FRAGILE CYCLES

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ABSTRACT. We study diffeomorphisms f with heterodimensional cycles, that is, heteroclinic cycles associated to saddles p and q with different indices. Such a cycle is called fragile if there is no diffeomorphism close to f with a robust cycle associated to hyperbolic sets containing the continuations of p and q. We construct a codimension one submanifold of $\mathrm{Diff}^r(\mathbb{S}^2\times\mathbb{S}^1)$, $r\geq 1$, that consists of diffeomorphisms with fragile heterodimensional cycles. Our construction holds for any manifold of dimension ≥ 4.

1. Introduction

In the late sixties, Newhouse constructed the first examples of C^2-open sets of non-hyperbolic surface diffeomorphisms. Any such set $\mathcal{U}$ consists of diffeomorphisms with *C^2-robust homoclinic tangencies*: every diffeomorphism $f\in\mathcal{U}$ has a hyperbolic set K_f (depending continuously on f) whose stable and unstable manifolds have non-transverse intersections, see [17].

Later, in [18], Newhouse proved that homoclinic tangencies of surface diffeomorphisms can be *stabilized*: given a diffeomorphism f with a homoclinic tangency associated to a saddle p_f, there is a C^2-open set whose closure contains f and which consists of diffeomorphisms g with robust homoclinic tangencies associated to hyperbolic sets K_g containing the continuation p_g of p_f. In particular, these results show that homoclinic tangencies always generate C^2-robust homoclinic tangencies. In fact, robust homoclinic tangencies are present in all known examples of open sets of non-hyperbolic surface diffeomorphisms. Let us observe that homoclinic tangencies of C^1-diffeomorphisms defined on surfaces cannot be stabilized, see [15].

Similarly, all known examples of C^1-open sets formed by non-hyperbolic diffeomorphisms exhibit C^1-robust *heterodimensional cycles,* that is, cycles relating the invariant manifolds of two hyperbolic sets of different s-indices (dimension of the stable bundle). Note that the existence of such cycles can only occur in dimension ≥ 3.

We wonder if, as in the case of homoclinic tangencies of C^2-diffeomorphisms, heterodimensional cycles can be made C^1-robust and can be C^1-stabilized. A first partial answer to this question is given in [8]: heterodimensional cycles associated to periodic saddles whose indices differ by one generate (by arbitrarily small C^1-perturbations) C^1-robust heterodimensional cycles. In some extend, the results in [8] are a version of the ones by Newhouse in [17, 18] in the context of C^1-heterodimensional cycles.

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However, compared with Newhouse's results for homoclinic tangencies, the ones in [8] have an important disadvantage: While the hyperbolic sets with the robust homoclinic tangencies in [17, 18] contain continuation of the saddle with the initial tangency, the hyperbolic sets involved in robust cycles in [8] do in general not contain the continuations of the saddles in the initial cycle. However this precise question can be important for understanding the global dynamics of non-hyperbolic diffeomorphisms. Let us discuss more this question in more detail.

Following Conley theory [11] and motivated by spectral decomposition theorems [16, 1], this global dynamics is structured using *homoclinic* or/and *chain recurrence classes* as "elementary" pieces of dynamics, see the definitions below. One aims to describe the dynamics of each piece and the relations between different pieces (cycles), for further details see [10, Chapter 10.3-4] and [5].

In general, the homoclinic class of a hyperbolic periodic point is contained in its chain recurrence class. An important property is that for C^1-generic diffeomorphisms homoclinic classes and chain recurrence classes of periodic points coincide, [6, Remarque 1.10]. However, in non-generic situations, two different homoclinic classes (even of saddles of different indices) may be joined by a cycle, hence they are contained in the same chain recurrence class. A question is when one can join them in a C^1-robust way by small perturbations. This occurs if the cycle can be stabilized. For instance, this is specially important for understanding the indices of the periodic points in an elementary piece of dynamics.

While the above explains why the stabilization of cycles is relevant, let us now provide the precise definitions of the concepts involved. First, recall that a hyperbolic basic set K of a diffeomorphism f has a (uniquely defined) *continuation* K_g for all g close to f: K_g is a hyperbolic basic set, close to K, and the dynamics of $f|_K$ and $g|_{K_g}$ are conjugate. The *s-index* of a hyperbolic transitive set is the dimension of its stable bundle.

Definition 1.1 (Robust continuations of cycles)**.**

• The diffeomorphism f is said to have a *heterodimensional cycle* associated to hyperbolic basic sets K and L if these sets have different s-indices and their stable and unstable manifolds meet cyclically:

$$W^s(K,f)\cap W^u(L,f)\neq\emptyset \quad \text{and} \quad W^u(K,f)\cap W^s(L,f)\neq\emptyset.$$

• The cycle associated to K and L is C^1-*robust* if there is a C^1-neighborhood $\mathcal{U}$ of f such that for all $g\in\mathcal{U}$ the hyperbolic continuations K_g and L_g of K and L have a heterodimensional cycle.

• A heterodimensional cycle associated to a pair of saddles p and q of f *can be* C^1-*stabilized* if every C^1-neighborhood $\mathcal{U}$ of f contains a diffeomorphism g with hyperbolic basic sets $K_g\ni p_g$ and $L_g\ni q_g$ having a robust heterodimensional cycle. Here p_g and q_g are the continuations of p and q for g.

Definition 1.2 (Fragile cycle)**.** A heterodimensional cycle associated to a pair of saddles is C^1-*fragile* if it cannot be C^1-stabilized.

The previous discussion leads to the following question that we address in this paper: *Can every heterodimensional cycle be* C^1-*stabilized?*

The results in [9] give a positive answer to this question for "most" types of *coindex one* heterodimensional cycles, that is, related to saddles whose s-indices differ by one. Indeed in [9] it is proved that fragile coindex one cycles associated to saddles p and q exhibit a quite specific geometry:

- The homoclinic classes of p and q are both trivial.
- The *central eigenvalues* of p and q are all real and positive.[1]
- There is a well-defined one-dimensional orientable central bundle E^c along the cycle (i.e. defined on some closed set containing the saddles p and q in the cycle and a pair of heteroclinic orbits $x \in W^s(p) \cap W^u(q)$ and $y \in W^u(p) \cap W^s(q)$), but the cycle diffeomorphism does not preserve the orientation of E^c. The cycle is *twisted* by the terminology in [4].

In this paper we provide examples of fragile coindex one cycles, see Theorem 1.

1.1. **Definitions and statement of results.** Recall that the *homoclinic class* of a hyperbolic periodic point p, denoted by $H(p, f)$, is the closure of the transverse intersections of the stable and unstable manifolds of the orbit of p. The homoclinic class $H(p, f)$ coincides with the closure of the set of all saddles q *homoclinically related with p,* i.e. the stable manifold of the orbit of q transversely meets the unstable manifold of the orbit of p and vice-versa. A homoclinic class is *non-trivial* if it contains at least two different orbits.

Let us now recall the definition of a *chain recurrence class.* A finite sequence of points $(x_i)_{i=0}^n$ is an *ϵ-pseudo-orbit* of a diffeomorphism f if $\mathrm{dist}(f(x_i), x_{i+1}) < \epsilon$ for all $i = 0, \dots, n-1$. A point x is *chain recurrent* for f if for every $\epsilon > 0$ there is an ϵ-pseudo-orbit $(x_i)_{i=0}^n$, $n \geq 1$, starting and ending at x (i.e. $x = x_0 = x_n$). The *chain recurrent set* $R(f)$ of f is the set of all chain recurrent points. This set splits into disjoint *chain recurrence classes*: the class $C(x, f)$ of $x \in R(f)$ is the set of points y such that for every $\epsilon > 0$ there are ϵ-pseudo-orbits joining x to y and y to x. A periodic point p of f is *isolated* if its *chain recurrence class* coincides with its orbits. In this case, the orbit of p is the maximal invariant set in some *filtrating neighborhood*. This implies that the homoclinic class of p is C^1-robustly trivial (i.e. the homoclinic class of p_g is trivial for every g close to f).

We are now ready to state our main result.

Theorem 1. *There is an open set $\mathcal{U}$ of* $\mathrm{Diff}^1(\mathbb{S}^2 \times \mathbb{S}^1)$ *and a codimension one submanifold Σ contained in $\mathcal{U}$ with the following property: For every $f \in \mathcal{U}$ there are hyperbolic saddles p_f and q_f with different s-indices depending continuously on f such that*

(1) every $f \in \Sigma$ has a heterodimensional cycle associated to p_f and q_f,
(2) the set $\mathcal{U} \setminus \Sigma$ is the union of two connected sets $\mathcal{U}^+$ and $\mathcal{U}^-$ such that
  - *for every $f \in \mathcal{U}^+$ the saddle p_f is isolated,*
  - *for every $f \in \mathcal{U}^-$ the saddle q_f is isolated.*

Note that if two hyperbolic basic sets K_f and L_f have a heterodimensional cycle then the chain recurrence classes of any pair of saddles $a_f \in K_f$ and $b_f \in L_f$ coincide. In particular, if the cycle associated to K_f and L_f is robust then the chain recurrence classes of a_g and b_g are the same for all g in some neighborhood of f. In particular, the chain recurrence class $C(a_g, g) = C(b_g, g)$ is non-trivial and the saddles a_g and b_g are both non-isolated for g. Thus Theorem 1 implies that the cycles in Σ cannot be made robust. This implies the following:

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[1]The definition of central eigenvalues is a little intricate. Assuming that the s-index of p is bigger than the one of q, the central eigenvalues correspond to the weakest contracting direction of p and the weakest expanding direction of q.

Corollary 1. *The submanifold Σ in Theorem 1 consists of diffeomorphisms having C^1-fragile heterodimensional cycles.*

As for any $n > 3$ the set $\mathbb{S}^2 \times \mathbb{S}^1$ can be embedded as a normally contracting manifold in a ball $\mathbb{B}^n$, we obtain the following.

Corollary 2. *Any compact manifold M with $\dim M > 3$ supports diffeomorphisms with C^1-fragile cycles.*

As our examples demand a somewhat specific topological configuration, the following question arises naturally.

Question 1. *Does every 3-manifold admit diffeomorphisms with C^1-fragile heterodimensional cycles?*

The examples presented in this paper display many interesting and somehow unexpected properties. There are also many important aspects of their dynamics yet unexplored. Thus, after completing our construction, in Section 6 we conclude with a discussion about the properties of our examples.

This paper is organized as follows. In the first step of our construction, in Section 2, we build an auxiliary Morse-Smale vector field X on the 3-sphere $\mathbb{S}^3$. In Section 3, we consider a surgery in $\mathbb{S}^3$ (associated to some identifications by a local diffeomorphism Ψ of $\mathbb{S}^3$). This surgery provides a diffeomorphism F_Ψ defined on $\mathbb{S}^2 \times \mathbb{S}^1$ induced by the time-one map $F_0 = X_1$ of the vector field X and the gluing map Ψ. We also see how the dynamics of F_Ψ depends on the gluing map Ψ. In Section 4, we study the dynamics of diffeomorphisms close to F_Ψ. Finally, in Section 5, we choose the gluing map Ψ to get a diffeomorphism F_Ψ with a fragile cycle and construct the submanifold Σ consisting of diffeomorphisms with fragile cycles. The paper is closed with a discussion section.

2. An auxiliary vector field on $\mathbb{S}^3$

In this section we construct a Morse-Smale vector field defined on the three sphere $\mathbb{S}^3$ whose non-wandering set consists of singular points. This vector field also satisfies some normally hyperbolic properties. We now go to the details of this construction.

We consider the sphere $\mathbb{S}^3$ as the union of two solid tori $\mathcal{T}_1$ and $\mathcal{T}_2$ with the same boundary $\partial\mathcal{T}_1 = \partial\mathcal{T}_2 = \mathbb{T}^2$. A simple closed curve of $\mathbb{T}^2$ is a $\mathcal{T}_i$*-meridian* if it is not 0-homotopic in $\mathbb{T}^2$ but is 0-homotopic in $\mathcal{T}_i$. We consider an identification of the boundaries of these solid tori that does not preserve the meridians: the $\mathcal{T}_2$-meridians are isotopic in $\mathcal{T}_1$ to the "central circle" of $\mathcal{T}_1$ and are classically called $\mathcal{T}_1$*-parallels*. Similarly, $\mathcal{T}_1$-meridians are $\mathcal{T}_2$-parallels.

2.1. An auxiliary Morse-Smale vector field X in $\mathbb{S}^3$.

Consider a Morse-Smale vector field X defined on $\mathbb{S}^3$ such that (see Figure 2.1):

(1) X is transverse to $\partial\mathcal{T}_1 = \partial\mathcal{T}_2 = \mathbb{T}^2$.
(2) The solid torus $\mathcal{T}_1$ is attracting and the solid torus $\mathcal{T}_2$ is repelling: The positive orbit of any point $x \in \mathbb{T}^2$ enters (and remains) in the interior of $\mathcal{T}_1$ and its negative orbits enters (and remains) in the interior of $\mathcal{T}_2$.
(3) The maximal invariant set of X in $\mathcal{T}_1$ is a normally hyperbolic (contracting) circle σ^s. Analogously, the maximal invariant set of X in $\mathcal{T}_2$ is a normally hyperbolic (repelling) circle σ^u.

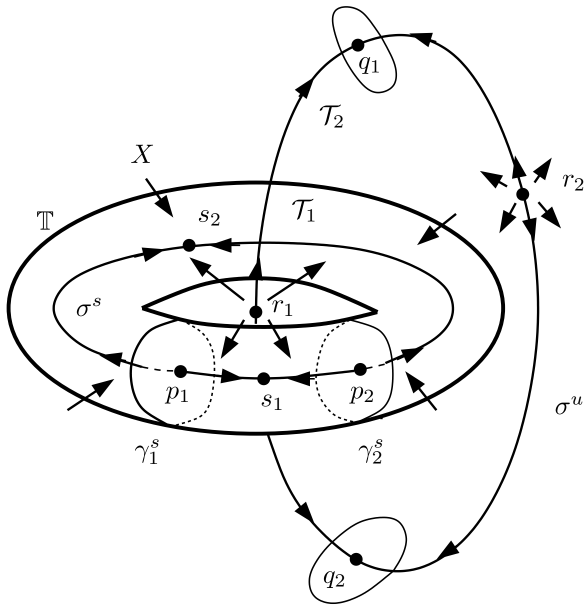

FIGURE 2.1. The Morse-Smale vector field X defined on $\mathbb{S}^3$.

(4) The limit set of X is $\{s_1, s_2, r_1, r_2, p_1, p_2, q_1, q_2\}$, where
- $s_1, s_2 \in \sigma^s$ are attracting singularities,
- $r_1, r_2 \in \sigma^u$ are repelling singularities,
- $p_1, p_2 \in \sigma^s$ are saddle singularities of s-index 2, and
- $q_1, q_2 \in \sigma^u$ are saddle singularities of s-index 1.

(5) The two (one-dimensional) separatrices of the unstable manifold of the singularity p_i, $i = 1, 2$, are contained in $W^s(s_1)$ and in $W^u(s_2)$. A similar assertion holds for the separatrices of the stable manifold of q_i, $i = 1, 2$, that are contained in $W^u(r_1)$ and $W^u(r_2)$.

(6) The local stable manifold of p_i, $i = 1, 2$, is a 2-disk contained in $\mathcal{T}_1$ whose boundary
$$\partial W^s_{loc}(p_i) = W^s_{loc}(p_i) \cap \partial \mathcal{T}_1 \stackrel{\text{def}}{=} \gamma_i^s$$
is a $\mathcal{T}_1$-meridian. Similarly, the local unstable manifold of q_i, $i = 1, 2$, is a 2-disk contained in $\mathcal{T}_2$ whose boundary
$$\partial W^u_{loc}(q_i) = W^u_{loc}(q_i) \cap \partial \mathcal{T}_2 \stackrel{\text{def}}{=} \gamma_i^u,$$
is a $\mathcal{T}_2$-meridian.

(7) For every $i, j \in \{1, 2\}$, the curve γ_i^s is transverse to γ_j^u and the intersection $\gamma_i^s \cap \gamma_j^u$ is exactly one point x_{ij}.

Remark 2.1 (Dynamics of the vector field X)**.**

(1) The boundary $\mathbb{T}^2$ of the solid torus $\mathcal{T}_1$ is the union of two cylinders $\mathbb{C}_1^s$ and $\mathbb{C}_2^s$ with disjoint interiors and the same boundary $\gamma_1^s \cup \gamma_2^s$. The notation is chosen such that $W^s(s_i) \cap \mathbb{T}^2$ is the interior of the cylinder $\mathbb{C}_i^s$.

Similarly, $\mathbb{T}^2 = \partial \mathcal{T}_2$ is the union of the cylinders $\mathbb{C}_1^u$ and $\mathbb{C}_2^u$ bounded by γ_1^u and γ_2^u and whose interiors are the intersections $\mathbb{T}^2 \cap W^u(r_1)$ and $\mathbb{T}^2 \cap W^u(r_2)$, respectively. See Figure 2.2.

(2) As a consequence of item (7) in the definition of the vector field X, the intersection $\overline{\mathbb{C}_1^u} \cap \overline{\mathbb{C}_1^s}$ is a "rectangle" R such that

$$R = \overline{W^s(s_1) \cap W^u(r_1) \cap \mathbb{T}^2}$$

and its boundary ∂R of R is the union of four curves a_1^s, a_2^s, b_1^u, b_2^u with disjoint interiors such that

$$a_1^s \subset \gamma_1^s, \quad a_2^s \subset \gamma_2, \quad b_1^u \subset \gamma_1^u, \quad b_2^u \subset \gamma_2^u. \tag{2.1}$$

See Figure 2.2. Note that the interiors of b_1^u and b_2^u are contained in $W^s(s_1)$ and the interiors of a_1^s and a_2^s are contained in $W^u(r_1)$.

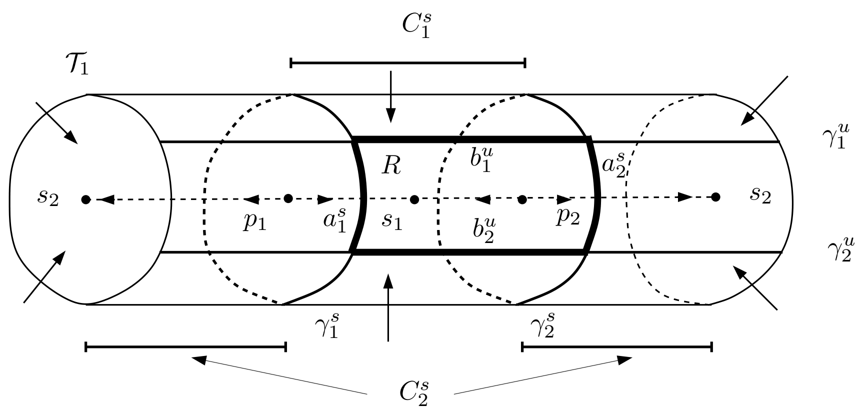

FIGURE 2.2. The rectangle R.

2.2. **Partially hyperbolicity of X.** We also assume that the vector field X satisfies the following partially hyperbolic conditions:

(1) There is a partially hyperbolic splitting of X over the circle σ^s (recall (3) in Section 2.1) of the form

$$T_{\sigma^s}\mathbb{S}^3 = E^{ss} \oplus E^{cs},$$

where E^{cs} is a 2-dimensional central bundle containing the X direction, and E^{ss} is a strong stable bundle that is oriented along the circle σ^s.

A similar condition holds for the circle σ^u: There is a partially hyperbolic splitting of X over σ^u of the form

$$T_{\sigma^u}\mathbb{S}^3 = E^{cu} \oplus E^{uu},$$

where E^{cu} is a 2-dimensional center bundle containing the X-direction and E^{uu} is a strong unstable bundle that is oriented along σ^u.

(2) Consider the two-dimensional strong stable manifold $W^{ss}(\sigma^s)$ of σ^s that is tangent to $X \oplus E^{ss}$ along σ^s. Define the local strong stable manifold of σ^s by $W^{ss}_{\text{loc}}(\sigma^s) = W^{ss}(\sigma^s) \cap \mathcal{T}_1$. Then the intersection between $W^{ss}_{\text{loc}}(\sigma^s)$ and $\mathbb{T}^2$ consists of two disjoint $\mathcal{T}_1$-parallels σ^s_- and σ^s_+. Similarly, the intersection $W^{uu}_{\text{loc}}(\sigma^u) \cap \mathbb{T}^2$ is the disjoint union of two $\mathcal{T}_2$-parallels σ^u_+ and σ^u_-. We require that

$$\mathbb{C}_1^u \cap \left(\sigma^s_- \cup \sigma^s_+\right) = \emptyset \quad \text{and} \quad \mathbb{C}_1^s \cap \left(\sigma^u_- \cup \sigma^u_+\right) = \emptyset.$$

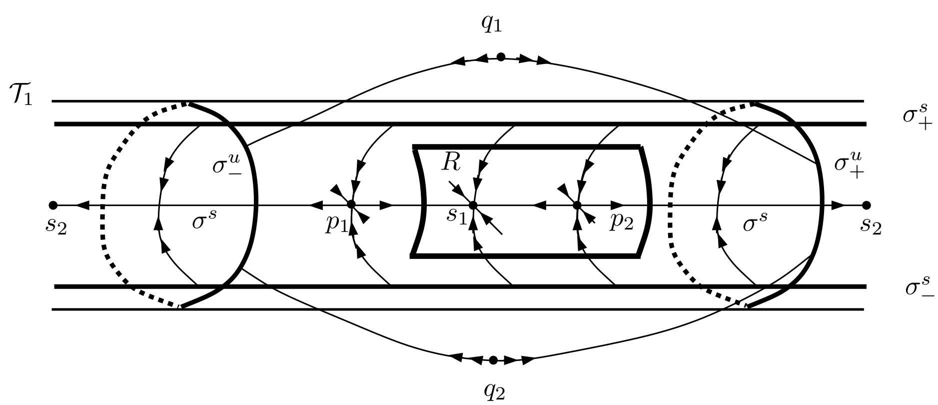

FIGURE 2.3. The rectangle R and the "strong" manifolds.

Let us explain how this property can be obtained. Recall that $W^u(r_1) \cap \mathbb{T}^2$ is the interior of the cylinder $\mathbb{C}_1^u$ bounded by γ_1^u and γ_2^u. Thus, since γ_1^u, γ_2^u, σ^s_-, and σ^s_+ are $\mathcal{T}_1$-parallels (or equivalently $\mathcal{T}_2$-meridians), we can assume that $\mathbb{C}_1^u \cap (\sigma^s_- \cup \sigma^s_+) = \emptyset$. See Figure 2.3. The condition for the cylinder $\mathbb{C}_1^s$ and the circles σ^u_+ and σ^u_- follows identically noting that γ_1^s, γ_2^s, σ^u_-, and σ^u_+ are $\mathcal{T}_2$-parallels.

By the partially hyperbolic conditions, the strong stable manifolds $W^{ss}(p_i)$, $i = 1, 2$, (tangent to E^{ss} at p_i) are well defined and has dimension one. Similarly, the strong unstable manifold $W^{uu}(q_1)$ and $W^{uu}(q_2)$ are well defined and have dimension one. As a consequence of item (2) above (see Figure 2.3) we have the following:

$$a_1^s \cap W^{ss}(p_1) = \emptyset, \quad a_2^s \cap W^{ss}(p_2) = \emptyset,$$
$$b_1^u \cap W^{uu}(q_1) = \emptyset, \quad b_2^u \cap W^{u}(q_2) = \emptyset. \tag{2.2}$$

2.3. **Transverse heteroclinic intersection.** Consider the "corner" points of the rectangle R,

$$a_1^s \cap b_1^u \stackrel{\text{def}}{=} x_{1,1}, \quad a_1^s \cap b_2^u \stackrel{\text{def}}{=} x_{1,2}, \quad a_2^s \cap b_1^u \stackrel{\text{def}}{=} x_{2,1}, \quad a_2^s \cap b_2^u \stackrel{\text{def}}{=} x_{2,2}. \tag{2.3}$$

By definition, the ω and α-limits of the point $x_{i,j}$ are the singularities p_i and q_j, respectively. Denote by $\Gamma_{i,j}$ the closure of the orbit of $x_{i,j}$ (see Figures 2.4 and 2.5). As the intersection between $W^s(p_i)$ and $W^u(q_j)$ is exactly the orbit of $x_{i,j}$, we have

$$\Gamma_{i,j} \stackrel{\text{def}}{=} \overline{W^s(p_i) \cap W^u(q_j)} = \{p_i\} \cup \{q_j\} \cup \big(W^s(p_i) \pitchfork W^u(q_j)\big). \tag{2.4}$$

Remark 2.2. The curve $\Gamma_{i,j}$ is a C^1-invariant normally hyperbolic compact segment.

This remark is a standard consequence of the following facts:

- The point $x_{i,j}$ is a transverse heteroclinic intersection associated to the singularities p_i and q_j.
- The partial hyperbolicity hypothesis at the singularities p_i and q_j implies that $W^{ss}(p_i)$ and $W^{uu}(q_j)$ are well defined.
- By construction, recall equation (2.2), $x_{i,j} \notin W^{ss}(p_i) \cup W^{uu}(q_j)$.
- The curve $\Gamma_{i,j}$ is the closure of the orbit of $x_{i,j}$.

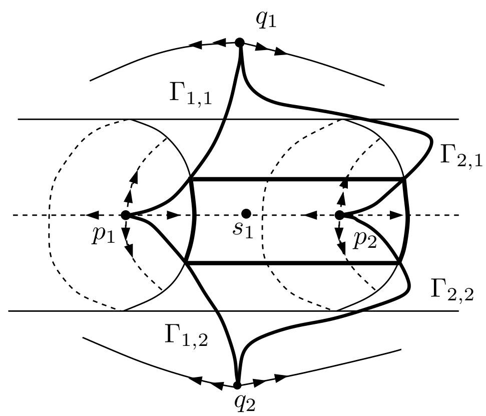

FIGURE 2.4. The curves $\Gamma_{i,j}$. Global Dynamics

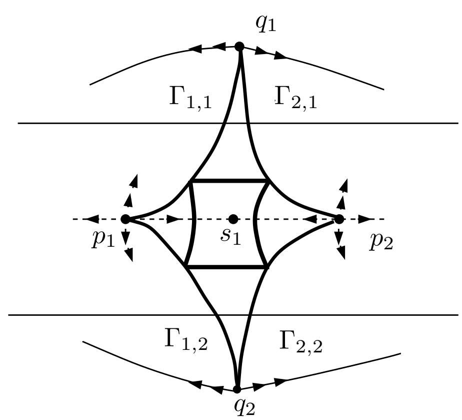

FIGURE 2.5. Outline of the curves $\Gamma_{i,j}$.

2.4. **Invariant manifolds of the segments $\Gamma_{i,j}$.** For each singularity q_j, we have that $\big(W^u(q_j) \setminus W^{uu}(q_j)\big)$ is the disjoint union of two connected invariant surfaces

$$\big(W^u(q_j) \setminus W^{uu}(q_j)\big) \stackrel{\text{def}}{=} W^{u,+}(q_j) \cup W^{u,-}(q_j),$$

where $W^{u,+}(q_j)$ contains the interior of the curves $\Gamma_{1,j}$ and $\Gamma_{2,j}$. With this notation, the invariant manifolds of the curve $\Gamma_{i,j}$ are

$$W^u(\Gamma_{i,j}) = W^u(p_i) \cup W^{u,+}(q_j) \cup W^{uu}(q_j),$$
$$W^s(\Gamma_{i,j}) = W^s(q_j) \cup W^{s,+}(p_i) \cup W^{ss}(p_i).$$

See Figures 2.5 and 2.6. Note that these manifolds are injective C^1-immersions of $[0,1] \times \mathbb{R}$.

Figures 2.4, 2.5, and 2.6 suggest that the curves $\Gamma_{1,j}$ and $\Gamma_{2,j}$ form a "cusp" at the point q_j. This geometric configuration will play a key role in our construction. So let us define precisely what we mean by a *cusp*.

A topological two-disk $\mathbb{G}$ contained in the interior of a smooth surface S *has a cusp* at a point $p \in \partial\mathbb{G}$ if for every $\varepsilon > 0$ there are a convex cone C_ε of angle ε at p and a neighborhood U_ε of p in S such that $\mathbb{G} \cap U_\varepsilon \subset C_\varepsilon$. Given a curve γ contained

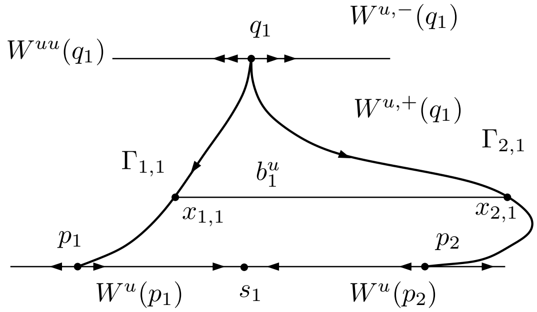

FIGURE 2.6. The unstable manifold of $\Gamma_{i,j}$.

in the interior of the surface S, a point q in the interior of γ is a *cusp* of γ if there is a topological disk $\mathbb{F}$ whose boundary contains γ and has a cusp at q.

Remark 2.3. Recall that the interior of the curves $\Gamma_{1,i}$ and $\Gamma_{2,i}$ are disjoint from $W^{uu}(q_i)$. Moreover, the interior of these curves are the orbits of the points $x_{1,i}$ and $x_{2,i}$, respectively. These two curves are connected by the segment $b_i^u \subset W^u(q_i)$ that is disjoint from $W^{uu}(q_i)$, recall (2.1) and see Figure 2.6. The partial hyperbolicity hypothesis now implies that $\Gamma_{1,i}$ and $\Gamma_{2,i}$ are "central curves" arriving to q_1 from the same side of $W^{uu}(q_i)$. These two conditions imply that

$$\Gamma(q_i) \stackrel{\text{def}}{=} \Gamma_{1,i} \cap \Gamma_{2,i}, \quad i = 1, 2, \tag{2.5}$$

is a curve with a *cusp singularity* at q_1, see Figures 2.4, 2.5, and 2.6.

The unstable manifold $W^u(\Gamma(q_i))$ of $\Gamma(q_i)$ is the set $W^u(\Gamma_{1,i}) \cup W^u(\Gamma_{2,i})$. Noting that the interior of the "strip" $W^u(\Gamma(q_i))$ is $W^{u,+}(q_i)$, $i = 1, 2$, we get

$$W^u(\Gamma(q_i)) = W^u(p_1) \cup W^u(p_2) \cup W^{u,+}(q_i) \cup W^{uu}(q_i). \tag{2.6}$$

We observe that the set $W^u(\Gamma(q_i))$ is an injective C^1-immersion of a connected surface with boundary. Equivalent statements hold for

$$\Gamma(p_i) \stackrel{\text{def}}{=} \Gamma_{i,1} \cup \Gamma_{i,2}, \quad i = 1, 2, \tag{2.7}$$

and its stable manifold

$$W^s(\Gamma(p_i)) = W^s(q_1) \cup W^s(q_2) \cup W^{s,+}(p_i) \cup W^{ss}(p_i), \tag{2.8}$$

where $W^{s,+}(p_i)$ is the component of $\big(W^s(p_i) \setminus W^{ss}(p_i)\big)$ containing the interior of the curves $\Gamma_{i,1}$ and $\Gamma_{i,2}$.

With this notation, the sides a_i^s and b_j^u of the rectangle R (see (2.1)) satisfy the following property

$$a_i^s \subset W^s(\Gamma(p_i)) \cap \mathbb{T}^2 \quad \text{and} \quad b_j^u \subset W^u(\Gamma(q_j)) \cap \mathbb{T}^2, \qquad i, j \in \{1, 2\}. \tag{2.9}$$

2.5. **Central bundles.** In this section, we see that the unstable manifolds of $\Gamma_{i,1}$ and $\Gamma_{i,2}$ touch each other at $W^u(p_i)$ tangentially "coming from the same side" of $W^u(p_i)$, see Figure 2.7. In the following we will precise what this means.

Lemma 2.4 (Center stable/unstable bundles)**.** *Given any singularity p of saddle type with a strong stable direction $W^{ss}(p)$ (tangent to a strong stable bundle $E^{ss}(p)$) there is a unique invariant "central" bundle E^c defined over the unstable manifold $W^u(p)$ of p that is transverse at p to the bundle E^{ss} and has codimension* $\dim(E^{ss})$.

A similar property holds for saddle singularities q with a strong unstable manifold tangent to some strong unstable bundle E^{uu}. In this case, there is a central bundle defined over $W^s(q)$ that is transverse to E^{uu} at q and has codimension $\dim(E^{uu})$.

Proof. We can assume that for every point $x \in W^u_{\mathrm{loc}}(p)$ there is defined a negatively invariant cone-field $\mathcal{C}^{ss}$ around the strong stable direction E^{ss}. Consider the complement $\mathcal{C}^c$ of $\mathcal{C}^{ss}$. Given $y \in W^u(p)$ there is $t(y) > 0$ such that $X_{-t}(y) \in W^u_{\mathrm{loc}}(p)$ for all $t \geq t(y)$. Given $y \in W^u(p)$ it is enough to define

$$E^c(y) \stackrel{\text{def}}{=} \{v : \, D_y X_{-t}(v) \in \mathcal{C}^c(X_{-t}(y)) \quad \text{for all } t \geq t(y)\}.$$

By construction, the bundle $E^c(y)$ is transverse to E^{ss} and its dimension is the codimension of E^{ss}. This completes the proof of the lemma. □

Remark 2.5. Applying Lemma 2.4 to the singularity p_i, we get a two dimensional bundle $E^c(p_i)$, $i = 1, 2$, coinciding with the bundle E^{cs} defined along the curve σ^s in Section 2.2. Analogously, the bundle E^c defined along $W^s(q_j)$ coincides with the bundle E^{cu} along σ^u.

Lemma 2.6. *The surfaces $W^u(\Gamma(q_1))$ and $W^u(\Gamma(q_2))$ are tangent to the bundle E^{cs} along their intersection $W^u(p_1) \cup W^u(p_2)$. Similarly, the surfaces $W^s(\Gamma(p_1))$ and $W^s(\Gamma(p_2))$ are tangent to the bundle E^{cu} along their intersection $W^s(q_1) \cup W^s(q_2)$.*

Proof. The boundary part of $W^u(\Gamma(q_j))$ has three components, $W^u(p_1)$, $W^u(p_2)$, and $W^{uu}(q_j)$, recall equation (2.6) and see Figure 2.6. Moreover, the surface $W^u(\Gamma(q_j))$ is transverse to $W^{ss}(p_i)$. The uniqueness of the central bundle E^c in Lemma 2.4 implies that for each $x \in W^u(p_i) \subset \sigma^s$ the fiber $E^c(x) = E^{cs}(x)$ (recall Remark 2.5) is the tangent space $T_x(W^u(\Gamma(q_j)))$. This implies the lemma. □

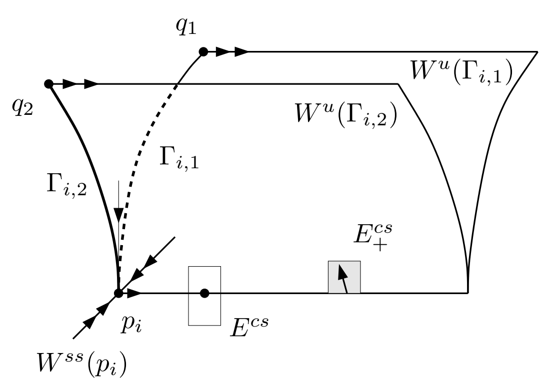

FIGURE 2.7. The open "half-planes" E^{cu}_+.

Remark 2.7 (The open "half-planes" E^{cs}_+ and E^{cu}_+)**.** The normally hyperbolic curves $\Gamma_{i,1}$ and $\Gamma_{i,2}$ are "central curves" contained in $W^s(p_i)$ arriving to p_i from the same side of $W^s(p_i) \setminus W^{ss}(p_i)$. Furthermore, the boundary surfaces $W^u(\Gamma_{i,1})$ and $W^u(\Gamma_{i,2})$ are tangent to E^{cs} along $W^u(p_i)$. This implies that, for every $x \in W^u(p_i)$, the vectors in $E^{cs}(x)$ pointing to the interior of $W^u(\Gamma(q_1)))$ form an open half-plane $E^{cs}_+(x)$. This half-plane coincides with the vectors of $E^{cs}(x)$ pointing to the interior of $W^u(\Gamma_{i,1})$ or (equivalently) of $W^u(\Gamma_{i,2})$. See Figure 2.7.

For points $y \in W^s(q_j)$, we similarly define the half-plane $E^{cu}_+(y)$ as the vectors in E^{cu} pointing to the interior of $W^s(\Gamma(p_i))$, $i = 1, 2$.

2.6. **Position of the invariant manifolds in the basins of r_1 and s_1.** Consider a "small" two-sphere $\mathbb{S}^s$ contained in the interior of the solid torus $\mathcal{T}_1$ that is transverse to the vector field X and bounds a three-ball $\mathbb{B}^s \subset W^s(s_1) \cap \mathcal{T}_1$ whose interior contains the singularity s_1. Let

$$\eta^s \stackrel{\text{def}}{=} \mathbb{S}^s \cap W^{ss}(\sigma^s). \tag{2.10}$$

Note that η^s is a circle that contains the points

$$y_1^u \stackrel{\text{def}}{=} W^u(p_1) \cap \mathbb{S}^s \quad \text{and} \quad y_2^u \stackrel{\text{def}}{=} W^u(p_2) \cap \mathbb{S}^s, \qquad y_1^u, y_2^u \in \eta^s. \tag{2.11}$$

We similarly define a "small" two-sphere $\mathbb{S}^u \subset \mathcal{T}_2$ transverse to X bounding a three-ball $\mathbb{B}^u \subset W^u(r_1)$ whose interior contains r_1. We define the circle

$$\eta^u \stackrel{\text{def}}{=} \mathbb{S}^u \cap W^{uu}(\sigma^u)$$

and the intersection points

$$y_1^s \stackrel{\text{def}}{=} W^s(q_1) \cap \mathbb{S}^u \quad \text{and} \quad y_2^s = W^s(q_2) \cap \mathbb{S}^u, \qquad y_1^s, y_2^s \in \eta^u. \tag{2.12}$$

Remark 2.8. Choosing the balls $\mathbb{B}^s$ and $\mathbb{B}^u$ small enough, we can assume that the minimum time that a point takes to go from $\mathbb{B}^u$ to $\mathbb{B}^s$ is arbitrarily large. In particular, this time is bigger than 10: $X_t(\mathbb{B}^u) \cap \mathbb{B}^s = \emptyset$ for all $t \in [0, 10]$.

Consider the sets $\mathbb{G}^u$ and $\mathbb{G}^s$ (the set $\mathbb{G}^u$ is depicted in Figure 2.8),

$$\mathbb{G}^u \stackrel{\text{def}}{=} \overline{W^u(r_1) \cap \mathbb{S}^s} \quad \text{and} \quad \mathbb{G}^s \stackrel{\text{def}}{=} \overline{W^s(s_1) \cap \mathbb{S}^u}.$$

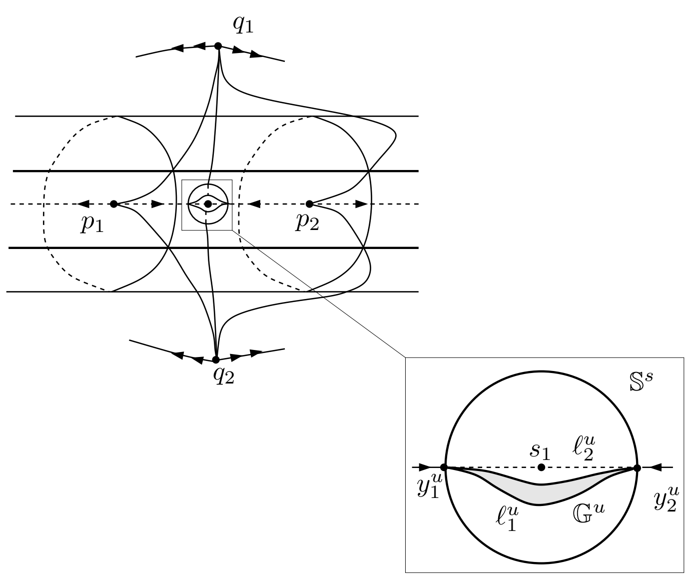

FIGURE 2.8. The set $\mathbb{G}^u$.

Lemma 2.9. *The set $\mathbb{G}^u$ is a topological two-disk bounded by $\ell_1^u \cup \ell_2^u \cup \{y_1^u\} \cup \{y_2^u\}$, where*

$$\ell_1^u \subset W^u(q_1) \quad \textit{and} \quad \ell_2^u \subset W^u(q_2)$$

are disjoint (open) simple curves whose endpoints are y_1^u *and* y_2^u*. The closed curves* $\overline{\ell_1^u}$ *and* $\overline{\ell_2^u}$ *have the same tangent direction at their endpoints* y_1^u *and* y_2^u*. The disk* $\mathbb{G}^u$ *has two cusps at the points* y_1^u *and* y_2^u*.*

Similarly, the set $\mathbb{G}^s$ *is a topological two-disk bounded by* $\ell_1^s \cup \ell_2^s \cup \{y_1^s\} \cup \{y_2^s\}$*, where*

$$\ell_1^s \subset W^s(p_1) \quad \textit{and} \quad \ell_2^s \subset W^s(p_2)$$

are disjoint (open) simple curves whose endpoints are y_1^s *and* y_2^s*. The closed curves* $\overline{\ell_1^s}$ *and* $\overline{\ell_2^s}$ *have the same tangent direction at their endpoints* y_1^s *and* y_2^s*. The disk* $\mathbb{G}^s$ *has two cusps at* y_1^s *and* y_2^s*.*

We consider the following notation, given an interval $[t_1, t_2]$ and a set A we let

$$X_{[t_1,t_2]}(A) \stackrel{\text{def}}{=} \bigcup_{t\in[t_1,t_2]} X_t(A).$$

Proof. We only prove the lemma for the set $\mathbb{G}^u$, the proof for $\mathbb{G}^s$ is identical.

Note that the sphere $\mathbb{S}^s$ intersects every orbit of the set $W^s(s_1)\setminus\{s_1\}$ in exactly one point. Thus, since the rectangle R in Remark 2.1 is the closure of $W^u(r_1)\cap W^s(s_1)\cap\mathbb{T}^2$, the positive orbit of any point in the interior of R intersects $\mathbb{S}^s$ in exactly one point. Hence the set $\mathbb{G}^u$ is the closure of the "projection" along the orbits of X of the interior of R into $\mathbb{S}^s$, that is,

$$\mathbb{G}^u = \overline{X_{[0,\infty)}\big(\mathrm{int}(R)\big)\cap\mathbb{S}^s}.$$

By construction the set $\mathbb{G}^u$ is a topological two-disk. We next describe its boundary.

By equation (2.1) the boundary of R consists of the segments $a_i^s \subset W^s(p_i)$ and $b_i^u \subset W^u(q_j)$, $i = 1, 2$. Furthermore, the interiors of the segments b_1^u and b_2^u are contained in $W^s(s_1)$. Denote by ℓ_1^u and ℓ_2^u the "projections" by the flow of X of these interiors into $\mathbb{S}^s$, that is,

$$\ell_i^u \stackrel{\text{def}}{=} X_{[0,\infty)}\big(\mathrm{int}(b_i^u)\big)\cap\mathbb{S}^s.$$

Consider any sequence (x_n) of points in the interior of R accumulating to the side a_i^s of R. Note that the (positive) orbit of x_n by the flow of X goes arbitrarily close to the saddle singularity p_i before intersecting $\mathbb{S}^s$ at a point y_n. By construction, the sequence (y_n) converges to $y_i^u = W^u(p_i)\cap\mathbb{S}^s$. Indeed, for any given curve $b\subset R$ transverse to X joining the sides a_1^s and a_2^s of R the intersection of the sphere $\mathbb{S}^s$ and the positive orbit of b by the flow X (i.e., the "projection" of b into $\mathbb{S}^s$ by the flow) is a curve ℓ_b joining y_2^u and y_2^u (these points are in the closure of ℓ_b). In particular, y_1^u and y_1^u are the endpoints of ℓ_i^u, $i = 1, 2$.

Bearing in mind equation (2.9) and the definitions of y_i^u, ℓ_i^u, and $\Gamma(q_i)$, $i = 1, 2$, we get the following:

$$\begin{aligned}\overline{\ell_i^u} &= \ell_i^u \cup \{y_1^u, y_2^u\} = W^u(\Gamma(q_i))\cap\mathbb{S}^s,\\ \partial\mathbb{G}^u &= \big(W^u(\Gamma(q_1))\cap\mathbb{S}^s\big)\cup\big(W^u(\Gamma(q_2))\cap\mathbb{S}^s\big) = \ell_1^u\cup\ell_2^u\cup\{y_1^u, y_2^u\}.\end{aligned}$$

This completes the description of the set $\partial\mathbb{G}^u$.

It remains to see that y_1^u and y_2^u are cusps of $\mathbb{G}^u$. By Lemma 2.6 the curves $\overline{\ell_1^u}$ and $\overline{\ell_2^u}$ are tangent at y_i^u to $E^{cs}(y_i^u)\cap T_{y_i^u}(\mathbb{S}^s)$. To see that the point y_i^u is a cusp of $\mathbb{G}^u$ it is enough to note that the interior of $\mathbb{G}^u$ is disjoint from the circle η^s. Thus the disk $\mathbb{G}^u$ is the "thin component" of $\mathbb{S}^s\setminus\partial\mathbb{G}^u$. This ends the proof of the lemma. $\square$

Remark 2.10. *With the notations above, the following inclusions hold*

$$(\mathbb{S}^s \setminus \mathbb{G}^u) \subset W^u(r_2) \quad \textit{and} \quad (\mathbb{S}^u \setminus \mathbb{G}^s \subset W^s(s_2)).$$

2.7. **The diffeomorphism time-one map X_1.** Let X_1 denote the time-one map of the vector field X and define the diffeomorphism $F_0 \stackrel{\text{def}}{=} X_1$. Note that F_0 is a Morse-Smale diffeomorphism whose non-wandering set consists of the sinks s_1 and s_2, the saddles of s-index two p_1 and p_2, the saddles of s-index one q_1 and q_2, and the sources r_1 and r_2. Note that the invariant manifolds of these points for the vector field X and for the diffeomorphism F_0 coincide. We only write $W^i(x, X)$ or $W^i(x, F_0)$ to emphasize the role of X or F_0, otherwise we just write $W^i(x)$.

Consider the fundamental domain Δ^s of $W^s(s_1)$ for F_0 bounded by $\mathbb{S}^s$ and $F_0(\mathbb{S}^s)$. Note that

$$\Delta^s \stackrel{\text{def}}{=} X_{[0,1]}(\mathbb{S}^s) = \mathbb{B}^s \setminus \text{int}(F_0(\mathbb{B}^s)) \simeq \mathbb{S}^s \times [0,1].$$

Let

$$\mathbb{E}^u \stackrel{\text{def}}{=} X_{[0,1]}(\mathbb{G}^u), \quad \mathbb{L}^u_i \stackrel{\text{def}}{=} X_{[0,1]}(\ell^u_i), \quad \mathbb{Y}^u_i \stackrel{\text{def}}{=} X_{[0,1]}(y^u_i).$$

These sets are depicted in Figure 2.9.

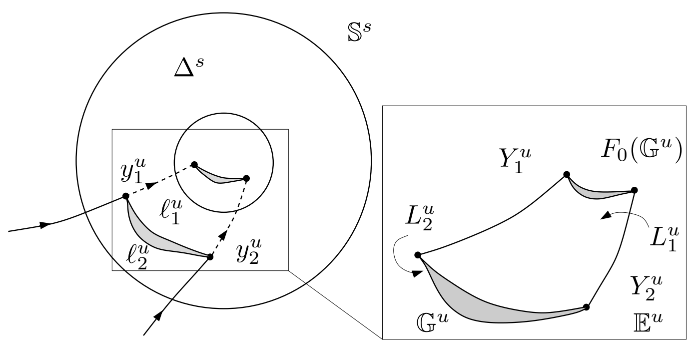

FIGURE 2.9. The sets $\mathbb{E}^u, \mathbb{L}^u_i$, and $\mathbb{Y}^u_i$.

By construction, the set $\mathbb{E}^u$ is a topological ball bounded by the disks $\mathbb{G}^u$ and $F_0(\mathbb{G}^u)$, the "rectangles" $\mathbb{L}^u_1$ and $\mathbb{L}^u_2$, and the curves $\mathbb{Y}^u_1$ and $\mathbb{Y}^u_2$. By the definition of $\mathbb{G}^u$ and of the cylinders $\mathbb{C}^u_1$ and $\mathbb{C}^u_2$ and by Lemma 2.9 (see also Remark 2.10), we have that

$$\overline{W^u(r_1) \cap \Delta^s} = \mathbb{E}^u \quad \text{and} \quad \Delta^s \setminus \mathbb{E}^u \subset W^u(r_2). \tag{2.13}$$

Since $\ell^u_i \subset W^u(q_i, X)$ and $y^u_i \subset W^u(p_i, X)$ we have

$$\mathbb{L}^u_i \subset W^u(q_i, F_0) \quad \text{and} \quad \mathbb{Y}^u_i \subset W^u(p_i, F_0). \tag{2.14}$$

Note that the common boundary of the rectangles $\mathbb{L}^u_1$ and $\mathbb{L}^u_2$ are the curves $\mathbb{Y}^u_1$ and $\mathbb{Y}^u_2$. Moreover, the closure of $\mathbb{L}^u_i$ is tangent to the center unstable bundle E^{cu} along the curves $\mathbb{Y}^u_1$ and $\mathbb{Y}^u_2$.

We similarly define the fundamental domain Δ^u of $W^u(r_1)$ for F_0 bounded by $\mathbb{S}^u$ and $F_0(\mathbb{S}^u)$. As before

$$\Delta^u \stackrel{\text{def}}{=} X_{[0,1]}(\mathbb{S}^u) \simeq \mathbb{S}^u \times [0,1].$$

We let

$$\mathbb{E}^s \stackrel{\text{def}}{=} X_{[0,1]}(\mathbb{G}^s), \quad \mathbb{L}^s_i \stackrel{\text{def}}{=} X_{[0,1]}(\ell^s_i), \quad \mathbb{Y}^s_i \stackrel{\text{def}}{=} X_{[0,1]}(y^s_i)$$

and as above we have

$$\mathbb{L}_i^s \subset W^s(p_i) \quad \text{and} \quad \mathbb{Y}_i^s \subset W^s(q_1). \tag{2.15}$$

Moreover, the "rectangles" $\mathbb{L}_1^s$ and $\mathbb{L}_2^s$ are tangent at the curves $\mathbb{Y}_1^s$ and $\mathbb{Y}_2^s$ to the center stable bundle E^{cs}. By construction, $\mathbb{E}^s$ is bounded by the disks $\mathbb{G}^s$ and $F_0(\mathbb{G}^s)$, the "rectangles" $\mathbb{L}_1^s$ and $\mathbb{L}_s^s$, and the curves $\mathbb{Y}_1^s$ and $\mathbb{Y}_2^s$. Finally,

$$\overline{W^s(s_1) \cap \Delta^u} = \mathbb{E}^s \quad \text{and} \quad \Delta^u \setminus \mathbb{E}^s \subset W^s(s_2). \tag{2.16}$$

3. A diffeomorphism on $\mathbb{S}^2 \times \mathbb{S}^1$ obtained by a surgery

3.1. **The surgery.** In this section identify some regions of $\mathbb{S}^3$ by a local diffeomorphism Ψ. This surgery provides a diffeomorphism F_Ψ defined on $\mathbb{S}^2 \times \mathbb{S}^1$ induced by $F_0 = X_1$ and the quotient of $\mathbb{S}^3$ by Ψ. We will see in Section 5 that for an appropriate choice of Ψ the diffeomorphism F_Ψ has fragile cycles.

With the notation in Section 2, consider a diffeomorphism $\Psi\colon \Delta^s \to \Delta^u$ such that

- $\Psi(\mathbb{S}^s) = \mathbb{S}^u$ and $\Psi(F_0(\mathbb{S}^s)) = F_0(\mathbb{S}^u)$,
- $\Psi \circ F_0|_{\mathbb{S}^s} = F_0|_{\mathbb{S}^u} \circ \Psi$,
- $D\Psi \circ DF_0|_{\mathbb{S}^s} = DF_0|_{\mathbb{S}^u} \circ D\Psi$.

Recall that $\mathbb{B}^s \subset W^s(s_1)$ and $\mathbb{B}^u \subset W^u(r_1)$ are small balls containing s_1 and r_1, respectively. In the set $\mathbb{S}^3 \setminus \Big(\mathrm{int}\big(F_0(\mathbb{B}^s)\big) \cup \mathrm{int}\big(\mathbb{B}^u\big)\Big)$ we identify the points $x \in \Delta^s$ and $\Psi(x) \in \Delta^u$ obtaining the quotient space

$$M \stackrel{\text{def}}{=} \Big(\mathbb{S}^3 \setminus \Big(\mathrm{int}\big(F_0(\mathbb{B}^s)\big) \cup \mathrm{int}\big(\mathbb{B}^u)\big)\Big)\Big) \Big/ \Psi.$$

The set M is a C^1-manifold diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$ and the diffeomorphism F_0 induces a diffeomorphism $F_\Psi\colon M \to M$.

Denote by π the projection $\pi\colon \mathbb{S}^3 \setminus \Big(\mathrm{int}\big(F_0(\mathbb{B}^s)\big) \cup \mathrm{int}\big(\mathbb{B}^u)\big)\Big) \to M$ that associates to x its class $\pi(x) = [x]$ by the equivalence relation induced by Ψ. For notational simplicity, if $x \notin \Delta^s \cup \Delta^u$ we simply write x instead of $\pi(x)$. Write

$$\Delta \stackrel{\text{def}}{=} \big(\Delta^s \cup \Delta^u\big)/\Psi = \pi(\Delta^s) = \pi(\Delta^u).$$

In what follows we write $\mathbb{K}_i^u(F_\Psi) \stackrel{\text{def}}{=} \pi(\mathbb{K}_i^u)$, where $\mathbb{K} = \mathbb{Y}, \mathbb{L}, \mathbb{G}, \mathbb{E}$ and $i =, 1, 2, \emptyset$.

Remark 3.1. The set

$$\Delta \setminus \big(\mathbb{L}_1^u(F_\Psi) \cup \mathbb{L}_2^u(F_\Psi) \cup \mathbb{Y}_1^u(F_\Psi) \cup \mathbb{Y}_2^u(F_\Psi) \cup \mathbb{G}^u(F_\Psi) \cup F_\Psi(\mathbb{G}^u(F_\Psi))\big)$$

has two connected components. The set $\mathbb{E}^u(F_\Psi)$ is the component that is diffeomomorphic to $\mathbb{G}^u \times [0,1]$. This set is a topological ball. There is a similar characterization for $\mathbb{E}^s(F_\Psi)$.

3.2. **The dynamics of F_Ψ.** Note that the orbits of F_Ψ disjoint from Δ are the projection of the orbits of the diffeomorphism F_0. Thus s_2 is a sink and r_2 is a source of F_Ψ (note that the sink s_1 and the source r_1 are removed in the construction of M). Similarly, the points p_1 and p_2 are saddles of s-index 2 of F_Ψ and the points q_1 and q_2 are saddles of s-index 1 of F_Ψ. Observe that F_Ψ can have further periodic points, but by Remark 2.8 these points have period larger than 10.

Using the identification by Ψ and the properties of F_0, we have the following characterization of the sets $\mathbb{L}_i^{u,s}(F_\Psi)$ and $\mathbb{Y}_i^{u,s}(F_\Psi)$, $i = 1, 2$:

$$
\begin{aligned}
\mathbb{L}_i^u(F_\Psi) &= \{x \in \Delta :\ x \in W^u(q_i, F_\Psi) \text{ and } F_\Psi^{-j}(x) \notin \Delta \text{ for all } j \geq 2\},\\
\mathbb{Y}_i^u(F_\Psi) &= \{x \in \Delta :\ x \in W^u(p_i, F_\Psi) \text{ and } F_\Psi^{-j}(x) \notin \Delta \text{ for all } j \geq 2\},\\
\mathbb{L}_i^s(F_\Psi) &= \{x \in \Delta :\ x \in W^s(p_i, F_\Psi) \text{ and } F_\Psi^{j}(x) \notin \Delta \text{ for all } j \geq 2\},\\
\mathbb{Y}_i^s(F_\Psi) &= \{x \in \Delta :\ x \in W^s(q_i, F_\Psi) \text{ and } F_\Psi^{j}(x) \notin \Delta \text{ for all } j \geq 2\}.
\end{aligned} \tag{3.1}
$$

Remark 3.2. The normally hyperbolic curves $\Gamma_{i,j}$ of F_0 (recall Remark 2.2) are disjoint from $\mathbb{B}^s \cup \mathbb{B}^u$, thus their projections on M (also denoted by $\Gamma_{i,j}$) are normally hyperbolic curves of F_Ψ . Observe also that by construction, the interior of $\Gamma_{i,j}$ is contained in $W^s(p_i, F_\Psi) \pitchfork W^u(q_j, F_\Psi)$, recall equation (2.4).

We continue to use the notation $\Gamma(q_j) = \Gamma_{1,j} \cup \Gamma_{2,j}$ and $\Gamma(p_i) = \Gamma_{i,1} \cup \Gamma_{i,2}$.

With the previous notation we have that

$$\mathbb{L}_j^u(F_\Psi) \cup \mathbb{Y}_1^u(F_\Psi) \cup \mathbb{Y}_2^u(F_\Psi) = \overline{\mathbb{L}_j^u(F_\Psi)}$$

is a connected component of $W^u(\Gamma(q_j), F_\Psi) \cap \Delta$.

Lemma 3.3 (Invariant manifolds and their intersections)**.** *Consider $x \in \Delta$.*

(1) If $x \notin \mathbb{E}^u(F_\Psi)$ then $x \in W^u(r_2, F_\Psi)$ and thus it is not chain recurrent,
(2) if $x \notin \mathbb{E}^s(F_\Psi)$ then $x \in W^s(s_2, F_\Psi)$ and thus it is not chain recurrent.
(3) if $x \in \mathbb{L}_i^u(F_\Psi)$ then $x \in W^u(q_i, F_\Psi)$,
(4) if $x \in \mathbb{Y}_i^u(F_\Psi)$ then $x \in W^u(p_i, F_\Psi)$,
(5) if $x \in \mathbb{L}_i^s(F_\Psi)$ then $x \in W^s(p_i, F_\Psi)$, and
(6) if $x \in \mathbb{Y}_i^s(F_\Psi)$ then $x \in W^s(q_i, F_\Psi)$,

Proof. The first item follows immediately from equation (2.13) and the definition of F_Ψ. Similarly, the second item follows from (2.16). Items (3)-(6) follow from equation (3.1). □

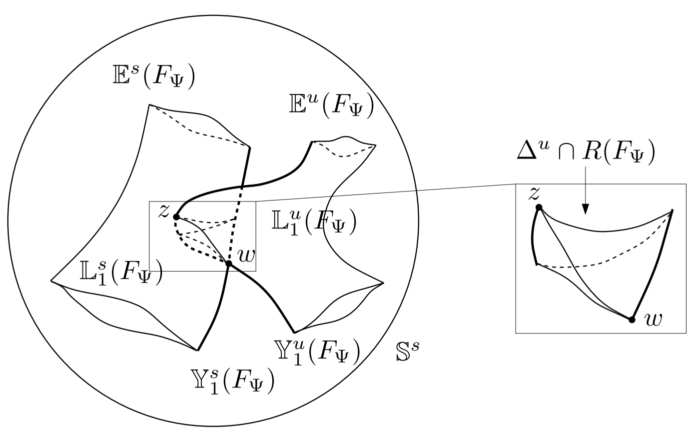

FIGURE 3.1. The recurrence region of F_Ψ.

An immediate consequence of the first two items of Lemma 3.3 is the following:

Corollary 3.4. *Let $x \in \Delta$ be a chain recurrent point for F_Ψ. Then $x \in \mathbb{E}^s(F_\Psi) \cap \mathbb{E}^u(F_\Psi)$.*

A straightforward consequence of Lemma 3.3 and Remark 3.2 is the following:

Corollary 3.5. *Consider $x \in \Delta$.*

(1) If $x \in \mathbb{L}^u_i(F_\Psi) \cap \mathbb{L}^s_j(F_\Psi)$ then $x \in W^u(q_i, F_\Psi) \pitchfork W^s(p_j, F_\Psi)$.
(2) If $x \in \mathbb{L}^u_i(F_\Psi) \cap \mathbb{Y}^s_j(F_\Psi)$ then $x \in W^u(q_i, F_\Psi) \cap W^s(q_j, F_\Psi)$.
(3) If $x \in \mathbb{Y}^u_i(F_\Psi) \cap \mathbb{L}^s_j(F_\Psi)$ then $x \in W^u(p_i, F_\Psi) \cap W^s(p_j, F_\Psi)$.
(4) If $x \in \mathbb{Y}^u_i(F_\Psi) \cap \mathbb{Y}^s_j(F_\Psi)$ then $x \in W^u(p_i, F_\Psi) \cap W^s(q_j, F_\Psi)$.
(5) The interior of $\Gamma_{i,j}$ is contained in $W^s(p_i, F_\Psi) \pitchfork W^u(q_j, F_\Psi)$.

Recall that a periodic point is called *isolated* if its chain recurrent class coincides with its (finite) orbit.

Lemma 3.6 (Heterodimensional cycles and trivial homoclinic classes)**.**

(1) If $\mathbb{Y}^u_1(F_\Psi) \cap \mathbb{Y}^s_1(F_\Psi) \neq \emptyset$ then F_Ψ has a heterodimensional cycle associated to p_1 and q_1.
(2) If $\mathbb{Y}^u_1(F_\Psi) \pitchfork \mathbb{L}^s_1(F_\Psi) \neq \emptyset$ then the homoclinic class of q_1 is non trivial.
(3) If $\mathbb{Y}^s_1(F_\Psi) \pitchfork \mathbb{L}^u_1(F_\Psi) \neq \emptyset$ then homoclinic class of p_1 is non trivial.
(4) If $\mathbb{Y}^u_1(F_\Psi) \cap \mathbb{E}^s(F_\Psi) = \emptyset$ then p_1 is an isolated saddle.
(5) If $\mathbb{Y}^s_1(F_\Psi) \cap \mathbb{E}^u(F_\Psi) = \emptyset$ then q_1 is an isolated saddle.

Proof. To prove the first item just recall that by item (5) in Corollary 3.5 the interior of $\Gamma_{1,1}$ is contained in $W^s(p_1, F_\Psi) \pitchfork W^u(q_1, F_\Psi)$. By item (4) in Corollary 3.5, if $\mathbb{Y}^u_1(F_\Psi) \cap \mathbb{Y}^s_1(F_\Psi) \neq \emptyset$ then $W^u(p_1, F_\Psi) \cap W^s(q_1, F_\Psi) \neq \emptyset$ and thus there is a heterodimensional cycle associated to p_1 and q_1.

Items (2) and (3) follow from Lemma 3.3.

To prove item (4) we use the following simple fact whose proof we omit.

Remark 3.7. Let p be a hyperbolic saddle that is non-isolated. Then its stable/unstable manifold contains points of its chain recurrence class that do not belong to its orbit. In particular, if a hyperbolic fixed point p is such that $W^u(p) \setminus \{p\}$ (respectively, $W^s(p) \setminus \{p\}$) is contained in the stable (resp. unstable) manifolds of some sinks (resp. sources) then it is isolated.

Note that every point $x \in W^u(p_1, F_\Psi)$, $x \neq p_1$, in the separatrix of $W^u(p_1, F_\Psi)$ that does not contain y^u_1 is contained in $W^s(s_2, F_\Psi)$. Thus this separatrix does not contain chain recurrent points. Hence it is enough to consider points in the separatrix of $W^u(p_1, F_\Psi)$ containing y^u_1. Note that $W^u(p_1, F_\Psi) \cap \Delta$ contains a fundamental domain of $W^u(p_1, F_\Psi)$ that is contained in $\mathbb{Y}^u_1(F_\Psi)$. Thus, by Remark 3.7, if p_1 is not isolated then $\mathbb{Y}^u_1(F_\Psi)$ must contain some point of the chain recurrence class of p_1. Thus in such a case $\mathbb{Y}^u_1(F_\Psi)$ cannot be contained in $W^s(s_2, F_\Psi)$.

Suppose that $\mathbb{Y}^u_1(F_\Psi) \cap \mathbb{E}^s(F_\Psi) = \emptyset$. Then, by item (2) in Lemma 3.3, one has that $\mathbb{Y}^u_1(F_\Psi) \subset W^s(s_2)$. By the above discussion this implies that p_1 is isolated, proving item (4).

The proof of item (5) is identical to the previous one and thus it is omitted. □

4. Dynamics in a neighborhood of F_Ψ

In this section we consider diffeomorphisms F in a small C^1-neighborhood of F_Ψ. The hyperbolic-like properties of the objects introduced in Section 3 for the diffeomorphism F_Ψ allows us to define their continuations for nearby diffeomorphisms and thus to repeat these constructions. In particular, the continuations of the hyperbolic points s_2, r_2, p_1, p_2, q_1, and q_2 of F_Ψ are defined. We will omit the dependence on F of these continuations.

As the arguments in this section are similar to those in Sections 2 and 3, some of these constructions will be just sketched. We now go to the details of our constructions.

Consider the spheres $\pi(\mathbb{S}^s)$ and $F(\pi(\mathbb{S}^s))$ and denote by Δ_F the closure of the connected component of $M \setminus \big(\pi(\mathbb{S}^s) \cup F(\pi(\mathbb{S}^s))\big)$ which is close to Δ (i.e., the component that is in the same side of $\pi(\mathbb{S}^s)$ as Δ). The set Δ_F is diffeomorphic to $\mathbb{S}^2 \times [0,1]$ and varies continuously with F in the C^1-topology. In particular, by Remark 2.8, if $x \in \Delta_F$ and $F^i(x) \in \Delta_F$ then $|i| \geq 9$.

Bearing in mind the definitions of the sets $\mathbb{L}_i^{s,u}(F_\Psi)$ and $\mathbb{Y}^{s,u}(F_\Psi)$ in (3.1), we define their "continuations" $\mathbb{L}_i^{s,u}(F)$ and $\mathbb{Y}^{s,u}(F)$ for F close to F_Ψ by

$$
\begin{aligned}
\mathbb{L}_i^u(F) &= \{x \in \Delta_F : \ x \in W^u(q_i, F) \text{ and } F^{-i}(x) \notin \Delta_F \text{ for all } i \geq 2\},\\
\mathbb{Y}_i^u(F) &= \{x \in \Delta_F : \ x \in W^u(p_i, F) \text{ and } F^{-i}(x) \notin \Delta_F \text{ for all } i \geq 2\},\\
\mathbb{L}_i^s(F) &= \{x \in \Delta_F : \ x \in W^s(p_i, F) \text{ and } F_\Psi^{i}(x) \notin \Delta_F \text{ for all } i \geq 2\},\\
\mathbb{Y}_i^s(F) &= \{x \in \Delta_F : \ x \in W^s(q_i, F) \text{ and } F^{i}(x) \notin \Delta_F \text{ for all } i \geq 2\}.
\end{aligned}
$$

Remark 4.1. The sets $\mathbb{L}_i^{s,u}(F)$ and $\mathbb{Y}_i^{s,u}(F)$, $i = 1, 2$, depend continuously on F.

Note that the closed curves $\Gamma_{i,j}$ are normally hyperbolic for F_Ψ (recall Remark 3.2). Thus for every F close to F_Ψ there are defined their continuations, denoted by $\Gamma_{i,j}(F)$, that depend continuously on F. These curves join the saddles p_i and q_j and their interiors are center stable manifolds of p_i and center unstable manifolds of q_j. Finally, from the normal hyperbolicity of $\Gamma_{i,j}(F)$, compact parts of the invariant manifolds $W^s(\Gamma_{i,j}(F))$ and $W^u(\Gamma_{i,j}(F))$ depend continuously on F.

Observe that $W^u(q_i, F) \setminus W^{uu}(q_i, F)$ (resp. $W^s(p_i, F) \setminus W^{ss}(p_i, F)$) has two connected components (separatrices), denoted by $W^{u,+}(q_i, F)$ and $W^{u,-}(q_i, F)$ (resp. $W^{s,\pm}(p_i, F)$). We choose these components such that the following holds:

Remark 4.2 (Invariant manifolds of $\Gamma_{i,j}(F)$)**.**

- $W^u(\Gamma_{1,j}(F)) \setminus W^u(p_1, F) = W^u(\Gamma_{2,j}(F)) \setminus W^u(p_2, F) = W^{u,+}(q_j, F) \cup W^{uu}(q_j, F)$.
- $W^s(\Gamma_{i,1}(F)) \setminus W^s(q_1, F) = W^s(\Gamma_{i,2}(F)) \setminus W^s(q_2, F) = W^{s,+}(p_i, F) \cup W^{ss}(p_i, F)$.
- $W^u(\Gamma_{i,1}(F)) \cap W^u(\Gamma_{i,2}(F)) = W^u(p_i, F)$.
- $W^s(\Gamma_{1,j}(F)) \cap W^s(\Gamma_{2,j}(F)) = W^s(q_j, F)$.

By Lemma 2.4 and using the notation in Remark 2.5, for every F close to F_Ψ there is a unique invariant central bundle E_F^{cs} (resp. E_F^{cu}) defined on $W^u(p_i, F)$ (resp. $W^s(q_j, F)$) and transverse to the strong stable (resp. unstable) direction at p_i (resp. q_j). The central bundles E_F^{cs} and E_F^{cu} depend continuously on F.

Remark 4.3. The manifolds with boundary $W^u(\Gamma_{i,1}(F))$ and $W^u(\Gamma_{i,2}(F))$ are tangent along $W^u(p_i, F)$ to the plane field E_F^{cs}. As in Remark 2.7, for $x \in W^u(p_i, F)$, the vectors of E_F^{cs} entering in the interior of $W^u(\Gamma_{i,j}(F))$ define an half plane $E_{+,F}^{cs}(x)$.

Analogously, the manifolds with boundary $W^s(\Gamma_{1,j}(F))$ and $W^u(\Gamma_{2,j}(F))$ are tangent along $W^s(q_j,F)$ to the plane field E^{cu}_F. For $x \in W^s(p_j,F)$, the vectors of E^{cu}_F entering in the interior of $W^(\Gamma_{i,j}(F))$ define a half plane $E^{cu}_{+,F}$.

As in (2.5) and (2.7), we define the sets

$$\Gamma(q_j,F) \stackrel{\text{def}}{=} \Gamma_{1,j}(F) \cup \Gamma_{2,j}(F) \quad \text{and} \quad \Gamma(p_i,F) \stackrel{\text{def}}{=} \Gamma_{i,1}(F) \cup \Gamma_{i,2}(F).$$

Then

$$W^u(\Gamma(q_j,F)) = W^u(\Gamma_{1,j}(F)) \cup W^u(\Gamma_{2,j}(F))$$

is a C^1-surface with boundary whose compact parts depend continuously on F. Moreover,

$$W^u(\Gamma(q_1,F)) \cap W^u(\Gamma(q_2,F)) = W^u(p_1,F) \cup W^u(p_2,F)$$

The surfaces $W^u(\Gamma(q_1,F))$ and $W^u(\Gamma(q_2,F))$ depend continuously on F and are tangent to E^{cs}_F along this intersection. This last assertion is just a version of Lemma 2.6 for F close to F_Ψ.

Similarly, the compact parts of the surfaces $W^s(\Gamma(p_1,F))$ and $W^s(\Gamma(p_2,F))$ depend continuously on F and they are tangent to E^{cu}_F along their intersection $W^s(q_1,F) \cup W^s(q_2,F)$.

Using the previous notation we get that

$$\overline{\mathbb{L}^u_j(F)} = \mathbb{L}^u_j(F) \cup \mathbb{Y}^u_1(F) \cup \mathbb{Y}^u_2(F),$$
$$\overline{\mathbb{L}^s_i(F)} = \mathbb{L}^s_i(F) \cup \mathbb{Y}^s_1(F) \cup \mathbb{Y}^s_2(F).$$

The set $\overline{\mathbb{L}^u_j(F)}$ is the connected component of $W^u(\Gamma(q_j,F)) \cap \Delta_F$ whose negative iterates $F^{-i}(\overline{\mathbb{L}^u_j(F)})$, $i \geq 2$, are disjoint from Δ_F. Similarly, the set $\overline{\mathbb{L}^s_i(F)}$ is the connected component of $W^u(\Gamma(p_i,F) \cap \Delta_F$ whose positive iterates larger than 2 are disjoint from Δ_F.

As a consequence of the previous constructions, we get

Lemma 4.4. *The sets $\overline{\mathbb{L}^u_j(F)}$ and $\overline{\mathbb{L}^s_i(F)}$ are "rectangles" depending continuously on F (for the C^1 topology).*

The sets $\mathbb{G}^{s,u}(F)$ and $\mathbb{E}^{s,u}(F)$ are defined similarly as in the case F_Ψ. The set $\mathbb{G}^u(F)$ is the topological disk with two cusps (these cuspidal points are in $W^u(p_1,F)$ and $W^u(p_2,F)$) whose boundary is the union of $W^u(\Gamma(q_1,F)) \cap \Delta_F$ and $W^u(\Gamma(q_2,F)) \cap \Delta_F$. There is an analogous definition for $\mathbb{G}^s(F)$. Note that by construction these sets depend continuously on F.

Finally, the set $\mathbb{E}^u(F)$ is the topological ball bounded by $\overline{\mathbb{L}^u_1(F)}$, $\overline{\mathbb{L}^u_2(F)}$, $\mathbb{G}^u(F)$ and $F(\mathbb{G}^u(F))$ that is close to $\mathbb{E}^u(F_\Psi)$. There is a similar definition for the set $\mathbb{E}^s(F)$. By construction the sets $\mathbb{E}^{s,u}(F)$ depend continuously on F.

There is the following reformulation of Corollary 3.4 and Lemma 3.6 for diffeomorphisms F close to F_Ψ.

Lemma 4.5. *Consider a diffeomorphism F close to F_Ψ.*

(1) *If $x \in \Delta_F$ is a chain recurrent point for F then $x \in \mathbb{E}^s(F) \cap \mathbb{E}^u(F)$.*
(2) *If $\mathbb{Y}^u_1(F) \cap \mathbb{Y}^s_1(F) \neq \emptyset$ then F has a heterodimensional cycle associated to p_1 and q_1.*
(3) *If $\mathbb{Y}^u_1(F) \pitchfork \mathbb{L}^s_1(F) \neq \emptyset$ then the homoclinic class of q_1 is non trivial.*
(4) *If $\mathbb{Y}^s_1(F) \pitchfork \mathbb{L}^u_1(F) \neq \emptyset$ then homoclinic class of p_1 is non trivial.*
(5) *If $\mathbb{Y}^u_1(F) \cap \mathbb{E}^s(F) = \emptyset$ then p_1 is isolated.*

(6) If $\mathbb{Y}_1^s(F) \cap \mathbb{E}^u(F) = \emptyset$ then q_1 is isolated.

Observe also that by construction

$$\Delta_F \subset \mathbb{E}^s(F) \subset W^s(s_2, F) \quad \text{and} \quad \Delta_F \setminus \mathbb{E}^u(F) \subset W^u(r_2, F).$$

As in Corollary 3.4, a consequence of these inclusions is the following.

Lemma 4.6. *For every F close to F_ψ, every point of Δ_F that is chain recurrent is contained in $\mathbb{E}^s(F) \cap \mathbb{E}^u(F)$.*

5. Choice of the local diffeomorphism Ψ

Lemmas 3.3 means that, for the diffeomorphism F_Ψ, the existence of heterodimensional cycles and homoclinic intersections for p_1, p_2, q_1, and q_2 depend on the intersections of the sets $\mathbb{L}_i^{u,s}(F_\Psi)$, $\mathbb{Y}_i^{u,s}(F_\Psi)$, $\mathbb{E}^u(F_\Psi)$, and $\mathbb{E}^s(F_\Psi)$. The choice of the identification map Ψ determines these intersections. Lemma 4.5 explains how these properties are translated for diffeomorphisms close to F_Ψ.

We assume that the local diffeomorphism Ψ is such that the diffeomorphism F_Ψ satisfies the following two conditions, see Figure 5.1:

(T) Topological hypothesis: There is a point $z \in \Delta$ such that

$$\mathbb{Y}_1^s(F_\Psi) \cap \mathbb{Y}_1^u(F_\Psi) = \mathbb{Y}_1^s(F_\Psi) \cap \mathbb{E}^u(F_\Psi) = \mathbb{Y}_1^u(F_\Psi) \cap \mathbb{E}^s(F_\Psi) = \{z\}.$$

(D) Differentiable hypothesis:

- The intersection of the semi-planes $E_+^{cs}(z) \cap E_+^{cu}(z)$ is a half straight line. This implies that the intersection $\mathbb{Y}_1^u(F_\Psi) \cap \mathbb{Y}_1^s(F_\Psi)$ at z is quasi-transverse.
- In a neighborhood of z, the sets $\mathbb{E}^s(F_\Psi)$ and $\mathbb{E}^u(F_\Psi)$ are locally in the same side of any locally defined surface containing the curves $\mathbb{Y}_1^u(F_\Psi)$ and $\mathbb{Y}_1^s(F_\Psi)$.

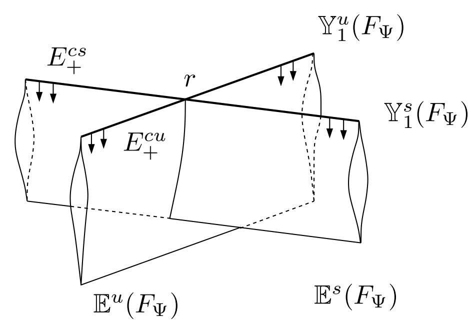

Figure 5.1. Conditions T and D

Proposition 5.1. *Suppose that that F_Ψ satisfies conditions (T) and (D). Then there are a C^1-neighborhood $\mathcal{U}_{F_\Psi}$ of F_Ψ and a codimension one submanifold Σ containing F_Ψ such that:*

(1) The set $\mathcal{U}_{F_\Psi} \setminus \Sigma$ is the union of two disjoint open sets $\mathcal{U}_\Sigma^+$ and $\mathcal{U}_\Sigma^-$ such that:
- *for every $G \in \mathcal{U}_\Sigma^+$ the saddle p_1 is isolated and the homoclinic class of q_1 is non-trivial, and*
- *for every $G \in \mathcal{U}_\Sigma^-$ the saddle q_1 is isolated and the homoclinic class of p_1 is non-trivial.*

(2) Every diffeomorphism $G \in \Sigma$ has a heterodimensional cycle associated to p_1 and q_1.

We have the following corollary:

Corollary 5.2. *The submanifold Σ consists of diffeomorphisms F having fragile cycles associated to p_1 and q_1.*

Note that Theorem 1 follows immediately from Proposition 5.1 and Corollary 5.2.

Corollary 5.2 is a consequence of Proposition 5.1 and the following simple fact about cycles and chain recurrence classes:

Remark 5.3 (Cycles and chain recurrence classes)**.** Let F be a diffeomorphism with a heterodimensional cycle associated to two transitive hyperbolic sets L and K. Then every pair of saddles $p \in L$ and $q \in K$ are in the same chain recurrence class of F. In particular, both saddles p and q are not isolated.

Proof of Corollary 5.2. We argue by contradiction. If there is $F \in \Sigma$ such that the cycle is not fragile then there is a diffeomorphism G close to F with a robust cycle associated to a pair of transitive hyperbolic sets $L \ni p_1$ and $K \ni q_1$. Then by Remark 5.3 the saddles p_1 and q_1 are not isolated for G. Finally, as the cycle is robust, we can assume that $G \notin \Sigma$, that is, $G \in \mathcal{U}_\Sigma^+ \cup \mathcal{U}_\Sigma^-$. Since p_1 is isolated if $G \in \mathcal{U}_\Sigma^+$ this implies that $G \notin \mathcal{U}_\Sigma^+$. Similarly, as q_1 is isolated if $G \in \mathcal{U}_\Sigma^-$ we have that $G \notin \mathcal{U}_\Sigma^-$. This contradicts the fact that the cycle associated to G is robust. □

5.1. **Proof of Proposition 5.1.** To define the submanifold Σ we take the unitary vector $\overrightarrow{n}$ normal to the plane $T_z(\mathbb{Y}_1^u(F_\Psi)) \oplus T_z(\mathbb{Y}_2^s(F_\Psi))$ pointing to the "opposite" direction of $\mathbb{E}^u(F_\Psi)$ and $\mathbb{E}^s(F_\Psi)$. Fix small $\epsilon > 0$ and for F close to F_Ψ consider the family of one-dimensional disks

$$\{\mathbb{Y}_1^s(F) + t\,\overrightarrow{n}\}_{t \in [-\epsilon, \epsilon]}\,.$$

The main step of the proof of the proposition if the following lemma whose proof we postpone.

Lemma 5.4. *There is a small neighborhood $\mathcal{U}_{F_\Psi}$ of F_ψ such that for every $F \in \mathcal{U}_{F_\Psi}$ there is a unique parameter $t = \tau_F$, depending continuously on F, such that*

$$\left(\mathbb{Y}_1^s(F) + \tau_F\,\overrightarrow{n}\right) \cap \mathbb{Y}_2^u(F) \neq \emptyset.$$

There are the following three possibilities according to the value of τ_F:

- *If $\tau_F = 0$ then the diffeomorphism F has a heterodimensional cycle associated to p_1 and q_1.*
- *If $\tau_F > 0$ then p_1 is isolated and the homoclinic class of q_1 is non-trivial.*
- *If $\tau_F < 0$ then q_1 is isolated and the homoclinic class of p_1 is non-trivial.*

Proof of Proposition 5.1. In view of Lemma 5.4 we let

$$\Sigma \stackrel{\text{def}}{=} \{F \in \mathcal{U}_{F_\Psi} \text{ such that } \tau_F = 0\}.$$

Then the set $\mathcal{U}_{F_\Psi} \setminus \Sigma$ has two components $\mathcal{U}_\Sigma^+$ and $\mathcal{U}_\Sigma^-$. The component $\mathcal{U}_\Sigma^+$ consists of the diffeomorphisms F such that $\tau_F > 0$ and $\mathcal{U}_\Sigma^-$ consists of the diffeomorphisms F with $\tau_F < 0$. The proposition now follows immediately from Lemma 5.4. □

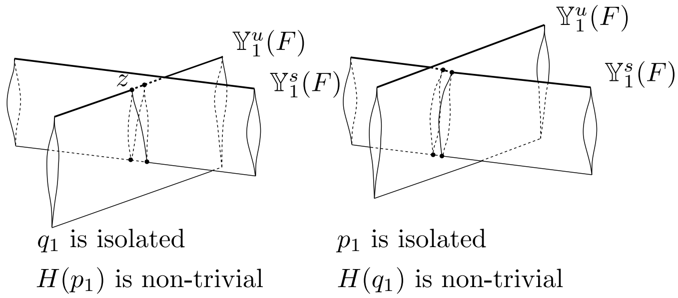

Figure 5.2. Lemma 5.4

Proof. Consider the surface

$$\mathcal{Y}^s_1(F) = \bigcup_{t\in[-\varepsilon,\varepsilon]} \mathbb{Y}^s_1(F) + t\,\overrightarrow{n}.$$

Recall that $\mathbb{Y}^s_1(F)$ and $\mathbb{Y}^u_1(F)$ depend continuously on F (see Remark 4.1). Thus the surface $\mathcal{Y}^s_1(F)$ also depends continuously on F. Note that, by hypothesis, $\mathcal{Y}^s_1(F_\Psi)$ is transverse to $\mathbb{Y}^u_1(F_\Psi)$ and this intersection is the point z in condition (T). Thus for F close to F_Ψ the intersection $\mathcal{Y}^s_1(F)\cap\mathbb{Y}^u_1(F)$ also consists of exactly one point z_F depending continuously on F. Thus there is exactly one parameter τ_F with

$$z_F \in \mathbb{Y}^s_1(F) + \tau_F\,\overrightarrow{n}.$$

Moreover, the parameter τ_F depends continuously on F.

If $\tau_F = 0$ then $\mathbb{Y}^u_1(F)\cap\mathbb{Y}^u_2(F)\neq\emptyset$. Thus by item (2) in Lemma 4.5 the diffeomorphism F has a heterodimensional cycles associated to p_1 and q_1.

The choice of $\overrightarrow{n}$ implies that for $\tau_F > 0$ the sets $\mathbb{Y}^u_1(F)$ and $\mathbb{E}^s(F)$ are disjoint. Thus by item (5) in Lemma 4.5 the point p_1 is isolated for F.

Similarly, for $\tau_F > 0$ one has that $\mathbb{Y}^s_1(F)$ intersects $\mathbb{E}^u(F)$. Thus, if τ_F is small enough, one has that $\mathbb{Y}^s_1(F)\pitchfork\mathbb{L}^u_1(F)$. Hence by item (3) in Lemma 4.5 the homoclinic class $H(q_1,F)$ is non-trivial.

A similar argument shows that for $\tau_F < 0$ the sets $\mathbb{Y}^u_1(F)$ and $\mathbb{E}^u(F)$ are disjoint. As above item (6) in Lemma 4.5 implies that the point q_1 is isolated. Also for $\tau_F < 0$ one has that $\mathbb{Y}^u_1(F)\pitchfork\mathbb{L}^s_1(F)$ and item (4) in Lemma 4.5 implies that the homoclinic class $H(p_1,F)$ is non-trivial. This completes the proof of the lemma. □

Remark 5.5. Our construction can be done such that the saddles q_2 and q_1 are homoclinically related for every small $t > 0$. For that it is enough to choose the diffeomorphisms Ψ such that $\mathbb{Y}^s_2(F_\Psi)\pitchfork\mathbb{L}^u_1(F_\Psi)$, see Corollary 3.5.

Observe that in this case one has $H(q_2,F_t) = H(q_1,F_t)$ for $t > 0$. However, for $t = 0$ the saddle q_1 escapes from the non-trivial homoclinic class of q_2. Surprisingly, in such a case one can generate robust cycles associated to q_2 and p_1 (this follows from [9]) but not associated to q_1 and p_1.

6. Discussion

Our construction provides examples of fragile cycles relating two saddles p_1 and q_1 of different indices (for simplicity we will omit the dependence on the diffeomorphisms). This construction has a "prescribed" part concerning the relative positions of the invariant manifolds of these saddles. But this prescribed dynamics involves only the "cuspidal" regions $\mathbb{Y}^s_1$ and $\mathbb{Y}^u_1$ of $\mathbb{E}^s$ and $\mathbb{E}^u$, respectively. The rest of the intersection $\mathbb{E}^s \cap \mathbb{E}^u$ can be chosen arbitrarily. This provides a lot of "freedom" for the global dynamics, for instance, for the behavior of the other saddles of the diffeomorphism and their invariant manifolds (recall Remark 5.5). Hence, without further assumptions on the dynamics outside the cycle the global dynamics cannot be described.

6.1. **Partial hyperbolicity, wild dynamics, and fragile cycles.** A first ingredient of our construction is the gluing map Ψ that plays a key role for determining the resulting global dynamics.

Question 6.1. *Can the resulting dynamics be partially hyperbolic? More precisely, does it exist a gluing map Ψ so that the chain recurrence class of p_1 for F_Ψ has a partially hyperbolic splitting with three* 1*-dimensional bundles?*

We expect a positive answer to this question. This would imply that the phenomena associated to these fragile cycles would also occur in the most rigid non-hyperbolic setting of partial hyperbolicity with one-dimensional center.

We next discuss how fragile cycles can be involved in the generation of *wild dynamics* (roughly, persistent coexistence of infinitely many homoclinic classes, see [5] for further details).

First note that perturbations of the diffeomorphism F in the fragile cycles submanifold Σ in Theorem 1 expels periodic points in the chain recurrence class that simultaneously contains p_1 and q_1: the saddle p_1 is expelled by diffeomorphisms in the component $\mathcal{U}_\Sigma^+$ and the saddle q_1 by diffeomorphisms in the component $\mathcal{U}_\Sigma^-$. Thus the chain recurrence class fall into pieces. On the other hand, there is a natural question about whether fragile cycles may generate a "cascade" of fragile cycles:

Question 6.2. *Can the submanifold of fragile cycles Σ be accumulated by (codimension one) submanifolds consisting of fragile cycles?*

If the answer to this question would be positive then a "fragile cycles configuration" could be repeated generating infinitely many different chain recurrence classes. Being very optimistic, positive answers to the two questions above could provide the first examples of wild dynamics in the partially hyperbolic (with one-dimensional center) setting. We will discuss further questions about wild dynamics later.

6.2. **Chain recurrence classes.** Consider a diffeomorphism F in the submanifold Σ of fragile cycles. By construction, the intersection $W^s(p_1) \pitchfork W^u(q_1)$ contains a curve $\Gamma_{1,1}$ joining p_1 to q_1. Moreover, the intersection $W^u(p_1) \cap W^u(q_1)$ is exactly the orbit of a quasi-transverse heteroclinic point x_F. This cyclic configuration implies that the chain recurrence classes of p_1 and q_1 coincide and contain the curve $\Gamma_{1,1}$ and the orbit of x_F, recall also Remark 5.3.

We can now perform perturbations of $F \in \Sigma$ preserving the cycle, that is, the resulting diffeomorphisms continue to belong to Σ. In this way, and using for instance the arguments in [8], one can slightly modify the central eigenvalues of the saddles p_1 and q_1 in the cycle (corresponding to the tangent direction of the connection $\Gamma_{1,1}$) to generate "new" periodic saddles r whose orbits pass arbitrarily close to p_1 and q_1 and belong to the chain recurrence class of $C(p_1) = C(q_1)$. The latter fact is a consequence of the geometry of the cycle that guarantees that indeed $W^u(r) \cap W^s(p_1) \neq \emptyset$ and $W^s(r) \cap W^u(q_1) \neq \emptyset$, see Figure 6.1.

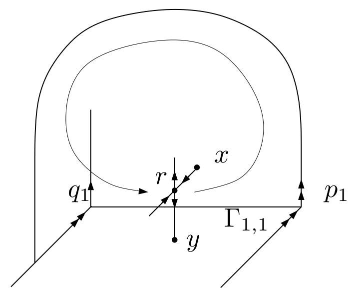

FIGURE 6.1. Periodic points in $C(p_1) = C(q_1)$.

Question 6.3. *For C^1-generic diffeomorphisms $F \in \Sigma$, is the heteroclinic curve $\Gamma_{1,1}$ contained in the closure of a set of periodic points? Can the periodic points in that set be chosen in the same homoclinic class or (even more) homoclinically related?*

Concerning this question, observe that our construction provides saddles that are in the same recurrence class (the one of p_1 and q_1). Note also that by [6], for C^1-generic diffeomorphisms, the chain recurrence classes of periodic points always coincide with their homoclinic classes. The difficulty here is that we consider C^1-generic diffeomorphisms in the codimension one submanifold Σ (which is a meager set). Thus the result in [6] cannot be applied. Indeed, for $F \in \Sigma$ the homoclinic classes of p_1 and q_1 are both trivial and hence different from their non-trivial chain recurrence classes since $p_1, q_1 \in C(p_1) = C(q_1)$.

We believe that the answer to Question 6.3 is positive. In such a case, it would be interesting to consider the central Lyapunov exponents of the periodic orbits in the chain recurrence class $C(p_1) = C(q_1)$. This is our next topic.

6.3. **Lyapunov exponents.** The papers [13, 14, 12] consider examples of diffeomorphisms having twisted heterodimensional cycles (recall Section 1) and analyze the spectrum of central Lyapunov exponents of the (non-trivial) homoclinic classes involved in the cycle. In the examples in [14, 12] this spectrum has a *gap*. The existence of this gap is related to the fact that the homoclinic class $H(p)$ considered contains a saddle q of different index that looks like a "cuspidal corner point" (exactly as the points p_1 and q_1 of our construction) and satisfies $q \in H(p)$ but $H(q) = \{q\}$. So the saddle q is topologically an extreme point but also dynamically is also an extreme point as it is not homoclinically related to other saddles in the class $H(p)$. We believe that precisely this is reflected by the Lyapunov spectrum that has one gap.

The previous comments and the fact that the chain recurrence classes considered in this paper have "two cuspidal corner points" (the saddles p_1 and q_1) that are also dynamically extremal lead to the following question.

Question 6.4. *Does there exist a gluing map Ψ such that there is a neighborhood $\mathcal{U}$ of F_Ψ such that "persistently" in $\mathcal{U} \cap \Sigma$ the diffeomorphisms have two (or more) gaps in the spectrum of central Lyapunov exponents of the periodic orbits in the chain recurrence class $C(p_1)$? Here the term "persistent" is purposely vague and may mean C^1-generic or C^1-dense, for instance.*

Observe that by [2] for a C^1-generic diffeomorphism the spectrum of Lyapunov exponents of homoclinic classes has no gaps. Hence, one would need to consider diffeomorphisms of non-generic type.

6.4. **Collision, collapse, and birth of classes.** Consider an arc of diffeomorphisms $(F_t)_{t\in[-1,1]}$ that intersects Σ transversely at $t=0$. Assume that $F_t \in \mathcal{U}_\Sigma^+$ for $t>0$ and $F_t \in \mathcal{U}_\Sigma^-$ for $t<0$. Our construction implies that for $t>0$ the class $H(q_1,F_t)$ is non-trivial, $H(p_1,F_t)=\{p_1\}$, and both classes are disjoint. Similarly, for $t<0$ the class $H(p_1,F_t)$ is non-trivial, $H(q_1,F_t)=\{q_1\}$, and both classes are disjoint. Since $F_0\in\Sigma$ and $H(q_1,F_0)=\{q_1\}$ and $H(p_1,F_0)=\{p_1\}$, each of these classes collapses to a single point at $t=0$. Similarly (or symmetrically), for $t=0$ the chain recurrence classes of p_1 and q_1 collide at $t=0$ and contain the heteroclinic segment $\Gamma_{1,1}$, when one of these classes collapses to a point for $t\neq 0$. This illustrates the lower semi-continuous dependence of homoclinic classes and the upper semi-continuity of chain recurrence classes.

Let us observe that [7] provides a locally C^1-dense set of diffeomorphisms where homoclinic classes are properly contained in a robustly isolated chain recurrence class. This construction is somewhat similar to the one in this paper and involves a heterodimensional cycle relating cuspidal corner points of the class.

Coming back to our construction, it would be interesting to understand how for $t>0$ the points of $H(q_1,F_t)$ escape from the class or "disappear" as $t\to 0^+$. As we discussed above, in some cases the saddle q_1 is accumulated by saddles of the same index when $t=0$. These saddles are not homoclinically related to q_1 but they are in its chain recurrence class. This indicates that there are (infinitely many) saddles that "escape" from the homoclinic class of q_1 but not from the chain recurrence class of q_1 as t evolves. Though we do not know how these saddles are homoclinically related. Note that there is completely symmetric scenery for the saddle p_1.

Question 6.5. *Is there a diffeomorphism $F\in\Sigma$ with infinitely many different homoclinic classes, all of them contained in the chain recurrence class of p_1 and q_1?*

In view of the previous discussion, a simpler question is if the diffeomorphisms in Σ can be chosen having wild dynamics close to the cycle. More precisely,

Question 6.6. *Does there exist $F\in\Sigma$ with infinitely many different chain recurrence classes accumulating to p_1 or to q_1?*

Finally, let us observe that the previous setting is somehow reminiscent to the setting of the geometrical Lorenz attractor: at the point of "bifurcation", when a singular cycle occurs, infinitely many orbits of the vector field transform into heteroclinic orbits of a singularity, see for instance [3].

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