

Non-hyperbolic ergodic measures with large support

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Abstract

We prove that there is a residual subset \mathcal{S} in $\text{Diff}^1(M)$ such that, for every $f \in \mathcal{S}$, any homoclinic class of f with invariant one dimensional central bundle containing saddles of different indices (i.e. with different dimensions of the stable invariant manifold) coincides with the support of some invariant ergodic non-hyperbolic (one of the Lyapunov exponents is equal to zero) measure of f .

1 Introduction

How to characterize the absence of uniform hyperbolicity? What dynamical structures can not exist in the uniformly hyperbolic setting but must be present in the complement? In this paper, we study how the non-hyperbolicity is detected in the ergodic level. Namely, we consider non-hyperbolic invariant measures as indication of non-hyperbolicity, and construct such measures with full support for partially hyperbolic homoclinic classes.

Some other candidates for the role of “non-hyperbolic structure” are the cycles (homoclinic tangencies and heterodimensional cycles), super-exponential growth of the number of periodic points [24, 7], absence of shadowing property [9, 37, 3, 34], and non-existence of symbolic extensions [18, 4, 16].

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Recall that if μ is an ergodic measure of a diffeomorphism $f : M \rightarrow M$, $\dim M = \mathbf{m}$, then there is a set Λ of full μ -measure and real numbers $\chi_\mu^1 \leq \chi_\mu^2 \leq \dots \leq \chi_\mu^{\mathbf{m}}$ such that, for every $x \in \Lambda$ and every non-zero vector $v \in T_x M$, one has $\lim_{n \rightarrow \infty} 1/n \log \|Df^n(v)\| = \chi_\mu^i$ for some $i = 1, \dots, \mathbf{m}$, see [30, 26]. The number χ_μ^i is the i -th *Lyapunov exponent* of the measure μ .

Definition 1.1. *An ergodic invariant measure of a diffeomorphism is called non-hyperbolic if at least one of its Lyapunov exponents is equal to zero.*

Inspired by Palis' density conjecture ([31]), we stated the following ([15]):

Conjecture 1. *In $\text{Diff}^r(M)$, $r \geq 1$, there exists an open and dense subset $\mathcal{U} \subset \text{Diff}^r(M)$ such that every diffeomorphism $f \in \mathcal{U}$ is either uniformly hyperbolic or has an ergodic non-hyperbolic invariant measure.*

We consider the question of the existence of non-hyperbolic measures for homoclinic classes. The *homoclinic class* of a hyperbolic periodic point P of a diffeomorphism f , denoted by $H(P, f)$, is the closure of the transverse intersections of the invariant manifolds of the orbit of P , see [29]. A homoclinic class is a *transitive set* (there exists a dense orbit) where periodic points also form a dense subset. Note that a homoclinic class may fail to be hyperbolic and may contain saddles of different *s-indices* (dimension of the stable bundle). This is precisely the setting we consider in this paper. In many important cases homoclinic classes are used to structure the global dynamics, playing a role similar to that of the basic sets in the hyperbolic theory (in fact, basic sets are a special case of homoclinic classes). In [15] we show the following:

Theorem. *There is a residual subset \mathcal{S} of $\text{Diff}^1(M)$ such that, for every $f \in \mathcal{S}$, any homoclinic class of f containing saddles of different *s-indices* (dimension of the stable bundle) contains also an uncountable support of an invariant ergodic non-hyperbolic measure of f .*

In view of this result, it is natural to consider the following question about the support of the ergodic non-hyperbolic measures in non-hyperbolic homoclinic class.

Question 1. *When does a non-hyperbolic homoclinic class equal to the support of a non-hyperbolic ergodic measure?*

The main result of this paper is the following (for an accurate formal statement see Theorem 1):

Main result. *For a C^1 -generic diffeomorphism f , every homoclinic class H with a 1-dimensional central direction and saddles of different indices is the support of an ergodic non-hyperbolic invariant measure μ of f , $H = \text{supp } \mu$.*

In an opposite direction, we would like to mention the results in [1]: for C^1 -generic diffeomorphisms generic measures supported on isolated homoclinic class are ergodic and hyperbolic (all Lyapunov exponents are non-zero).

The constructions in [15] translate to the setting of homoclinic classes the arguments in [21] and [25] about the existence of non-hyperbolic ergodic measures for skew products defined over the circle \mathbb{S}^1 and the corresponding smooth realizations. There the circle \mathbb{S}^1 corresponds to the center direction and the fact that the central direction is one-dimensional is essential. This condition can be precisely formulated in terms of dominated splittings.

Definition 1.2 (Dominated splitting and partial hyperbolicity). *Consider a diffeomorphism f and a compact f -invariant set Λ . A Df -invariant splitting $T_\Lambda M = E \oplus F$ over Λ is dominated if the fibers E_x and F_x of E and F have constant dimension and there is a constant $k \in \mathbb{N}$ such that*

$$\frac{\|D_x f^k(u)\|}{\|D_x f^k(w)\|} < \frac{1}{2},$$

for every $x \in \Lambda$ and every pair of unitary vectors $u \in E_x$ and $w \in F_x$.

In some cases, we consider splittings with three bundles. A Df -invariant splitting

$$T_\Lambda M = E \oplus F \oplus G$$

over Λ is dominated if both splittings $(E \oplus F) \oplus G$ and $E \oplus (F \oplus G)$ are dominated.

The dominated splitting $T_\Lambda M = E \oplus F \oplus G$ is partially hyperbolic if E and G are both uniformly hyperbolic and at least one of them is not empty. We say that F is the central direction of the splitting.

In [15] general (non-hyperbolic) homoclinic classes are considered (i.e., there is no hyperbolic-like assumptions). A key step in [15] is to identify a partially hyperbolic region Λ of the homoclinic class where the (non-hyperbolic) central direction has dimension one: there is a partially hyperbolic splitting $T_\Lambda M = E \oplus F \oplus G$, where F is one-dimensional and non-hyperbolic and E and G are uniformly hyperbolic.

In general, this partially hyperbolic region Λ is properly contained in the whole homoclinic class, thus the support of the obtained measure is not the

whole homoclinic class. Prototypical examples of this case are the Derived from Anosov diffeomorphisms defined on \mathbb{T}^3 obtained via Hopf bifurcations, see [12, 11], where the homoclinic classes are the whole torus and its dominated splitting is of the form $T\mathbb{T}^3 = E \oplus E^u$, where E is two-dimensional, non-hyperbolic, and indecomposable (it does not contain one-dimensional invariant directions). Thus in these cases the arguments in [15] do not provide a non-hyperbolic measure supported on the whole homoclinic class.

On the other hand, when the non-hyperbolic central bundle has a one-dimensional invariant direction one can prove the following:

Theorem 1. *Let M be a closed manifold, $\dim M \geq 3$. There is a residual subset \mathcal{R} of $\text{Diff}^1(M)$ such that for every $f \in \mathcal{R}$ and every homoclinic class $H(f)$ of f such that*

- *$H(f)$ has a dominated splitting $T_{H(f)}M = E \oplus F \oplus G$, where F has dimension one,*
- *$H(f)$ contains saddles of s -indices $\dim(E)$ and $\dim(E \oplus F) = \dim(E) + 1$,*

there is a non-hyperbolic ergodic f -invariant measure $\mu_{H(f)}$ such that

$$\text{supp } \mu_{H(f)} = H(f).$$

The simplest setting for our result is the one of homoclinic classes having a partially hyperbolic splitting with a one-dimensional central direction. There are two main sorts of such homoclinic classes. On the one hand, there are the non-hyperbolic homoclinic classes generated by unfolding heterodimensional cycles, see [14, 17]. On the other hand, there are robustly transitive and non-hyperbolic diffeomorphisms (in this case, the homoclinic class is the whole ambient manifold). Recall that a diffeomorphism f is *robustly transitive* if there is a neighborhood \mathcal{U}_f of f in $\text{Diff}^1(M)$ consisting of *transitive diffeomorphisms*: every $g \in \mathcal{U}_f$ has a dense orbit in M .

Among the robustly transitive diffeomorphisms with one-dimensional central direction, we mention the Derived from Anosov diffeomorphisms (via saddle-node or fork bifurcations) in [26], the time-one maps of transitive Anosov vector fields, see [5], and the perturbations of products of Anosov diffeomorphisms and maps defined on the circle (skew products), [5, 35, 33]. For these cases, Theorem 1 just can be read as follows:

Corollary 1. *Let M be a closed manifold, $\dim M \geq 3$. There is a residual subset \mathcal{R} of $\text{Diff}^1(M)$ such that for every $f \in \mathcal{R}$ and every partially hyperbolic homoclinic class of f having saddles of different indices and an one-dimensional central bundle there is a non-hyperbolic ergodic f -invariant measure whose support is the whole homoclinic class.*

We next discuss the interplay between the results above, the sorts of dominated splittings of homoclinic classes, and the occurrence of homoclinic tangencies. Consider the subset $\text{HT}^1(M) \subset \text{Diff}^1(M)$ of diffeomorphisms f having a *homoclinic tangency* associated to some saddle (i.e., the invariant manifolds of the saddle have some non-transverse intersection). Define the (open) set of *diffeomorphisms far from homoclinic tangencies* by

$$\text{FT}^1(M) = \text{Diff}^1(M) \setminus \overline{\text{HT}^1(M)}.$$

A recent result by Yang states the following dichotomy for C^1 -generic diffeomorphisms f far from homoclinic tangencies (i.e., diffeomorphisms f in a residual subset of $\text{FT}^1(M)$): any homoclinic class of f is either hyperbolic or it supports a non-hyperbolic ergodic measure, see [36]. Note that, in principle, there is the possibility for a generic diffeomorphism f to have non-hyperbolic homoclinic classes such that all saddles have the same s -index. For homoclinic classes of diffeomorphisms in $\text{FT}^1(M)$ containing saddles of different indices, Theorem 1 implies that there is an ergodic non-hyperbolic measure whose support is the whole homoclinic class.

Finally, we state a refinement of Theorem 1 about the zero Lyapunov exponents of the non-hyperbolic measure. Consider a homoclinic class $H(f)$ of f , its *s -index (variation) interval* is the interval $s\text{-ind}(H(f)) = [i, j]$, where i and j are the minimum and the maximum of the *s -indices* of the periodic points in $H(f)$. The homoclinic class $H(f)$ has *index variation* if $i < j$.

Corollary 2. *There is a residual subset \mathcal{R} of $\text{Diff}^1(M)$ such that for every diffeomorphism $f \in \text{FT}^1(M) \cap \mathcal{R}$, every homoclinic class $H(f)$ of f with s -index variation interval $[i, j]$, and every $k \in [i, j]$ there is an ergodic measure $\mu_{H(f)}^k$ whose support is the whole homoclinic class $H(f)$ such that its k -th Lyapunov exponent $\chi^k(\mu_{H(f)}^k)$ is zero.*

Indeed, fix $k \in [i, j]$. Theorem 1.1 in [22] claims that if $H(f)$ is a homoclinic class such that $k, k+1$ belong to the index interval $s\text{-ind}(H(f))$ and there is no dominated splitting defined over $H(f)$ of the form $E \oplus F \oplus G$ with $\dim(E) = k$

and $\dim(F) = 1$, then there is a diffeomorphism g arbitrarily C^1 -close to f having a homoclinic tangency associated to a saddle of s-index k . Therefore, since $f \in \text{FT}^1(M)$, the homoclinic class $H(f)$ has a dominated splitting of the form $E \oplus F \oplus G$, where $\dim(E) = k$ and F is one-dimensional. On the other hand, by [2], as f is a C^1 -generic diffeomorphism the homoclinic class $H(f)$ contains saddles of s-indices $k, k + 1 \in [i, j]$. Now we can apply Theorem 1 to get Corollary 2.

We close this introduction discussing the special case of robustly non-hyperbolic and transitive diffeomorphisms. We point out that the proof of our results makes a systematic use of the machinery recently developed about the dynamics and generic properties of homoclinic classes of C^1 diffeomorphisms, see Section 3 for details. Therefore our results are formulated for C^1 -generic diffeomorphisms.

On the other hand, the setting of robustly transitive and non-hyperbolic diffeomorphisms is a natural framework for considering Question 1 for open sets of non-hyperbolic diffeomorphisms. Moreover, in a recent paper, Nalsky gave examples of open sets of diffeomorphisms exhibiting non-hyperbolic ergodic measures supported on the whole homoclinic classes, [28]. In his case, one essentially has a skew product over a hyperbolic diffeomorphism in a base and a circle as a fiber. This example improves the construction in [21] mentioned before. Motivated by these results we want to formulate the partial case of the Conjecture 1 that looks more approachable:

Conjecture 2. *Let M be a closed manifold with $\dim(M) \geq 3$. Consider an open subset $\mathcal{T}(M)$ of $\text{Diff}^1(M)$ consisting of non-hyperbolic transitive diffeomorphisms f such that there is a Df -invariant dominated splitting $TM = E \oplus F \oplus G$, where F has dimension one and is non-hyperbolic. Then there is an open and dense subset $\mathcal{N}(M)$ of $\mathcal{T}(M)$ consisting of diffeomorphisms f having a non-hyperbolic ergodic measure μ_f with full support, $\text{supp } \mu_f = M$.*

Notice that the approach used in this paper cannot be used straightforwardly to settle this conjecture. Indeed, our construction involves measures supported on periodic orbits, and therefore periodic orbits are dense in the support of the measure that we obtain. It is an open question whether periodic orbits are dense in a nonwandering set for an open and dense subset of the set of robustly transitive diffeomorphisms; C^1 -Closing Lemma [32] implies this property only for a residual set of C^1 -diffeomorphisms.

This paper is organized as follows. In Section 2 we develop the idea initially suggested by Ilyashenko in [21] for constructing ergodic invariant measures as limit of measures supported on periodic orbits. In Section 3 we state properties of homoclinic classes of C^1 -generic diffeomorphisms and recall results about heterodimensional cycles. Finally, in Section 4 we construct a collection of atomic measures supported on periodic orbits whose limit is a non-hyperbolic ergodic measure with full support, thus proving our main result.

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2 Non-hyperbolic ergodic measures as limits of atomic measures

The approach suggested in [21] allows to construct non-hyperbolic ergodic invariant measures as limits of measures supported on special sequences of periodic points. Here we show that the support of the limit measure constructed in this way is a “topological limit” of the sequence of periodic points, see Proposition 2.7 for details.

2.1 Ergodicity, invariant direction fields, and Lyapunov exponents

Consider a diffeomorphism $f : M \rightarrow M$ and a closed invariant set $\Delta \subset M$. Assume that there is a Df -invariant continuous direction field $E = (E_x)_{x \in \Delta}$ in Δ . Then for every invariant measure μ whose support is contained in Δ one of

the Lyapunov exponents of μ is associated to E (denote it by χ^E). Namely, for μ -a.e. $x \in M$ and for every non-zero vector $v \in E_x \subset T_x M$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log |Df^n(v)| = \chi^E(\mu).$$

In this section, by convergence of a sequence of measures we mean $*$ -weak convergence: μ_n converges to μ , if for any continuous function $\varphi: M \rightarrow \mathbb{R}$ it holds

$$\int \varphi d\mu_n \rightarrow \int \varphi d\mu, \quad \text{as } n \rightarrow \infty.$$

To obtain non-hyperbolic measures we will use the following statement.

Lemma 2.1 ([15], Lemma 2.1, or [21], Lemma 1). *Let diffeomorphism $f: M \rightarrow M$ have an invariant continuous direction field E in an f -invariant closed set $\Delta \subset M$. Let μ_n and μ be ergodic probability measures with supports in Δ , and $\mu_n \rightarrow \mu$ as $n \rightarrow \infty$. Then $\chi^E(\mu_n) \rightarrow \chi^E(\mu)$.*

2.2 Sufficient conditions for ergodicity and convergence

In this section we state an improved version of Lemma 2 from [21]. We need the following definition.

Definition 2.2 (n -measure). *Let \mathfrak{G} be a continuous map of a metric compact space \mathfrak{Q} into itself. A n -measure of the point x_0 is an atomic measure uniformly distributed on n subsequent iterations of the point x_0 under the map \mathfrak{G} :*

$$\nu_n(x_0) = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{\mathfrak{G}^i(x_0)},$$

where δ_x is δ -measure supported at point x .

Lemma 2.3. *Let $\{X_n\}$ be a sequence of periodic orbits with increasing periods $\pi(X_n)$ of a continuous map \mathfrak{G} of a compact metric space \mathfrak{Q} into itself. For each n , let μ_n be the probability atomic measure uniformly distributed on the orbit X_n .*

Assume that for every $\varepsilon > 0$ and every $m \in \mathbb{N}$ there exists a subset $\tilde{X}_{m,\varepsilon} \subset X_m$ such that the following holds. For each continuous function φ on \mathfrak{Q} there exists $N = N(\varepsilon, \varphi) \in \mathbb{N}$ such that for all $m > N$ the following conditions are satisfied:

1. $\mu_m(\tilde{X}_{m,\varepsilon}) = \frac{\#\tilde{X}_{m,\varepsilon}}{\#X_m} > 1 - \varepsilon$ ($\#X$ the cardinality of the finite set X) and

2. for any n , such that $m > n \geq N$, and for all $x \in \tilde{X}_{m,\varepsilon}$ it holds

$$\left| \int \varphi d\nu_{\pi(X_n)}(x) - \int \varphi d\mu_n \right| < \varepsilon.$$

Then the sequence of atomic measures $\{\mu_n\}$ has a limit, and the limit measure is ergodic.

Proof. Lemma 2 from [21] claims that under conditions of Lemma 2.3 every limit point μ of the sequence $\{\mu_n\}$ is an ergodic measure. Therefore we just need to show that the sequence of measures $\{\mu_n\}$ has a $*$ -weak limit. In order to do that it is enough to show that for any continuous function φ on \mathfrak{Q} the sequence $\{\int \varphi d\mu_n\}$ is Cauchy and thus converges. Indeed, since $\{\mu_n\}$ has a convergent subsequence, $\mu_{n_k} \rightarrow \mu$ as $k \rightarrow \infty$, this implies that $\int \varphi d\mu_{n_k} \rightarrow \int \varphi d\mu$ and therefore $\mu_n \rightarrow \mu$ as $n \rightarrow \infty$.

Fix a continuous function φ on \mathfrak{Q} . It is enough to show that the sequence $\{\int \varphi d\mu_n\}$ is Cauchy. Since \mathfrak{Q} is compact, φ is bounded, $|\varphi| < M$ for some $M > 0$. Fix small $\varepsilon > 0$ and consider the sets $\tilde{X}_{m,\varepsilon} \subset X_m$ such that for all $m > N = N(\varepsilon, \varphi)$ properties (1) and (2) hold. Now, for a given $m > n > N(\varepsilon, \varphi)$, we have

$$\begin{aligned} \sum_{x \in X_m} \int \varphi d\nu_{\pi(X_n)}(x) &= \sum_{x \in X_m} \left(\frac{1}{\pi(X_n)} \sum_{i=0}^{\pi(X_n)-1} \varphi(\mathfrak{G}^i(x)) \right) = \\ &= \frac{1}{\pi(X_n)} \sum_{x \in X_m} \sum_{i=0}^{\pi(X_n)-1} \varphi(\mathfrak{G}^i(x)) = \frac{1}{\pi(X_n)} \sum_{x \in X_m} \pi(X_n) \varphi(x) = \\ &= \sum_{x \in X_m} \varphi(x) = \pi(X_m) \int \varphi d\mu_m. \end{aligned}$$

Therefore we can estimate

$$\begin{aligned}
\left| \int \varphi d\mu_m - \int \varphi d\mu_n \right| &= \frac{1}{\pi(X_m)} \left| \pi(X_m) \int \varphi d\mu_m - \pi(X_m) \int \varphi d\mu_n \right| = \\
&= \frac{1}{\pi(X_m)} \left| \sum_{x \in X_m} \int \varphi d\nu_{\pi(X_m)}(x) - \pi(X_m) \int \varphi d\mu_n \right| \leq \\
&\leq \frac{1}{\pi(X_m)} \left| \sum_{x \in \tilde{X}_{m,\varepsilon}} \left(\int \varphi d\nu_{\pi(X_m)}(x) - \int \varphi d\mu_n \right) \right| + \\
&\quad + \frac{1}{\pi(X_m)} \left| \sum_{x \notin \tilde{X}_{m,\varepsilon}} \left(\int \varphi d\nu_{\pi(X_m)}(x) - \int \varphi d\mu_n \right) \right|.
\end{aligned}$$

Finally, from condition (2) in the lemma, $|\varphi| < M$, and condition (1), we have

$$\left| \int \varphi d\mu_m - \int \varphi d\mu_n \right| \leq \frac{1}{\pi(X_m)} (\pi(X_m)\varepsilon + 2M\pi(X_m)\varepsilon) = (1 + 2M)\varepsilon.$$

So the sequence $\{\int \varphi d\mu_n\}$ is Cauchy, and Lemma 2.3 is proved. \square

2.3 Sufficient conditions for existence of an invariant non-hyperbolic measure with large support

In this section, we state an improved version of Proposition 2.5 from [15], where we add a description of the support of the resulting limit measure. We need the following definition.

Definition 2.4 (Good approximations). *A periodic orbit Y of a map \mathfrak{G} of a compact metric space \mathfrak{Q} into itself is a (γ, \varkappa) -good approximation of the periodic orbit X of \mathfrak{G} if the following holds.*

- *There exists a subset Γ of Y and a projection $\rho: \Gamma \rightarrow X$ such that*

$$\text{dist}(\mathfrak{G}^j(y), \mathfrak{G}^j(\rho(y))) < \gamma,$$

for every $y \in \Gamma$ and every $j = 0, 1, \dots, \pi(X) - 1$;

- $\frac{\#\Gamma}{\#Y} \geq \varkappa$;
- $\#\rho^{-1}(x)$ is the same for all $x \in X$.

Lemma 2.5. *Let $\{X_n\}$ be a sequence of periodic orbits with increasing periods $\pi(X_n)$ of a continuous map \mathfrak{G} of a compact metric space Ω into itself. For each n , let μ_n be the probability atomic measure uniformly distributed on the orbit X_n .*

Assume that there are sequences of numbers $\{\gamma_n\}_{n \in \mathbb{N}}$, $\gamma_n > 0$, and $\{\varkappa_n\}_{n \in \mathbb{N}}$, $\varkappa_n \in (0, 1]$, such that

1. *for each $n \in \mathbb{N}$ the orbit X_{n+1} is a (γ_n, \varkappa_n) -good approximation of X_n ;*
2. $\sum_{n=1}^{\infty} \gamma_n < \infty$;
3. $\prod_{n=1}^{\infty} \varkappa_n \in (0, 1]$.

Then the sequence of atomic measures $\{\mu_n\}$ has a limit μ . The limit measure μ is ergodic, and

$$\text{supp } \mu = \bigcap_{k=1}^{\infty} \left(\overline{\bigcup_{l=k}^{\infty} X_l} \right) \equiv \mathbf{X}.$$

In other words, $\text{supp } \mu$ is a topological limit of the sequence of orbits X_n ,

$$\text{supp } \mu = \{y \in \Omega \mid \exists m_i \rightarrow \infty \text{ and } x_i \in X_{m_i} \text{ such that } \lim_{i \rightarrow \infty} x_i = y\}.$$

Proof. Let us check first that the conditions of the Lemma 2.3 are satisfied by the sequence of orbits X_n .

Take arbitrary $\varepsilon > 0$ and continuous map $\varphi: \Omega \rightarrow \mathbb{R}$. By assumption 1), for orbits $\{X_n\}$ a sequence of subsets $\tilde{X}_n \subset X_n$ and a sequence of projections $\rho_n: \tilde{X}_{n+1} \rightarrow X_n$ are defined such that:

$$\prod_{n=1}^{\infty} \frac{\#\tilde{X}_{n+1}}{\#X_{n+1}} \geq \prod_{n=1}^{\infty} \varkappa_n > 0. \quad (1)$$

Choose $\delta = \delta(\varepsilon, \varphi)$ such that:

$$\omega_{\delta}(\varphi) := \sup_{\text{dist}(x,y) < \delta} |\varphi(x) - \varphi(y)| < \varepsilon.$$

By assumption 2), we have $\sum_{n=1}^{\infty} \gamma_n < \infty$. Choose $N = N(\varepsilon, \varphi)$ such that the following holds:

$$\sum_N^{\infty} \gamma_k < \delta(\varepsilon, \varphi) \quad \text{and} \quad \prod_N^{\infty} \varkappa_k > 1 - \varepsilon. \quad (2)$$

Since the number of points in a pre-image for projections ρ_n does not depend on a point in the image, a set $\tilde{X}_{m,\varepsilon} \subset X_m$ where the total projection

$$\rho_{m,N} = \rho_{m-1} \circ \cdots \circ \rho_N: \tilde{X}_{m,\varepsilon} \rightarrow X_N$$

is defined contains most of the orbit X_m :

$$\frac{\#\tilde{X}_{m,\varepsilon}}{\#X_m} \geq \prod_{k=N}^{m-1} \varkappa_k \geq \prod_N^{\infty} \varkappa_k > 1 - \varepsilon.$$

This implies 1) in Lemma 2.3.

Take arbitrary m and n with $m > n > N(\varepsilon, \varphi)$. By construction, on the set $\tilde{X}_{m,\varepsilon}$ the total projection $\rho_{m,n} = \rho_{m-1} \circ \cdots \circ \rho_n$ is defined and for every point x from the set $\tilde{X}_{m,\varepsilon} \subset X_m$ from the first part of equation (2) we have

$$\text{dist}(\mathfrak{G}^j(x), \mathfrak{G}^j(\rho_{m,n}(x))) < \delta(\varepsilon, \varphi), \quad \text{for all } j = 0, 1, \dots, \pi(X_n) - 1. \quad (3)$$

Hence for $x \in \tilde{X}_{m,\varepsilon}$ we have:

$$\left| \int \varphi d\nu_{\pi(X_n)}(x) - \int \varphi d\mu_n \right| < \omega_\delta(\varphi) < \varepsilon.$$

Thus all conditions of Lemma 2.3 are verified. Therefore the sequence $\{\mu_n\}$ has a limit μ , and the limit measure μ is ergodic.

Now we need the following lemma. Denote by $U_\delta(x)$ the ball of radius δ centered at x .

Lemma 2.6. *Set $r_n = \sum_{k=n}^{\infty} \gamma_k$. For every point $x \in X_n$ one has $\mu(\overline{U_{r_n}(x)}) > 0$.*

Proof. Take any point $x \in X_n$. Notice that in its γ_n -neighborhood there are at least $\frac{\#\tilde{X}_{n+1,\varepsilon}}{\pi(X_n)}$ points of the orbit of X_{n+1} , where

$$\frac{\#\tilde{X}_{n+1,\varepsilon}}{\pi(X_n)} = \frac{\#\tilde{X}_{n+1,\varepsilon}}{\pi(X_{n+1})} \frac{\pi(X_{n+1})}{\pi(X_n)} \geq \varkappa_n \frac{\pi(X_{n+1})}{\pi(X_n)} \equiv \bar{\varkappa}_n. \quad (4)$$

Therefore,

$$\mu_{n+1}(U_{\gamma_n}(x)) \geq \frac{\varkappa_n \frac{\pi(X_{n+1})}{\pi(X_n)}}{\pi(X_{n+1})} = \varkappa_n \frac{1}{\pi(X_n)} = \varkappa_n \mu_n(\{x\}). \quad (5)$$

In the neighborhood $U_{\gamma_n}(x)$ there are p different points x_1, \dots, x_p of the orbit of X_{n+1} , where $p \geq \bar{\varkappa}_n$. Notice that, by equation (4), for each of these points x_i the pre-image $\rho_{n+2}^{-1}(x_i)$ consists of at least

$$\frac{\#\tilde{X}_{n+2,\varepsilon}}{\pi(X_{n+1})} \geq \varkappa_{n+1} \frac{\pi(X_{n+2})}{\pi(X_{n+1})}$$

points. The sets $\rho_{n+2}^{-1}(x_i)$ and $\rho_{n+2}^{-1}(x_j)$ are disjoint for $i \neq j$. Therefore we have

$$\begin{aligned} \mu_{n+2}(U_{\gamma_{n+1}+\gamma_n}(x)) &\geq \left[p \varkappa_{n+1} \frac{\pi(X_{n+2})}{\pi(X_{n+1})} \right] \frac{1}{\pi(X_{n+2})} \geq \\ &\geq \left(\varkappa_n \frac{\pi(X_{n+1})}{\pi(X_n)} \right) \varkappa_{n+1} \frac{1}{\pi(X_{n+1})} = \frac{\varkappa_n \varkappa_{n+1}}{\pi(X_n)} = \\ &= \varkappa_n \varkappa_{n+1} \mu_n(\{x\}). \end{aligned}$$

Thus arguing inductively we have, for every $x \in X_n$,

$$\mu_{n+\ell}(U_{\gamma_{n+\ell}+\dots+\gamma_{n+1}+\gamma_n}(x)) \geq (\varkappa_{n+\ell} \cdots \varkappa_{n+1} \varkappa_n) \mu_n(\{x\}).$$

Taking a limit and taking into account the condition \mathcal{B} in Lemma 2.5, we have:

$$\mu(\overline{U_{r_n}(x)}) \geq \left(\prod_{k=n}^{\infty} \varkappa_k \right) \mu_n(\{x\}) > 0, \quad \text{where } r_n = \sum_{k=n}^{\infty} \gamma_k.$$

Therefore, Lemma 2.6 holds. \square

Now let us show that the support of μ is the topological limit \mathbf{X} of the sequence of orbits X_n . It is a general fact that for any convergent sequence of measures the support of the limit is a subset of the topological limit of the sequence of supports, therefore $\text{supp } \mu \subseteq \mathbf{X}$. We need to show that in our case we also have $\mathbf{X} \subseteq \text{supp } \mu$.

Take any point $z \in \mathbf{X}$. It is enough to show that arbitrary small $\varepsilon > 0$ one has $\mu(U_\varepsilon(z)) > 0$. Choose $n' \in \mathbb{N}$ large enough to guarantee that

$$X_{n'} \cap U_{\varepsilon/2}(z) \neq \emptyset \quad \text{and} \quad r_{n'} < \varepsilon/2.$$

Take any $z' \in X_{n'} \cap U_{\varepsilon/2}(z)$. We have $U_\varepsilon(z) \supset \overline{U_{r_{n'}}(z')}$, and due to Lemma 2.6 $\mu(\overline{U_{r_{n'}}(z')}) > 0$. Hence $\mu(U_\varepsilon(z)) > 0$, and therefore $z \in \text{supp } \mu$.

This completes the proof of Lemma 2.5. \square

Proposition 2.7. *Assume that a diffeomorphism $f : M \rightarrow M$ has the following properties:*

- 1) *there exists an f -invariant closed set $\Delta \subset M$ such that f has an invariant continuous direction field E in Δ ;*
- 2) *there exists a sequence of periodic orbits $\{X_n\}_{n=1}^\infty$, $X_n \subset \Delta$, of f whose periods $\pi(X_n)$ tend to infinity as $n \rightarrow \infty$.*

Denote by $\chi^E(X)$ the Lyapunov exponent along the periodic orbit X with respect to the invariant direction field E .

- 3) *There exists a sequence of numbers $\{\gamma_n\}_{n=1}^\infty$, $\gamma_n > 0$, and a constant $C > 0$ such that for each n the orbit X_{n+1} is a $(\gamma_n, 1 - C|\chi^E(X_n)|)$ -good approximation of the orbit X_n ;*
- 4) *let d_n be the minimal distance between the points of the orbit X_n , then*

$$\gamma_n < \frac{\min_{1 \leq i \leq n} d_i}{3 \cdot 2^n};$$

- 5) *there exists a constant $\xi \in (0, 1)$ such that for every n*

$$|\chi^E(X_{n+1})| < \xi |\chi^E(X_n)|.$$

Then the sequence of atomic measures μ_n supported on periodic orbits X_n has a ($*$ -weak) limit μ . The measure μ is a non-hyperbolic ergodic invariant measure of f , and $\text{supp } \mu$ is uncountable. Moreover, $\text{supp } \mu$ is a topological limit of the sequence of orbits X_n , i.e.

$$\text{supp } \mu = \bigcap_{k=1}^{\infty} \left(\overline{\bigcup_{l=k}^{\infty} X_l} \right).$$

Proof. Assumptions **1)**–**5)** of Proposition 2.7 imply that conditions of Lemma 2.5 are satisfied. Therefore there exists a limit $\lim_{n \rightarrow \infty} \mu_n = \mu$, the measure μ is an ergodic invariant measure of f , and $\text{supp } \mu$ is a topological limit of the sequence of orbits X_n . Due to Lemma 2.1 and property **5)**, measure μ is non-hyperbolic. Finally, Proposition 2.5 from [15] implies that $\text{supp } \mu$ is uncountable. \square

3 Homoclinic classes: C^1 -generic properties and generation of cycles

In this section we recall some known results on homoclinic classes and heterodimensional cycles that will be used later in our construction.

3.1 Generic properties

Here we state properties of homoclinic classes of C^1 -generic diffeomorphisms. There is a residual subset \mathcal{G} of $\text{Diff}^1(M)$ such that every diffeomorphism $f \in \mathcal{G}$ satisfies properties **R1)**–**R5)** below.

- R1)** Every homoclinic class of $f \in \mathcal{G}$ depends continuously on f in Hausdorff metric, see [13]. Moreover, if the homoclinic class contains saddles of indices a and b , $a < b$, it also contains saddles of index c , for every $c \in (a, b) \cap \mathbb{N}$. See [2, Theorem 1].

Consider a hyperbolic periodic point P_f of a diffeomorphism f . It is well known that there are open neighborhoods U of P_f in the manifold and \mathcal{U} of f in $\text{Diff}^1(M)$ such that every $g \in \mathcal{U}$ has a unique hyperbolic periodic point P_g of the same period as P_f in U . The point P_g is called the *continuation* of P_f .

Recall that two saddles P_f and Q_f are *homoclinically related* if their invariant manifolds $W^s(P_f)$ and $W^u(Q_f)$ and $W^u(P_f)$ and $W^s(Q_f)$ have non-empty transverse intersections. In this case, the homoclinic classes of P_f and of Q_f coincide and the saddles have the same index. Moreover, the continuations P_g and Q_g are also homoclinically related for all g close to f .

Definition 3.1 (Persistently linked saddles). *Consider a pair of hyperbolic saddles P_f and Q_f whose continuations are defined for every f in an open set \mathcal{U} of $\text{Diff}^1(M)$. The saddles P_f and Q_f are persistently linked in \mathcal{U} if there is a residual subset \mathcal{R} of \mathcal{U} such that $H(P_f, f) = H(Q_f, f)$ for all $f \in \mathcal{U} \cap \mathcal{R}$.*

R2) Given any $f \in \mathcal{G}$ and any pair of saddles P_f and Q_f of f , there is a neighborhood \mathcal{U}_f of f such that either P_f and Q_f are persistently linked in \mathcal{U}_f or $H(P_g, g) \cap H(Q_g, g) = \emptyset$ for all $g \in \mathcal{U}_f \cap \mathcal{G}$. Moreover, If the saddles P_f and Q_f have the same index they are homoclinically related. See [2, Lemma 2.1].

Definition 3.2 (Saddle with real multipliers). *Let P be a periodic point of period $\pi(P)$ of a diffeomorphism f . We say that P has real multipliers if every eigenvalue λ of $Df^{\pi(P)}(P)$ is real and has multiplicity one, and two different eigenvalues of $Df^{\pi(P)}(P)$ have different absolute values. We order the eigenvalues of $Df^{\pi(P)}(P)$ in increasing ordering according their absolute values $|\lambda^1(P)| < \dots < |\lambda^{\mathbf{m}}(P)|$ and say that $\lambda^k(P)$ is the k -th multiplier of P .*

Definition 3.3 (s- and u-index). *Let P be a hyperbolic saddle. By s-index of P we mean the number of multipliers of P with absolute value less than one, and by u-index the number of multipliers of P with absolute value greater than one.*

Let P be a saddle with real multipliers, and suppose that $\text{s-index}(P) = t + 1$. Consider the bundle $E^{\text{ss}} \subset T_P M$ corresponding to the first t contracting eigenvalues of P and the *strong stable manifold* $W^{\text{ss}}(P)$ of P (the only f -invariant manifold of dimension t tangent to the strong stable direction E^{ss}).

Definition 3.4 (s- and u-biaccumulation). *A hyperbolic saddle P with real multipliers is s-biaccumulated (by its homoclinic points) if both connected components of $W_{\text{loc}}^s(P) \setminus W_{\text{loc}}^{\text{ss}}(P)$ contain transverse homoclinic points of P . Define also u-biaccumulation by homoclinic points in a similar way.*

Note that s- and u-biaccumulation are open properties.

Given a saddle P , we denote by $\text{Per}_{\mathbb{R}}(H(P, f))$ the saddles homoclinically related to P having real multipliers. Clearly, $\text{Per}_{\mathbb{R}}(H(P, f)) \subset H(P, f)$.

R3) For every diffeomorphism $f \in \mathcal{G}$ and every saddle P of f whose homoclinic class is non-trivial the set $\text{Per}_{\mathbb{R}}(H(P, f))$ is dense in the whole homoclinic class $H(P, f)$. Moreover, the saddles from $\text{Per}_{\mathbb{R}}(H(P, f))$ which are s- (or u-) biaccumulated are also dense in $H(P, f)$. See [2, Proposition 2.3] and [15, Lemma 3.4].

R4) Consider $f \in \mathcal{G}$ and a saddle P of f whose homoclinic class is non-trivial. For every $\epsilon > 0$ and every $k = 1, \dots, \mathbf{m}$, one has that

$$\overline{\{S \in \text{Per}_{\mathbb{R}}(H(P, f)) : |\chi^k(S) - \chi^k(P)| < \epsilon\}} = H(P, f).$$

Assume now that $\chi^k(P) < 0$ and suppose that $H(P, f)$ contains a saddle Q with $\chi^k(Q) > 0$. Then one has that

$$\overline{\{S \in \text{Per}_{\mathbb{R}}(H(P, f)) : \chi^k(S) \in (-\epsilon, 0)\}} = H(P, f).$$

Moreover, we can additionally assume that the dense subsets of $H(P, f)$ above consist of saddles with the biaccumulation properties. For details see [2, 7, 15].

R5) We will actually need a slightly stronger property.

Lemma 3.5. *Consider any $f \in \mathcal{G}$ and any hyperbolic saddle P_f whose homoclinic class is non-trivial. For every $\epsilon > 0$ and $k = 1, \dots, \mathbf{m}$, the homoclinic class $H(P_f, f)$ of f contains a saddle $Y_f \in H(P_f, f)$ with real multipliers such that Y_f is homoclinically related to P_f , its orbit is ϵ -dense in the homoclinic class $H(P_f, f)$, and the Lyapunov exponent $\chi^k(Y_f)$ is ϵ -close to $\chi^k(P_f)$.*

Proof. Indeed, for a given $\varepsilon > 0$, by **R4)** there exists a finite collection of periodic saddles $\{P_i\}_{i=1,2,\dots,m}$ which is $\varepsilon/2$ -dense in $H(P_f, f)$, and their Lyapunov exponents $\chi^k(P_i)$ are ε -close to $\chi^k(P_f)$. Moreover, as $f \in \mathcal{G}$, by **R2)** these saddles are homoclinically related. This implies that there exists a locally maximal transitive hyperbolic set Λ_f with $P_i \in \Lambda_f$, for all $i = 1, \dots, m$. Then there is a periodic saddle $Y_f \in \Lambda_f$ that spends an arbitrary large portion of time in an arbitrary small neighborhood of the initial collection of saddles, and is $\varepsilon/2$ -dense in Λ_f . This implies that the orbit of Y_f is ε -dense in $H(P_f, f)$ and its Lyapunov exponent $\chi^k(Y_f)$ is close to the Lyapunov exponent $\chi^k(P_f)$, completing the proof of the lemma. \square

3.2 Heterodimensional cycles

In this section, we state results that allow us to generate heterodimensional cycles for saddles in a non-hyperbolic homoclinic classes. Roughly speaking, we need to be able to create a cycle associated with a given pair of homoclinically linked saddles of different indices (Proposition 3.6); to produce a locally dense set of diffeomorphisms with these type of cycles (Proposition 3.7); and, finally, to use these cycles to generate periodic saddles with some special properties

(Proposition 3.8) that will allow to use Proposition 2.7 later to obtain non-hyperbolic ergodic measures with large support.

The following result, which is a consequence of the Connecting Lemma in [23], can be found in [15, Proposition 3.5]. Recall that two saddles P and Q of different indices have a *heterodimensional cycle* if its invariant manifolds meet cyclically, $W^s(P) \cap W^u(Q) \neq \emptyset$ and $W^u(P) \cap W^s(Q) \neq \emptyset$. The cycle has *coindex one* if $\text{s-index}(P) = \text{s-index}Q \pm 1$.

Proposition 3.6. *Let \mathcal{U} be an open set of $\text{Diff}^1(M)$ such that there are saddles P_f and Q_f (depending continuously on $f \in \mathcal{U}$) with consecutive indices which are persistently linked in \mathcal{U} . Then there is a dense subset \mathcal{H} of \mathcal{U} such that every diffeomorphism $f \in \mathcal{H}$ has a coindex one heterodimensional cycle associated to saddles $A_f \in \text{Per}_{\mathbb{R}}(H(P_f, f))$ and $B_f \in \text{Per}_{\mathbb{R}}(H(Q_f, f))$.*

The following statements are straightforward reformulations of Propositions 4.3 and 4.5 from [15] better suited for our case.

Proposition 3.7. *Let f have a cycle associated to saddles A_f and B_f such that*

- A_f and B_f have real multipliers and $\text{s-index}(A_f) = \text{s-index}(B_f) + 1$;
- A_f is s-biaccumulated and B_f is u-biaccumulated ;

Then arbitrarily C^1 -close to f there are an open set $\mathcal{E} \subset \text{Diff}^1(M)$ and dense subset $\mathcal{D} \subset \mathcal{E}$ such that every $g \in \mathcal{D}$ has a cycle associated with A_g and B_g .

Proposition 3.8. ([15, Proposition 4.5]) *Let f be a diffeomorphism with a heterodimensional cycle associated to saddles A_f and B_f such that*

- (i) *the saddles A_f and B_f have real multipliers;*
- (ii) *$\text{s-index}(A_f) = i + 1$ and $\text{s-index}(B_f) = i$;*
- (iii) *A_f is s-biaccumulated , and B_f is u-biaccumulated .*

Fix neighborhoods U_B of the orbit of B_f and U_A of the orbit of A_f . Then there are sequences of natural numbers l_k, m_k , that tend to infinity as $k \rightarrow \infty$, and a sequence of diffeomorphisms f_k , $f_k \rightarrow f$ as $k \rightarrow \infty$, such that f_k coincides with f along the orbits of A_f and B_f , and has a hyperbolic saddle R_k having real multipliers with the following properties:

- (1) *the orbit of the saddle R_k spends a fixed number $t_{(a,b)}$ (independent of k) of iterates to go from U_B to U_A , then it remains $\ell_k \pi(A_f)$ iterates in U_A , then it takes a fixed number of iterates $t_{(b,a)}$ (independent of k) to go from U_A to U_B , and finally it remains $m_k \pi(B_f)$ iterates in U_B . In particular, there is a constant $t \in \mathbb{N}$ independent of k such that the period of R_k is $\pi(R_k) = m_k \pi(B_f) + \ell_k \pi(A_f) + t$;*
- (2) *there is a constant $\Theta > 0$ independent of k such that the central multiplier of R_k satisfies $\Theta^{-1} < |\lambda^{i+1}(R_k)| < \Theta$.*

Suppose also that $|\lambda^{i+1}(A_f)| \in \left(\frac{1}{\sqrt{2}}, 1\right)^1$. Then

- (3) *R_k has the same index as B_f and is homoclinically related to B_f ;*
- (4) *$W^s(R_k)$ intersects $W^{uu}(R_k)$, and $W^{uu}(R_k)$ intersects $W^s(B_f)$. Moreover, these intersections are quasi-transverse;*
- (5) *there is a heterodimensional cycle associated to R_k and A_f .*

Remark 3.9. *Notice that the condition (1) implies that there exists a sequence $\delta_k \rightarrow 0^+$ such that*

$$\mathcal{O}(A_f), \mathcal{O}(B_f) \subseteq B_{\delta_k}(\mathcal{O}(R_k)).$$

Here $\mathcal{O}(X)$ denotes the full orbit of the point X .

Remark 3.10. *In [15] we introduced V -related cycles in order to guarantee that all of the sets we consider are in the domain where a central direction field is defined. Here this property is one of the assumptions of the main result, so we do not need to focus on V -related cycles. Propositions 3.7 and 3.8 above are therefore weaker versions of Propositions 4.3 and 4.5 from [15].*

4 Main construction: tree of periodic points and limit measures

In this section we use the results of previous sections to complete the proof of Theorem 1. Let \mathcal{G} be the residual set of $\text{Diff}^1(M)$ described in Section 3.1. The statement below is a local version of Theorem 1.

¹In [15] here just assumed that the $(i+1)$ -th multiplier $\lambda^{i+1}(A_f)$ of A_f is close to one. To be more accurate, here we replaced it by the condition that we actually needed for the proof, see equation (8) in [15].

Theorem 2. *Let $f \in \mathcal{G}$ have a homoclinic class $H(f)$ of f such that*

- *$H(f)$ has a dominated splitting of the form $T_{H(f)}M = E \oplus F \oplus G$, where F has dimension one,*
- *$H(f) = H(P_f, f) = H(Q_f, f)$, where the saddles P_f and Q_f have s-indices $\dim(E \oplus F) = \dim(E) + 1$ and $\dim(E)$.*

Then arbitrarily C^1 -close to f there exists a C^1 -open set $\mathcal{Z} \subset \text{Diff}^1(M)$ and a residual subset $\mathcal{R} \subset \mathcal{Z}$ such that every $g \in \mathcal{R}$ has a non-hyperbolic ergodic invariant measure whose support is the whole homoclinic class $H(P_g, g)$.

As in [15] (see also [2]), standard genericity arguments imply that Theorem 1 follows from Theorem 2.

Before we continue, let us remind that existence of the dominated splitting is preserved by small perturbations.

Lemma 4.1 (Dominated splittings, Appendix B.1.1 in [10]). *Assume that the homoclinic class $H(P_f, f)$ has a dominated splitting $T_{H(P_f, f)}M = E_f \oplus F_f \oplus G_f$. There are neighborhoods V of $H(P_f, f)$ and \mathcal{U}_f of f such that for every $g \in \mathcal{U}_f$, the maximal invariant set $\Lambda_g(V)$ has a dominated splitting $T_{\Lambda_g(V)}M = E_g \oplus F_g \oplus G_g$, where $\dim K_f = \dim K_g$, $K = E, F, G$. With a slight abuse of notation, we will omit the dependence on g of this splitting.*

4.1 Construction of the sequences of periodic orbits

Consider a homoclinic class $H(f) = H(P_f, f) = H(Q_f, f)$ of $f \in \mathcal{G}$ that satisfy the conditions of Theorem 2.

The genericity hypotheses **R2)–R3)** imply that the homoclinic class $H(f)$ contains two saddles A_f and B_f with real multipliers such that

$$\text{s-index}(A_f) = \text{s-index}(P_f), \quad \text{s-index}(B_f) = \text{s-index}(Q_f),$$

and

$$H(f) = H(A_f, f) = H(B_f, f) = H(P_f, f) = H(Q_f, f).$$

The homoclinic class $H(f)$ has a splitting $E \oplus F \oplus G$, where F is one-dimensional. We can assume that the saddle A_f is s-biaccumulated, the saddle B_f is u-biaccumulated, and that the central multiplier of A_f is close to one (it is enough to have $|\lambda^F(A_f)| \in (0.9, 1)$), see generic conditions **R3)–R4)**.

Recall that $\chi^F(A_f)$ denotes the central Lyapunov exponent of saddle A_f corresponding to the central direction F . Fix a constant C such that

$$C > \frac{32}{\nu}, \text{ where } \nu = |\chi^F(A_f, f)|. \quad (6)$$

Lemma 4.2. *Let f satisfy the conditions of Theorem 2 and constants $C, \nu > 0$ be as in (6). Given $\epsilon > 0$, arbitrary close to f there exists an open set $\mathcal{Z} \subset \text{Diff}^1(M)$ with the following properties.*

1) For every $g \in \mathcal{Z} \cap \mathcal{G}$

$$H(f) \subset B_{\epsilon/2}(H(P_g, g)), \quad H(P_g, g) \subset B_{\epsilon/2}(H(f)).$$

2) For every $g \in \mathcal{Z} \cap \mathcal{G}$, we have

- $H(P_g, g) = H(Q_g, g) = H(A_g, g) = H(B_g, g)$,
- there is a dominated splitting (with one-dimensional central direction) on $H(g) = H(P_g, g)$.

3) For each $g \in \mathcal{Z}$ there is a saddle $Y_g \subset H(P_g, g)$ with real multipliers such that the following holds.

- the orbit of Y_g is ϵ -dense in $H(g)$,
- Y_g and P_g are homoclinically related, hence $H(Y_g, g) = H(P_g, g)$,
- $\chi_{Y_g}^F \in [-2\nu, -\nu/2]$,
- the saddle Y_g is s-biaccumulated.

4) There is a dense in \mathcal{Z} countable subset $\mathcal{D} \subset \mathcal{Z}$ such that every $\phi \in \mathcal{D}$ has a cycle associated with the saddles Y_ϕ and B_ϕ .

Proof. Since $f \in \mathcal{G}$, by property **R1**), continuous dependence of homoclinic classes of diffeomorphisms in \mathcal{G} , there is an open set $\mathcal{W} \subset \text{Diff}^1(M)$, $f \in \mathcal{W}$, such that for every $g \in \mathcal{W} \cap \mathcal{G}$ we have

$$H(f) \subset B_{\epsilon/2}(H(P_g, g)), \quad H(P_g, g) \subset B_{\epsilon/2}(H(f)),$$

obtaining property **1**).

By Lemma 4.1 there is a dominated splitting (with one-dimensional central direction) on $H(g) = H(P_g, g)$ which is a continuation of $E \oplus F \oplus G$. In

particular, such a splitting also has one dimensional central direction. Slightly abusing the notation, we will continue to denote this splitting by $E \oplus F \oplus G$. If \mathcal{W} is small enough then for every $g \in \mathcal{W} \cap \mathcal{G}$, we have $H(P_g, g) = H(Q_g, g) = H(A_g, g) = H(B_g, g)$, see **R2**). This gives **2**).

Due to **R5**) there is g arbitrary C^1 close to f and a saddle $Y_g \subset H(P_g, g)$ with real multipliers such that its orbit is $\epsilon/3$ -dense in $H(g)$, $H(Y_g, g) = H(P_g, g)$, $\chi_{Y_g}^F \in [-2\nu, -\nu/2]$, and which is s-biaccumulated, hence we have **3**).

Finally, Proposition 3.6 implies that C^1 -arbitrarily close to g there is h with a cycle corresponding to the saddles Y_h and B_h . Therefore, by Proposition 3.7, there is an open set $\mathcal{Z} \subset \mathcal{W}$ and a dense in \mathcal{Z} countable subset $\mathcal{D} \subset \mathcal{Z}$ such that every $\phi \in \mathcal{D}$ has a cycle associated with the saddles Y_ϕ and B_ϕ . This completes the proof of Lemma 4.2. \square

Next proposition can be considered as an improved version of Proposition 5.3 from [15]. Notice that properties **Z3**) and **Z4**) allow to “spread” the constructed sequence of periodic orbits around the whole homoclinic class.

Proposition 4.3. *Let f satisfy the conditions of Theorem 2 and constants $C, \nu > 0$ be as in (6). Fix a decreasing sequence $\{\epsilon_N\}_{N \in \mathbb{N}}$ of positive numbers, $\epsilon_N \rightarrow 0^+$ as $N \rightarrow \infty$. Arbitrarily C^1 -close to f there exists a nested sequence of open sets*

$$\dots \mathcal{Z}_N \subset \mathcal{Z}_{N-1} \subset \dots \mathcal{Z}_2 \subset \mathcal{Z}_1 \subset \text{Diff}^1(M)$$

such that the following holds.

Z1) *A set \mathcal{Z}_N is a dense open subset of \mathcal{Z}_{N-1} .*

Z2) *Every diffeomorphism $g \in \mathcal{Z}_N$ has a finite sequence of periodic saddles homoclinically related to B_g (thus of the same index as B_g)*

$$\{Q_{1,g}, Q_{2,g}, \dots, Q_{N,g}\} \subset H(B_g, g)$$

having real multipliers, satisfying the u-biaccumulation property, and of growing periods, $\pi(B_g) = \pi(Q_{1,g}) < \dots < \pi(Q_{N,g})$. Moreover, saddles $\{Q_{1,g}, Q_{2,g}, \dots, Q_{N,g}\}$ depend continuously on g when g varies over \mathcal{Z}_N .

Z3) *For any $N \in \mathbb{N}$ there exists a countable dense subset $\mathcal{D}_N \subset \mathcal{Z}_N$ such that every $g \in \mathcal{D}_N$ has a saddle $Y_{N,g}$ depending continuously on g such that*

- *the g -orbit of $Y_{N,g}$ is ϵ_{N+1} -dense in $H(B_g, g)$,*

- $\chi^F(Y_{N,g}) \in [-2\nu, -\nu/2]$,
- there is a heterodimensional cycle associated to the saddles $Y_{N,g}$ and $Q_{N,g}$.

Z4) There is a sequence of locally constant functions $\gamma_N : \mathcal{Z}_N \rightarrow (0, +\infty)$ such that for any $N \in \mathbb{N}$ and any $g \in \mathcal{Z}_{N+1}$ the orbit of $Q_{N+1,g}$ is a $(\gamma_N(g), 1 - C|\chi^F(Q_{N,g})|)$ -good approximation of the orbit of $Q_{N,g}$ (recall Definition 2.4), where C is the constant in (6). Moreover,

$$B_{\epsilon_{N+1}}(\mathcal{O}(Q_{N+1,g})) \supset \mathcal{O}(Y_{N,g}).$$

Z5) Take any $g \in \mathcal{Z}_N$. Let $d_k, 1 \leq k \leq N$, be the minimal distance between the points of the g -orbit of $Q_{k,g}$. Then

$$\gamma_N(g) < \frac{\min_{1 \leq k \leq N} d_k}{3 \cdot 2^N}.$$

Z6) For each $N \in \mathbb{N}$ and every $g \in \mathcal{Z}_{N+1}$

$$|\chi^F(Q_{N+1,g})| < \frac{1}{2} |\chi^F(Q_{N,g})|.$$

Proof. We prove Proposition 4.3 by induction.

As a set \mathcal{Z}_1 we take the set \mathcal{Z} from Lemma 4.2 (where we take $\epsilon = \epsilon_2$). Taking $Q_{1,g} = B_g$, $\mathcal{D}_1 = \mathcal{D}$, and $Y_{1,g} = Y_g$, we see that properties **Z2)** and **Z3)** for \mathcal{Z}_1 are satisfied. This form the base of induction. Notice that here we do not need to check the properties **Z4)** - **Z6)**.

Now assume that the sets $\mathcal{Z}_1 \supset \mathcal{Z}_2 \supset \dots \mathcal{Z}_N$ together with periodic orbits $\{Q_{1,g}, \dots, Q_{N,g}\}$, $\{Y_{1,g}, \dots, Y_{N,g}\}$, functions $\gamma_i : \mathcal{Z}_i \rightarrow (0, +\infty)$, and sets \mathcal{D}_i are constructed, $i = 1, \dots, N$. Let us construct the set $\mathcal{Z}_{N+1} \subset \mathcal{Z}_N$.

Notice that it is enough to construct an open dense subset of each connected component of \mathcal{Z}_N (together with periodic orbits $Q_{N+1,g}, Y_{N+1,g}$, function γ_{N+1} , and a dense subset \mathcal{D}_{N+1}) that satisfies conditions **Z2)** - **Z6)**. Take one of the connected components of the set \mathcal{Z}_N , denote it by \mathcal{Z}^* .

Let us enumerate diffeomorphisms from $\mathcal{D}_N \cap \mathcal{Z}^* = \{g_i\}_{i \in \mathbb{N}}$. Take one of these diffeomorphisms, say $g_i \in \mathcal{D}_N \cap \mathcal{Z}^*$.

Proposition 3.8 (applied to the saddles Q_{N,g_i} and Y_{N,g_i}) allows to obtain a sequence of diffeomorphisms $g_{ik}, g_{ik} \rightarrow g_i$ as $k \rightarrow \infty$, such that each diffeomorphism g_{ik} has a periodic saddle S_{ik} with real multipliers (denoted by R_k in Proposition 3.8), having the following properties:

- S1)** the saddle S_{ik} is homoclinically related to $Q_{N,g_{ik}}$, thus has the same index as B_{g_i} ;
- S2)** for some constant Θ that does not depend on k and for each $k \in \mathbb{N}$ we have $1 < |\lambda^F(S_{ik})| < \Theta$;
- S3)** the map g_{ik} has a heterodimensional cycle associated to S_{ik} and $Y_{N,g_{ik}}$;
- S4)** take a real number $\gamma_N = \gamma_N(\mathcal{Z}^*) > 0$ small enough (the choice of γ_N will be clear from the construction below) and γ_N -neighborhoods $U_{\mathcal{O}(Y_{N,g_i})}$ and $U_{\mathcal{O}(Q_{N,g_i})}$ of the orbits of Y_{N,g_i} and Q_{N,g_i} , and sequences of natural numbers ℓ_k, m_k that tend to infinity as $k \rightarrow \infty$, such that under the iterates of g_{ik} it takes to the saddle S_{ik} a fixed number of iterates (independent of k) to go from $U_{\mathcal{O}(Y_{N,g_i})}$ to $U_{\mathcal{O}(Q_{N,g_i})}$, then it remains $\ell_k \pi(Y_{N,g_i})$ iterates in $U_{\mathcal{O}(Y_{N,g_i})}$, then it needs a fixed number of iterates to go from $U_{\mathcal{O}(Y_{N,g_i})}$ to $U_{\mathcal{O}(Q_{N,g_i})}$, and finally it remains $m_k \pi(Q_{N,g_i})$ iterates in $U_{\mathcal{O}(Q_{N,g_i})}$. In particular, there is a constant $t \in \mathbb{N}$ independent of k such that

$$\pi(S_{ik}) = m_k \pi(Q_{N,g_i}) + \ell_k \pi(Y_{N,g_i}) + t.$$

Exactly as in [15], properties (3) and (4) from Proposition 3.8 guarantee that making an arbitrary small perturbation of g_{ik} (preserving properties **S1** - **S4**) we also have:

- S5)** the saddle S_{ik} has the u-biaccumulation property.

We need the following quantitative estimates:

Lemma 4.4 (Lemma 5.4 from [15]). *For every large $k \in \mathbb{N}$ the saddles S_{ik} satisfy:*

$$0 < \chi^F(S_{ik}) < \frac{1}{2} \chi^F(Q_{N,g_{ik}}), \quad (7)$$

$$\frac{m_k \pi(Q_{N,g_i})}{\pi(S_{i,k})} = \frac{m_k \pi(Q_{N,g_i})}{m_k \pi(Q_{N,g_i}) + \ell_k \pi(Y_{N,g_i}) + t} > 1 - C \chi^F(Q_{N,g_{ik}}). \quad (8)$$

Consider now the set

$$\mathcal{D}' = \{g_{ik} \mid S_{ik} \text{ satisfies conditions (7), (8), } i \geq 0 \text{ and } k \geq 0\} \subset \mathcal{Z}^*.$$

By construction, the set \mathcal{D}' is a countable dense subset of \mathcal{Z}^* . Let us enumerate the elements of $\mathcal{D}' = \{h_n\}_{n \in \mathbb{N}}$. Let us also denote by $Q_{N+1,g}(n)$ the continuation of the periodic saddle S_{ik} of the map $h_n \equiv g_{ik}$. In particular, $Q_{N+1,g_{ik}}(n) = S_{ik}$.

Now an application of Lemma 3.5 and Proposition 3.7 implies that for each h_n there is an open set \mathcal{U}_n , which is $\frac{1}{n}$ -close to h_n , and for each $g \in \mathcal{U}_n \cap \mathcal{G}$ there is a saddle $Y_{N+1,g}(n)$ such that

Y1) the saddle $Y_{N+1,g}(n)$ is in $H(g) = H(P_g, g)$ and its orbit is ϵ_{N+2} -dense in $H(g)$,

Y2) the saddles $Q_{N+1,g}(n)$ and $Y_{N+1,g}(n)$ have different indices,

Y3) the Lyapunov exponent $\chi^F(Y_{N+1,g}(n))$ is close to $(-\nu)$, and

Y4) there is a dense subset $\tilde{\mathcal{D}}(n)$ of maps in \mathcal{U}_n having a heterodimensional cycle between $Q_{N+1,g}(n)$ and $Y_{N+1,g}(n)$.

We can take \mathcal{U}_n small enough and close enough to h_n to guarantee that for every $g \in \mathcal{U}_n$ one has $0 < \chi^F(Q_{N+1,g}(n)) < \frac{1}{2}\chi^F(Q_{N,g})$. Indeed, due to (7) this inequality holds for h_n . Since Lyapunov exponents of a hyperbolic saddle depend continuously on a diffeomorphism, the inequality holds also for all g sufficiently C^1 -close to h_n .

Let us now define inductively

$$\mathcal{Z}(1) = \mathcal{U}_1, \quad \mathcal{Z}(2) = \mathcal{U}_2 \setminus \overline{\mathcal{Z}(1)}, \dots, \quad \mathcal{Z}(n) = \mathcal{U}_n \setminus \overline{\mathcal{Z}(n-1)}, \dots$$

and

$$\mathcal{D}(n) = \tilde{\mathcal{D}}(n) \cap \mathcal{Z}(n), \quad n \in \mathbb{N}.$$

Also we set

$$\mathcal{Z}_{N+1}^* = \bigcup_{n \in \mathbb{N}} \mathcal{Z}(n), \quad \mathcal{D}_{N+1}^* = \bigcup_{n \in \mathbb{N}} \tilde{\mathcal{D}}(n).$$

Besides, for each $g \in \mathcal{Z}(n)$ we set $Q_{N+1,g} = Q_{N+1,g}(n)$ and $Y_{N+1,g} = Y_{N+1,g}(n)$. Finally, we define \mathcal{Z}_{N+1} and \mathcal{D}_{N+1} as the union of constructed sets \mathcal{Z}_{N+1}^* and \mathcal{D}_{N+1}^* over all connected components of \mathcal{Z}_N . We claim that the set \mathcal{Z}_{N+1} satisfies the required properties **Z1) - Z6)**.

- Since the set $\{h_n\}_{n \in \mathbb{N}}$ is dense in \mathcal{Z}^* (the chosen connected component of \mathcal{Z}_N), the union $\bigcup_{n \in \mathbb{N}} \mathcal{U}_n$ is dense in \mathcal{Z}^* , and hence $\bigcup_{n \in \mathbb{N}} \mathcal{Z}(n)$ is also dense in \mathcal{Z}^* , so **Z1)** holds.
- For each $g \in \mathcal{Z}(n)$ the saddle $Q_{N+1,g}(n)$ is homoclinically related to B_g (recall **S1)**), has real multipliers, u-biaccumulation property, and $\pi(Q_{N,g}) < \pi(Q_{N+1,g}(n))$, so **Z2)** holds.

- The sets $\tilde{\mathcal{D}}(n) \subset \mathcal{U}_n$ and the saddles $Y_{N+1,g}(n)$ were constructed to satisfy **Z3**). More precisely, the estimate of the Lyapunov exponent $\chi^F(Y_{N+1,g}(n))$ follows from the choice of $Y_{N+1,g}(n)$ in **Y3**), the ϵ_{N+2} -density of the orbit of $Y_{N+1,g}(n)$ in $H(g)$ follows from **Y1**), and the existence of the heterodimensional cycle follows from **Y4**).
- Take $g \in \mathcal{Z}_{N+1}$, and denote by Γ the part of the orbit of $Q_{N+1,g}$ that belongs to the neighborhood $U_{Q_{N,g}}$. Define the projection

$$\rho : \Gamma \rightarrow \mathcal{O}(Q_{N,g}), \quad \rho(x) = \{\text{the point of } \mathcal{O}(Q_{N,g}) \text{ nearest to } x\}.$$

By construction (recall **S4**)),

$$\#\Gamma = m_k \pi(Q_{N,g}) \quad \text{and} \quad \#(\mathcal{O}(Q_{N+1,g})) = m_k \pi(Q_{N,g}) + \ell_k \pi(Y_{N,g}) + t.$$

Recall that here k and n are related due to the enumeration $h_n = g_{ik}$; notice that in fact integers m_k , ℓ_k , and t depend also on the index i , but our notations do not reflect this dependence. Now the first part of **Z4**) follows from the inequality (8). The second part, $B_{\epsilon_{N+2}}(\mathcal{O}(Q_{N+1,g})) \supset \mathcal{O}(Y_{N,g})$, follows from the fact that γ_N can be chosen arbitrarily small, in particular, smaller than ϵ_{N+2} .

- Condition **Z5**) is obtained by choosing sufficiently small $\gamma_N > 0$.
- The last property **Z6**) follows directly from inequality (7).

We completed the step of induction, and, thus, proved Proposition 4.3. \square

4.2 Infinite sequences of periodic orbits

Here we conclude the proof of Theorem 2 and, therefore, of the main result.

Proposition 4.5. *Let f satisfy the conditions of Theorem 2. Arbitrary close to f there exists an open set $\mathcal{Z} \subset \text{Diff}^1(M)$ such that generic diffeomorphism g from \mathcal{Z} have a sequence of periodic saddles which satisfies the assumptions of Proposition 2.7, belongs to the homoclinic class $H(P_g, g)$, and the union of their orbits is dense in the homoclinic class $H(P_g, g)$.*

Note that Propositions 4.5 and 2.7 give non-hyperbolic ergodic measures supported on the whole homoclinic class for generic diffeomorphisms from \mathcal{Z} , and, thus, Theorem 2.

Proof. Apply Proposition 4.3, and set $\mathcal{Z} = \mathcal{Z}_1$. Due to Property **Z1**), for any $N \in \mathbb{N}$ the set \mathcal{Z}_N is an open and dense subset of \mathcal{Z} . Therefore the intersection

$$\mathcal{R} = \mathcal{G} \cap \left(\bigcap_{N \in \mathbb{N}} \mathcal{Z}_N \right).$$

is a residual subset of \mathcal{Z} .

Take any $g \in \mathcal{R}$. Due to **Z2**), for the diffeomorphism $g \in \mathcal{R}$ a sequence of periodic points

$$\{B_g, Q_{1,g}, Q_{2,g}, \dots, Q_{N,g}, \dots\} \subset H(P_g, g)$$

is well defined. We claim that this sequence satisfies the assumptions of Proposition 2.7.

Indeed, assumption **1**) holds since $g \in \mathcal{G}$ and due to the choice of \mathcal{Z} (existence of a one dimensional center direction for $\Delta = H(P_g, g)$), and **2**) follows from property **Z2**). Assumptions **3**), **4**), and **5**) follow from **Z4**), **Z5**), and **Z6**), respectively.

Finally, since $\epsilon_N \rightarrow 0$ as $N \rightarrow \infty$, from **Z3**), **Z4**) we have that the nonhyperbolic ergodic measure given by Proposition 2.7 satisfy

$$\text{supp}(\mu) = \bigcap_{N \in \mathbb{N}} \left(\overline{\bigcup_{K \geq N} \mathcal{O}(Q_{K,g})} \right) = H(P_g, g). \quad (9)$$

This proves Proposition 4.5. □

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