# Supergrowth of number of periodic orbits for non-hyperbolic homoclinic classes

Ch. Bonatti, L. J. Díaz, and T. Fisher *

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**Abstract**

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## 1 Introduction

A a diffeomorphism $f$ is Artin-Mazur (A-M for short) if the number of isolated periodic points of period $n$ of $f$, denoted by $\mathbb{P}_n(f)$, grows at most exponentially fast: there is a constant $C > 0$ such that

$$\# \mathbb{P}_n(f) \leq \exp(C\, n), \quad \text{for all } n \in \mathbb{N}.$$

Artin and Mazur proved in [3] that the A-M maps are dense in the space of $C^r$ maps endowed with the uniform topology. Later, Kaloshin proved in [17] that the A-M diffeomorphisms having only hyperbolic periodic points are dense in the space of $C^r$-diffeomorphisms.

Later, in [18], Kaloshin proved that the A-M diffeomorphisms are not topologically generic in the space of $C^r$-diffeomorphisms ($r \geq 2$). The proof of this result involves the so-called Newhouse domains (i.e., an open set where the diffeomorphisms with homoclinic tangencies are dense). The standard way to get Newhouse domains is by unfold a homoclinic tangency of a $C^2$-diffeomorphism, see [22]. Moreover, the existence of such domains it is only known in the $C^2$-topology.

The main technical step in [18] is the following. An open set $\mathcal{K}$ has a *supergrowth for the number of periodic points* if for every arbitrary sequence of positive integers $a = (a_n)_{n=1}^{\infty}$ there is a residual subset $\mathcal{R}_a$ of $\mathcal{K}$ such that $\limsup \#\mathbb{P}_n(f)/a_n = \infty$ for any diffeomorphism $f \in \mathcal{R}_a$. [18] proved that Newhouse domains have supergrowth. Moreover, given any Newhouse domain $\mathcal{N}$, there is a dense subset $\mathcal{D}$ of $\mathcal{N}$ of diffeomorphisms having a curve of periodic points.

To get in the $C^1$-topology a dynamical configuration generating (with some persistence) curves of periodic points is quite simple. One can proceed as follows, consider a diffeomorphism $f$ defined on a manifold $M$ of dimension $n$, $n \geq 3$, having a non-hyperbolic homoclinic class $H(P_f, f)$ such

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that for every $g$ in a $C^1$-neighbourhood $\mathcal{U}_f$ of $f$ the homoclinic class $H(P_g, g)$ ($P_g$ is the continuation of $P_f$ for $g$) contains a saddle $Q_g$ whose *index* (dimension of the unstable bundle) is different from the one of $P_f$. Then there is a dense subset $\mathcal{D}$ of $\mathcal{U}_f$ consisting of diffeomorphisms $g$ having a saddle-node. Moreover, the period of such a saddle-node can be taken arbitrarily big. Using this fact, and noting that the behaviour of a saddle-node in the non-hyperbolic direction is close to the identity, one gets after a new perturbation, a diffeomorphism having an interval of fixed periodic points. The proof of these properties follows straighforwardly using the arguments in [2], which are refinament of the constructions in [8] (we will review this construction in Section **??**). Examples of diffeomorphisms (and homoclinic classes) with this dynamical feature can be found, for instance, in [6, 12, 13, 14].

A more interesting fact is that in some cases the periodic points generating the supergrowth of the number of periodic points can be obtained inside in the homoclinic class, thus generating homoclinic classes whose number of periodic points have supergrowth.

Before stating our main result let us recall that *homoclinic classes* were introduced by Newhouse in [20] as a generalization of the basic sets in the Smale Decomposition Theorem (see [25]). The homoclinic class of a hyperbolic saddle $p$ of a diffeomorphism $f$, denoted by $H(p, f)$, is the closure of the transverse intersections of the invariant manifolds (stable and unstable ones) of the orbit of $p$. A homoclinic class can be also (equivalently) defined as the closure of the set of hyperbolic saddles $q$ *homoclinically related* to $p$ (the stable manifold of the orbit of $q$ transversely meets the unstable one of the orbit of $p$ and vice-versa).

**Theorem 1.** *There is a residual subset $\mathcal{S}(M)$ of* $\mathrm{Diff}^1(\mathrm{M})$ *of diffeomorphisms $f$ such that, for every $f \in \mathcal{S}(M)$, any homoclinic class of $f$ containing saddles of different indices has superexponential growth of the number of periodic points.*

In fact, a stronger version (a bit more technical) of this theorem holds:

**Theorem 2.** *The residual subset $\mathcal{S}(M)$ of* $\mathrm{Diff}^1(\mathrm{M})$ *in Theorem 1 can be chosen as follows: Consider $f \in \mathcal{S}(M)$ and any homoclinic class $H(p, f)$ of $f$ containing saddles of different indices $\alpha$ and $\beta$, $\alpha < \beta$. Then for every natural number $\gamma \in [\alpha, \beta]$ the number of (hyperbolic) periodic points of index $\gamma$ of $H(p, f)$ has superxponential growth.*

Finally, these two results are consequences from the following result. Consider an open set $\mathcal{U}$ of $\mathrm{Diff}^1(\mathrm{M})$ such that for every $f$ in $\mathcal{U}$ are defined hyperbolic periodic saddles $p_f$ and $q_f$, depending continuously on $f$, having different indices. By [2, Lemma 2.1], there is a residual subset $\mathcal{G}$ of $\mathcal{U}$ such that

- either $H(p_f, f) = H(q_f, f)$ for all $f \in \mathcal{G}$,
- or $H(p_f, f) \cap H(q_f, f) = \emptyset$ for all $f \in \mathcal{G}$.

In the first case, we say that the saddles $p_f$ and $q_f$ are *genercially homoclinically linked* in $\mathcal{U}$.

Let

$$\mathbb{P}_n^\gamma(H(p, f)) = \{x \in H(p, f),\ f^n(x) = x,\ x \text{ hyperbolic, and index } (x) = \gamma\}.$$

Motivated by the definitions above, we say that the *saddles of index $\gamma$ of the homoclinic class $H(p_f, f)$ has a supergrowth in $\mathcal{U}$* if for every sequence of positive integers $a = (a_n)_{n=1}^\infty$ there is a residual subset $\mathcal{R}_a$ of $\mathcal{U}$ such that

$$\limsup \frac{\#\,\mathbb{P}_n^\gamma(H(p, f))}{a_n} = \infty, \quad \text{for every diffeomorphism } f \in \mathcal{R}_a.$$

In this case, we say that the *growth of the number of saddles of index* $\gamma$ *of the class in lower bounded by the sequence* $a = (a_k)$.

**Proposition 1.1.** *Consider an open set* $\mathcal{U}$ *of* $\mathrm{Diff}^1(\mathrm{M})$ *and a pair of saddles* $p_f$ *and* $q_f$ *which are generically homoclinically linked in* $\mathcal{U}$*. Suppose that the indices of the saddles are* $p_f$ *and* $q_f$ *are* $\alpha$ *and* $\beta$*,* $\alpha < \beta$*. Then, for every natural number* $\gamma \in [\alpha, \beta]$*, the saddles of index* $\gamma$ *of* $H(p_f, f)$ *has supergrowth in* $\mathcal{U}$*.*

We now need that notion of *chain recurrence class.* A point $y$ is $f$*-chain attainable* from the point $x$ if for every $\varepsilon > 0$ there is an $\varepsilon$-pseudo-orbit going from $x$ to $y$. The points $x$ and $y$ are $f$*-bi-chain attainable* if $x$ is chain attainable from $y$ and vice-versa. The bi-chain attainability relation defines an equivalence relation on the *chain recurrent set* $R(f)$ of $f$ (i.e., the set of points $x$ which are chain attainable from themselves). The *chain recurrence classes* are the equivalence classes of $R(f)$ for the bi-chain attainability relation.

A diffeomorphism is *tame* if every chain recurrence class of $f$ is robustly isolated. In this case, for diffeomorphisms in a residual subset of $\mathrm{Diff}^1(\mathrm{M})$, these chain recurrence classes are in fact homoclinic classes. In this case, one also has that non-hyperbolic homoclinic classes of $C^1$-generic tame diffeomorphisms contains saddles of different indices. Thus Proposition 1.1 implies the following:

**Corollary 1.2.** *Every non-hyperbolic homoclinic class of a* $C^1$*-generic tame diffeomorphism has supergrothw of the number of periodic points.*

This paper can be viewed as a continuation of [2], where it is proved that, for $C^1$-generic diffeomorphisms, the indices of the saddles in a homoclinic class form an interval in $\mathbb{N}$: there is a residual subset $\mathcal{R}$ of $\mathrm{Diff}^1(\mathrm{M})$ such that for every $f \in \mathcal{R}$ and every homoclinic class $H(p, f)$ of $f$, if the class contains saddles of indices $\alpha$ and $\beta$, $\alpha < \beta$, it is also contains saddles of indices $\gamma$ for all $\gamma \in [\alpha, \beta] \cap \mathbb{N}$. In fact, our proof follows from the constructions in [2] (we need to check the steps in the construction of saddles of intermediate index). We proceed as follows, suppose that the homoclinic class $H(p_f, f)$ $H(p_f, f)$ contains saddles $q_f$ of index $\alpha$ and $r_f$ of index $\alpha + 1$. We can assume that $H(p_f, f) = H(q_f, f) = H(r_f, f)$. we see that, after a perturbation, we can assume that there is a saddle-node $s_f$ (or arbitrarily large period) with $s_f \in H(q_f, f) = H(r_f, f)$. This saddle-node has a strong stable direction of the same dimension as the one as $r_f$ (i.e., of dimension $n - \alpha - 1$, $n$ is the dimension of the ambient) a strong unstable direction of the same dimension as the one of $q_f$ (i.e., dimension $\alpha$), and a one dimensional central direction. We see that the strong stable manifold of $s_f$ transversely meets the unstable manifold of $r_f$ and that the strong unstable manifold of $s_f$ transversely meets the stable manifold of $q_f$.

Consider now a sequence $a = (a_k)_k$ of strictly positive integers. The previous constructions, specially the fact that the period of $s_f$ can be taken arbitrarily large, imply that, after a small perturbation, we can assume that the dynamics of $f$ at $s_f$ in the central direction is exactly the identity. Suppose that the period of the saddle-node is $k$ (large $k$). We fix $a_k$ and perturb the diffeomorphism $f$ to get a new diffeomorphism $g$ such that $g$ has $k\, a_k$ saddles $q_g^1, \dots, q_g^{k\, a_k}$ of index $\alpha$ and $a_k$ saddles $r_g^1, \dots, r_g^{k\, a_k}$ of index $\alpha + 1$ of the same period as $s_f$ and arbitrarily close to $s_f$. By a continuity argument, this implies that, for every $i$, the unstable manifolds of $q_g^i$ and $r_g^i$ meets the stable manifold of $q_g$ and the stable manifolds of $q_g^i$ and $r_g^i$ meets the unstable manifold of $q_g$. Next step is to check the saddles $q_g^i$ and $r_g^i$ are in the chain recurrence class of $r_g$ and $q_g$ (see Section **??** for the precise definition). The argument now follows noting that for $C^1$-generic diffeomorphisms every chain recurrence class containing a saddle is the homoclinic class of such a saddle, see [5, Remarque 1.10].

The previous arguments show that there is a subsequence $n_k \to \infty$ such that for every $a_{n_k}$ the set $\mathcal{A}_{n_k}$ of diffeomorphisms $g$ such that

$$\frac{\# \mathbb{P}^{\gamma}_{n_k}(H(p_g, g))}{a_{n_k}} \geq n_k, \quad \gamma = \alpha, \, \alpha + 1$$

is open and dense in $\mathcal{U}$. Consider now the set $\mathcal{R}_a$ defined as the intersection of the sets $\mathcal{A}_{n_k}$. By construction, the set $\mathcal{R}_a$ is residual in $\mathcal{U}$ and consists of diffeomorphisms such that the growth of saddles of indices $\alpha$ and $\alpha + 1$ is lower bounded by the sequence $a = (a_k)$.

# 2 Generic properties of $C^1$-diffeomorphisms and perturbation lemmas

Using recent results on the dynamics of $C^1$-generic diffeomorphisms, one gets a residual subset $\mathcal{G}$ of $\mathrm{Diff}^1(\mathrm{M})$ of diffeomorphisms $f$ verifying the following properties:

**(G1)** The periodic points of $f$ are dense in the chain recurrent set of $f$. In particular (recall that for $C^1$-generic diffeomorphisms $f$ the non-wandering set of $f$ is the closure of the periodic points of $f$, [24]), the non-wandering set and the chain recurrent set coincide, see [5, Corollaire 1.2]

**(G2)** Every chain recurrence class $\Lambda$ of $f$ containing a (hyperbolic) periodic point $p$ satisfies $\Lambda = H(p, f)$. In particular, for any pair of saddles $p$ and $q$ of $f$, either $H(p, f) = H(q, f)$ or $H(p, f) \cap H(q, f) = \emptyset$, see [5, Remarque 1.10] (for previous results see [4, 11]).

In what follows, if $p_f$ is a hyperbolic periodic point of a diffeomorphism $f$, we denote by $p_g$ the continuation of $p_f$ for $g$ close to $f$.

**(G3)** For every pair of saddles $p_f$ and $q_f$ of $f$, there is a neighborhood $\mathcal{U}_f$ of $f$ in $\mathcal{G}$ such that either $H(p_g, g) = H(q_g, g)$ for all $g \in \mathcal{U}_f$, or $H(p_g, g) \cap H(q_g, g) = \emptyset$ for all $g \in \mathcal{U}_f$, see [2, Lemma 2.1].

**(G4)** For every saddle $p_f$ of $f$ whose homoclinic class $H(p_f, f)$ contains saddles of indices $s$ and $s + k$, there is a neighborhood $\mathcal{U}_f$ of $f$ in $\mathcal{G}$ such that, for every $g \in \mathcal{U}_f$, the homoclinic class $H(p_g, g)$ contains saddles $q^s_g, q^{s+1}_g, \cdots, q^{s+k}_g$ of indices $s, s+1$, and $s + k$, see [2, Theorem 1].

Given a homoclinic class $H(p, f)$, denote by $\mathrm{Per}_h(H(p, f))$ the set of hyperbolic saddles $q$ *homoclinically related* to $p$, that is, the stable manifold of the orbit of $q$ meets the unstable manifold of the orbit of $p$ and vice-versa. Note that homoclinically related saddles have the same index. The set $\mathrm{Per}_h(H(p, f))$ is dense in $H(p, f)$, see, for instance, [20].

We say that a periodic point $p$ of period $\pi(p)$ of a diffeomorphism $f$ has *real multipliers* if every eigenvalue of the linear isomorphism $Df^{\pi(p)}(p) \colon T_pM \to T_pM$ is real, positive, and has multiplicity one. We denote by $\mathrm{Per}_{\mathbb{R}}(H(p, f))$ the subset of $\mathrm{Per}_h(H(p, f))$ of periodic points with real multipliers.

**(G5)** For every diffeomorphisms $f \in \mathcal{G}$ and every nontrivial homoclinic class $H(p, f)$ of $f$, the set $\mathrm{Per}_{\mathbb{R}}(H(p, f))$ is dense in $H(p, f)$, see [2, Proposition 2.3], which is just a dynamical reformulation of [7, Lemmas 1.9 and 4.16].

We close this section quoting some standard perturbations lemmas in $C^1$ dynamics. The first one allows us to perform dynamically perturbations of cocycles:

**Lemma 2.1. (Franks, [15])**. *Consider a diffeomorphism $f$ and an $f$-invariant finite set $\Sigma$. Let $A$ be an $\varepsilon$-perturbation of the derivative of $f$ in $\Sigma$ (i.e., the linear maps $Df(x)$ and $A(x)$ are $\varepsilon$-close for all $x \in \Sigma$). Then, for every neighborhood $U$ of $\Sigma$, there is $g$ $\varepsilon$-$C^1$-close to $f$ such that*

- *$f(x) = g(x)$ for every $x \in \Sigma$ and every $x \notin U$,*
- *$Dg(x) = A(x)$ for all $x \in \Sigma$.*

Next lemma allows us to obtaining intersection between invariant manifolds of periodic points:

**Lemma 2.2. (Hayashi's Connecting Lemma, [16])** *Let $p_f$ and $q_f$ be a pair of saddles of a diffeomorphism $f$ such that there are sequences of points $y_n$ and of natural numbers $k_n$ such that*

- *$y_n \to y \in W^u_{loc}(p_f, f)$, $y \neq p_f$, and*
- *$f^{k_n}(y_n) \to z \in W^s_{loc}(q_f, f)$, $z \neq p_f$.*

*Then there is a diffeomorphism $g$ arbitrarily $C^1$-close to $f$ such that $W^u(p_g, g)$ and $W^s(q_g, g)$ have an intersection arbitrarily close to $y$.*

Recall that a diffeomorphism $f$ has a *heterodimensional cycle* associated to the saddles $p$ and $q$ if $p$ and $q$ have different indices and both intersections $W^s(\mathcal{O}_p) \cap W^u(\mathcal{O}_q)$ and $W^s(\mathcal{O}_q) \cap W^u(\mathcal{O}_p)$ are non-empty. An immediate consequence of the connecting lemma is the following:

**Lemma 2.3.** *Let $\mathcal{U}$ be an open set of $\mathrm{Diff}^1(\mathrm{M})$ such that the saddles $p_f$ and $q_f$ of indices $\alpha$ and $\beta$, $\alpha < \beta$, are generically homoclinically linked in $\mathcal{U}$. There there is a dense subset $\mathcal{D}$ of $\mathcal{U}$ of diffeomorphisms $f$ having a heterodimensional cycle associated to $p_f$ and $q_f$.*

This result follows applying twice the Connecting Lemma to the diffeomorphisms in $\mathcal{U}$. First, using the transitivity of the homoclinic class $H(p_f, f)$, one gets a dense subset $\mathcal{T}$ of $\mathcal{U}$ such that the unstable manifold of the orbit of $q_f$ and the stable manifold of the orbit of $p_f$ have non-empty intersection. As the sum of the dimensions of these manifolds is strictly greater than the dimension of the ambient, one can assume that this intersection is transverse. Thus such an intersection persists by perturbations. Hence, the set $\mathcal{T}$ contains an open and dense subset $\mathcal{S}$ of $\mathcal{U}$ such that $W^s(\mathcal{O}_{p_f}) \cap W^u(\mathcal{O}_{q_f}) \neq \emptyset$, for every $f \in \mathcal{S}$. A new application of the Connecting Lemma, now interchanging the roles of $p_f$ and $q_f$, gives a dense subset $\mathcal{D}$ of $\mathcal{S}$ (thus a dense subset of $\mathcal{U}$) such that $W^s(\mathcal{O}_{q_f}) \cap W^u(\mathcal{O}_{p_f}) \neq \emptyset$. Thus every diffeomorphisms $f \in \mathcal{D}$ has a heterodimensional cycle associated to $p_f$ and $q_f$. For details see [2, Section 2.4]

# 3 Super-growth of periodic orbits in non-hyperbolic homoclinic classes

In this section, we prove Proposition 1.1. The main step of this proof is the following:

**Proposition 3.1.** *Let $(a_k)_k$ be a sequence of natural numbers and $\mathcal{U}$ an open set of* $\mathrm{Diff}^1(\mathrm{M})$ *such that there are saddles $p_f$ and $q_f$ generically homoclinically linked in $\mathcal{U}$.*

*Let $\alpha$ and $\beta$, $\alpha < \beta$, be the indices of the saddles $p_f$ and $q_f$. Then, for every $\gamma \in [\alpha, \beta] \cap \mathbb{N}$ and every $k \in \mathbb{N}$, there is a residual subset $\mathcal{G}^\gamma(k)$ of $\mathcal{U}$ such that for every diffeomorphism $\varphi \in \mathcal{G}^\gamma(k)$ there is $n_\varphi(k) \geq k$ such that the homoclinic class $H(p_\varphi, \varphi)$ contains at least $(n_\varphi(k)\, a_{n_\varphi(k)})$ different periodic orbits of period $n_\varphi(k)$ and index $\gamma$.*

**Proof of Proposition 1.1:** Proposition 1.1 is a straightforward consequence of Proposition 3.1. Consider any sequence $(a_k)_k$ of natural numbers and fix a natural number $\gamma \in [\alpha, \beta]$. Consider the intersection

$$\mathcal{R}^\gamma = \mathcal{R}^\gamma((a_k)_k) = \bigcap_k \mathcal{G}^\gamma(k).$$

By construction, this set is residual in $\mathcal{U}$. We claim that, for every $\varphi \in \mathcal{R}^\gamma$, it holds

$$\limsup \frac{\#\, \mathbb{P}_k^\gamma(H(p_\varphi, \varphi))}{a_k} = \infty.$$

Since $\varphi \in \mathcal{R}^\gamma$, one has that $\varphi \in \mathcal{G}^\gamma(k)$ for all $k \in \mathbb{N}$. Thus, for each $k$, there is $n_k(\varphi) \geq k$ such that the homoclinic class of $H(p_\varphi, \varphi)$ contains at least $(n_\varphi(k)\, a_{n_\varphi(k)})$ periodic orbits of index $\gamma$ and period $n_\varphi(k)$. As $n_k(\varphi) \to \infty$, there is a strictly increasing subsequence $(n_{k_j})$ of $(n_k(\varphi))$ with $(n_{k_j}) \to \infty$ and such that

$$\frac{\#\, \mathbb{P}_{n_{k_j}}^\gamma(H(p_\varphi, \varphi))}{a_{n_{k_j}}} \geq n_{k_j}.$$

This implies our claim.

Taking the residual subset $\mathcal{R}((a_k))$ of $\mathcal{U}$ defined by

$$\mathcal{R}((a_k)) = \bigcap_{\gamma=\alpha}^{\beta} \mathcal{R}^\gamma((a_k))$$

one concludes the proof of the proposition. $\square$

**Proof of Proposition 3.1:** Let $\mathcal{G}$ be the residual subset of $\mathrm{Diff}^1(\mathrm{M})$ in Section 2 and write $\mathcal{G}_\mathcal{U} = \mathcal{G} \cap \mathcal{U}$ (this set is residual in $\mathcal{U}$).

**Lemma 3.2.** *Let $\mathcal{U}$ be an open set such that there are saddles $p_f$ and $q_f$ of indices $\alpha$ and $\beta$, $\alpha < \beta$, which are generically homoclinically linked in $\mathcal{U}$. Then there for every $g \in \mathcal{G}_\mathcal{U}$ there is a neighbourhood $\mathcal{V}_g$ in $\mathcal{G}_\mathcal{U}$ such that for every $\varphi \in \mathcal{V}_g$ there are saddles $q_\varphi^\alpha, q_\varphi^{\alpha+1}, \dots, q_\varphi^\beta$ such that:*

- $H(p_g, g) = H(q_g^\alpha, g) = \cdots = H(q_g^\beta, g) = H(q_g, g)$,
- *the index of $q_\varphi^i$ is $i$,*

- *every saddle $q^i_\varphi$ has real multipliers,*
- *the saddles $q^i_\varphi$ depend continuously on $\varphi$.*

**Proof:** By generic properties (G2), (G3), and (G4), for every $\varphi \in \mathcal{G}_{\mathcal{U}}$ there are saddles $q^\alpha_\varphi, q^{\alpha+1}_\varphi, \dots, q^\beta_\varphi$ of indices $\alpha, \alpha+1, \dots, \beta$ such that

$$H(p_\varphi, \varphi) = H(q^\alpha_\varphi, \varphi) = \cdots = H(q^\beta_\varphi, \varphi) = H(q_\varphi, \varphi).$$

Moreover, by (G3), these saddles can be chosen depending continuously on a small neighbourhood of $\varphi$. Finally, by (G5), we can assume that these saddles have real multipliers. This concludes the proof of the lemma. □

Given a pair of hyperbolic periodic points $p$ and $q$, we write $p <_{\rm us} q$ if the unstable manifold $W^u(\mathcal{O}_p)$ of the orbit $\mathcal{O}_p$ of $p$ intersects transversally the stable manifold $W^s(\mathcal{O}_q)$ of the orbit $\mathcal{O}_q$ of $q$: there exists a point $x \in W^u(\mathcal{O}_p) \cap W^s(\mathcal{O}_q)$ such that $T_xM = T_xW^u(\mathcal{O}_p) + T_yW^s(\mathcal{O}_q)$.

**Remark 3.3.** *The property $<_{\rm us}$ is open in* $\mathrm{Diff}^1(\mathrm{M})$*: let $p_f$ and $q_f$ be hyperbolic periodic points of a diffeomorphism $f$ with $p_f <_{\rm us} q_f$, then there is a neighborhood $\mathcal{V}_f$ of $f$ in* $\mathrm{Diff}^1(\mathrm{M})$ *such that $p_g <_{\rm us} q_g$ for every $g \in \mathcal{V}_f$.*

The main step of the proof of Proposition 3.1 is the following:

**Proposition 3.4.** *Let $(a_k)_k$ a sequence of natural numbers. Let $f$ be a diffeomorphism having a pair of hyperbolic periodic saddles $p_f$ and $q_f$ with real multipliers. Assume that the indices of $p_f$ and $q_f$ are $\gamma$ and $\gamma+1$ and that $f$ has a heterodimensional cycle associated to $p_f$ and $q_f$.*

*Then for every $k \in \mathbb{N}$ there are $n_k \geq k$ and a diffeomorphism $g$ arbitrarily $C^1$-close to $f$ having $(n_k\, a_{n_k})$ saddles $r^\gamma_1, \dots, r^\gamma_{n_k\, a_{n_k}}$ of period $n_k$ and index $s$ and $(n_k\, a_{n_k})$ saddles $r^{\gamma+1}_1, \dots, r^{\gamma+1}_{n_k\, a_{n_k}}$ of period $n_k$ and index $\gamma+1$ such that*

$$p_g <_{\rm us} r^\gamma_i <_{\rm us} q_g \quad \textit{and} \quad p_g <_{\rm us} r^{\gamma+1}_i <_{\rm us} q_g, \quad \textit{for all } i = 1, \dots, n_k\, a_{n_k}.$$

*Moreover, the orbits of the saddles $r^\gamma_1, \dots, r^\gamma_{n_k\, a_{n_k}}, r^{\gamma+1}_1, \dots, r^{\gamma+1}_{n_k\, a_{n_k}}$ are different.*

We postpone the proof of this proposition to Section 4. We know prove Proposition 3.1 assuming it. We need the following result (we give the proof for completeness).

**Lemma 3.5 (Claim 4.3 in [2]).** *Let $f$ be a diffeomorphism having a pair of saddles $p$ and $q$ such that $H(p,f) = H(q,f)$. Consider a saddle $r$ of $f$ such that $p <_{\rm us} r <_{\rm us} q$. Then the saddles $p, r,$ and $q$ are in the same chain recurrent class.*

**Proof:** It suffices to see that for every $\varepsilon > 0$ there is a closed $\varepsilon$-pseudo-orbit containing $q, r$ and $p$. First, as $H(p,f) = H(q,f)$ and this set is transitive, there is $x_1$ $\varepsilon/2$-close to $f(q)$ such that $f^{n_1}(x_1)$ is $\varepsilon/2$-close to $p$.

Since $p <_{\rm us} r$, there is some $x_2 \in W^u(\mathcal{O}_p, f) \cap W^s(\mathcal{O}_r, f)$. Therefore there are positive numbers $n_2$ and $m_2$ such that $f^{-n_2}(x_2)$ is $\varepsilon/2$-close to $f(p)$ and $f^{m_2}(x_2)$ is $\varepsilon/2$-close to $r$. Similarly, $r <_{\rm us} q$ gives $x_3 \in W^u(\mathcal{O}_r, f) \cap W^s(\mathcal{O}_q, f)$ and positive numbers $n_3$ and $m_3$ such that $f^{-n_3}(x_3)$ is $\varepsilon/2$-close to $f(r)$ and $f^{m_3}(x_3)$ is $\varepsilon/2$-close to $q$.

The announced closed $\varepsilon$-pseudo-orbit containing $p$, $r$, and $q$ is obtained concatenating the segments of orbits above:

$$q, x_1, \dots, f^{n_1-1}(x), p, f^{-n_2}(x_2), \dots, f^{m_2-1}(x_2), r, f^{-n_3}(x_3), \dots, f^{m_3-1}(x_3), q.$$

The proof of the lemma is now complete. □

To conclude the proof of Proposition 3.1. Fix $\gamma \in [\alpha, \beta - 1] \cap \mathbb{N}$ and $g \in \mathcal{G}_{\mathcal{U}} = \mathcal{G} \cap \mathcal{U}$.

**Lemma 3.6.** *Let $\gamma \in [\alpha, \beta-1] \cap \mathbb{N}$, $\varphi$ a diffeomorphism in $\mathcal{G}_{\mathcal{U}}$, and $q^\gamma_\varphi$ and $q^{\gamma+1}_\varphi$ saddles as in Lemma 3.2 (real multipliers and indices $\gamma$ and $\gamma+1$). There is a dense subset $\mathcal{D}^\gamma_{\mathbb{R}}$ of $\mathcal{U}$ of diffeomorphisms $\phi$ having a heterodimensional cycle associated to $q^\gamma_\phi$ and $q^{\gamma+1}_\phi$.*

**Proof:** Take any $\varphi \in \mathcal{G}_{\mathcal{U}}$. By Lemma 3.2, the diffeomorphism $\varphi$ has saddles $q^\gamma_\varphi$ and $q^{\gamma+1}_\varphi$ such that $H(q^\gamma_g) = H(q^{\gamma+1}_g)$ for every $g \in \mathcal{G}$ close to $\varphi$. By Lemma 2.3, there is $\phi$ arbitrarily close to $\varphi$ with a heterodimensional cycle associated to $q^\gamma_\phi$ and $q^{\gamma+1}_\phi$. This ends the proof of the lemma. □

Fix $k \in \mathbb{N}$. Take $\phi \in \mathcal{D}^\gamma_{\mathbb{R}}$ and consider the cycle associated to $q^\gamma_\phi$ and $q^{\gamma+1}_\phi$. This cycle satisfies the hypothesis of Proposition 3.4. Thus, by Proposition 3.4 and Remark 3.3, there is an open set $\mathcal{V}_\phi(k)$ such that

- $\phi$ is in the closure of $\mathcal{V}_\phi(k)$;
- for every diffeomorphism $g \in \mathcal{V}_\phi(k)$, there is $n_g(k) > k$ such that $g$ has $(n_g(k)\, a_{n_g(k)})$ different orbits $\mathcal{O}_{r^\gamma_1}, \dots, \mathcal{O}_{r^\gamma_{n_g(k)\, a_{n_g(k)}}}$ of period $n_g(k)$ and index $\gamma$ and $(n_g(k)\, a_{n_g(k)})$ different orbits $\mathcal{O}_{r^{\gamma+1}_1}, \dots, \mathcal{O}_{r^{\gamma+1}_{n_g(k)\, a_{n_g(k)}}}$ of period $n_g(k)$ and index $(\gamma+1)$;
- the saddles verify $p_g <_{\mathrm{us}} r^\gamma_i <_{\mathrm{us}} q_g$ and $p_g <_{\mathrm{us}} r^{\gamma+1}_i <_{\mathrm{us}} q_g$, for all $i = 1, \dots, n_g(k)\, a_{n_g(k)}$.

Consider the set

$$\mathcal{V}^\gamma(k) = \bigcup_{\phi \in \mathcal{D}^\gamma_{\mathbb{R}}} \mathcal{V}_\phi(k).$$

By construction, the set $\mathcal{V}^\gamma(k)$ is open and dense in $\mathcal{U}$. Consider now the set

$$\mathcal{G}^\gamma(k) = \mathcal{G}_{\mathcal{U}} \cap \mathcal{V}^\gamma(k) \subset \mathcal{G}.$$

This set is residual in $\mathcal{U}$. By construction, $p_g <_{\mathrm{us}} r^\gamma_g <_{\mathrm{us}} q_g$ and $p_g <_{\mathrm{us}} r^{\gamma+1}_g <_{\mathrm{us}} q_g$, thus Lemma 3.5 implies that chain recurrence classes of $p_g, r^\gamma_g, r^{\gamma+1}_g$, and $q_g$ coincide for all $g \in \mathcal{G}^\gamma(k) \subset \mathcal{G}$. By (G2), the homoclinic classes of these saddles coincide for all $g \in \mathcal{G}^\gamma(k)$ and all $i = 1, \dots, n_g(k)\, a_{n_g(k)}$. Since $n_g(k) \geq k$, the set $\mathcal{G}^\gamma(k)$ verifies the conclusion in the Proposition 3.1. □

# 4 Generation of saddles at heterodimensional cycles

In this section, we prove Proof of Proposition 3.4. We need the following preparatory result:

**Proposition 4.1.** *Let $f$ be a diffeomorphism having a pair of periodic saddles $p_f$ and $q_f$ of indices $\gamma$ and $\gamma+1$, respectively, and with real multipliers. Assume that the diffeomorphism $f$ has a heterodimensional cycle associated to $p_f$ and $q_f$.*

*Then for every $C^1$-neighborhood $\mathcal{U}$ of $f$ there are constants $k_0 \in \mathbb{N}$ and $C > 0$ such that for every pair of natural numbers $\ell$ and $m$ large enough there is a diffeomorphism $g_{\ell,m} \in \mathcal{U}$ having a periodic point $r_{\ell,m}$ such that*

1. *the period $\pi(r_{\ell,m})$ of $r_{\ell,m}$ is $\ell\,\pi(p_f) + m\,\pi(q_f) + k_0$;*
2. *the point $r_{\ell,m}$ is partially hyperbolic, there is a $Df^{\pi(r_{\ell,m})}(r_{\ell,m})$-invariant splitting $T_{r_{\ell,m}}M = E^{ss} \oplus E^c \oplus E^{uu}$ such that $E^{ss}$ and $E^{uu}$ are uniformly hyperbolic (contracting and expanding, respectively), $\dim E^c = 1$, and $\dim E^{ss} = \gamma$;*
3. *the eigenvalue $\lambda_c(r_{\ell,m})$ of $Df^{\pi(r_{\ell,m})}(r_{\ell,m})$ corresponding to the central direction $E^c$ satisfies*
$$1/C < \lambda_c(r_{\ell,m}) < C;$$
4. $p_g <_{\mathrm{us}} r_{\ell,m} <_{\mathrm{us}} q_g$.

**Sketch of the proof of Proposition 4.1:** This proposition follows form the arguments in [2, Theorem 3.2]. For completeness, we outline the main steps and ingredients of the proof of this proposition. For details, see [2, 8].

By hypothesis, the saddles $p_f$ and $q_f$ have real eigenvalues, thus there are eigenvalues $\lambda_c$ of $Df^{\pi(p_f)}(p_f)$ and $\beta_c$ of $Df^{\pi(q_f)}(q_f)$ such that $1 < \lambda_c < \lambda$ for every expanding eigenvalue $\lambda$ of $Df^{\pi(p_f)}(p_f)$ and $1 > \beta_c > \beta$ for every contracting eigenvalue $\beta$ of $Df^{\pi(q_f)}(q_f)$. The eigenvalues $\lambda_c$ and $\beta_c$ are the *central eigenvalues of the cycle*.

The fact that the saddles $p_f$ and $q_f$ have real multipliers also implies that there is a (unique) $Df$-invariant dominated splitting[1] defined on the union of the orbits $\mathcal{O}_{p_f}$ of $p_f$ and $\mathcal{O}_{q_f}$ of $q_f$,
$$T_xM = E^{ss}_x \oplus E^c_x \oplus E^{uu}_x, \qquad x \in \mathcal{O}_{p_f} \cup \mathcal{O}_{q_f},$$
where $\dim E^c_x = 1$ and $\dim E^{ss}_x = \gamma$. We let $\nu = \dim E^{uu}_x$.

After a $C^1$-perturbation of $f$, one gets a new heterodimensional cycle (associated to the same saddles $p_f$ and $q_f$ with real multipliers) and local coordinates at these saddles such that the dynamics of $f$ in a neighborhood of the cycle is affine (this corresponds to the notion of *affine heterodimensional cycle* in [2, Section 3.1]). Let us explain this point more precisely. For that we introduce some notations. The elements are depicted in the figure.

We fix small neighborhoods of $U_p$ and $U_q$ of the orbits of $p_f$ and $q_f$ and heteroclinic points $x \in W^s(p_f, f) \cap W^u(q_f, f)$ and $y \in W^u(p_f, f) \cap W^s(q_f, f)$. After a perturbation, we can assume that

[1] A $Df$-invariant splitting $E \oplus F$ of $TM$ over an $f$-invariant set $\Lambda$ is *dominated* if the fibers of the bundles have constant dimension and there are a metric $||\cdot||$ and a natural number $n \in \mathbb{N}$ such that
$$||Df^n(x)_E|| \cdot ||Df^{-n}(x)_F|| < \frac{1}{2}, \quad \text{for all } x \in \Lambda.$$
For splittings with three bundles $E \oplus F \oplus G$, domination means that the splittings $(E \oplus F) \oplus G$ and $E \oplus (F \oplus G)$ are both dominated.

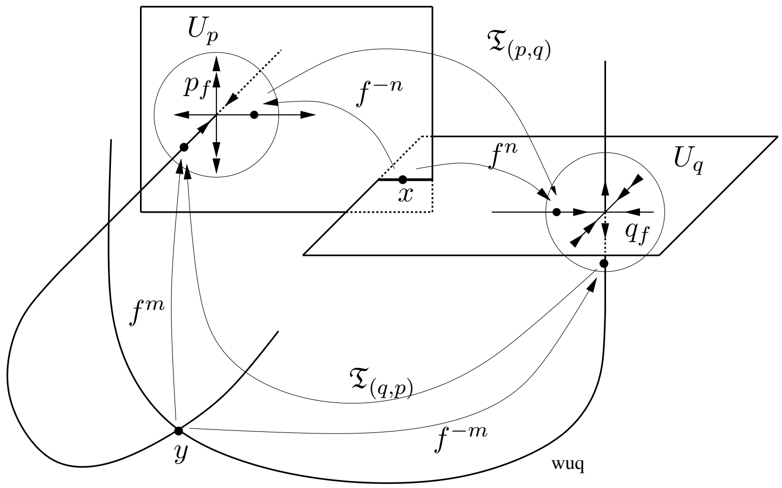


Figure 1: An affine heterodimensional cycle

- the intersection between $W^u(p_f, f)$ and $W^s(q_f, f)$ at $x$ is transverse, and
- the intersection between $W^s(p_f, f)$ and $W^u(q_f, f)$ at $y$ is quasi-transverse (i.e., $T_yW^s(p_f, f)+T_yW^u(q_f, f) = T_yW^s(p_f, f) \oplus T_uW^s(q_f, f)$ and this sum has dimension $n-1$, where $n$ is the dimension of the ambient).

Then there are neighborhoods $U_x$ of $x$ and $U_y$ of $y$ and natural numbers $n$ and $m$ such that

$$f^{-n}(U_x) \subset U_p, \quad f^n(U_x) \subset U_q, \quad f^m(U_y) \subset U_p, \quad \text{and} \qquad f^{-m}(U_y) \subset U_q.$$

We say that $t_{(p,q)} = 2\,n$ and $t_{(q,p)} = 2\,m$ are *transition times* from $U_p$ to $U_q$ and from $U_q$ to $U_p$, respectively. The maps

$$\mathfrak{T}_{(p,q)} = f^{t_{(p,q)}} \quad \text{and} \quad \mathfrak{T}_{(q,p)} = f^{t_{(q,p)}}$$

are *transition maps* from $U_p$ to $U_q$ and from $U_q$ to $U_p$. These maps are defined on small neighborhoods $U_x^-$ of $f^{-n}(x)$ and $U_y^-$ of $f^{-m}(y)$.

Using the domination, by increasing $n$ and $m$ and after a small perturbation, one can assume that the transition maps preserve the dominated splitting $E^{ss} \oplus E^c \oplus E^{uu}$ defined above. More precisely, in the neighborhoods $U_p$ and $U_q$ the expressions of $f^{\pi(p_f)}$ and $f^{\pi(q_f)}$ are linear. Then, in these local charts, the splitting $E^{ss} \oplus E^c \oplus E^{uu}$ is of the form

$$E^{ss} = \mathbb{R}^\gamma \times \{(0, 0^\nu)\}, \quad E^c = \{0^\gamma\} \times \mathbb{R} \times \{0^\nu\}, \quad E^{uu} = \{(0^\gamma, 0)\} \times \mathbb{R}^\nu.$$

In this way, one also has that the maps

$$\mathfrak{T}_{(p,q)} = f^{t_{(p,q)}} : U_x^- \to U_q \qquad \text{and} \qquad \mathfrak{T}_{(q,p)} = f^{t_{(q,p)}} : U_y^- \to U_p$$

are affine maps preserving the splitting $E^{ss} \oplus E^c \oplus E^{uu}$. More precisely,

$$\mathfrak{T}_{(i,j)} = (T^s_{(i,j)}, T_{(i,j)}, T^u_{(i,j)}), \quad i, j = p, q \quad \text{or} \quad i, j = q, p,$$

where

$$T^s_{(i,j)}\colon \mathbb{R}^\gamma \to \mathbb{R}^\gamma, \qquad T^c_{(i,j)}\colon \mathbb{R} \to \mathbb{R}, \qquad T^u_{(i,j)}\colon \mathbb{R}^\nu \to \mathbb{R}^\nu,$$

are affine maps such that $T^s_{(i,j)}$ is a contraction (i.e., its norm is less than one), $T^u_{(i,j)}$ is an expansion (i.e., $(T^u_{(i,j)})^{-1}$ is a contraction). We let $\tau_{(i,j)}$ the derivative of $T^c_{(i,j)}$. In fact, $T^c_{(q,p)}$ is a linear map.

The previous construction gives the dynamics of $f$ in a neighbourhood $V$ of the cycle,

$$V = U_p \cup U_q \cup \left(\cup_{i=-n}^{n} f^i(U_x)\right) \cup \left(\cup_{i=-m}^{m} f^i(U_y)\right).$$

Next step consists in considering the unfolding of this cycle. For this we consider a one-parameter family of transitions $(\mathfrak{T}_{(q,p),\rho})_\rho$ from $q_f$ to $p_f$ defined as follows

$$\mathfrak{T}_{(q,p),\rho} = \mathfrak{T}_{(q,p)} + (0^\gamma, \rho, 0^\nu).$$

To each small $\rho$ corresponds a diffeomorphism $f_\rho$ (which is a local perturbation of $f$ at the heteroclinic point $y$) such that, for every large $\ell$ and $m$, there is a small subset $U_{\ell,m}$ of $U_x^-$ such that

$$f_\rho^{\pi_{\ell,m}}|_{U_{\ell,m}} = f^{\ell\,\pi(p_f)} \circ \mathfrak{T}_{(q,p),\rho} \circ f^{\ell\,\pi(q_f)} \circ \mathfrak{T}_{(p,q)}, \quad \pi_{\ell,m} = \ell\,\pi(p_f) + t_{(q,p)} + m\,\pi(q_f) + t_{(p,q)}.$$

Moreover, $f_\rho^{\pi_{\ell,m}}$ maps $U_{\ell,m}$ into $U_p$.

Next step is to find a parameter $\rho = \rho_{\ell,m}$ such that $f_\rho$ has a periodic point of period $\pi_{\ell,m}$. Note that, by construction, in a neighbourhood of the cycle, the diffeomorphism $f_\rho$ keeps invariant the codimension one foliation generated by the sum of the strong stable and strong unstable directions (hyperplanes parallel to $\mathbb{R}^\gamma \times \{0\} \times \mathbb{R}^\nu$). Also note that $f_\rho$ acts hyperbolically on these hyperplanes. We consider the quotient dynamics of $f_\rho$ by this strong stable/strong unstable foliation, obtaining a one-dimensional map. Fixed points of this quotient dynamics will correspond to periodic points of the diffeomorphism $f_\rho$. Let us explain this point more precisely.

Suppose for simplicity that, in local coordinates,

$$x^- = f^{-n}(x) = (0^\gamma, 1, 0^\nu) \in U_p \quad \text{and} \quad x^+ = f^n(x) = (0^\gamma, -1, 0^\nu) \in U_q.$$

Assume, for instance, that $T^c_{(q,p)}(z) = \tau_{(q,p)}\, z$ (in fact, the case $T^c_{(q,p)}(z) = -\tau_{(q,p)}\, z$ is simpler). Fix now large $\ell$ and $m$ and take $\rho_{\ell,m}$ such that

$$\lambda^\ell\,(-\tau_{(q,p)}\,\beta^m + \rho_{\ell,m}) = 1, \qquad \rho_{\ell,m} = \lambda^{-\ell} + \tau_{(q,p)}\,\beta^m.$$

This choice and $T^c_{(p,q)}(1) = -1$ imply that the quotient dynamics satisfy:

$$\lambda^\ell \circ T^c_{(q,p),\rho} \circ \beta^m \circ T^c_{(p,q)}(1) = \lambda^\ell\,(-\tau_{(q,p)}\,\beta^m + \rho_{\ell,m}) = 1$$

Thus 1 is a fixed point of the quotient dynamics.

Since $f_{\ell,m} = f_{\rho_{\ell,m}}$ preserves the $E^{ss}$, $E^{uu}$, and $E^c$ directions, the hyperbolicity of the directions $E^{ss}$ and $E^{uu}$ implies that the map

$$f^{\ell\,\pi(p_f)} \circ \mathfrak{T}_{(q,p),\rho_{\ell,m}} \circ f^{m\,\pi(q_f)} \circ \mathfrak{T}_{(p,q)}$$

has a fixed point $r_{\ell,m}$ of the form $r_{\ell,m} = (r^\gamma_{\ell,m}, 1, r^\nu_{\ell,m})$ in $U_{\ell,m}$. The point $r_{\ell,m}$ is a periodic point of period $\pi_{\ell,m}$ of $f_{\ell,m}$.

By construction, the periodic point $r_{\ell,m}$ is uniformly expanding in the $E^{uu}$ direction, uniformly contracting in the $E^{ss}$ direction, and the derivative of $Df_{\ell,m}^{\pi_{\ell,m}}(r_{\ell,m})$ the central direction is

$$\kappa_{\ell,m} = \lambda^{\ell}\,\tau_{(p,q)}\,\beta^{m}\,\tau_{(q,p)}.$$

Note that we can choose large $\ell$ and $m$ with

$$\lambda^{-1} \le \beta^{m}\,\tau_{(p,q)}\,\lambda^{\ell}\,\tau_{(q,p)} \le \lambda.$$

Taking $C = \lambda$ and $k_0 = t_{(p,q)} + t_{(q,p)}$, we get the first three items in the proposition.

The last item of the proposition, $p_g <_{\mathrm{us}}< r_{\ell,m} <_{\mathrm{us}} q_g$ is exactly [2, Proposition 3.10]. Consider the points $h_{\ell,m}$ and $d_{\ell,m}$ of the $f_{\ell,m}$ orbit of $r_{\ell,m}$,

$$\begin{aligned} h_{\ell,m} &= f^{-\ell\,\pi(p_f)}(r_{\ell,m}) = (h^{\gamma}_{\ell,m}, \lambda^{-\ell}, h^{\nu}_{\ell,m}), \\ d_{\ell,m} &= f_{\ell,m}^{-t_{(q,p)}}(h_{\ell,m}) = \mathfrak{T}^{-1}_{(q,p),\rho_{\ell,m}}(h_{\ell,m}) = f_{\ell,m}^{m+t_{(p,q)}}(r_{\ell,m}) = (d^{\gamma}_{\ell,m}, \beta^{m}, d^{\nu}_{\ell,m}). \end{aligned}$$

The key step is to observe that, by construction,

$$\begin{aligned} \Delta^{s}_{\ell,m} &= [-1,1]^{\gamma} \times \{(\lambda^{-\ell}, h^{\nu}_{\ell,m})\} \subset W^{s}(h_{\ell,m}, f_{\ell,m}) \subset U_p, \\ \Delta^{u}_{\ell,m} &= \{(d^{\gamma}_{\ell,m}, \beta^{m})\} \times [-1,1]^{\nu} \subset W^{u}(d_{\ell,m}, f_{\ell,m}) \subset U_q, \end{aligned}$$

where $h^{\gamma}_{\ell,m} \to 0^{\gamma}$ and $d^{\nu}_{\ell,m} \to 0^{\nu}$. For details, see [2, Lemma 3.11].

Noting that, in the the coordinates in $U_p$, $\{0^{\gamma}\} \times [-1,1]^{\nu+1}$ is contained in the unstable manifold of the orbit of $p_f = p_{f_{\ell,m}}$, one has $p_{f_{\ell,m}} <_{\mathrm{us}} r_{\ell,m}$. The relation $r_{\ell,m} <_{\mathrm{us}} q_{f_{\ell,m}}$ follows noting that, in the local coordinates in $U_q$, $[-1,1]^{\gamma+1} \times \{0^{\nu}\}$ is contained in the unstable manifold of the orbit of $q_f = q_{f_{\ell,m}}$. $\square$

## 4.1 Proof of Proposition 3.4

Using Lemma 2.1, we next consider a perturbation of the dynamics of $f_{\ell,m}$ along the orbit of $r_{\ell,m}$ consisting in a multiplication of the derivative of $f_{\ell,m}$ along the central direction by a factor

$$(\kappa_{\ell,m})^{1/\pi_{\ell,m}}$$

In this way, we have a diffeomorphism $g_{\ell,m}$ such that $r_{\ell,m}$ is a partially hyperbolic periodic point of period $\pi_{\ell,m}$ whose derivative in the central direction is the identity. Moreover, if $W^{ss}(\mathcal{O}_{r_{\ell,m}}, g_{\ell,m})$ is the strong stable manifold of the orbit of $r_{\ell,m}$ and $W^{uu}(\mathcal{O}_{r_{\ell,m}}, g_{\ell,m})$ is the strong unstable manifold of the orbit of $r_{\ell,m}$ we have that

$$W^{u}(\mathcal{O}_p, g_{\ell,m}) \pitchfork W^{ss}(\mathcal{O}_{r_{\ell,m}}, g_{\ell,m}) \neq \emptyset \quad \text{and} \quad W^{uu}(\mathcal{O}_{r_{\ell,m}}, g_{\ell,m}) \pitchfork W^{s}(\mathcal{O}_q, g_{\ell,m}) \neq \emptyset,$$

where $\pitchfork$ means transverse intersection. We fix disk $\Upsilon^{s} \subset W^{ss}(\mathcal{O}_{r_{\ell,m}}, g_{\ell,m})$ and $\Upsilon^{u} \subset W^{uu}(\mathcal{O}_{r_{\ell,m}}, g_{\ell,m})$ such that $\Upsilon^{s} \pitchfork W^{u}(\mathcal{O}_p, g_{\ell,m}) \neq \emptyset$ and $\Upsilon^{u} \pitchfork W^{s}(\mathcal{O}_p, g_{\ell,m}) \neq \emptyset$.

Given now any $\kappa$, $\kappa > \pi_{\ell,m}\, a_{\pi_{\ell,m}}$ ($(a_k)$ is the sequence of natural numbers in the proposition), there is $\phi^{\kappa}_{\ell,m}$ arbitrarily close to $g_{\ell,m}$ such that $\phi^{\kappa}_{\ell,m}$ has $\kappa$-saddles $r^{s}_{1}, \dots, r^{s}_{\kappa}$ of index $\gamma + 1$ and $\kappa$-saddles $r^{u}_{1}, \dots, r^{u}_{\kappa}$ of index $\gamma$, all of them of period $\pi_{\ell,m}$, whose orbits are arbitrarily close to the $g_{\ell,m}$-orbit of $r_{\ell,m}$. This implies that

- The stable manifold of the orbit of $r_i^s$ contains a disk close to $\Upsilon^s$. Thus $W^u(\mathcal{O}_p, \phi^\kappa_{\ell,m}) \pitchfork W^s(\mathcal{O}_{r_i^s}, \phi^k_{\ell,m}) \neq \emptyset$, and $p <_{\rm us} r_i^s$.
- The unstable manifold of the orbit of $r_i^s$ contains a disk close to $\Upsilon^u$. Thus $W^s(\mathcal{O}_q, \phi^k_{\ell,m}) \pitchfork W^u(\mathcal{O}_{r_i^s}, \phi^k_{\ell,m}) \neq \emptyset$, and $r_i^s <_{\rm us} q$.

A similar argument holds for the saddles $r_1^u, \dots, r_\kappa^u$ of index $(\gamma+1)$. Therefore, $p <_{\rm us}< r_i^u <_{\rm us} q$, for all $i = 1, \dots \kappa$. This completes the proof of the proposition. □

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**Christian Bonatti** (bonatti@u-bourgogne.fr)
Institut de Mathématiques de Bourgogne
B.P. 47 870
21078 Dijon Cedex
France

**Lorenzo J. Díaz** (lodiaz@mat.puc-rio.br)
Depto. Matemática, PUC-Rio
Marquês de S. Vicente 225
22453-900 Rio de Janeiro RJ
Brazil

**Todd Fisher** (tfisher@math.umd.edu)
Department of Mathematics
University of Maryland
College Park, MD 20742-4015
USA