Generic Symplectic Diffeomorphisms and Partial Hyperbolicity

Workshop on Symplectic Dynamics
Institute for Advanced Study (Princeton)

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Plan of the talk

- Recall basic definitions.
- State a theorem on partial hyperbolicity of generic symplectic diffeomorphisms.
- Discuss consequences and further developments of that theorem (including the ergodicity result).
- Compare with a cousin theorem for volume-preserving diffeomorphisms.
- Sketch the main ideas of the proof, and so explain:
  - why symplectic is more difficult than volume-preserving;
  - the probabilistic method for constructing the perturbations.
Lyapunov exponents, Oseledets splitting

$f : M \rightarrow M$ diffeomorphism of a compact manifold of dimension $d$.

By the **Oseledets theorem**, there exists a full probability set $R \subset M$ such that for every (*regular point*) $x \in R$ there is a (**Oseledets**) splitting

$$T_x M = E^1(x) \oplus \cdots \oplus E^{k(x)}(x), \quad \text{(each } \neq \{0\})$$

and numbers (**Lyapunov exponents**) $\Theta_1(x) > \cdots > \Theta_{k(x)}(x)$ such that

$$\frac{1}{n} \log \|Df^n(x) \cdot v\| \xrightarrow{n \to \pm \infty} \Theta_i(x) \quad \forall v \in E^i(x) \setminus \{0\}.$$
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$$\frac{1}{n} \log \|Df^n(x) \cdot v\| \xrightarrow[n \to \pm \infty]{} \Theta_i(x) \quad \forall v \in E^i(x) \setminus \{0\}.$$

The **zipped Oseledets splitting** is obtained by summing together all spaces with exponents of the same sign:

$$T_x M = E^+(x) \oplus E^0(x) \oplus E^-(x).$$

($E^*(x) = \{0\}$ now allowed, of course.)
Lyapunov exponents, Oseledets splitting (continued)

The multiplicity of each Lyapunov exponent $\Theta_j(x)$ is $\dim E^j(x)$ (by definition).

Indicate the Lyapunov exponents repeated according to multiplicity by:

$$\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_d(x), \quad (d = \dim M).$$

If $f$ preserves a symplectic form $\omega$ on $M$ then ($d$ is even and) the exponents are symmetric:

$$\lambda_1 = -\lambda_d, \quad \lambda_2 = -\lambda_{d-1}, \quad \ldots, \quad \lambda_{\frac{d}{2}} = -\lambda_{\frac{d}{2}+1}.$$  

In particular, if $T_x M = E^+(x) \oplus E^0(x) \oplus E^-(x)$ is the zipped Oseledets splitting then

$$\dim E^+(x) = \dim E^-(x), \quad \dim E^0(x) = \text{even}.$$
A quote

Mañé ICM lecture (1983):

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Invariant splittings with uniform properties

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A splitting $T_{\Lambda}M = E^u \oplus E^s$ is called \textit{uniformly hyperbolic} if $E^s$ is uniformly contracted and $E^u$ is uniformly expanded (i.e. contracted in the past): there are constants $C > 0$, $\tau > 1$ such that

\[
\forall n \geq 0, \begin{cases}
\| Df^n(x) \cdot v^s \| \leq C \tau^{-n} \| v^s \| & \forall v^s \in E^s(x) \setminus \{0\} \\
\| Df^{-n}(x) \cdot v^u \| \leq C \tau^{-n} \| v^u \| & \forall v^u \in E^u(x) \setminus \{0\}
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\end{cases}
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2. A splitting $T_{\Lambda}M = E^u \oplus E^c \oplus E^s$ is called **partially hyperbolic** if:
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   - $E^u$ dominates $E^c$, and $E^c$ dominates $E^s$. 
Invariant splittings with uniform properties

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A splitting $T_{\Lambda}M = E^u \oplus E^s$ is called *uniformly hyperbolic* if $E^s$ is uniformly contracted and $E^u$ is uniformly expanded (i.e. contracted in the past): there are constants $C > 0$, $\tau > 1$ such that

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Remark: These bundles are automatically uniformly continuous, and thus extend (with the same hyperbolicity properties) to the closure $\Psi$. 

Jairo Bochi (PUC-Rio)  
Generic Symplectic Diffeomorphisms  
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Domination: the weakest uniform form of hyperbolicity

If $E$, $F$ are $Df$-invariant subbundles of $T_{\Lambda}M$ then we say that $E$ dominates $F$ (in symbols, $E > F$), if there are constants $c > 0$, $\tau > 1$ such that for all unit vectors $\vec{e} \in E(x)$, $\vec{f} \in F(x)$ and all $n \geq 0$

$$\frac{\|Df^n(x) \cdot \vec{e}\|}{\|Df^n(x) \cdot \vec{f}\|} > c \tau^n.$$
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\]

**Dominated splitting:** $T_{\Lambda}M = E^1 \oplus \cdots \oplus E^k$ with $E_1 > E_2 > \cdots > E_k$.

“Morse–Smale-like” dynamics on projective space:

Domination is a.k.a. (in ODE theory) as exponential separation. It dates back to Perron (rediscovered by Mañé.)
Mañé’s statement

\(\text{Diff}^1_\omega(M)\) is the set of symplectic \(C^1\)-diffeomorphisms of \((M, \omega)\), endowed with the \(C^1\) topology. We consider on \(M\) the volume measure \(\mu\) induced by \(\omega\).
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**Theorem (B. ’10)**

For every $f$ in a residual (dense $G_\delta$) subset $\mathcal{R}$ of $\text{Diff}_\omega^1(M)$, the following properties hold:

1. either trivial $T_{\Lambda}^1_M = E_0$, i.e., all Lyapunov exponents at $x$ are zero;
2. or uniformly hyperbolic with $E_u = E^+$, $E_s = E^-$;
3. or partially hyperbolic with $E_u = E^+$, $E_c = E_0$, $E_s = E^-$ (all $\neq \{0\}$).

Moreover, the 2nd alternative occurs for a positive $\mu$-measure set of points $x \in M$ if and only if $f \in \mathcal{R}$ is Anosov.

Rem.: A weaker version (with no PH) was proved earlier in [B., Viana ’05].
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**Rem.**: A weaker version (with no PH) was proved earlier in [B., Viana ’05].
If $\dim M = 2$ then the 3rd alternative in the theorem (partial hyperbolicity with 3 bundles) is impossible, so we get:

**Corollary (B. ’02)**

$C^1$-generic area-preserving diffeomorphisms are either Anosov or have zero Lyapunov exponents almost everywhere.

**Rem.:** The proof of this result appeared much before, and relied heavily on Mañé’s ideas.
Discussion: dim $M > 2$

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For \( \dim M > 2 \), the picture is not necessarily so nice... 

One can break \( M \) (minus a zero set) invariantly:

\[
M = Z \sqcup \bigsqcup \Lambda_n \mod 0 \text{ where } \begin{cases}
Z = \{ \text{all } \lambda_i = 0 \}, \\
\Lambda_n = \text{partially hyperbolic sets}
\end{cases}
\]

- Each \( \Lambda_n \) (or its closure \( \Psi_n \)) has of course its hyperbolicity constants \( c_n, \tau_n \).
- However, these constants become weaker and weaker as \( n \) grows.
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- However, these constants become weaker and weaker as \( n \) grows.

Please note that this is much stronger than what’s is given by Oseledets theorem, which gives no uniformity along the orbits.

Also note that if \( f \in \mathcal{R} \) is ergodic then the situation is much simpler...
The case of globally partially hyperbolic maps

Let $\text{PH}_\omega^1 (M)$ be the subset of $\text{Diff}_\omega^1 (M)$ formed by the diffeos that have a partially hyperbolic splitting over the whole tangent bundle.
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**Theorem (B. ’10)**

For the generic $f \in \text{PH}^1_\omega(M)$, all Lyapunov exponents in the center bundle are zero almost everywhere.

**Rem:** It may be necessary to pass to a different global partially hyperbolic splitting.
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**Proof of the theorem:** Just combine the previous theorem with this:

**Theorem (Corollary of Dolgopyat–Wilkinson)**

*Generic $f \in \text{PH}_\omega^1(M)$ are (accessible and) weakly ergodic (i.e., almost every point has a dense orbit).*
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**Nice thing:** The zero exponents in the center give a *nonuniform version of Burns–Wilkinson’s center bunching*. 
Ergodicity

Indeed many of Burns–Wilkinson’s arguments work with *nonuniform center bunching*.
Putting these together with other [non obvious!] arguments, we get:

**Theorem (Avila, B., Wilkinson ’09)**

The generic $f \in \text{PH}^1_\omega(M)$ is ergodic.

*(Curiosity: This paper was published before its ancestors [BW’10] and [B’10].)*

This gives a $C^1$-generic, symplectic version of the Pugh–Shub ergodicity conjecture.
Proofs?

Now let’s give an idea of the proof of the theorem stated by Mañé:

**Theorem**

For every \( f \) in a residual (dense \( G_\delta \)) subset \( \mathcal{R} \) of \( \text{Diff}_\omega^1(M) \), the following properties hold: For \( \mu \)-a.e. \( x \in M \), the zipped Oseledets splitting \( T_\Lambda M = E^+ \oplus E^0 \oplus E^- \) on \( \Lambda = \text{orb}(x) = \{f^n(x); n \in \mathbb{Z}\} \) is:

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Moreover, the 2nd alternative occurs for a positive $\mu$-measure set of points $x \in M$ if and only if $f \in \mathcal{R}$ is Anosov.

For the “moreover” part, we show that for $C^2$ (and hence $C^1$-generic) diffeos, hyperbolic sets have either zero or full measure. [B., Viana ’04].
Back to basics: domination, and the symplectic case

In fact, to prove the result above, one “only” needs to show the following:

**Theorem (Main Theorem)**

*If f is generic in $\text{Diff}^1_\omega(M)$ then for almost every $x \in M$, the Oseledets splitting along the orbit of $x$ is either trivial or dominated.*

A $Df$-invariant splitting $T_\Lambda M = E^1 \oplus \cdots \oplus E^k$ is called

- **trivial** if $k = 1$;
- **dominated** if each $E_i$ dominates $E_{i+1}$.
In fact, to prove the result above, one “only” needs to show the following:

**Theorem (Main Theorem)**

If $f$ is generic in $\text{Diff}^1_\omega(M)$ then for almost every $x \in M$, the Oseledets splitting along the orbit of $x$ is either trivial or dominated.

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- _trivial_ if $k = 1$;
- _dominated_ if each $E_i$ dominates $E_{i+1}$.

Then, to conclude the proof of Mañé’s statement, one has to use that for symplectic maps, “domination implies partially hyperbolicity”. [B., Viana ’04].

**Rem.**: Actually we obtain more information than in Mañé’s statement, since also get domination between different exponents of the same sign.
Comparison with the volume-preserving case

The “Main Theorem” just stated is also true replacing “symplectic” by “volume-preserving”:

**Theorem (B. Viana ’05)**

If \( f \) is generic in \( \text{Diff}^1_{\text{vol}}(M) \) then for almost every \( x \in M \), the Oseledets splitting along the orbit of \( x \) is either trivial or dominated.
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The proof of the two results have many things in common; however the symplectic result is more difficult, and we will see...
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**Rem.:** There are some recent “global” improvements of this result: Avila–B. (ArXiv 2010), Jana Rodriguez–Hertz (in preparation), but that’s another story...
Another reduction

Integrated summed Lyapunov exponent:

\[ L_p(f) = \int_M (\lambda_1 + \cdots + \lambda_p) \, d\mu \]
\[ = \int \left( \lim_{n \to \infty} \log \| \wedge^p (Df^n) \| \right) \, d\mu. \]

**Easy fact:** \( L_p : \text{Diff}^1_\omega(M) \to \mathbb{R} \) is upper-semicontinuous, and thus continuous on a residual subset.
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Let \( N = d/2 \).

**Theorem**

*If \( f \) is a point of continuity of \( L_1, \ldots \) and \( L_N \) then the Oseledets splitting is trivial or dominated along almost every orbit.*
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**Strategy of the proof:** Suppose that one can detect non-domination of the Oseledets splitting on a positive measure set. Then produce a perturbation of \( f \) for which some \( L_p \) drops.
Setup for the proof

Fix \( f, p \in \{1, \ldots, N = d/2\} \). Assume that the following set has positive measure:

\[
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Setup for the proof

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Another “zipped” Oseledets splitting:

\[
T_{\Sigma_p} M = E^u \oplus E^c \oplus E^s \quad \text{where} \quad \begin{cases} 
\dim E^u = \dim E^s = p \\
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\end{cases}
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Denote \( E^{uc} = E^u \oplus E^c \) etc. Then

\[
\omega(E^u, E^{uc}) \equiv \omega(E^c, E^{us}) \equiv \omega(E^s, E^{cs}) \equiv 0.
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$$\omega(E^u, E^{uc}) \equiv \omega(E^c, E^{us}) \equiv \omega(E^s, E^{cs}) \equiv 0.$$  

**Strategy:** Assume $E^u \nRightarrow E^{cs}$ and perturb $f$ so that $L_p = \int (\lambda_1 + \cdots + \lambda_p)$ drops.
Main steps (sketchy)

- If a point in $\Sigma_p$ "sees" non-domination $E^u \not\succ E^{cs}$ (for example, if $\prec(E^u, E^{cs})$ is small) then we can find a perturbation $g$ of $f$ that sends a vector from $E^u_f$ to $E^{cs}_f$. 

$\hat{x}$ Something $= \frac{1}{2} (\lambda^p(f, x) + \lambda^{p+1}(f, x))$ (which is POSITIVE). $\hat{x}$ In fact we should obtain the inequality not only for $x$ (a zero measure set is useless), but for most $z$ around $x$ in the support of the perturbation.
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- Do that in the *middle* of a long segment of orbit $\{x, \ldots, f^n x\}$. Then one gets

$$\frac{1}{n} \log \| \wedge^p Dg^n(x) \| < \lambda_1(f, x) + \cdots + \lambda_p(f, x) - \text{Something}.$$
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Main steps (continued)

Example: \( \dim M = 2 \)

\[
Df^n(x) = \left( \begin{array}{cc} 2 & 0 \\ 0 & 1/2 \end{array} \right)^k \text{Id}^m \left( \begin{array}{cc} 2 & 0 \\ 0 & 1/2 \end{array} \right)^k, \quad k \approx n/2 \gg m \gg 1.
\]

[draw a figure with the solution]
Main steps (continued)

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[draw a figure with the solution]

Back to the steps of the general proof:

- Around a segment of orbit $\{x, \ldots, f^n x\}$ that sees nondomination we find a thin and long tower $U \sqcup f(U) \sqcup \cdots \sqcup f^n(U)$, and find a perturbation supported in the tower so that the “finite-time” summed exponent drops (as explained above).
- Cover the a (“large”) positive measure set of the manifold with these towers.
- This causes a significant drop of $L_p = \int (\lambda_1 + \cdots + \lambda_p)$, as we wanted.
The 4 types of non-dominance

Assume \( x \in \Sigma \) and the segment of orbit \( \{x, \ldots, f^m x\} \) is very long \((m \gg 1)\) and “sees” non-domination \( E^u \not\succ E^{cs} \); more precisely:

\[
\frac{\|Df^m(x)|E^{cs}(x)\|}{m(Df^m(x)|E^u(x))} \geq \frac{1}{2}.
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Lemma (of Symplectic Linear Algebra): one of the following 4 cases occurs:
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**Case I: small angle.** There is a point \( y \in \{ x, \ldots, f^m x \} \) such that

\( \angle(E^{cs}(y), E^u(y)) \ll 1. \)

**How to perturb it:** Compose with a single small rotation.

**Case II: inverted behaviors.** There are unit vectors \( v^{cs} \in E^{cs}(x), v^u \in E^u(x) \) such that

\[ \| Df^m(x) \cdot v^{cs} \| \gg \| Df^m(x) \cdot v^u \|. \]

(Replace if necessary the whole segment of orbit by a subsegment.)

**How to perturb it:** Use two small rotations, one at the beginning and the other at the end.
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Case III: Identity on a symplectic $(\omega \neq 0)$ plane.
There is a long subsegment of orbit such the following holds: There is a (2-dim) plane spanned by a vector in $E^u$ and a vector in $E^s$ such that the restriction of $Df$ to this plane “looks like the identity” (or more precisely, becomes an isometry after a bounded change of coordinates).

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Notice that $\omega \neq 0$ in $P$.

How to perturb it: This case is essentially 2-dimensional. Use several “nested” rotations, as explained in a previous example.
The 4 types of non-dominance (continued)

Case IV: Expansion on a null ($\omega \equiv 0$) plane.
There is a (2-dim) plane $P$ spanned by a vector in $E^u$ and a vector in $E^c$ which is uniformly expanded and conformal (along the segment of orbit). That is, after a bounded change of coordinates we have

$$Df(f^i x)|P = \begin{pmatrix} \tau_i & 0 \\ 0 & \tau_i \end{pmatrix}, \quad \tau_i > c > 1.$$  

The plane is necessarily null ($\omega \equiv 0$).

How to perturb it?
The 4 types of non-dominance (continued)

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How to perturb it?

The 1st idea would be to imitate the previous case: rotate the plane \(P\) around a complementary codimension-2 axis (where the rotation is the identity).
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The 1st idea would be to imitate the previous case: rotate the plane \(P\) around a complementary codimension-2 axis (where the rotation is the identity).

However, this perturbation is not symplectic!

(In the volume-preserving case this idea would work; there are only 3 cases to be considered there.)
4-dimensional problem

There are standard symplectic coordinates \( p_1, \ldots, p_N, q_1, \ldots, q_N \) (so \( \omega = \sum_i dp_i \wedge dq_i \)) such that

\[
P = \left\langle \frac{\partial}{\partial p_1}, \frac{\partial}{\partial p_2} \right\rangle \quad \in E^u \\
\quad \in E^c
\]

\[
Q = \left\langle \frac{\partial}{\partial q_2}, \frac{\partial}{\partial p_1} \right\rangle \quad \in E^c \\
\quad \in E^s
\]

\[
Df|P \oplus Q = \begin{pmatrix}
\tau_i & \tau_i \\
\tau_i^{-1} & \tau_i^{-1}
\end{pmatrix}
\]

(order: \( p_1, p_2, q_2, q_1 \)).

If we rotate \( P \) we also need to rotate \( Q \).
4-dimensional problem

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If we rotate $P$ we also need to rotate $Q$. **Nested rotations don’t work!** The problem is that hyperbolicity of $Df$ quickly distorts the domain where the perturbation should be supported.
Solution of Case IV

- Start with a box $D$ as a perturbation domain. (We can pretend $M = \mathbb{R}^4$.)

Choose any symplectic perturbation of the identity $h : D \rightarrow D$ that doesn't leave the field of directions $\frac{\partial}{\partial p_1}$ invariant. Let $g = f \circ h$ be the perturbation of $f$.

Let $\Theta_0$ be the angle (in the $p_1 p_2$ projection) that the field of directions $v_0 = \frac{\partial}{\partial p_1}$ is rotated as we apply $h$. View $\Theta_0$ as a random variable. (Normalize measure $\mu(D) = 1$).

Look the image $g(D)$ and the image $v_1$ of the field $v_0 = \frac{\partial}{\partial p_1}$ by $Dg$. Using Vitali Lemma, cover most of $g(D)$ by many disjoint tiny boxes $D_i$ that look like (basically after a change of scale) the original box.
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Solution of Case IV (continued)

- Inside each small box $D_i$, we make a perturbation which is just a rescaling of $h$. 
Solution of Case IV (continued)

• Inside each small box $D_i$, we make a perturbation which is just a rescaling of $h$.

• Let $\Theta_1$ be the angle (in the $p_1 p_2$ projection) that $v_1$ is rotated at this step. We view $\Theta_1$ as a random variable ($\mu(g(D)) = 1$).
Solution of Case IV (continued)

- Inside each small box $D_i$, we make a perturbation which is just a rescaling of $h$.
- Let $\Theta_1$ be the angle (in the $p_1 p_2$ projection) that $v_1$ is rotated at this step. We view $\Theta_1$ as a random variable ($\mu(g(D)) = 1$).
- Then **THE RANDOM VARIABLES $\Theta_0$ AND $\Theta_1$ ARE (approximately) INDEPENDENT AND IDENTICALLY DISTRIBUTED!**
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Then THE RANDOM VARIABLES $\Theta_0$ AND $\Theta_1$ ARE (approximately) INDEPENDENT AND IDENTICALLY DISTRIBUTED!

Continuing this process we obtain a RANDOM WALK $S_n = \Theta_0 + \Theta_1 + \cdots + \Theta_n$.

Since every (non-stopped) random walk is transient, most orbits eventually reach $\pm \pi/2$.

Thus we succeeded in sending the direction $\frac{\partial}{\partial p_1} \in E^u$ to the direction $\frac{\partial}{\partial q_1} \in E^c$ (for most points).
The end

THANK YOU!