega_k} : \Omega_k \to \mathbb{C}$ is an univalent map for any k.

For $z \in \Omega_k$, $k = 0, \dots, 2m+1$, we set $w_k := W(z)$. By abuse of notation we just write $w = W(z)$. The range $\widetilde{\Omega}_k := W_k(\Omega_k)$ is a simply connected domain in $\mathbb{C}$ satisfying:

(1) If k is an even number, then $\widetilde{\Omega}_k$ is the complementary of a horizontal half-strip. The nonhorizontal component of $\partial\widetilde{\Omega}_k$ is a compact arc and $\mathrm{Im}w$ is strictly monotone along this arc. Moreover $\mathrm{Re}w$ is bounded from above by a real number a_k along $\partial\widetilde{\Omega}_k$, see [5, Figure 3 (a)].

(2) If k is an odd number, then $\widetilde{\Omega}_k$ is the complementary of a horizontal half-strip. The nonhorizontal component of $\partial\widetilde{\Omega}_k$ is a compact arc and $\mathrm{Im}w$ is strictly monotone along this arc. Moreover $\mathrm{Re}w$ is bounded from below by a real number b_k along $\partial\widetilde{\Omega}_k$, see [5, Figure 3 (b)].

Since $dw = |\sqrt{\phi(z)}|\,|dz|$ on each Ω_k, we deduce from formula (3) that the metric induced by the immersion X on each $\widetilde{\Omega}_k$ is:

$$d\widetilde{s}^2 = 4\cosh^2 \widetilde{\omega}(w)\, |dw|^2,$$

where, by abuse of notation, we set $\widetilde{\omega}(w) = (\omega \circ {W_k}^{-1})(w)$.
Let $\Gamma \subset U$ be a smooth Jordan curve nonhomologous to zero and let $C >> 0$ be a large number satisfying $|W(z)| << C$ for any $z \in \Gamma$.
Using [5, Lemma 2.3] and the description of each domain $\widetilde{\Omega}_k$, we can construct smooth compact and simple arcs $\widetilde{R}_k(C), \widetilde{I}_k(C)$, $k = 0, \dots, 2m+1$ such that the following conditions are satisfied.

- We have $\widetilde{R}_k(C) \subset \widetilde{\Omega}_k$ and $\widetilde{I}_k(C) \subset \widetilde{\Omega}_k$.
- If k is an even number then: $\mathrm{Re}w = C$ and $\mathrm{Im}w$ is increasing from $-C$ up to C along $\widetilde{R}_k(C)$. Also, $\mathrm{Im}w = C$ and $\mathrm{Re}w$ is increasing from $-C$ up to C along $\widetilde{I}_k(C)$. Moreover, the arcs $\widetilde{R}_k(C)$ and $\widetilde{I}_k(C)$ make a right angle at the point $w = C + iC$.

- If k is an odd number then: $\mathrm{Re}w = -C$ and $\mathrm{Im}w$ is increasing from $-C$ up to C along $\widetilde{R}_k(C)$, $\mathrm{Im}w = -C$ and $\mathrm{Re}w$ is increasing from $-C$ up to C along $\widetilde{I}_k(C)$. Moreover, the arcs $\widetilde{R}_k(C)$ and $\widetilde{I}_k(C)$ make a right angle at the point $w = -C - iC$.
- Setting $R_k(C) := W_k{}^{-1}\big(\widetilde{R}_k(C)\big)$ and $I_k(C) := W_k{}^{-1}\big(\widetilde{I}_k(C)\big)$, the curve $\Gamma(C) := \bigcup_{k=0}^{2m+1}\Big(R_k(C) \cup I_k(C)\Big)$ is a piecewise smooth Jordan curve nonhomologous to zero, and any of the $4m+4$ interior angles is equal to $\pi/2$.

Since $|W(z)| << C$ along Γ, we have $\Gamma \cap \Gamma(C) = \emptyset$. We denote by $U(C)$ the annulus in U bounded by Γ and $\Gamma(C)$. To prove that the end E has finite total curvature it suffices to show that $\int_{U(C)} K\, dA$ has finite limit as $C \to +\infty$.
Since the boundary component Γ is smooth and the other boundary component $\Gamma(C)$ has exactly $4m+4$ interior angles, each one being equal to $\pi/2$, the Gauss-Bonnet formula gives

$$\int_{U(C)} K\, dA + \int_\Gamma k_g\, ds + \int_{\Gamma(C)} k_g\, ds = -2(m+1)\pi,$$

where k_g denotes the geodesic curvature. Therefore it suffices to show that $\int_{\Gamma(C)} k_g\, ds \to 0$ when $C \to +\infty$.
First we prove that $\int_{I_k(C)} k_g\, ds \to 0$ when $C \to +\infty$, $k = 0, \dots, 2m+1$. Since a similar argument also shows $\int_{R_k(C)} k_g\, ds \to 0$, we will be done.
Since $W_k : (\Omega_k, ds^2) \to (\widetilde{\Omega}_k, d\tilde{s}^2)$ is an isometry, we have

$$\int_{I_k(C)} k_g\, ds = \int_{\widetilde{I}_k(C)} k_g\, d\tilde{s}.$$

Assume that k is even. We set $w = u + iv$ and we consider the parametrization of $\widetilde{I}_k(C)$ given by $w(t) = t + iC$, $t \in [-C, C]$. Using [8, Formula (42.8)] we derive the geodesic curvature:

$$k_g(w(t)) = \pm \frac{\sinh \widetilde{\omega}}{2\cosh^2 \widetilde{\omega}}\, \frac{\partial \widetilde{\omega}}{\partial v}\big(w(t)\big).$$

Therefore,

$$|k_g(w(t))| \leqslant \frac{1}{2\cosh \widetilde{\omega}(w(t))}\, |\nabla \widetilde{\omega}(w(t))|$$

where ∇ means the Euclidean gradient. It is shown in the proof of [5, Proposition 2.3] that there exists a positive constant δ such that, outside a compact part of $\widetilde{\Omega}_k$, we have:

$$|\nabla \widetilde{\omega}(w)| < \delta e^{-d(w, \partial \widetilde{\Omega}_k)},$$

where $d(w, \partial\widetilde{\Omega}_k)$ is the Euclidean distance between w and $\partial\widetilde{\Omega}_k$. Since, by construction, we have $|\mathrm{Im}(w)| \leqslant C_0$ along $\partial\widetilde{\Omega}_k$ for some $C_0 < C$, we get

$$|\nabla \widetilde{\omega}(w)| < \delta e^{C_0}\, e^{-C}, \quad \text{on} \quad \widetilde{I}_k(C).$$

Therefore,

$$
\begin{aligned}
\Big| \int_{I_k(C)} k_g\, ds \Big| &\leqslant \int_{\widetilde{I}_k(C)} |k_g(w)|\, d\tilde{s} \\
&= \int_{-C}^{C} |k_g(w(t))|\, 2\cosh \widetilde{\omega}(w(t))\, dt \\
&\leqslant 2C\delta e^{C_0}\, e^{-C}.
\end{aligned}
$$

We deduce that $\int_{I_k(C)} k_g\, ds \to 0$ when $C \to +\infty$.
If k is an odd number, then the argument is the same.
Along the curves $\widetilde{R}_k$, the geodesic curvature is given by $k_g(w(t)) = \pm \dfrac{\sinh \widetilde{\omega}}{2\cosh^2 \widetilde{\omega}} \dfrac{\partial \widetilde{\omega}}{\partial u}\big(w(t)\big)$.
Moreover we have along $\partial \widetilde{\Omega}_k$ that $\mathrm{Re} w \leqslant a_k$ if k is an even number and $\mathrm{Re} w \geqslant b_k$ if k is an odd number. Therefore we can proceed as before to show that $\int_{R_k(C)} k_g\, ds \to 0$ when $C \to +\infty$. This proves that the end E has finite total curvature. □

Step 6. We have $\partial_\infty E = \partial_\infty P$. Thus combining with Step 2, E converges to the vertical plane P.

Proof. We keep the notation of the proof of step 5.
For $k = 0, \dots, 2m+1$, the map $F_k := F \circ {W_k}^{-1} : \widetilde{\Omega}_k \to \mathbb{H}^2$ is a harmonic map. Assume that k is an even number. It is proved in [5, Theorem 2.1] that $\lim_{u \to +\infty} F_k(u + iC)$ exists and does not depend on $C \in \mathbb{R}$. Also, $\lim_{u \to -\infty} F_{k+1}(u + iC)$ exists and does not depend on $C \in \mathbb{R}$, moreover the two limits are different.
Since $\partial_\infty F(U) = \{p_\infty, q_\infty\}$, we can assume that $\lim_{u \to +\infty} F_k(u + iC) = p_\infty$, and $\lim_{u \to -\infty} F_{k+1}(u + iC) = q_\infty$ for any C.
Let t_0 be any real number. We want to prove that $(p_\infty, t_0) \in \partial_\infty E$. Since this will be true for any $t_0 \in \mathbb{R}$ and since we can show the same property for q_∞, we could conclude, using step 2 that $\partial_\infty E = \partial_\infty P$.
Let $C > 0$ be large enough so that $C >> |t_0|$. Let $\widetilde{R}_k(C) \subset \widetilde{\Omega}_k$ be the compact arc as in the proof of Step 5. If C is large enough then $(X \circ {W_k}^{-1})(\widetilde{R}_k(C))$ is a compact arc of E very close of $p_\infty \times [-2C, 2C]$ in the Euclidean sense. Moreover the height is increasing from $-2C$ to $2C$ along this arc Letting $C \to +\infty$ we can extract a sequence (p_n) on $E \cap \{t = t_0\}$ such that $p_n \to (p_\infty, t_0)$. We have therefore $\partial_\infty P \cap (\partial_\infty \mathbb{H}^2 \times \mathbb{R}) \subset \partial_\infty E$. Taking into account Step 2 we conclude that $\partial_\infty P = \partial_\infty E$ and the end E converges to the vertical plane P. □

Step 7. Assume that the end E is embedded. Then, up to a compact part, the end E is a horizontal graph.

Proof. The end E is conformally parametrized by $U := \{z \in \mathbb{C} \mid 1 \leqslant |z|\}$. The conformal immersion $X : U \to E \subset \mathbb{H}^2 \times \mathbb{R}$ is given by $X = (F, h)$ where $F : U \to \mathbb{H}^2$ is a harmonic map and $h : U \to \mathbb{R}$ is a harmonic function. Thus, $\phi = -{h_z}^2$ is holomorphic. Moreover ϕ has the following form

$$\phi(z) = (\sum_{k\geqslant 1} \frac{a_{-k}}{z^k} + P(z))^2, \tag{5}$$

where P is a polynomial function. We know from Lemma 2.1 that P is not identically zero. Let $m \in \mathbb{N}$ be the degree of P. Since X is an embedding and since $\partial_\infty F(U) = \{p_\infty, q_\infty\}$, we deduce from [5, Theorem 2.1] that $m = 0$.

Let $\varepsilon > 0$ be a small number, $\varepsilon << 1$. Since ϕ does not vanish on U, we have $|n_3| \neq 1$ throughout E, and therefore the end E is transversal to any slice $\mathbb{H}^2 \times \{t\}$. For any $p \in E$ we denote by $\kappa(p)$ the geodesic curvature, in $\mathbb{H}^2$, of the intersection curve between E and the horizontal slice through p. We deduce from [5, Proposition 2.3] that, there exists a compact part K of E such that

$$|\kappa(p)| < \varepsilon, \tag{6}$$

for any $p \in E \setminus K$.

Let $\gamma_1 \subset \mathbb{H}^2$ be a geodesic orthogonal to γ, thus $\partial_\infty\gamma \cap \partial_\infty\gamma_1 = \emptyset$. Let $\rho > 0$, we denote by ${L_\rho}^+$ and ${L_\rho}^-$ the two equidistant curves of γ_1 at distance ρ. Let ${Z_\rho}'$ be the connected component of $(\mathbb{H}^2 \times \mathbb{R}) \setminus ({L_\rho}^+ \cup {L_\rho}^-) \times \mathbb{R}$ containing the vertical plane P_{γ_1}. We deduce from [5, Proposition 4.3] and (6) that for ρ large enough $E \setminus {Z_\rho}'$ is a horizontal graph with respect to γ_1.

We infer from [5, Theorem 2.1] that there exists $C_0 > 0$ such that for any t satisfying $|t| > C_0$, the intersection $E \cap \mathbb{H}^2 \times \{t\}$ is a complete and connected curve L_t which is C^1-close to γ. Therefore, $L_t \cap {Z_\rho}'$ is a horizontal graph with respect to γ_1.

Thus, up to a compact part, E is a horizontal graph. This achieves the proof of Step 7 and concludes the proof of Theorem 2.1. $\square$

3. Appendix.

Proposition 3.1. *Let $U \subset \mathbb{R}^3$ be an open set and let $S \subset U$ be an immersed C^2-surface without boundary. Let g be a C^1-metric on U, and denote by g_{euc} the Euclidean metric. Let A and $\overline{A}$ be the second fundamental forms of S for, respectively, the metrics g_{euc} and g.*

Assume there exist positive constants C_1, C_2 such that the following conditions hold:

- $|\overline{A}| < C_1$ *on* S
- $|g_{ij} - g_{euc,ij}|_{C^1(U)} < C_2$ *and* $|g^{ij} - g_{euc,ij}|_{C^0(U)} < C_2$, $1 \leqslant i,j \leqslant 3$.

Then, there is a constant $C_3 > 0$, depending on C_1 and C_2 and not on S, such that $|A| < C_3$ on S.

Proof. Let $p \in S$ and let $v \in T_pS$ be a nonzero tangent vector. We choose Euclidean coordinates (x_1, x_2, x_3) on U so that $p = (0,0,0)$, the tangent plane T_pS coincides with the plane $\{x_3 = 0\}$ and v is tangent to the x_1-axis.

With these new coordinates we certainly have $|g_{ij} - g_{euc,ij}|_{C^1(U)} < 9C_2$ and also $|g^{ij} - g_{euc,ij}|_{C^0(U)} < 9C_2$, $1 \leqslant i,j \leqslant 3$.

Thus, a part of S is the graph of a function u defined in a neighborhood of the origin in the plane $\{x_3 = 0\}$.
Let $\lambda(v)$ (resp. $\overline{\lambda}(v)$) be the normal curvature of S at p in the direction v for the metric g_{euc}, resp. g. Both curvatures are computed with respect to normals inducing the same transversal orientation along S.
A straightforward computation shows that

$$\lambda(v) = u_{11}(0)$$
$$\overline{\lambda}(v) = \frac{1}{g_{11}\sqrt{g^{33}}}\Big[g\Big(\overline{\nabla}_{\partial_{x_1}}\partial_{x_1}, g^{13}\partial_{x_1} + g^{23}\partial_{x_2} + g^{33}\partial_{x_3}\Big)(0) + u_{11}(0)\Big],$$

where $\overline{\nabla}$ denotes the Riemannian connection of (U, g). Therefore,

$$\lambda(v) = \big(g_{11}\sqrt{g^{33}}\big)(0)\,\overline{\lambda}(v) - g\Big(\overline{\nabla}_{\partial_{x_1}}\partial_{x_1}, g^{13}\partial_{x_1} + g^{23}\partial_{x_2} + g^{33}\partial_{x_3}\Big)(0).$$

Since $|g_{ij} - g_{euc,ij}|_{C^1(U)} < 9C_2$ and $|g^{ij} - g_{euc,ij}|_{C^0(U)} < 9C_2$, $1 \leqslant i, j \leqslant 3$, there is a constant $M > 0$, depending only on C_2 and not on S, such that

$$|g\Big(\overline{\nabla}_{\partial_{x_1}}\partial_{x_1}, g^{13}\partial_{x_1} + g^{23}\partial_{x_2} + g^{33}\partial_{x_3}\Big)(0)| < M \quad \text{and} \quad g_{11}\sqrt{g^{33}}(0) < M.$$

Therefore we obtain

$$|\lambda(v)| \leqslant M|\overline{\lambda}(v)| + M \leqslant M(|\overline{A}(p)| + 1) < M(C_1 + 1),$$

so that it suffices to choose $C_3 = 2M(C_1 + 1)$. □

References

[1] P. Bérard and R. Sa Earp, Minimal hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$, total curvature and index, arXiv: 0808.3838v3 [Math DG].

[2] P. Eberlein and B. O'Neill, Visibility manifolds, *Pacific Jour. of Math* **46** (1) (1973) 45–109.

[3] D. Fisher-Colbrie, On complete minimal surfaces with finite Morse index in three manifolds, *Inventiones Mathematicae* **82** (1) (1985), 121–132.

[4] K.R. Frensel, Stable complete surfaces with constant mean curvature, *Bulletin of the Brazilian Mathematical Society* **27** (2) (1996), 129–144.

[5] L. Hauswirth, B. Nelli, R. Sa Earp and E. Toubiana, A Schoen theorem for minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$, *Advances in Mathematics* **274** (2015) 199–240.

[6] L. Hauswirth, R. Sa Earp and E. Toubiana, Associate and conjugate minimal immersions in $M \times \mathbb{R}$, *Tohoku Mathematical Journal* **60** (2) (2008), 267–286.

[7] L. Hauswirth and H. Rosenberg, Minimal surfaces of finite total curvature in $\mathbb{H}^2 \times \mathbb{R}$, *Matematica Contemporanea* **31** (2006), 65–80.

[8] E. Kreyszig, *Introduction to Differential Geometry and Riemannian Geometry*, Translated from the German, Mathematical Expositions **16**, University of Toronto Press, Toronto, 1968.

[9] F. Morabito and M. Rodriguez, Saddle towers and minimal k-noids in $\mathbb{H}^2 \times \mathbb{R}$, *Journal of the Institute of Mathematics of Jussieu* **11** (2) (2012), 1–17.

[10] B. Nelli and H. Rosenberg, Minimal Surfaces in $\mathbb{H}^2 \times \mathbb{R}$, *Bulletin of the Brazilian Mathematical Society* **33** (2002), 263–292; Errata Minimal Surfaces in $\mathbb{H}^2 \times \mathbb{R}$, *Bulletin of the Brazilian Mathematical Society, New Series* **38** (4) (2007),1–4.

[11] B. Nelli, R. Sa Earp and E. Toubiana, Maximum Principle and Symmetry for Minimal Hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$, *Annali della Scuola Normale Superiore di Pisa, Classe di Scienze.* Vol. XIV (2015) 1–14.

[12] R. Osserman, *A survey on minimal surfaces,* Dover Publications, New York, 1986.

[13] J. Pyo, New Complete Embedded Minimal Surfaces in $\mathbb{H}^2 \times \mathbb{R}$, *Annals of Global Analysis and Geometry* **40** (2) (2011), 167–176.

[14] H. Rosenberg, R. Souam and E. Toubiana, General curvature estimates for stable H-surfaces in 3-manifolds and applications, *Journal of Differential Geometry* **84** (2010), 623–648.

[15] R. Sa Earp, Parabolic and Hyperbolic Screw motion in $\mathbb{H}^2 \times \mathbb{R}$, *Journal of the Australian Mathematical Society* **85** (2008), 113–143.

[16] R. Sa Earp and E. Toubiana, Screw motion surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$, *Illinois Journal of Mathematics* **49** (2005), 1323–1362.

[17] R. Sa Earp and E. Toubiana, An asymptotic theorem for minimal surfaces and existence results for minimal graphs in $\mathbb{H}^2 \times \mathbb{R}$, *Mathematische Annalen*, **342** (2) (2008), 309–331.

[18] R. Sa Earp and E. Toubiana, Minimal graphs in $\mathbb{H}^n \times \mathbb{R}$ and $\mathbb{R}^{n+1}$, *Annales de l'Institut Fourier* **60** (7) (2010), 2373–2402.

Departamento de Matemática
Pontifícia Universidade Católica do Rio de Janeiro
Rio de Janeiro
22453-900 RJ
Brazil
E-mail address: rsaearp@gmail.com

Institut de Mathématiques de Jussieu - Paris Rive Gauche
Université Paris Diderot - Paris 7
Equipe Géométrie et Dynamique, UMR 7586
Bâtiment Sophie Germain
Case 7012
75205 Paris Cedex 13
France
E-mail address: eric.toubiana@imj-prg.fr