C^1- Perturbation Techniques in the Neighborhood of Periodic Orbits

Lecture 3

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Beyond Uniform Hyperbolicity

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"Linear cocycles" (or "vector bundle automorphisms")

X = compact space

$T: X \to X$ homeo

E = vector bundle over X (with Riem. norm $\|\cdot\|$)

$A: E \to E$ bundle automorphism over T, i.e.,

1)
$$\begin{tikzcd} E \arrow[r, "A"] \arrow[d, "\pi"'] & E \arrow[d, "\pi"] \\ X \arrow[r, "T"] & X \end{tikzcd}$$
commutes

2) Restrictions $A: \underbrace{\pi^{-1}(x)}_{\substack{\text{fiber} \\ E_x}} \to \underbrace{\pi^{-1}(Tx)}_{\substack{\text{fiber} \\ E_{Tx}}}$ are linear isomorphisms.

Examples:

1) X = manifold, T = diffeo, $E = TM$ tangent bundle,

$A = Df$

2) $E = X \times \mathbb{R}^d$ trivial bundle,

$A(x,v) = (Tx, M(x)\cdot v)$ (skew-product),

where $M : X \to GL(d, \mathbb{R})$ is the "generator" of the cocycle.

3) If T is a cyclic permutation of $X = \{x_0, x_1, \ldots, x_{p-1}\}$ then the cocycle is called <u>cyclic</u>.

$$\begin{tikzcd} \mathbb{R}^d \arrow[r, "A_0"] & \mathbb{R}^d \arrow[r, "A_1"] & \cdots \arrow[r] & \mathbb{R}^d \arrow[lll, bend left, "A_{p-1}"] \end{tikzcd}$$

$A : E \to E$ cocycle

$E = F \oplus G$ A-invariant splitting

is called (pointwise) ℓ-dominated if

$$\forall x \in X, \quad m\left(A^\ell|_{F_x}\right) \geq 2 \cdot \|A^\ell|_{G_x}\|.$$

- $m(\cdot) = \|\cdot^{-1}\|^{-1}$ is the "mininorm".
- ℓ is the weakness of domination.
- Notation: $F \underset{>}{\oplus} G$, $F \geq_\ell G$.
- Rem: It's convenient to assume ℓ = power of 2.

  Then $F \geq_\ell G$, $\ell' > \ell \Rightarrow F \geq_{\ell'} G$.

Let A = cocycle, d = dim fibers.

If μ is an ergodic prob. measure for $T: X \to X$ then the <u>Lyapunov exponents</u> are the numbers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{d-1} \geq \lambda_d$ s.t. for μ-a.e. x, for all non-zero $v \in E_x$, $\exists i$ s.t. $\lim_{n\to\infty} \frac{1}{n} \log \|A^n(v)\| = \lambda_i$.

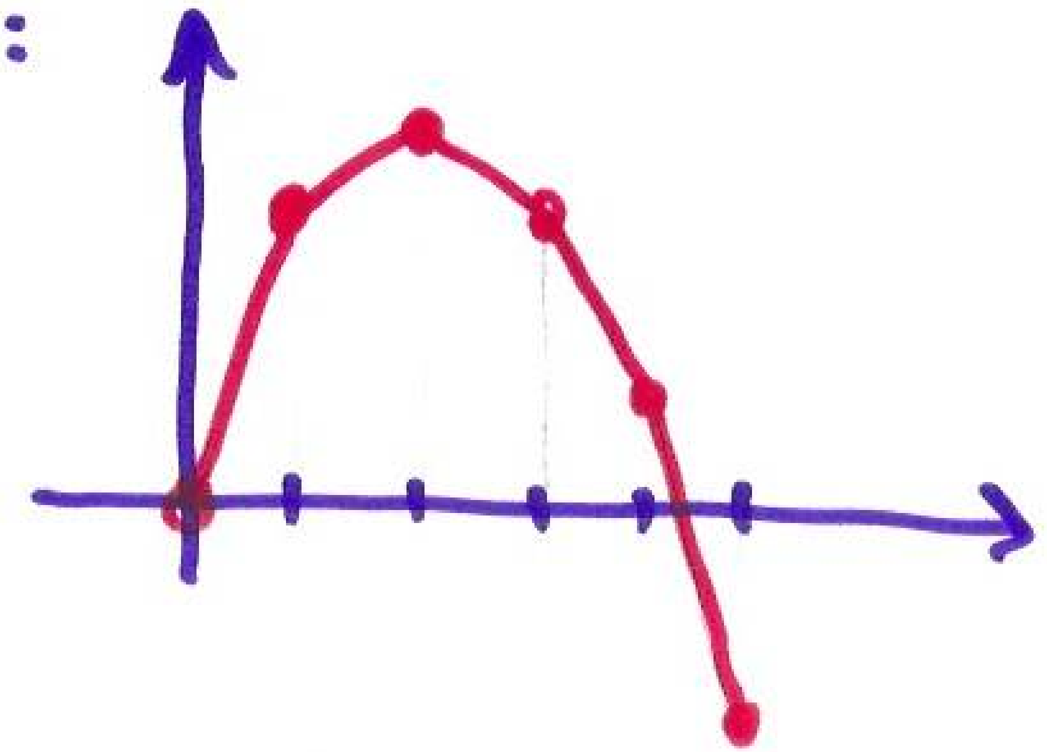
(Repetitions according to multiplicity...)

Consider sums of Lyapunov exponents (w.r.t. μ)

$\boxed{\sigma_i = \lambda_1 + \lambda_2 + \cdots + \lambda_i}$ $(i = 1, 2, \ldots, d)$

Also let $\sigma_0 := 0$.

Since $\lambda_1 \geq \cdots \geq \lambda_d$, the graph of $i \mapsto \sigma_i$ is **concave**:



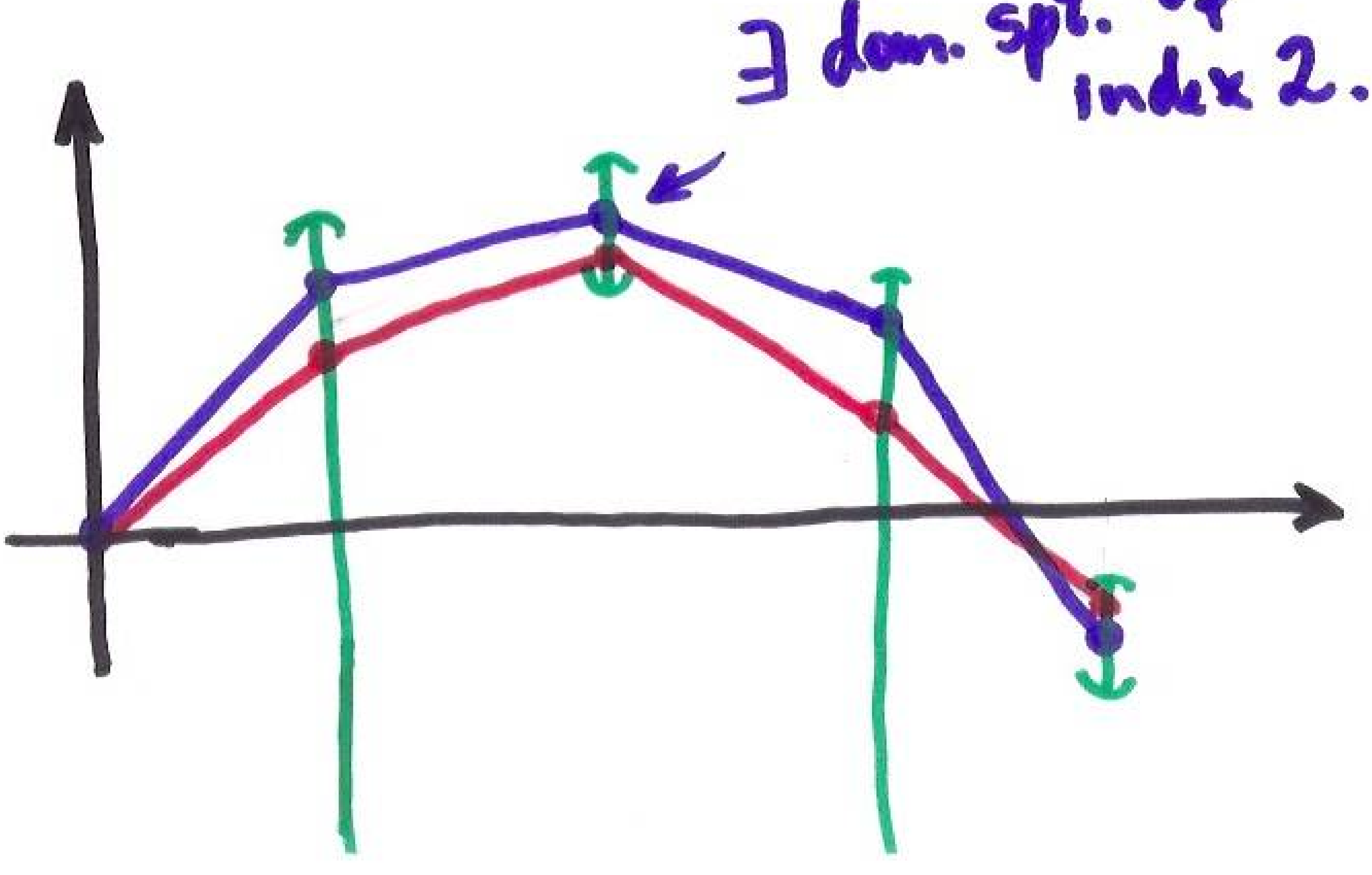
<u>Rem:</u> The σ_i's are "natural":

$$\sigma_i = \lambda_1 + \dots + \lambda_i = \underbrace{\lim_{n\to\infty}}_{\text{also an inf}} \frac{1}{n} \int \underbrace{\log \|\Lambda^i A^n\|}_{\text{subadditive seq. of functions}} d\mu$$

- In general, λ_i's and σ_i's are <u>not</u> continuous as functions of μ, A, T...
- BUT the σ_i's are upper semicontinuous.
- $\sigma_d = \int \log|\det A|\, d\mu$ is continuous.

(Also, σ_i is cont. if i is a domination index.)

Summary: possible effect of perturbations

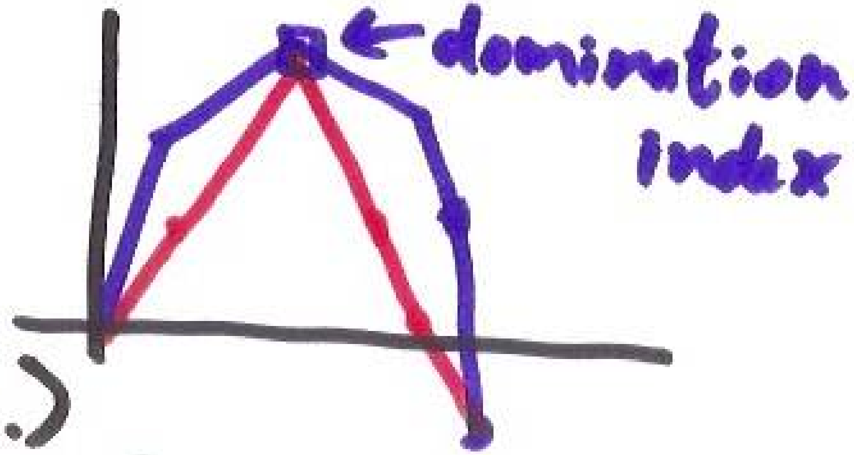


based on Mañé's conjectures...

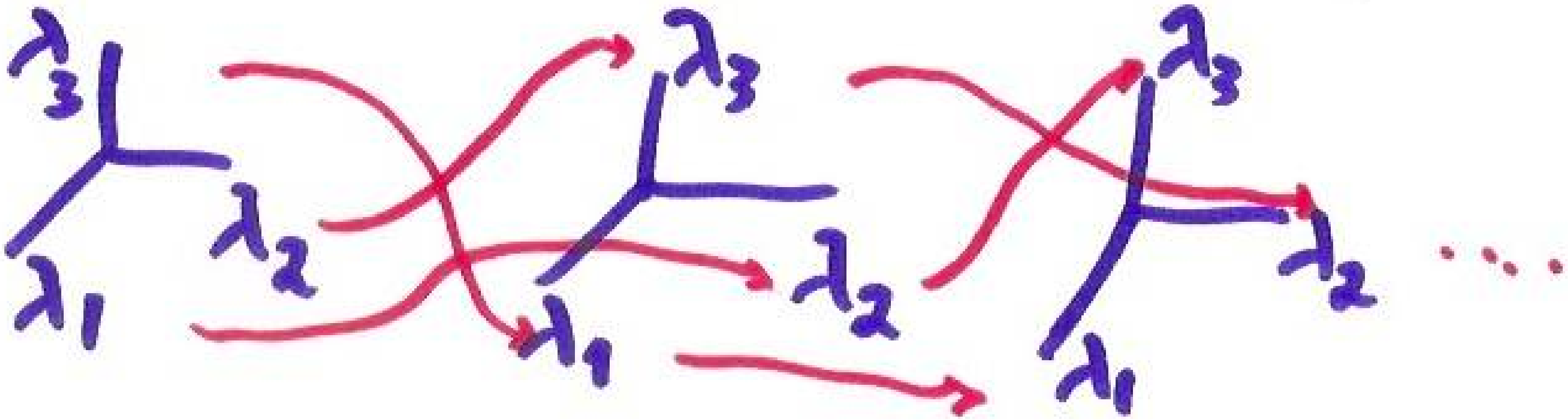
Thrm. (B.–Viana) Let A be a cocycle. Fix μ ergodic, non-atomic.

- If A has no dominated splitting then there is $\tilde{A}$ C^0-close to A whose Lyapunov graph $\sigma(\tilde{A})$ is flat: (i.e., all L.E. are equal.)

- In general, we can find $\tilde{A} \simeq A$ whose graph has "flat pieces": (i.e., all L.E. inside each bundle of the finest D.S. are $\simeq$.)



MAIN IDEA: Lack of domination allows us to mix Lyapunov exponents (under well-chosen perturbations).



Abstract result on "mixing" (or "redistribution of wealth"...)

<u>Def.</u> (Hardy-Littlewood-Pólya)

$x, x' \in \mathbb{R}^d$ $\quad \boxed{x \succsim x'}$ (x majorizes x')

$\Updownarrow$ def

$\exists$ P doubly stochastic matrix such that $x' = Px$

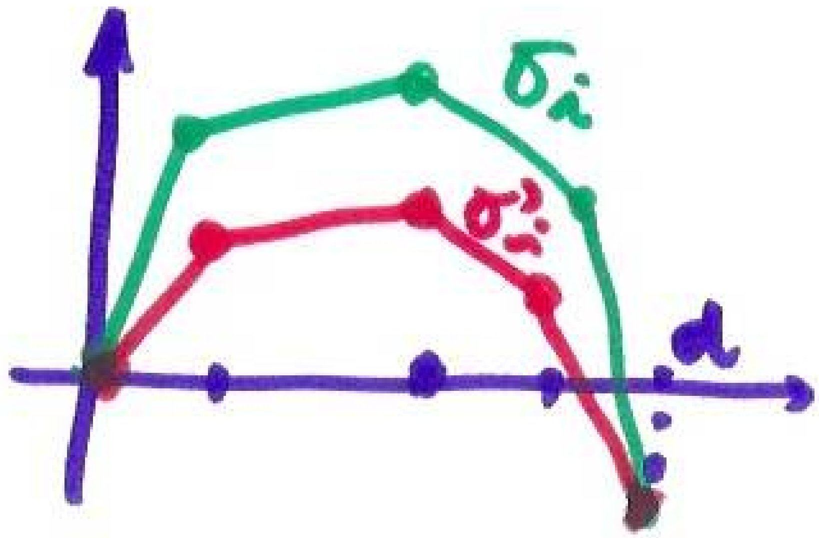
: $P = (p_{ij})$, $p_{ij} \geq 0$, $\sum_i p_{ij} = 1 = \sum_j p_{ij}$

This is called the majorization partial order.

Thrm [HLP] $x \succsim x'$ iff:

$$\text{Let}\begin{cases}(\lambda_1,\dots,\lambda_d) = \text{reordered } (x_1,\dots,x_d) \\ (\lambda_1',\dots,\lambda_d') = \text{———} \quad (x_1',\dots,x_d') \\ \sigma_i := \lambda_1 + \dots + \lambda_i \;,\; \sigma_i' := \lambda_1' + \dots + \lambda_i' \end{cases}$$

$$\text{Then}\begin{cases}\sigma_i \geq \sigma_i' \quad \forall i \\ \sigma_d = \sigma_d'\end{cases}$$



[HLP] suggests that it should be possible to improve the BV Thrm so to obtain more general graphs under perturbations....

(Ok if $d=2$ - unwritten)

That's exactly what we are going to do, assuming that the dynamics is a single periodic orbit (i.e., cocycle is cyclic).

Theorem (B.–Bonatti)

$\forall d \geq 2$ $\forall K > 1$ $\forall \varepsilon > 0$ $\exists L$ such that

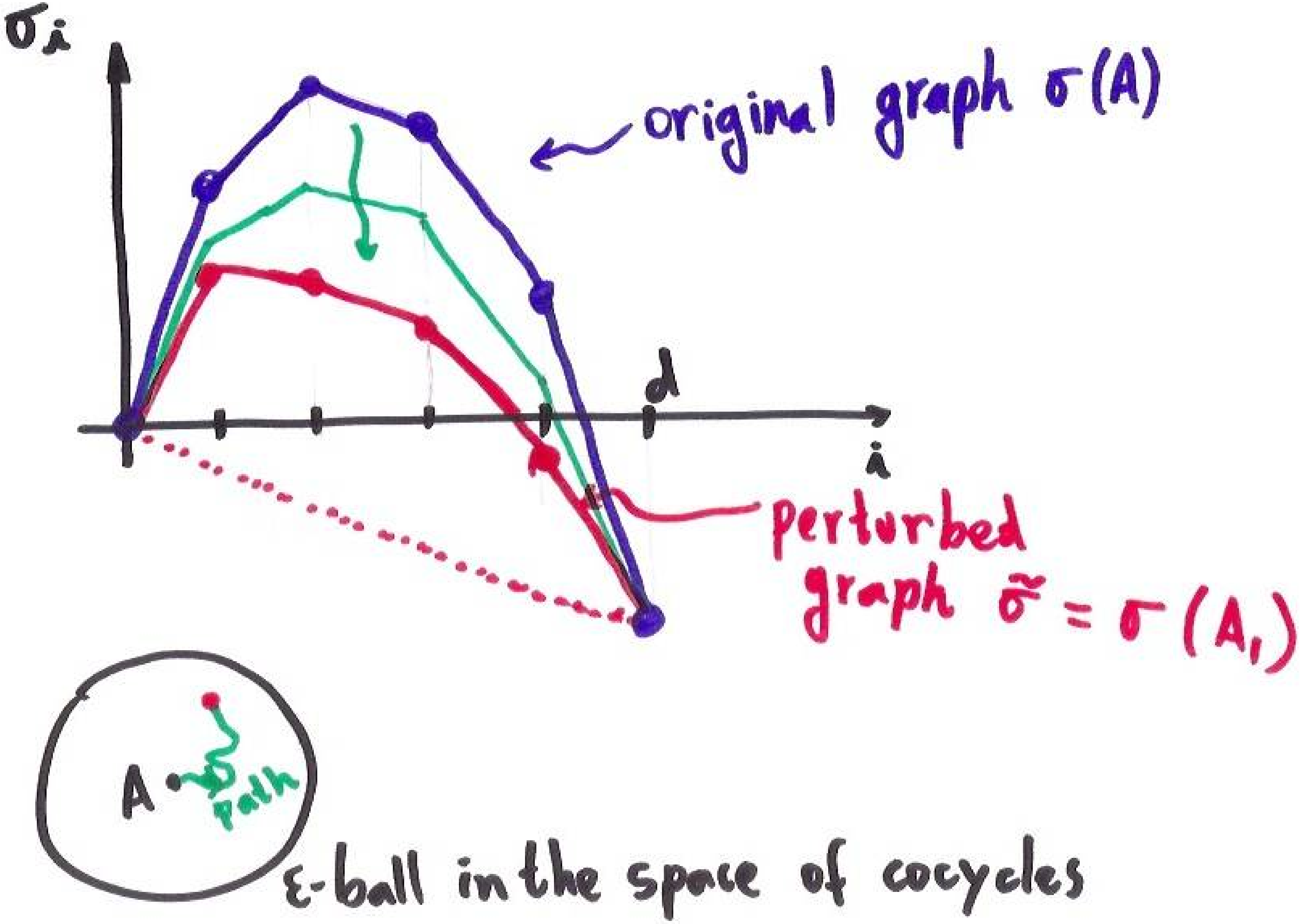
$\forall$ cyclic cocycle A of dim. d, with bounds $\|A^{\pm 1}\| \leq K$,

of period $\geq L$, and with no L-dominated splitting,

$\forall$ concave graph $\tilde{\sigma} \leq \sigma(A)$ with $\tilde{\sigma}_d = \sigma_d(A)$,

$\exists$ ε-short path of cocycles $\{A_t\}_{t \in [0,1]}$ such that:

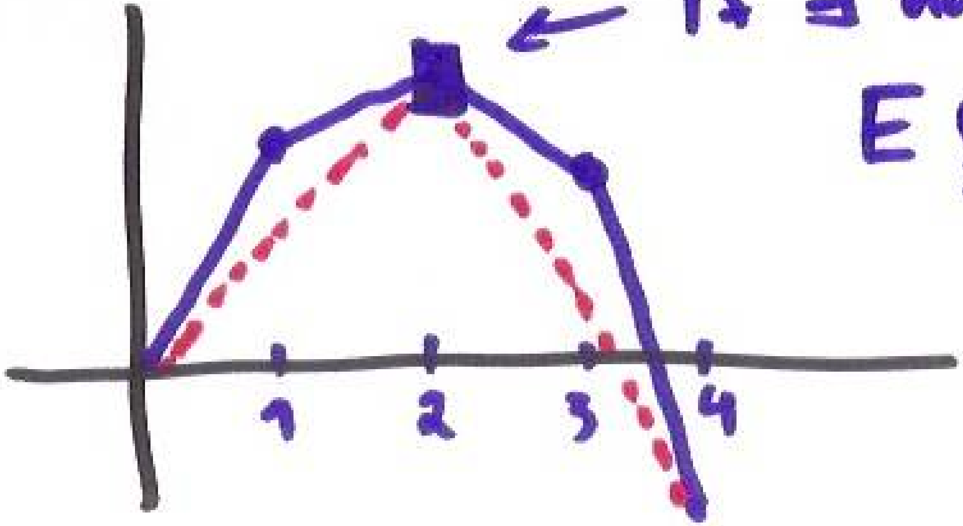
- $A_0 = A$;
- the path of ~~graphs~~ graphs $\{\sigma_t\}_{t \in [0,1]}$ is non-decreasing;
- $\sigma(A_1) = \tilde{\sigma}$.



σ_i
original graph $\sigma(A)$
d
i
perturbed graph $\tilde{\sigma} = \sigma(A_1)$
A
path
ε-ball in the space of cocycles

More generally:

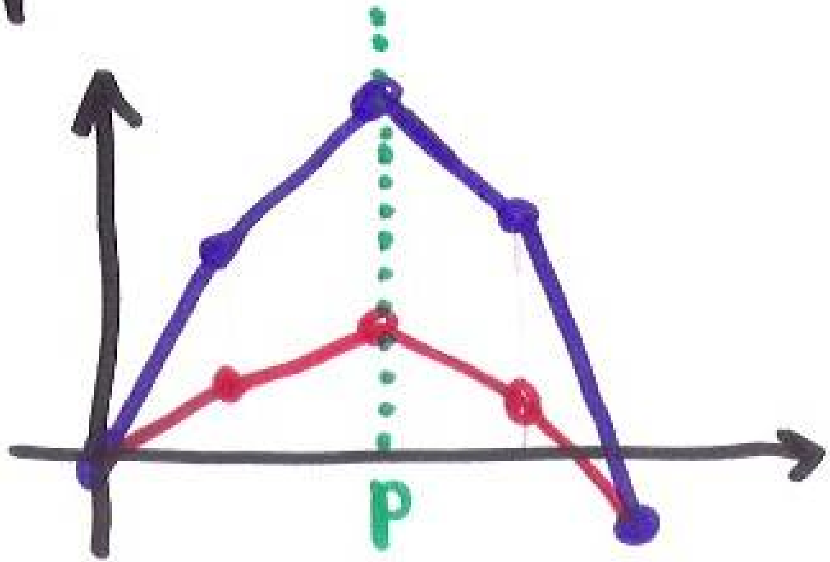
If we allow "some" domination for A, then the Thrm. also holds provided we don't try to move the points on the graph corresponding to the domination indexes:



Is $\exists$ dom. spl.? $E \oplus_{>} F$ with $\dim E = 2$ then this point ■ doesn't move.

Thrm. [Bo-Bo] continued

Moreover, if $\sigma(A)$ and $\tilde{\sigma}$ are both hyperbolic with index p, then we can choose the path of cocycles $\{A_t\}_{t\in[0,1]}$ such that each graph $\sigma(A_t)$ is hyperbolic with index p.



Although the Thrm. is stated for "cyclic linear cocycles", we are more interested in (periodic orbits of) diffeos.

Combined with Franks' Lemma, our Thrm [redacted] has applications to perturbative results on $\mathrm{Diff}^1(M)$.

(<u>Rem</u>: Related results by Gourmelon in $\mathrm{Diff}^r(M)$.)

(Rem: Previous related work by Bonatti-Gourmelon-Vivier.)

Thrm. [Bonatti – Díaz – Pujals]

If a diffeo $f: M \to M$ is C^1-robustly transitive then it admits a (global) dominated splitting.

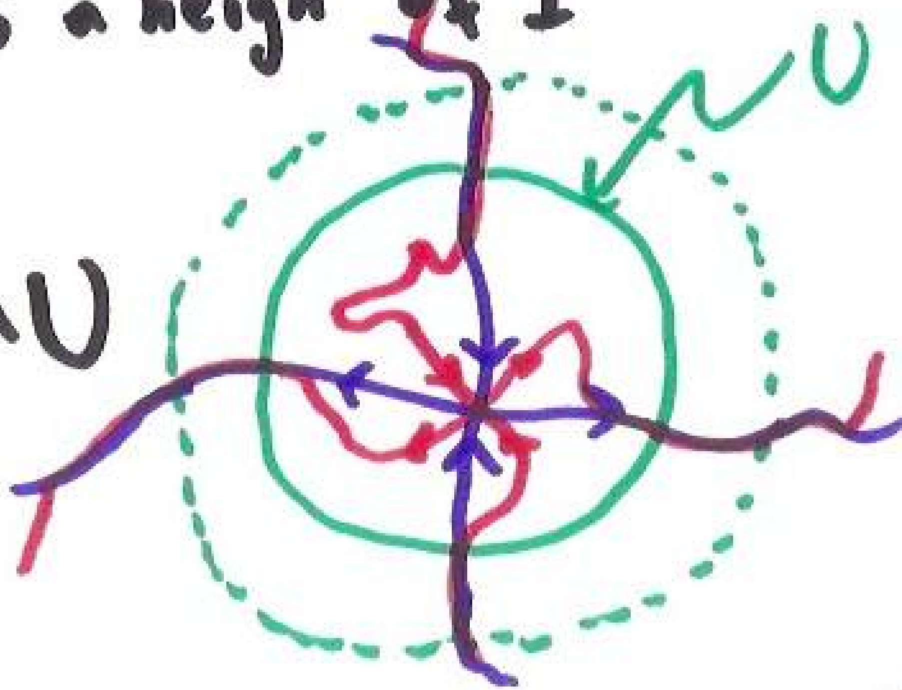
- For $d=2$, this is Mañé's thrm. proved by Christian.
- **Proof**: - No $g \in V_\varepsilon(f)$ has a periodic sink or source.
  - By [Bo-Bo] (or [BV]...), $\exists L(\varepsilon)$ s.t. every g-periodic orbit of per. $\geq L$ has an L-dominated splitting.
  - By Pugh, $\exists g_n \to f$ s.t. $\overline{Per(g_n)} = M$.

  So f has a global L-dom. splitting.

Why do we care about paths / indexes ... ?

Thrm. [Gourmelon's Franks-type lemma]
$f: M \to M$ difeo, $P \subset M$ hyperbolic per. orbit, $p \in P$. $A_0 = Df|_P$. Given any short path $\{A_t\}_{t\in[0,1]}$ of hyperbolic cocycles, $\exists\, g$ C^1-close to f s.t.

- $f = g$ on $P \cup U^c$, where U is a neigh^d of P
- $Dg|_P = A_1$
- $W^s_{loc}(P,g) \setminus U = W^s_{loc}(P,f) \setminus U$
- Analogously for W^u.



(Actually the result holds for strong s/u manifolds...)

Another application/extension:

Let $f: M \to M$ be a diffeo.

Let $\Lambda \subset M$ be compact, f-invariant s.t.

$$\Lambda = \lim_{n\to\infty} P_n \text{ (Hausdorff), where}$$

P_n are periodic orbits with $|P_n| \to \infty$

<u>Question</u>: Which Lyapunov graphs may $D\tilde{f}|_{P_n}$ have (where $n \gg 1$ and $\tilde{f}$ is a Franks-type perturbation)?

Let μ_n be the ergodic meas. supported on P_n.

Passing to a subsequence, assume $\mu_n \to \mu$ (so $\operatorname{supp} \mu \subset \Lambda$).

Rem: μ is not necessarily ergodic.

Let $\sigma(\mu)$ be the Lyapunov graph (use averaged exponents).

Mark the vertices corresponding to the indexes of the finest dom. splt. of $f|\Lambda$

Thrm. [Bo-Bo 2]

The following statements about a concave graph $\tilde{\sigma}$ are equivalent:

① $\tilde{\sigma} \leq \sigma(\mu)$, and the two graphs touch at the marked points (domination)

② $\exists\, g_n \xrightarrow{C^1} f$ s.t. g_n preserves P_n and $\sigma(g_n, P_n) \longrightarrow \tilde{\sigma}$.

Rem: Not a conseq. of previous Thrm.

Rem: $\exists$ C^1-generic consequences. related to [Abdenur-Bonatti-Crovisier].

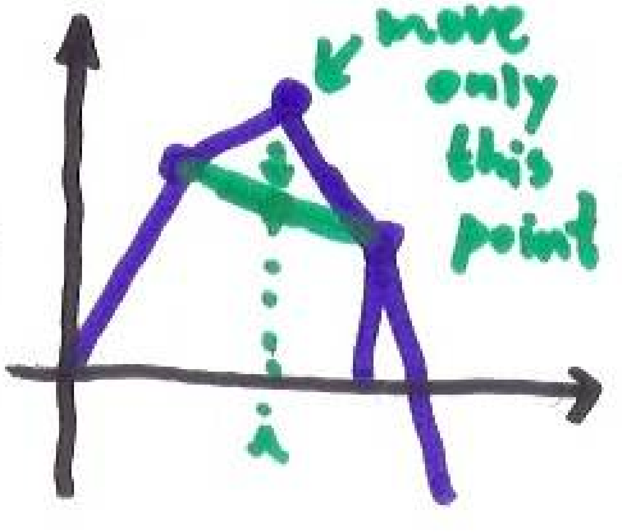
PROOF OF BO-BO THRM:

Main Proposition – particular case of the Thm.

$\forall d, K, \varepsilon\ \exists \ell$ such that $\forall$ cyclic cocycle $A_0 = A$ of dim. d, bound K, period $\geq \ell$, only real eigenvalues, and no ℓ-dominated splitting of index i

Then $\exists\ \varepsilon$-short path of cocycles $\{A_t\}_{t\in[0,1]}$

s.t. $\begin{cases} t \mapsto \sigma_j(A_t) \text{ is const. } \forall j \neq i \\ t \mapsto \sigma_i(A_t) \text{ is non-increasing} \\ \sigma_i(A_1) = \dfrac{\sigma_{i-1}(A) + \sigma_{i+1}(A)}{2} \end{cases}$



Notice that:

$$\sigma_i(A_1) = \frac{1}{2}\left(\sigma_{i-1}(A_0) + \sigma_{i+1}(A_0)\right), \text{ others unchanged}$$

$$\Updownarrow$$

$$\lambda_i(A_1) = \lambda_{i+1}(A_1) = \frac{\lambda_i(A_0) + \lambda_{i+1}(A_0)}{2}, \text{ others unchanged}$$

Effect of perturbation on the spectrum:

If $\underline{d=2}$, then the Main Prop. follows from Mañé's argument (yesterday's lecture).

Can we reduce the general case to the 2-dim. case? Idea: Restrict to the subbundle "containing" λ_i and λ_{i+1} and apply Mañé...

WRONG! The fact that $A: E \to E$ has no (strong) dominated splitting does not exclude the existence of invariant subbundles $F \subset E$ with (strong) D.S. $F = F_1 \oplus F_2$

Example: $E = E_1 \oplus E_2 \oplus E_3$ $\quad$ $\dim E_i = 1$

$(\lambda_1 > \lambda_2 > \lambda_3)$

$$\left.\begin{array}{ll} (1) & E_1 \not> E_2 \oplus E_3 \\ (2) & E_1 \oplus E_2 \not> E_3 \end{array}\right\} \Rightarrow \boxed{E \text{ has no (strong) dominated spl.}}$$

but $\boxed{E_2 > E_3 \text{ (strong)}}$.

$\measuredangle(E_1, E_2) \ll 1$ $\quad$ $\measuredangle(E_1, E_3) \ll 1$

$\Rightarrow (1)$ $\quad$ $\Rightarrow (2)$

Even so, to prove the Main Prop., we will try to reduce the dimension: actually we'll argue by induction on d.

Restriction and Quotient

$A : E \to E$ cocycle, F = invariant subbundle

$$A = \begin{pmatrix} A|F & * \\ 0 & A/F \end{pmatrix} \text{ w.r.t. splitting } E = F \oplus F^{\perp}$$

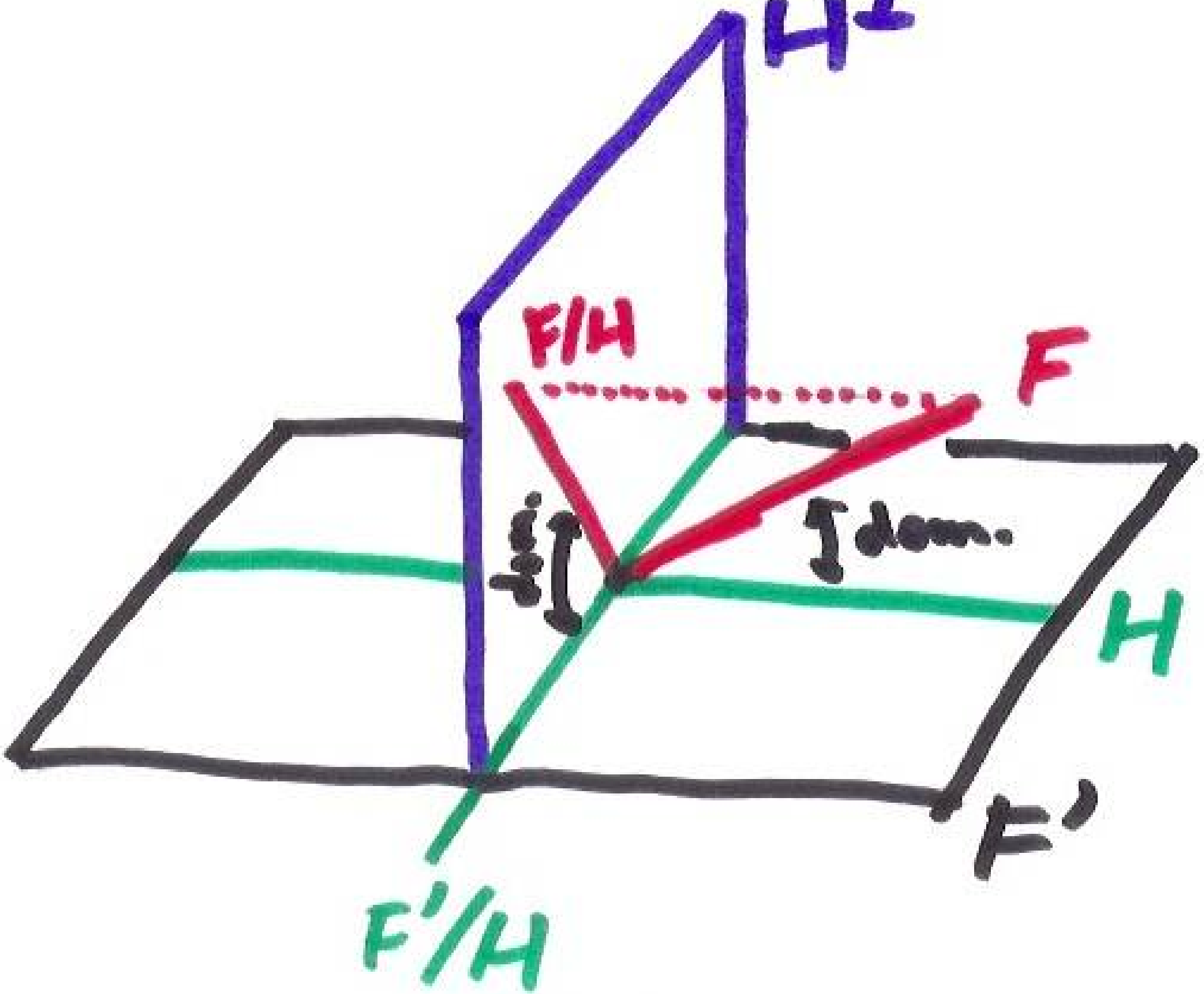
- We can perturb $A|F$ and A/F indep'ly, (keeping F invariant).

Lemma. (Improved version of a lemma in [BDP].)

$\forall d, K, \ell\ \exists L \gg \ell$ such that $\forall$ cyclic cocycle of dim. d, bound K, period $> L$,

$$\left.\begin{array}{l} F' \oplus F \text{ invariant spl.} \\ H \subset F' \text{ invariant sub-bundle} \\ H \underset{\ell}{>} F \\ F'/H \underset{\ell}{>} F/H \end{array}\right\} \Rightarrow F' \underset{L}{>} F$$

$$\left.\begin{array}{l} H > F \\ F'/H > F/H \end{array}\right\} \Longrightarrow F' > F$$

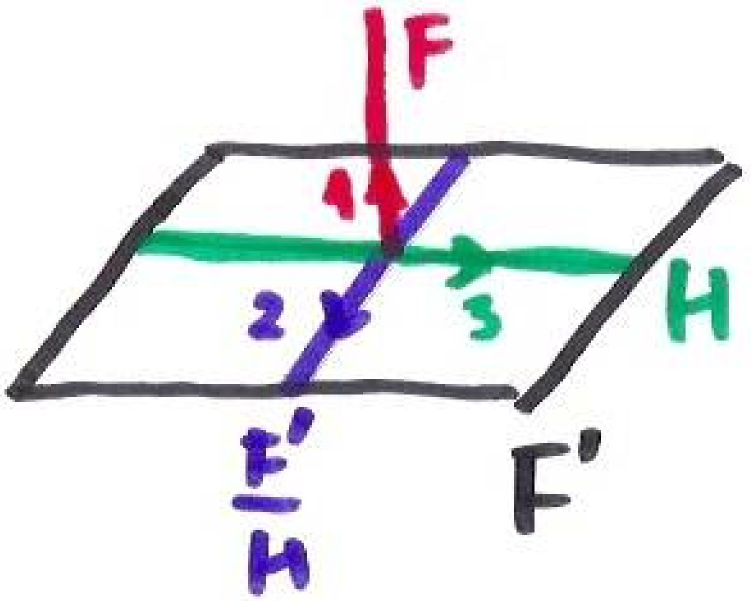


Proof of the Lemma:

<u>1st. step</u>: $\measuredangle(F', F)$ is not small. Indeed:

$$\sin \measuredangle(F, F') \geq \sin \measuredangle(F, H) \cdot \sin \measuredangle(F/_H, F'/_H)$$

In particular, we may change bases and assume that $F' \perp F$.



$$A^{-1} = \begin{pmatrix} M & 0 & 0 \\ 0 & \underline{P} & 0 \\ 0 & Q & R \end{pmatrix}$$

- M dominates $\underline{P}$ and R
- Typical product in the off-diag. pstn:

$RRR\underline{Q} + RR\underline{Q}P + R\underline{Q}PP + \underline{Q}PPP$

These are dominated by the corresponding products of M.

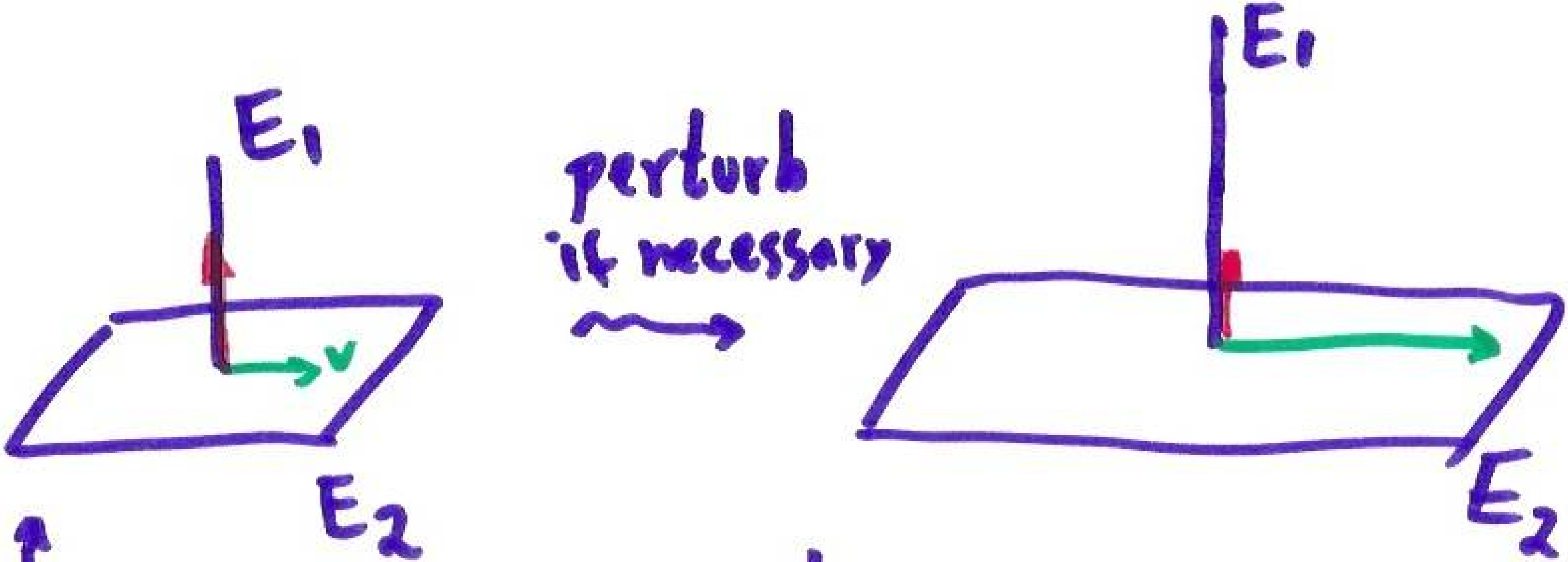
Recall the part of Mañé' argument about creating small angles:

<u>Lemma</u>. $\forall d, K, \varepsilon, \alpha \; \exists \ell$ such that $\forall$ cocycle A...

- $E_1 \oplus E_2$ invariant
- "dominated at the period"
- $E_1 \not>_{\ell} E_2$
- $\dim E_1 = 1$ ~~and~~ OR $\dim E_2 = 1$

$\Rightarrow$

$\exists$ ε-short path of cocycles s.t. no eigenv.s change, and at the end $\measuredangle(E_1^{new}, E_2^{new}) < \alpha$

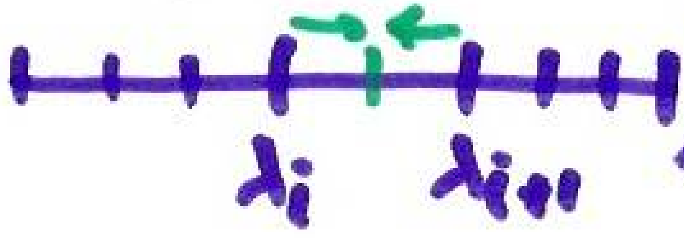


↑ point "where" $E_1 \not> E_2$

v = unit vector in E_2 whose expansion in time ℓ is $\gtrsim$ expansion along E_1

Apply shear perturbation (as Christian explained).

Proof of the Main Prop. (



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Induction on d.

$\boxed{d=2}$: Mañé argument
(non-domination $\rightsquigarrow$ small angle $\rightsquigarrow$ rotation)

$\boxed{\text{Inductive step}}$: Assume M.P. true for dim $2, 3, \ldots, d-1$.

Assume: • $i > 1$ (otherwise reverse time)

• $\lambda_i > \lambda_{i+1}$ (otherwise there is nothing to do)

Here we consider a cocycle A with only real eigv. and no ℓ-dominated spl. of index i (where $\ell \gg 1$).

Assume also $\boxed{\lambda_1 > \lambda_2}$ (The case $\lambda_1 = \lambda_2$ is simpler).

Invariant subbundles:

$$\underbrace{\underbrace{\lambda_1}_{H} > \underbrace{\lambda_2 \geq \cdots \geq \lambda_i}_{G}}_{F'} > \underbrace{\lambda_{i+1} \geq \cdots \geq \lambda_d}_{F}$$

Main assumption: $F' \not> F$.

Notice that if

$$(1)\ G \not\succ F \quad \text{or} \quad (2)\ \frac{G}{H} \not\succ \frac{F}{H}$$

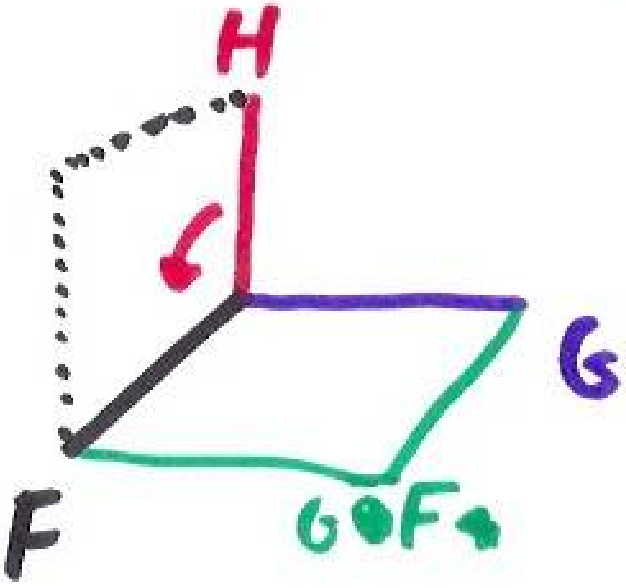
then we can use the induction hypotheses to conclude:

$$\left.\begin{array}{l} (1) \Rightarrow \text{perturb } A|_{G \oplus F} \\ (2) \Rightarrow \text{perturb } A/_{H} \end{array}\right\} \begin{array}{l} \text{and obtain} \\ \lambda_i^{new} = \lambda_{i+1}^{new} \end{array}$$

Is (1) or (2) automatic? I.e.,

does $H \oplus G \not\pitchfork F$ imply $G \not\pitchfork F$ or $\frac{G}{H} \not\pitchfork \frac{F}{H}$?

<u>Answer: NO.</u>



But the counterexample is too delicate: we'll show that it can be destroyed by perturbation — this will prove the Main Proposition.

C^1- perturbation techniques in the neighborhood of periodic points

Lecture 4

As explained before, the proof of the Main Prop. reduces to prove:

Lemma. A = cyclic cocycle (with only real eigenvalues) with L.E.

$$\underbrace{\lambda_1}_{H} > \underbrace{\lambda_2 \geq \cdots \geq \lambda_i}_{G} > \underbrace{\lambda_{i+1} \geq \cdots \geq \lambda_d}_{F}$$

If $H \oplus G \not>_{\ell} F$ then after an ε-short path of perturb. (without changing eigenvalues) we have $G \not>_{\ell_0} F$ or $\frac{G}{H} \not>_{\ell_0} \frac{F}{H}$. $\boxed{\ell = \ell(\ell_0, \varepsilon, d, \ldots)}$

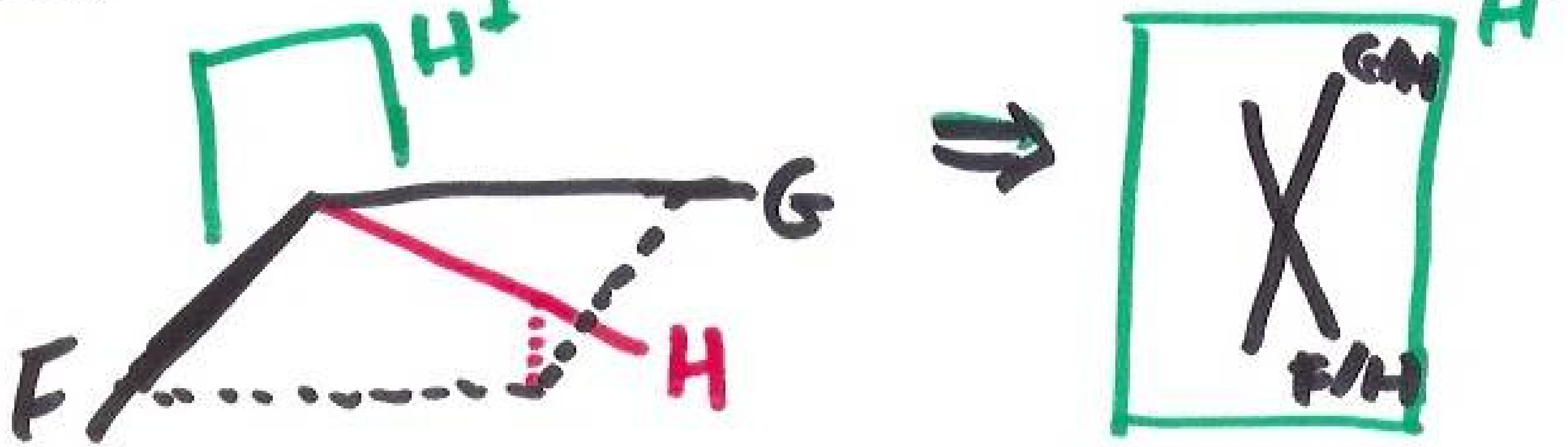
Proof of the Lemma:

Assume $\underbrace{H \oplus G \not> F}_{(1)}$, $\underbrace{G > F}_{(2)}$ and $\underbrace{\frac{G}{H} > \frac{F}{H}}_{(3)}$.

- By a previous lemma, $(1)+(3) \Longrightarrow$

  $H \not> F$ (4)

Suppose that at some point, $\measuredangle(H, G \oplus F)$ is small, but $\measuredangle(H,G)$ and $\measuredangle(H,F)$ are not:



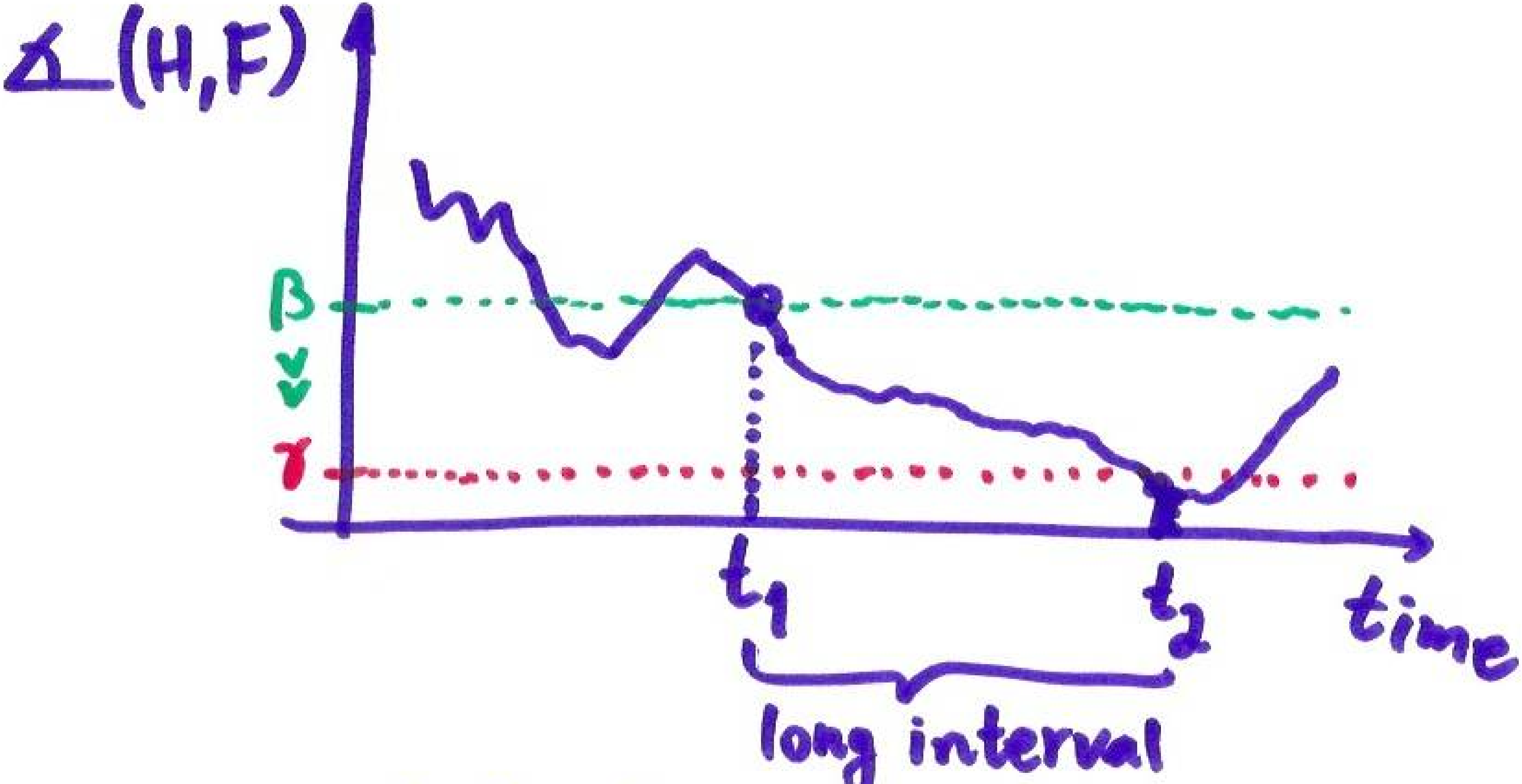
$\measuredangle\left(\frac{G}{H}, \frac{F}{H}\right)$ small, contradiction.

So each time that $\measuredangle(H, G \oplus F)$ becomes small, either $\measuredangle(H,F)$ or $\measuredangle(H,G)$ is small, but not both (because $G > F$).

We will show that by perturbation, we can find this situation (H close to $G \oplus F$, but far from $G \cup F$), thus completing the proof.

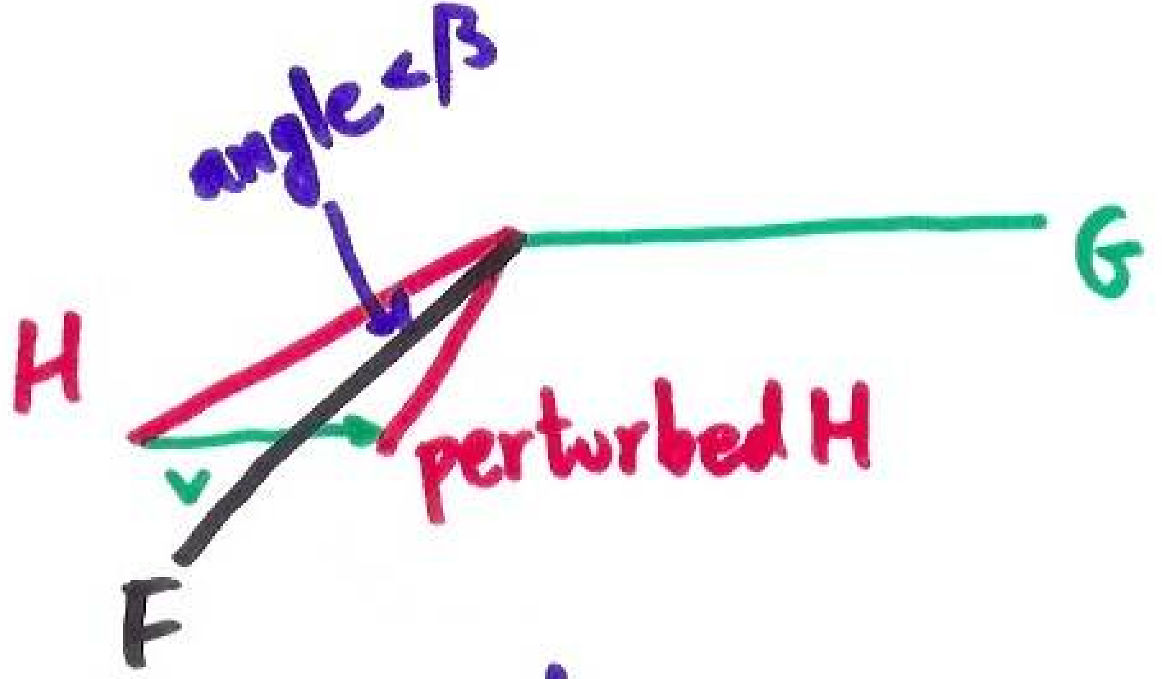
First, since (4) $H \not\succ F$, by a previous lemma (Mañé argument), we can (perturbing if necessary), assume that

$$\Delta(H,F) < \underbrace{\gamma \lll 1}_{\gamma \ll \beta \ll 1}$$ at some point.



We will perturb at time t_1:

At time t_1, "add to H" a small G component (green vector).



Rem: it is convenient to choose $v \in G$ an eigenvector.

More precisely, we compose the cocycle with a linear map close to Id that fixes $G \oplus F$ and moves H as described.

$\|v\|$ is of the order of β (small, but not extremely small.)

Project orthogonally onto $G \oplus F$:

(time t_1) (some time $< t_2$)

(Use that $\boxed{G > F}$ and $t_2 - t_1 \gg 1$.)

Full picture:

- The last thing is to "close" the path of the perturbed space H, so that it becomes an eigenspace.
- That's easy: since λ_1 is the biggest L.E. (and $\boxed{v}$ is an eigenvector), the perturbed H returns nearby, and so we can close it.
- This proves the lemma.
- By induction, the main proposition follows.

The Main Proposition assumes that all eigenvalues are real.

How to get rid of that assumption?

Lemma. [Bochi – Crovisier / Avila ...]

$\forall \varepsilon > 0\ \exists n_0$ s.t. $\forall n \geq n_0$,

$\forall A_1, A_2, \ldots, A_n \in GL^+(2, \mathbb{R})$

$\exists \theta \in (-\varepsilon, \varepsilon)$ s.t.

$R_\theta A_n R_\theta A_{n-1} \cdots R_\theta A_1$ has real eigenvalues.

Proof. Let v be the direction which is most contracted by $A_n \cdots A_1$. Let

$$I_j := \{ R_\theta A_j R_\theta A_{j-1} \cdots A_1 R_\theta v \,;\ \theta \in [-\varepsilon, \varepsilon] \}$$

$I_j \subset \mathbb{P}^1$. $\qquad I_0 := \{ R_\theta v \} = [v - \varepsilon, v + \varepsilon]$

If $\exists j$ s.t. $I_j = \mathbb{P}^1$ then $I_n = \mathbb{P}^1$, so $v \in I_n$, so $R_\theta A_n \cdots R_\theta A_1 R_\theta$ has an eigendirection and we are done.

So assume $I_j \neq \mathbb{P}^1$ $\forall j$.

$I_{j+1} = \varepsilon$-neigh. of $A_j I_j$

In each I_j we take the Hilbert metric.

- The maps

$$I_0 \xrightarrow{A_0} I_1 \xrightarrow{A_1} I_2 \longrightarrow \cdots \xrightarrow{A_n} I_{n+1}$$

are uniform contractions (by a factor $\lambda(\varepsilon) < 1$), w.r.t. the Hilbert metric

- So $A_n \cdots A_1 : I_0 \to I_{n+1}$ is a strong contraction.
- This is impossible, because $v \in I_0$ is the most contracted direction.

Remark: The same argument can be used to create complex eigenvalues when $\|A_n \cdots A_1\| \gg 1$ (Avila's lemma)

Now, the <u>Proof of the Bo-Bo (1st) Theorem</u>

- Given a cyclic cocycle A with no strong domination, we want to perturb it to obtain a prescribed graph $\tilde{\sigma} \leq \sigma(A)$:

- We have already solved the problem for $\tilde{\sigma}$ of this kind:

or

Call this an "elementary (flattening) operation".

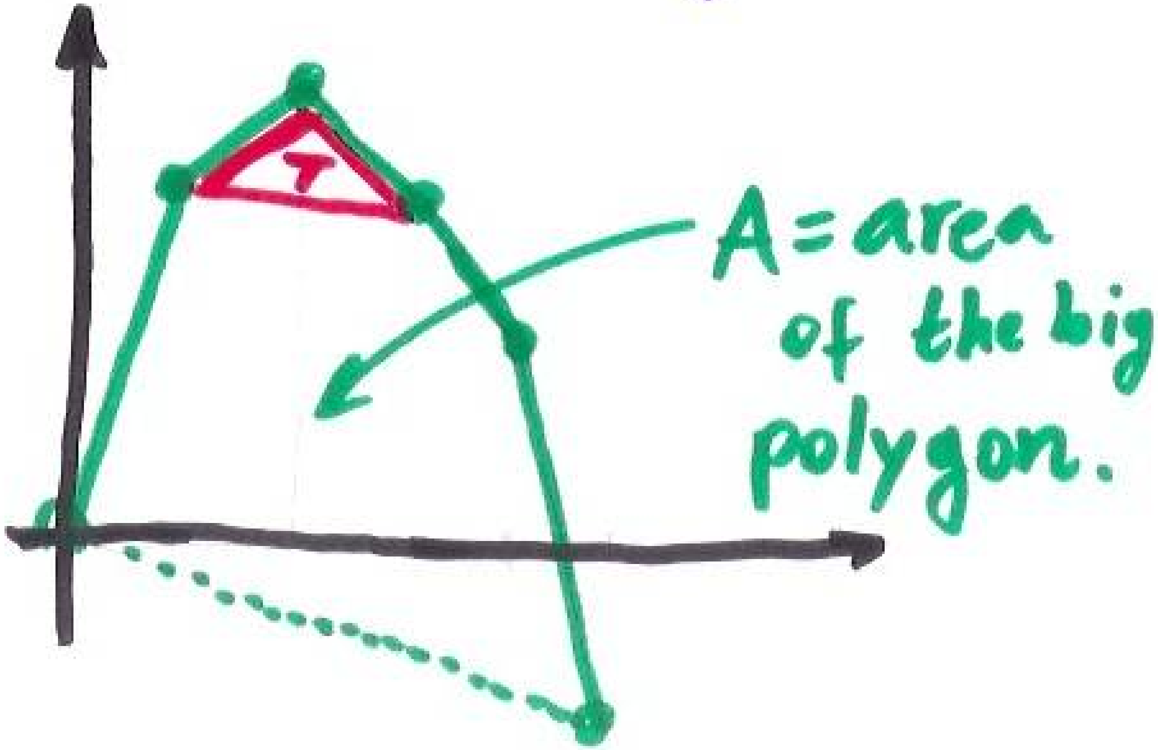
It is clear that with a sequence of elementary operations, we can approach any given graph $\tilde{\sigma}$.

However, it's necessary to bound the numbers of elementary operations we're going to use.

That's because:
- we need to control the size of perturb;
- after an operation, we may gain(!) some (weak) domination, which is bad.

Lemma. $\forall d \geq 2 \ \exists\, c = c(d) \in (0,1]$ $\left(c = \frac{6}{d(d^2-1)} \text{ works}\right)$ such that for any concave graph $\sigma : \{0,1,2,\dots,d\} \to \mathbb{R}$ there are 3 consecutive vertices forming a triangle of area

$$\boxed{T \geq c \cdot A}$$



- So, choosing which triangles to flatten according to a <u>greedy strategy</u>, the area to be flattened converges to zero exponentially fast.
- Once the area becomes sufficiently small, it's trivial to make a final correction.
- Therefore we have an a priori bound on the # of elementary operations.

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The index preserving case:

- We have to be more careful in the choice of the elementary flattening operations: we are not allowed to change the index.
- Again we need an a priori bound on the number of elementary operations.
- Again we will find an strategy s.t. the area to be flattened converges exponentially to zero (w.r.t. # of operations).

Strategy: (p = index)

1) Flatten as much as possible both sides (according to the previous greedy strategy), keeping the vertex p fixed.
2) Flatten p as much as possible.
3) Go to step 1.

It's possible to show that step 2 either wins the game or kills a definite proportion of area!

C^1-Perturbation techniques in the neighborhood of periodic orbits

Lecture 5

pages 20-22

The 2nd Bo-Bo Theorem. $f: M \to M$

$\mathbb{P}_n = (\text{periodic orbits}) \xrightarrow{\text{Haus.}} \Lambda$

$\nu_n = (\text{measures on } \mathbb{P}_n) \longrightarrow \mu$

We want to perturb the cocycle $Df|\mathbb{P}_n$ so that we obtain a prescribed Lyapunov graph

$\boxed{\tilde{\sigma} \leq \sigma(\mu)}$ which:
- is concave
- "respects" the finest dom. spl. of $f|\Lambda$

Actually it's sufficient to prove the Thrm. in the case $\tilde{\sigma} = \sigma(\mu)$; then the general case follows from the 1st Thrm. So we only need to prove the following cocycle version:

<u>Thrm.</u> Take a linear cocycle

$$\begin{array}{ccc} E & \xrightarrow{A} & E \\ \downarrow & & \downarrow \\ X & \xrightarrow{T} & X \end{array}$$

Let P_k be a sequence of periodic orbits with $|P_k| \to \infty$. Assume $\nu_k \to \mu$, where ν_k = ergodic measure on P_k.

<u>Then</u> there exists a sequence of cocycles $B_k \to A$ such that

$$\sigma(B_k, \overset{\nu_k}{\cancel{\mu_k}}) = \sigma(A, \mu) \quad \forall k.$$

<u>Rem.</u> Kalinin (Annals'11) has a $\boxed{\text{non-perturbative version}}$ of this result, assuming:

- T = "hyperbolic"
- A Hölder
- μ ergodic

→ $\forall \mu \ \exists P_n$ (sequence of periodic orbits) such that $\sigma(P_n) \longrightarrow \sigma(\mu)$.

Proof of our result:

Start with $d=2$. We only need to care about one exponent:

$$\sigma_1(\mu) = \lim_{m\to\infty} \frac{1}{m} \int \log \|A^m\| \, d\mu .$$

Take $m \gg 1$ s.t.

$$\frac{1}{m} \int \log \|A^m\| \, d\mu \simeq \sigma_1(\mu).$$

Now take $\boxed{k} \gg 1$ such that:

- the periodic orb. P_k has period $N \gg m$
- the measure $\boxed{\nu_k}$ on P_k is close to $\boxed{\mu}$

and so

$$\frac{1}{m}\int \log \|A^m\| \, d\nu_k \simeq \frac{1}{m}\int \log\|A^m\| \, d\mu$$

$$\simeq \sigma_1(A, \mu).$$

For simplicity, assume that the period $N \gg m$ is a multiple of m.

Let $Z_1(y) := \dfrac{1}{N/m} \displaystyle\sum_{i=0}^{N/m-1} \frac{1}{m} \log \| A^m(T^{mi} y) \|$

$(N = |P_k|)$

for $y \in P_k$.

Then:

$$\begin{array}{l}\text{Average value}\\ \text{of } Z_1 \text{ on } P_k\end{array} = \int \frac{1}{m} \log \| A^m \| \, d\nu_k \simeq \sigma_1(A, \mu)$$

Therefore $\exists$ a good pt. $y \in P_k$ s.t.

$Z_1(y) \gtrsim \sigma_1(A,\mu)$.

By def.,

$$\|\overbrace{A^m(T^{N-m}y)}^{M_n}\| \cdots \|\overbrace{A^m(T^m y)}^{M_2}\| \cdot \|\overbrace{A^m(y)}^{M_1}\|$$

$$= e^{N Z_1(y)} \gtrsim e^{N \sigma_1(A,\mu)}.$$

Notice that $M_n \cdots M_2 M_1 = A^N(y)$.

$(n = N/m)$

$$e^{N\sigma_1(A,\nu_k)} = \begin{bmatrix} |\text{largest } |\text{eigenvalue}| \\ \text{of } M_n \cdots M_1 = A^N(y) \end{bmatrix}$$

$$\overset{(1)}{\leq} \|M_n \cdots M_1\|$$

$$\overset{(2)}{\leq} \|M_n\| \cdots \|M_1\| = e^{N Z_1(y)}$$

<u>Claim</u>: Both inequalities (1) and (2) become approx. equalities under perturb. of M_i's.

It follows from the Claim that

$\exists\, \tilde{A}$ = perturb. of A s.t.

$$\sigma_1(\tilde{A}, \nu_k) \approx Z_1(y) \underbrace{\gtrsim \sigma_1(A, \mu)}_{(y \text{ is good}).}$$

By semicontinuity, $\gtrsim$ is actually $\approx$.

This proves the Thrm. in $d=2$, modulo the Claim.

will be useful for $d>2$.

<u>Rem.</u> Actually for all the good pts. y, which by defn. have $Z_1(y) \gtrsim \sigma_1(A,\mu)$, we actually have $Z_1(y) \simeq \sigma_1(A,\mu)$ (otherwise we would break semicontinuity). Since $\frac{1}{|P_k|} \sum_{y \in P_k} Z_1(y) \simeq \sigma_1(A,\mu)$, we conclude that

$$\boxed{\text{most points in } P_k \text{ are good.}}$$

Claim (1). $\forall M \in GL(2,\mathbb{R})\ \exists\, \theta \simeq 0$ s.t. |largest eigenvalue| of $R_\theta M$ is $\approx \|M\|$.

Example: $M = R_{\pi/2}\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1}\end{pmatrix} = \begin{pmatrix} 0 & \lambda^{-1} \\ -\lambda & 0 \end{pmatrix}$ $(\lambda \gg 1)$

has $|\text{eigen.}| = 1$. But $R_\theta M = \begin{pmatrix} \lambda \sin\theta & * \\ * & \lambda^{-1}\sin\theta \end{pmatrix}$

has $|\text{eigenv.}| \simeq \lambda|\sin\theta| \approx \lambda = \|M\|$

Proof of the Claim (1): same idea...

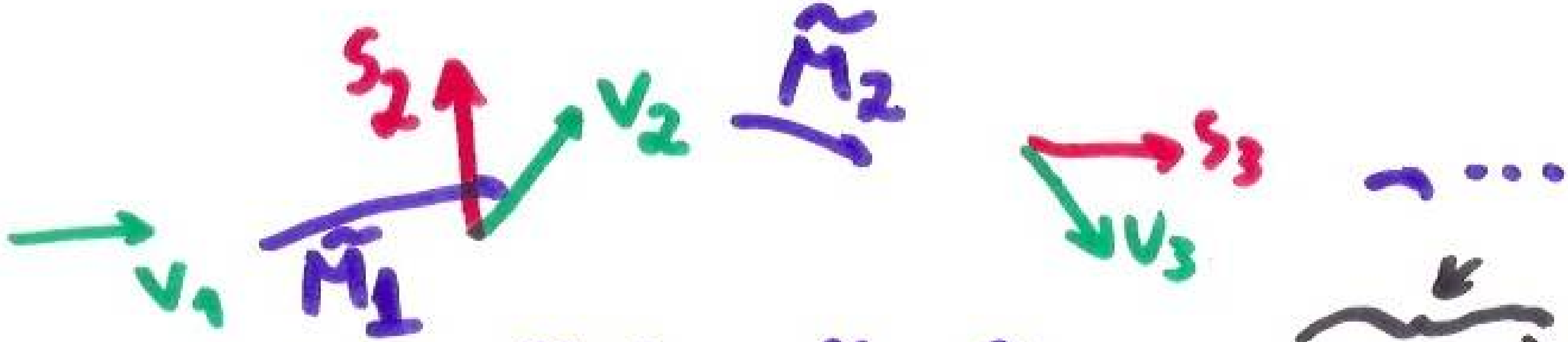
<u>Claim(2)</u>. $\forall M_1, M_2, \dots, M_n \in GL(2,\mathbb{R})$, $\exists\, \theta_1, \theta_2, \dots, \theta_{n-1} \approx 0$ s.t.

$$\|M_n R_{\theta_{n-1}} \cdots R_{\theta_2} M_2 R_{\theta_1} M_1\| \approx \|M_n\| \cdots \|M_2\| \cdot \|M_1\|$$

<u>Proof</u>. For each i, let s_i be the direction most contracted by M_i (or by $\tilde{M}_i = R_{\theta_i} M_i$):

The angles $\theta_1, \dots, \theta_{n-1} \in [-\varepsilon, \varepsilon]$ are chosen in this order (recursively) so that: $\measuredangle(v_i, s_i) \geq \varepsilon \quad \forall i$

where $\begin{cases} v_0 := \text{the most expanded } \cancel{\text{dir.}} \text{ unit vector by } M_1 \\ v_{i+1} := \tilde{M}_i \, v_i \end{cases}$, $\quad \tilde{M}_i = R_{\theta_i} M_i$



Then $\|\tilde{M}_n \cdots \tilde{M}_1\| \geq \|\tilde{M}_n \cdots \tilde{M}_1 v_1\| \geq \overbrace{(\sin\varepsilon)^n} \|M_n\| \cdots \|M$

The term $(\sin\varepsilon)^n \simeq \varepsilon^n$ is negligible, because

$$\frac{1}{N}\left|\log \varepsilon^n\right| = \frac{n}{N}\left|\log\varepsilon\right| = \underbrace{\frac{1}{m}}_{m \gg 1.}\left|\log\varepsilon\right| \ll 1$$

This proves Claim (2), and hence the Thrm. for $d=2$.

Proof in any dimension d: similar...

- $\sigma_i(\mu) = \lim\limits_{m\to\infty} \frac{1}{m} \int \log \|\Lambda^i A^m\| \, d\mu.$

- Choose m, k, $N = |P_k|$ as before.

- New functions

$$Z_i(y) := \frac{1}{N/m} \sum_{i=0}^{N/m-1} \frac{1}{m} \log \|\Lambda^i A^m(T^{mi} y)\|$$

$(y \in P_k)$, with average value $\simeq \sigma_i(A,\mu)$.

- A point is called i-good if $Z_i(y) \gtrsim \sigma_i(A, \mu)$. (actually $\simeq$)
- For these pts. it's possible to insert small rotations, avoid cancellations, and so obtain the desired value for $\sigma_i(\tilde{A}, P_K)$.

(Not difficult, but tricky.)

exterior algebra.

- As before, for each i, most pts. in P_k are i-good.
- Therefore $\exists$ a point which is i-good for every $i = 1, 2, \ldots, d-1$.
- For this point we can avoid cancellations in each dimension.
- Done!