

Robust heterodimensional cycles and C^1 -generic dynamics

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Dedicated to Welington de Melo for his sixtieth birthday

Abstract

A diffeomorphism f has a heterodimensional cycle if there are (transitive) hyperbolic sets Λ and Σ having different indices (dimension of the unstable bundle) such that the unstable manifold of Λ meets the stable one of Σ and vice-versa. This cycle has co-index one if $\text{index}(\Lambda) = \text{index}(\Sigma) \pm 1$. This cycle is robust if, for every g close to f , the continuations of Λ and Σ for g have a heterodimensional cycle.

We prove that any co-index one heterodimensional cycle associated to a pair of hyperbolic saddles generates C^1 -robust heterodimensional cycles. Therefore, in dimension three, every heterodimensional cycle generates robust cycles.

We also derive some consequences from this result for C^1 -generic dynamics (in any dimension). Two of such consequences are the following. For tame diffeomorphisms (generic diffeomorphisms with finitely many chain recurrence classes) there is the following dichotomy: either the system is hyperbolic or it has a robust heterodimensional cycle. Moreover, any chain recurrence class containing saddles having different indices has a robust cycle.

keywords: Axiom A, chain recurrence class, cycle, dominated splitting, heterodimensional cycle, homoclinic class, hyperbolicity, shadowing property.

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1 Introduction

1.1 Motivations and main result

Spoiling Smale's dream that the theory of hyperbolicity could describe an open and dense set of dynamical systems, Abraham and Smale [AS] constructed in 1970, by the first time, open sets in the space of C^1 -diffeomorphisms whose elements do not satisfy the Axiom A property. In contrast, generically, all periodic points of diffeomorphisms are hyperbolic. Recall that a diffeomorphism f satisfies the *Axiom A* if the hyperbolic structures on its hyperbolic points are compatible and fit together coherently: the non-wandering set $\Omega(f)$ of f is hyperbolic and

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coincides with the closure of its periodic points. In this case, by the *spectral decomposition theorem* (see [Ne₂]), the non-wandering set of f is the union of finitely many pairwise disjoint hyperbolic basic sets, $\Omega(f) = \Lambda_1 \cup \dots \cup \Lambda_m$, called the *basic pieces* of $\Omega(f)$.

However, [AS] shows that this global hyperbolic structure is too rigid to be generic: there is a C^1 -open set \mathcal{U} of non-Axiom A diffeomorphisms defined on a manifold of dimension four (the product of a two torus \mathbb{T}^2 and a two sphere \mathbb{S}^2) such that every diffeomorphism in \mathcal{U} has two hyperbolic sets Γ and Σ whose *indices* (dimension of the unstable bundle) are different and which are related by a C^1 -robust heterodimensional cycle. These cycles are defined as follows.

Definition 1.1 (Robust heterodimensional cycles). *A diffeomorphism f has a heterodimensional cycle associated to the (transitive) hyperbolic sets Γ and Σ of f if:*

1. *the indices (dimension of the unstable bundle) of the sets Γ and Σ are different;*
2. *the stable manifold of Γ meets the unstable manifold of Σ and the same holds for the stable manifold of Σ and the unstable manifold of Γ .*

The heterodimensional cycle of f associated to the sets Γ and Σ above is C^1 -robust if there is a C^1 -neighborhood \mathcal{U} of f such that every diffeomorphism $g \in \mathcal{U}$ has a heterodimensional cycle associated to the hyperbolic sets Γ_g and Σ_g , where Γ_g and Σ_g are the continuations of Γ and Σ for g .

Clearly, heterodimensional cycles can only occur in dimensions greater than or equal to three. However, Newhouse constructed in [Ne₁] C^2 -open sets of non-Axiom A surface diffeomorphisms. This construction relies on the notion of C^2 -robust homoclinic tangency associated to a hyperbolic set.

After the Abraham-Smale construction numerous examples of C^1 -robustly non-Axiom A diffeomorphisms were constructed by several authors. First, [Si] strengthens the ideas in [AS] to get robustly non-Axiom A diffeomorphisms in the three dimensional torus. Later, [Sh, Ma₁, BD₁] gave examples of a special type of C^1 -robustly non-Axiom A diffeomorphisms, the so-called *robustly non-hyperbolic transitive diffeomorphisms*: these diffeomorphisms are *transitive* (existence of a dense orbit in the whole manifold) and have hyperbolic saddles with different indices. Transitivity implies that the non-wandering set is the whole manifold. The existence of saddles having different indices now prevents the Axiom A property. Otherwise, by transitivity, the whole ambient manifold should be a hyperbolic transitive set, therefore all the saddles should have the same index, which is a contradiction.

The examples of robustly non-Axiom A diffeomorphisms in [AS, Si, BD₁] rely on the construction of robust heterodimensional cycles (although this terminology is not used there), while the constructions in [Sh, Ma₁] do not involve explicitly robust cycles. However, see Remark 2, it follows from our main result that in *all known examples of C^1 -robustly non-Axiom A diffeomorphisms those having C^1 -robust heterodimensional cycles form a dense and open subset*. Thus it seems natural to ask how generally robust heterodimensional cycles appear for diffeomorphisms far from hyperbolic ones:

Question 1. *Let M be closed manifold. Does it exist a C^1 -open and dense subset $\mathcal{O} \subset \text{Diff}^1(M)$ such that every $f \in \mathcal{O}$ either verifies the Axiom A and the no-cycles condition or has a C^1 -robust heterodimensional cycle?*

Note that a positive answer to this question implies the C^1 -density of hyperbolic surface diffeomorphisms. See the discussion in Section 1.3 about the Smale density conjecture. We will see that Theorem 2 gives a partial positive answer to this question for the so-called *tame diffeomorphisms* (diffeomorphisms finitely many homoclinic classes, see the precise definition in Section 1.2).

The examples by Abraham-Smale of non-Axiom A diffeomorphisms involves a hyperbolic set Γ whose unstable manifold has dimension strictly greater than the dimension of its unstable bundle. Note that a normally hyperbolic extension of transitive Anosov diffeomorphisms on a torus \mathbb{T}^2 gives an example of this configuration.

The construction in [BD₁] gives a slightly different mechanism for constructing non-Axiom A diffeomorphisms and robust heterodimensional cycles, based on the notion of *blender*. Roughly speaking, a blender is a hyperbolic set whose embedding in the ambient manifold verifies some specific geometric properties, whose effect is that, as in the Abraham-Smale example, the unstable manifold of a blender looks like a manifold of higher dimension. We review the construction and main properties of blenders in Section 4.1.3. See also [BDV₂, Chapter 6.1] for a discussion of this notion.

One of the goals of this paper is to show that *blenders* (and as a consequence robust heterodimensional cycles) appear in a natural way in the unfolding of *heterodimensional cycles* associated to two saddles.

Definition 1.2 (Heterodimensional cycle and co-index one cycle). *A diffeomorphism f has a heterodimensional cycle associated to two hyperbolic periodic saddles P and Q of f if the saddles P and Q have different indices, the stable manifold of the orbit of P meets the unstable manifold of the orbit of Q , and the same holds for the stable manifold of the orbit of Q and the unstable manifold of the orbit of P .*

A co-index one cycle is a heterodimensional cycle associated to saddles P and Q whose indices p and q satisfy of $q = p \pm 1$.

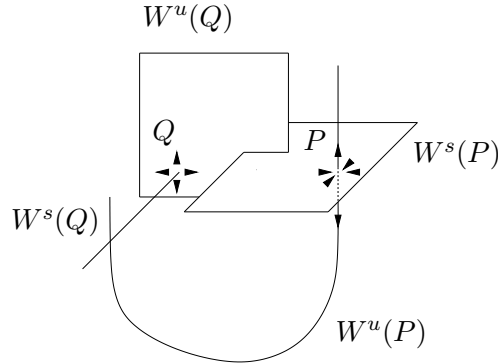


Figure 1: A heterodimensional cycle

Note that, by Kupka-Smale's theorem, heterodimensional cycles associated to saddles occur in the complement of a residual set of diffeomorphisms, thus they never are robust. Therefore robust cycles should involve at least one non-trivial hyperbolic set.

The study of heterodimensional cycles was initiated in [NP] in the context of bifurcation theory. Thereafter a systematic analysis of co-index one cycles was done in the series of papers

[DR₁, DU, D₁, D₂, BD₁, DR₂, DR₃, BDPR], where heterodimensional cycles are studied from the point of view of bifurcation theory as well as a mechanism generating robustly non-hyperbolic transitive sets and robust cycles. These results lead to the following question:

Question 2. *Let f be a diffeomorphism with a heterodimensional cycle (associated to a pair of saddles). Does every C^1 -neighborhood of f contain diffeomorphisms with C^1 -robust heterodimensional cycles?*

Our main result gives a positive answer to this question in the case of co-index one cycles.

Theorem 1. *Let f be a C^1 -diffeomorphism having a co-index one cycle associated to a pair of saddles. Then there are diffeomorphisms arbitrarily C^1 -close to f having robust (heterodimensional) co-index one cycles.*

Let f be a diffeomorphism defined on a manifold of dimension 3 with a heterodimensional cycle related to saddles P and Q . In this case, either $\text{index}(P) = 1$ and $\text{index}(Q) = 2$ or vice-versa, so such heterodimensional cycles are co-index one cycles. Therefore, we have

Corollary 1. *Every diffeomorphism f defined on a 3-manifold with a heterodimensional cycle associated to a pair of saddles belongs to the C^1 -closure of the set of diffeomorphisms having C^1 -robust heterodimensional cycles.*

Remark 1. *In [Ne₃], Newhouse proved that the unfolding of any homoclinic tangency of a C^2 -surface diffeomorphisms generates C^2 -robust tangencies associated to hyperbolic sets. Theorem 1 can be viewed as a version of this result for heterodimensional cycles in the C^1 -topology.*

Remark 2. *The approximation by C^1 -robust cycles holds for all known examples of C^1 -robustly non-Axiom A diffeomorphisms. The diffeomorphisms in [AS, Si] exhibit robust cycles by construction. The diffeomorphisms in [Sh, Ma₁, BD₁] are robustly non-hyperbolic and robustly transitive. By [BDPR], open and densely, these diffeomorphisms have saddles of different consecutive indices. The transitivity and the Connecting Lemma in [Ha] (see also Lemmas 5.6 and 6.5) allow us to create cycles associated to these saddles, obtaining co-index one cycles. Theorem 1 now implies the assertion.*

Let us pose two questions related to the theorem and the corollary above. Consider a diffeomorphism f with a co-index one cycle associated to a pair of saddles. Theorem 1 gives diffeomorphisms g arbitrarily C^1 -close to f with C^1 -robust cycles associated to hyperbolic sets. However, our proof does not give any relation between these hyperbolic sets and the initial saddles in the cycle. Thus a natural question is the following:

Question 3. *Let f be a diffeomorphism with a co-index one cycle associated to saddles P and Q . Can the diffeomorphism f be C^1 -approximated by diffeomorphisms g with a robust cycle associated to hyperbolic sets containing the continuations P_g and Q_g of P and Q ?*

Another natural question concerns the degree of differentiability required in Theorem 1:

Question 4. *Let f be a C^r -diffeomorphism, $r > 1$, with a co-index one cycle. Can the diffeomorphism f be C^r -approximated by diffeomorphisms with robust heterodimensional cycles?*

The results in [DR₂] give a partial (positive) answer to Question 4 for some special heterodimensional cycles. Clearly, Questions 3 and 4 can be formulated for heterodimensional cycles of co-index greater than 1.

Concerning Question 2, a natural strategy for solving it is to see that any diffeomorphism with a heterodimensional cycle can be approximated by diffeomorphisms with co-index one cycles. However, while the arguments in the proof of Theorem 1 are semi-local (involving only the dynamics in a neighborhood of the two periodic saddles and of two heteroclinic orbits defining the cycle), Gourmelon convinced us that the higher co-index case exhibits some additional difficulties, requiring a global analysis of the dynamics. On the other hand, the approximation of heterodimensional cycles (not necessarily of co-index one type) by co-index one cycles is true for robust cycles:

Corollary 2. *Every diffeomorphism with a C^1 -robust heterodimensional cycle is C^1 -approximated by diffeomorphisms with C^1 -robust co-index one cycles.*

The proof of this corollary (see Section 6.6) follows from Theorem 1, the properties of homoclinic classes (see the precise definition in Section 1.2) of C^1 -generic diffeomorphisms¹ in [CMP, BC, ABCDW], and the Connecting Lemma ([Ha]).

In view of Corollary 2, Question 1 now is equivalent to the following one:

Question 5 (Question 1 reformulated). *Can any C^1 -robustly non-Axiom A diffeomorphism be C^1 -approximated by diffeomorphisms with co-index one cycles?*

In fact, this question is a stronger version of the following conjecture:

Conjecture 1 (Palis, [Pa]). *Every diffeomorphism in $\text{Diff}^1(M)$ can be C^1 -approximated either by an Axiom A diffeomorphism or by a diffeomorphism with a homoclinic tangency or a heterodimensional cycle.*

This conjecture was proved for surface diffeomorphisms by Pujals and Sambarino in [PS] (note that for surface diffeomorphism heterodimensional cycles can be omitted). We will discuss this conjecture, the previous questions, and our results in Section 1.3.

1.2 Consequences of Theorem 1

In this section, we deduce some consequences from our main result. In particular, we give some partial positive answers to Question 1. We begin by recalling some definitions and results.

The *homoclinic class* of a saddle P of a diffeomorphism f , denoted by $H(P, f)$, is the transitive f -invariant compact set defined as the closure of the transverse intersections between the invariant manifolds (stable and unstable) of the orbit of the saddle P . This set coincides with the closure of the set of saddles *homoclinically related with P* (i.e., the saddles whose stable and unstable manifolds transversely meet the unstable and the stable manifolds of P).

The *chain recurrent set* of a diffeomorphism f , denoted by $\mathcal{R}(f)$, is the set of points x such that, for every $\varepsilon > 0$, there is a closed ε -pseudo orbit joining x to itself: there is a finite sequence $x = x_0, x_1, \dots, x_n = x$ such that $d(f(x_i), x_{i+1}) < \varepsilon$. By definition, the chain recurrent set is closed and contains the set of periodic points. Two points x and y are in the same *chain recurrence class* if, for every $\varepsilon > 0$, there are ε -pseudo orbits going from x to y and vice-versa.

¹By C^1 -generic diffeomorphisms we mean diffeomorphisms forming a residual subset of $\text{Diff}^1(M)$.

By [BC, Corollaire 1.2], there is a residual set \mathcal{R}_1 of $\text{Diff}^1(M)$ of diffeomorphisms whose chain recurrence sets coincide with the closure of their hyperbolic periodic points. Moreover, for every $f \in \mathcal{R}_1$, any chain recurrence class containing a periodic point P coincides with the homoclinic class of P , see [BC, Remarque 1.10]. Furthermore, by [BC, Corollaire 1.13], any isolated chain recurrence class $\mathcal{C}(f)$ of a diffeomorphism $f \in \mathcal{R}_1$ is *robustly isolated*. This means that there are neighborhoods \mathcal{U} of f in $\text{Diff}^1(M)$ and \mathcal{O} of the chain recurrence class $\mathcal{C}(f)$ in M such that, for every $g \in \mathcal{U}$, the intersection $\mathcal{R}(g) \cap \mathcal{O}$ is a unique chain recurrence class of g . For the precise statement of the C^1 -generic properties of homoclinic and chain recurrence classes we use in this paper see conditions (G1)–(G5) in Section 6.1.

We say that a diffeomorphism is *tame* if every chain recurrence class of it is robustly isolated. Thus tame diffeomorphisms have finitely many chain recurrence classes and the number of such classes is locally constant. We denote the set of tame diffeomorphisms by $\mathcal{T} \subset \text{Diff}^1(M)$; this set is C^1 -open. Furthermore, for generic tame diffeomorphisms, chain recurrence classes are homoclinic classes.²

Theorem 2 (Hyperbolicity versus robust cycles). *There is an open and dense subset \mathcal{O} of the set \mathcal{T} of tame diffeomorphisms such that every $f \in \mathcal{O}$ is either hyperbolic (Axiom A and the no-cycles condition) or it has a C^1 -robust heterodimensional cycle.*

Recall that an Axiom A diffeomorphism f has a *cycle* if there are basic sets $\Lambda_{i_1}, \dots, \Lambda_{i_n}$ of the spectral decomposition of the non-wandering set of f such that $W^u(\Lambda_{i_k}) \cap W^s(\Lambda_{i_{k+1}}) \neq \emptyset$, for all $k = 1, \dots, n$, where $i_{n+1} = i_1$. We prove Theorem 2 in Section 6.2.

We also have the following local formulation of the theorem above (see Section 6.3 for the details of the proof).

Corollary 3. *There is a residual subset \mathcal{R} of $\text{Diff}^1(M)$ such that for every diffeomorphism $f \in \mathcal{R}$ and every isolated chain recurrence class $\mathcal{C}(f)$ of f there are two possibilities: either $\mathcal{C}(f)$ is hyperbolic or it has a robust heterodimensional cycle.*

[ABCDW] claims that, for C^1 -generic diffeomorphisms, the set of indices of the (hyperbolic) periodic points in a chain recurrence class (in fact, such classes are homoclinic ones) form an interval in \mathbb{N} . This result and the transitivity of chain recurrence classes with periodic points (for generic diffeomorphisms) imply that if a chain recurrence class has two saddles having different indices then one can obtain (after an arbitrarily small perturbation) a co-index one cycle. Theorem 1 now implies (see Section 6.4):

Theorem 3. *There is a residual subset \mathcal{R} of $\text{Diff}^1(M)$ such that any $f \in \mathcal{R}$ having a chain recurrence class with periodic saddles of different indices has a robust heterodimensional cycle.*

A diffeomorphism f satisfies the *shadowing property* if for any $\delta > 0$ there is $\varepsilon > 0$ such that any finite ε -pseudo-orbit of f is δ -shadowed by a true orbit: if $(x_i)_{i=0}^n$ is a δ -pseudo-orbit there is x such that $d(f^i(x), x_i) < \varepsilon$ for all $i = 0, \dots, n$. A consequence of the existence of C^1 -robust co-index one cycles in terms of the shadowing property is the following:

²According to [CMP], there is residual subset \mathcal{R}_0 of $\text{Diff}^1(M)$ such that, for every $f \in \mathcal{R}_0$, any pair of homoclinic classes of f are either disjoint or coincide. Thus, for $f \in \mathcal{R}_0$, one can consider the number (in $\mathbb{N} \cup \{+\infty\}$) of (different) homoclinic classes of f . This number is locally constant in \mathcal{R}_0 , see [Ab]. A diffeomorphism $f \in \mathcal{R}_0$ is *tame* if this number is finite and we say that it is *wild* if otherwise.

Theorem 4. *Let f be a diffeomorphism with a co-index 1 cycle. Then there is an open set \mathcal{U} of $\text{Diff}^1(M)$ whose closure contains f consisting of diffeomorphisms which do not satisfy the shadowing property.*

This theorem is motivated by Remark 2 and the following result in [AD] (in fact, the proof of Theorem 4 follows using the arguments there): among the C^1 -robustly non-hyperbolic and robustly transitive diffeomorphisms those which do not satisfy the shadowing property form an open and dense subset. The proof of Theorem 4 is in Section 6.5.

1.3 Discussion: C^1 -Robust homoclinic tangencies and heterodimensional cycles

The main difference between Question 5 and Conjecture 1 above is that the conjecture involves, besides heterodimensional cycles, homoclinic tangencies. Let us make a small digression about the role of homoclinic tangencies for C^1 -generic diffeomorphisms. Following Definition 1.1, a diffeomorphism f has a C^1 -robust homoclinic tangency if there is a C^1 -neighbourhood \mathcal{U} of f such that every $g \in \mathcal{U}$ has a hyperbolic set Λ_g whose unstable and stable manifolds have non-transverse intersections (here we do not impose continuous dependence on the diffeomorphisms g of the hyperbolic sets Λ_g).

First, there are not known examples of surface diffeomorphisms with C^1 -robust homoclinic tangencies. On the other hand, most of examples of C^1 -persistent tangencies (in dimension three or higher) yields robust heterodimensional cycles and involves the notion of blender, see [BD₂, BD₃, DNP]. Finally, [As] constructs C^1 -diffeomorphisms with robust homoclinic tangencies considering deformations of the product of a Plykin attractor and a hyperbolic dynamics of saddle type. It is not known if the construction in [As] yields heterodimensional cycles. Thus a key question is to decide whether there are diffeomorphisms with C^1 -robust homoclinic tangencies far from the ones having heterodimensional cycles. Since surface diffeomorphisms cannot display heterodimensional cycles, the simplest version of this question is about the existence of C^1 -robust tangencies for surface diffeomorphisms. This last problem is closely related to Smale's conjecture of C^1 -density of hyperbolic dynamics for surface diffeomorphisms (this conjecture remains open), see [Sm₂]³.

Recall that in [Ne₁] Newhouse constructed surface diffeomorphisms having C^2 -robust homoclinic tangencies. Later, [Ne₃] stated that, in the C^2 -topology, homoclinic tangencies of a surface diffeomorphism yields C^2 -robust tangencies. See also [Ro, PV] for generalizations to higher dimensions of this result. These results relies on the construction of *thick hyperbolic sets* (see [PT]).

In [Ur], Ures showed that the arguments in the Newhouse's construction cannot be carried out to the C^1 -topology. Moreover, Moreira recently presented evidences showing that hyperbolic sets of C^1 -diffeomorphisms cannot exhibit robust tangencies, [Mo]. We interpret these results as indications that homoclinic tangencies of surface diffeomorphisms cannot be C^1 -persistent. Thus robust heterodimensional cycles seems to be a key ingredient in the generation of C^1 -robust non-hyperbolic dynamics. This suggests that the answers to Questions 1 and 5 should be both positive.

³For a discussion on the current state of the Smale's density conjecture we refer to [ABCD]. Briefly, there are two sort of obstacles to the C^1 -density of hyperbolic dynamics: (i) persistence of infinitely many hyperbolic homoclinic classes and (ii) existence of a single homoclinic class with a robust homoclinic tangency. The discussion here is related to the second obstacle.

We close this discussion noting that the topological dimension property of blenders (the dimension of the unstable manifold of the blender is greater than its index) is a C^1 -robust property. This property plays a role similar to the thick hyperbolic sets in the construction of C^1 -robust heterodimensional cycles.

2 Plan of the proof of Theorem 1

In this section, we outline the proof of Theorem 1 and emphasize its main ingredients.

2.1 Cycles with real central eigenvalues

We begin by considering a special sort of heterodimensional cycles, called *cycles with real central eigenvalues*.

Consider a diffeomorphism f with a co-index one cycle associated to saddles P and Q , say of periods $\pi(P)$ and $\pi(Q)$ and of indices u and $u + 1$. We say that a contracting eigenvalue λ of $Df^{\pi(P)}(P)$ and an expanding eigenvalue β of $Df^{\pi(Q)}(Q)$ are a pair of *central eigenvalues of the cycle* if $|\lambda| \geq |\sigma|$ for every contracting eigenvalue σ of $Df^{\pi(P)}(P)$ and $|\beta| \leq |\eta|$ for every expanding eigenvalue η of $Df^{\pi(Q)}(Q)$. The *cycle has real central eigenvalues* if there is only one pair of central eigenvalues: there are a contracting real eigenvalue λ of $Df^{\pi(P)}(P)$ and an expanding real eigenvalue β of $Df^{\pi(Q)}(Q)$ such that λ and β have multiplicity one, $|\lambda| > |\sigma|$ for every contracting eigenvalue σ of $Df^{\pi(P)}(P)$, and $|\beta| < |\eta|$ for every expanding eigenvalue η of $Df^{\pi(Q)}(Q)$, see Definition 3.1.

Next theorem states the approximation of diffeomorphism with co-index one cycles by diffeomorphisms having cycles with real central eigenvalues:

Theorem 5.1 *Let f be a diffeomorphism having a co-index one cycle associated to saddles P and Q . Then every C^1 -neighborhood \mathcal{U} of f contains a diffeomorphism g with a co-index one cycle with real central eigenvalues. Moreover, this cycle can be taken associated to saddles P'_g and Q'_g homoclinically related to the continuations P_g and Q_g of P and Q .*

This theorem is proved in Section 5, it implies that it is enough to prove Theorem 1 for cycles with real central eigenvalues:

Theorem 2.1. *Let f be a diffeomorphism with a co-index one cycle having real central eigenvalues. Then there are diffeomorphisms arbitrarily C^1 -close to f with C^1 -robust heterodimensional cycles.*

To prove Theorem 2.1, we need the notion of *strong homoclinic intersection associated to a saddle-node or a flip periodic point*. Let f be a diffeomorphism and S a periodic point of f of period $\pi(S)$. We say that S is a *saddle-node* (resp. a *flip*) of f if the derivative $Df^{\pi(S)}(S)$ has an eigenvalue equal to 1 (resp. -1) and all others have modulus different from one. Then the tangent space $T_S M$ splits into three $Df^{\pi(S)}$ -invariant directions $T_S M = E^{ss}(S) \oplus E^c(S) \oplus E^{uu}(S)$, where $E^{ss}(S)$ and $E^{uu}(S)$ are the strong stable and strong unstable bundles and $E^c(S)$ is the one-dimensional center bundle (associated to the eigenvalue of modulus one). In our case, the strong stable and strong unstable bundles are both non-trivial. The *strong stable manifold* $W^{ss}(S)$ of S is the unique $f^{\pi(S)}$ -invariant manifold tangent to $E^{ss}(S)$ having the same dimension as $E^{ss}(S)$. This manifold is well and uniquely defined, see [HPS]. The *strong unstable manifold* of S , $W^{uu}(S)$, is defined similarly considering the bundle $E^{uu}(S)$. We say

that a saddle-node or a flip S has a *strong homoclinic intersection* if there is some point $X \neq S$ with $X \in W^{ss}(S) \cap W^{uu}(S)$. The point X is a *strong homoclinic point* of S . Strong homoclinic intersections for saddles having a partially hyperbolic splitting $E^{ss} \oplus E^c \oplus E^{uu}$ are defined in the same way.

Theorem 2.1 follows from the following two results (the proofs are in Sections 3 and 4)

Theorem 3.3. *Let f be a diffeomorphism with a co-index one cycle with real central eigenvalues. Then there are diffeomorphisms arbitrarily C^1 -close to f having strong homoclinic intersections associated to saddle-nodes or to flips.*

Theorem 4.1. *Let f be a diffeomorphism with a strong homoclinic intersection associated to a saddle-node or to a flip. Then every C^1 -neighborhood \mathcal{U} of f contains diffeomorphisms with C^1 -robust heterodimensional cycles.*

2.2 Ingredients of the proofs of Theorems 3.3 and 4.1

Sketch the proof of Theorem 3.3: We first perturb the diffeomorphism f having the cycle to get a new cycle whose *relevant dynamics* is as simple as possible (this corresponds to the notion of *simple cycle*, see Definition 3.5). This is done in Section 3.1, let us explain the main ingredients of this construction.

Consider a diffeomorphism f with a co-index one cycle with real central eigenvalues associated to saddles P and Q , say of indices u and $u + 1$. We first select heteroclinic points $X \in W^s(P) \cap W^u(Q)$ and $Y \in W^u(P) \cap W^s(Q)$. After a perturbation, we can assume that the heteroclinic intersection at $X \in W^s(P) \cap W^u(Q)$ is transverse (note that $\dim W^s(P) + \dim W^u(Q) = (n - u) + (u + 1) = n + 1$, where n is the dimension of the ambient) and that the heteroclinic intersection at $Y \in W^u(P) \cap W^s(Q)$ is quasi-transverse, i.e., $T_Y W^u(P) \oplus T_Y W^s(Q)$.

Using the heteroclinic points X and Y and following [BDPR, Section 3.1], we consider a pair of *transition maps* (corresponding to iterations of the diffeomorphism), the first one T_{PQ} goes from a neighborhood of P to a neighborhood of Q following the orbit of the heteroclinic point Y , and the second one T_{QP} goes from a neighborhood of Q to a neighborhood of P following the orbit of X . These transitions are depicted in Figure 2.

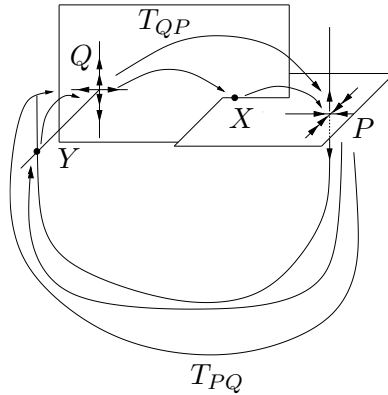


Figure 2: Transitions

We next focus on the dynamics in a small *neighborhood of the cycle*, that is, a neighborhood of the orbits of the saddles P and Q and of the selected heteroclinic points X and Y above.

A relevant part of the dynamics in this neighborhood (shortly, the *dynamics of the cycle*) is obtained by considering (suitable) compositions of the transition maps T_{PQ} and T_{QP} above and the restrictions of the diffeomorphism to neighborhoods of the saddles P and Q . The goal is to turn this semi-local dynamics of the cycle as simple as possible (in fact, composition of affine maps). For that we perform the following local and small C^1 -perturbations:

- *Linearization.* Using the C^1 -topology, we linearize (after a perturbation) the dynamics of f in a neighborhood of the cycle: the restrictions of $f^{\pi(P)}$ and $f^{\pi(Q)}$ to neighborhoods of P and Q are both linear and the transition maps are affine maps.
- *Preservation of dominance.* Using that the cycle has real central eigenvalues, one can assume (after a perturbation) that the dynamics of the cycle is dominated. More precisely, there is a locally constant *dominating splitting* $E^{ss} \oplus E^c \oplus E^{uu}$ which is preserved by the transitions and such that the dimension of E^c is 1. One has that $E^s(P) = E^{ss}(P) + E^c(P)$, and $E^u(Q) = E^{uu}(Q) + E^c(Q)$.

In Proposition 3.6, we obtain (after a small C^1 -perturbation) cycles in simplified form (satisfying the linearization and preservation of dominance properties above). The analysis of the dynamics of these cycles is essentially one-dimensional (reduction to the central direction) and depends on the central eigenvalues of P and Q . The proof of Theorem 3.3 now goes as follows:

- We consider the unfolding of simple cycles preserving their affine structures (associated to the dominated splittings). This leads to a time re-scaling of simple cycles and their unfoldings, called *model maps* and *model unfolding maps* (see Section 3.2). The model unfolding families $F_{\lambda,\beta,t}$ depend on three parameters: the parameters λ and β correspond to the central eigenvalues of the cycle and the parameter t corresponds to its unfolding. A key fact is that to each unfolding of a simple cycle preserving its affine structure corresponds a model unfolding family (and vice-versa).
- By construction, the model family keeps invariant the co-dimension one foliation generated by the sum of the strong stable and the strong unstable bundles of the dominated splitting. We consider the quotient of the dynamics of the model family by this foliation, obtaining a three-parameter family of systems of iterated functions defined on an interval (see Section 3.3). Proposition 3.9 gives a dictionary which translates properties from the one dimensional maps to the model families. Later, we will translate these properties of the model families to the *true* diffeomorphisms. The key word of the dictionary is that periodic points of the one-dimensional maps having two different periodic itineraries correspond to saddles of the model family with strong homoclinic intersections (i.e., there is a saddle whose strong stable manifold tangent to E^{ss} and strong unstable manifold tangent to E^{uu} meet quasi-transversely).
- In Section 3.4, using Proposition 3.9 and analyzing the dynamics of the system of one-dimensional iterated functions, we translate some properties of the one-dimensional dynamics to the model family. The key property guarantees the existence of strong homoclinic intersections associated to periodic points. In Propositions 3.16 and 3.17 we get a sequence of parameters (t_n) , $t_n \rightarrow 0$, such that for each t_n the corresponding model map F_{λ,β,t_n} has a periodic point A_n with a strong homoclinic intersection. The sequence of

periods ($\pi(A_n)$) of the saddles A_n go to infinity and the modulus of the central eigenvalue of $DF_{\lambda, \beta, t_n}^{\pi(A_n)}(A_n)$ (corresponding to the central bundle) is uniformly bounded. One has that the same properties (existence of strong homoclinic intersections and uniformly bounded central eigenvalues) hold for a sequence of diffeomorphisms g_n converging to f in the C^1 -topology (see Proposition 3.4).

- We conclude the proof of Theorem 3.3 by noting that a saddle (with arbitrarily large period and whose central eigenvalue has modulus uniformly bounded) having a strong homoclinic intersection, can be turned, by a small C^1 -perturbation, into a saddle-node or flip with a strong homoclinic intersection.

This ends the outline of the proof of Theorem 3.3

Sketch the proof of Theorem 4.1: Theorem 4.1 is proved in Section 4. The proof of this theorem follows from the results in [BD₁, BDV₁], which provide C^1 -robust cycles via the construction of blenders. We first consider in Section 4.1 strong homoclinic intersections associated to saddle-nodes. In Section 4.2, we reduce the case of strong homoclinic points associated to flips to the saddle-node case.

The proof for saddle-node has two main steps. We first introduce (see Section 4.1.1) the affine saddle-node cycles: a translation of the notion of simple cycle to the context of strong homoclinic intersections associated to saddle-nodes. We next see that strong homoclinic intersections generate affine saddle-node cycles. These constructions are similar to the construction of simple cycles in Section 3.1.

Thereafter, by applying a series of local perturbations to an affine saddle-node cycle, we get a blender, see Section 4.1.2. Finally, in Section 4.1.3, we review the notion of blender and deduce the generation of C^1 -robust cycles from the existence of such blenders.

2.3 Cycles with non-real central eigenvalues. Ingredients of Theorem 5.1

Let us now explain the main steps of the proof of Theorem 5.1 (see Section 5). Suppose that f has a co-index one cycle associated to saddles P and Q . Assume that the index of Q is greater than the index of P . We prove that there is g arbitrarily C^1 -close to f having a cycle with real central eigenvalues associated to new saddles homoclinically related with P and Q .

We first need a definition. A saddle A of period $\pi(A)$ of a diffeomorphism f has *real eigenvalues* if the eigenvalues of $Df^{\pi(A)}(A)$ are real and different in modulus and have multiplicity one. By [BDP], there is a residual subset of $\text{Diff}^1(M)$ of diffeomorphisms f such that, for every non-trivial homoclinic class $H(P, f)$ of f , the saddles of $H(P, f)$ whose eigenvalues are real form a dense subset of $H(P, f)$.

Using the previous result and the transitivity of a homoclinic class, one proves the following: Consider a diffeomorphism f with a co-index one cycle associated to saddles P and Q such that the homoclinic class of P is non-trivial. Then there is a new heterodimensional cycle associated to Q and to some saddle P' in the homoclinic class of P having at least one real central eigenvalue (the one corresponding to P'). See Theorem 5.3 in Section 5.1. In particular, this theorem implies that if the homoclinic classes of P and Q are both non-trivial, one can generate a new cycle associated to saddles (in the homoclinic classes of P and Q) with real eigenvalues. In this way, one gets a cycle with real central eigenvalues.

In view of Theorem 5.3, to prove Theorem 5.1 it is enough to see that given any diffeomorphism with a co-index one cycle there are two possibilities: either the cycle has real central eigenvalues (in this case there is nothing to do) or the diffeomorphism is approximated by diffeomorphisms with co-index one cycles associated to a pair of saddles whose homoclinic classes are both non-trivial.

Consider a diffeomorphism f with a co-index one cycle as above and a pair of central eigenvalues λ and β of the cycle (λ is a contracting eigenvalue of $Df^{\pi(P)}(P)$ and β is an expanding eigenvalue of $Df^{\pi(Q)}(Q)$). After a perturbation, we can assume that these central eigenvalues have multiplicity one and the only eigenvalues of the same modulus as λ and β are $\bar{\lambda}$ and $\bar{\beta}$ (assuming that they are non-real). There are three cases: (i) the central eigenvalues λ and β are both non-real, (ii) there is exactly one non-real central eigenvalue, and (iii) the central eigenvalues are both real, this is the case of central real eigenvalues. Thus it remains to consider cases (i) and (ii).

First, one proves that if a saddle in the cycle, say the saddle Q , has a pair of conjugate non-real central eigenvalues then there are diffeomorphisms g close to f with cycles associated to P and Q such that the homoclinic class $H(P, g)$ of P is non-trivial, see Proposition 5.7 in Section 5.2. The proof of this result is relatively easy and only involves linearizations and the assumption that the non-real central eigenvalue of the saddle Q has irrational argument. These assumptions are obtained after perturbations.

The previous result (Proposition 5.7) implies that if both saddles in the cycle have non-real central eigenvalues, we can assume, after a perturbation, that the homoclinic classes of both saddles P and Q are non-trivial. Hence, applying Theorem 5.3, one gets co-index one cycles with real central eigenvalues, see Lemma 5.9 in Section 5.3.

Finally, in Section 5.4 we study the remainder case, when only one central eigenvalue of the cycle is non-real (say the central eigenvalue of P). Thus, by Proposition 5.7, we can assume that the homoclinic class of Q is non-trivial. Hence the stable manifold of Q accumulates to the heteroclinic intersection of $W^u(P) \cap W^s(Q)$. The cycle configuration also implies that stable manifold of P also accumulates to the heteroclinic intersection. These two facts allow us to perform a perturbation, destroying the initial cycle, which simultaneously generates transverse homoclinic points of P as well as a new quasi-transverse intersection between $W^u(P)$ and $W^s(Q)$, see Lemma 5.10. Thus we get a new cycle associated to two saddles with non-trivial homoclinic classes. Once more, using Theorem 5.3, we get co-index one cycles with real central eigenvalues.

This concludes the outline of the proof of Theorem 5.1: every diffeomorphism with a co-index one cycle is C^1 -approximated by diffeomorphisms with cycles with real eigenvalues.

Standing notation: Throughout this paper we use the following notations.

- Given a periodic point P of a diffeomorphism f we denote by $\pi(P)$ the period of P .
- If P is hyperbolic, there is defined its continuation for every diffeomorphisms g close to f . We denote such a continuation by P_g .
- The perturbations we consider are always arbitrarily small. Thus the sentence *there is a C^1 -perturbation g of f such that* means *there is g arbitrarily C^1 -close to f such that*.

3 Co-index one cycles with real central eigenvalues

The goal of this section is to prove Theorem 3.3. We begin this section with two definitions.

Definition 3.1 (Central eigenvalues of a co-index one cycle). *Let f be a diffeomorphism with a co-index one cycle associated to the saddles P and Q (of periods $\pi(P)$ and $\pi(Q)$). Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of $Df^{\pi(P)}(P)$ enumerated with multiplicity, where $|\lambda_i| \leq |\lambda_{i+1}|$ (n is the dimension of the ambient manifold). Similarly, β_1, \dots, β_n , $|\beta_i| \leq |\beta_{i+1}|$, are the eigenvalues of $Df^{\pi(Q)}(Q)$. Suppose that the stable manifolds of P and Q have dimensions $s+1$ and s , respectively.*

- An eigenvalue λ_i of $Df^{\pi(P)}(P)$ is a central eigenvalue of the cycle (associated to P) if $|\lambda_i| = |\lambda_{s+1}| < 1$. Similarly, an eigenvalue β_j of $Df^{\pi(Q)}(Q)$ is a central eigenvalue of the cycle associated to Q if $|\beta_j| = |\beta_{s+1}| > 1$.
- The central eigenvalue of the cycle associated to P (resp., Q) is real if $|\lambda_{s+1}| > |\lambda_s|$ (resp., $|\beta_{s+1}| < |\beta_{s+2}|$). In this case, we write $\lambda_{s+1} = \lambda_c$ (resp., $\beta_{s+1} = \beta_c$).
- The cycle has real central eigenvalues if the central eigenvalues associated to P and Q are both real.

Consider a diffeomorphism f with a co-index one cycle with real central eigenvalues. The following properties hold:

- There is a (unique) Df -invariant dominated splitting⁴ defined on the union of the orbits \mathcal{O}_P of P and \mathcal{O}_Q of Q ,

$$T_A M = E_A^{ss} \oplus E_A^c \oplus E_A^{uu}, \quad A \in \mathcal{O}_P \cup \mathcal{O}_Q.$$

such that $\dim E_A^{ss} = s$, $\dim E_A^c = 1$, and $\dim E_A^{uu} = u$, where u is the index of P .

- The central eigenvalues λ_c and β_c of the cycle are the eigenvalues of $Df^{\pi(P)}(P)$ and $Df^{\pi(Q)}(Q)$ corresponding to the (central) bundle E^c , respectively.
- If $A \in \mathcal{O}_P$ then $E_A^s = E_A^{ss} \oplus E_A^c$ and if $A \in \mathcal{O}_Q$ then $E_A^u = E_A^c \oplus E_A^{uu}$.

We say that the splitting $E^{ss} \oplus E^c \oplus E^{uu}$ is the *partially hyperbolic splitting of the cycle*.

Definition 3.2 (Strong homoclinic intersections). *Let P be a periodic point of period $\pi(P)$ of a diffeomorphism f such that there is a Df -invariant partially hyperbolic splitting defined over the orbit \mathcal{O}_P of P ,*

$$T_{\mathcal{O}(P)} M = E^{ss} \oplus E^c \oplus E^{uu},$$

⁴A Df -invariant splitting $E \oplus F$ of TM over an f -invariant set Λ is *dominated* if the fibers of the bundles have constant dimension and there are a metric $\|\cdot\|$ and a natural number $n \in \mathbb{N}$ such that

$$\|Df^n(x)_E\| \cdot \|Df^{-n}(x)_F\| < \frac{1}{2}, \quad \text{for all } x \in \Lambda.$$

For splittings with three bundles $E \oplus F \oplus G$, domination means that the splittings $(E \oplus F) \oplus G$ and $E \oplus (F \oplus G)$ are both dominated. A dominated splitting is *partially hyperbolic* if at least one of the bundles is uniformly hyperbolic. We consider partially hyperbolic splittings $E \oplus F \oplus G$ such that E is uniformly contracting and G is uniformly expanding.

such that E^c has dimension one, every eigenvalue λ of $Df^{\pi(P)}(P)$ corresponding to E^{ss} satisfies $|\lambda| < 1$, and every eigenvalue β of $Df^{\pi(P)}(P)$ corresponding to E^{uu} satisfies $|\beta| > 1$ (i.e., E^{ss} is uniformly contracting and E^{uu} is uniformly expanding).

Let $W^{ss}(P, f)$ be the orbit of the unique $f^{\pi(P)}$ -invariant manifold tangent to $E^{ss}(P)$. Similarly, $W^{uu}(P, f)$ is the orbit of the unique $f^{\pi(P)}$ -invariant manifold tangent to $E^{uu}(P)$ ⁵

The periodic point P has a strong homoclinic intersection if there is $X \in W^{ss}(P, f) \cap W^{uu}(P, f)$, where $X \neq P$. We say that the point X is a strong homoclinic point of P . The point X is quasi-transverse if $T_X W^{ss}(P, f) + T_X W^{uu}(P, f) = T_X W^{ss}(P, f) \oplus T_X W^{uu}(P, f)$.

In this definition, the partial hyperbolicity implies that if λ_c is the eigenvalue of $Df^{\pi(P)}(P)$ corresponding to E^c then $|\lambda| < |\lambda_c| < |\beta|$, for every eigenvalue λ corresponding to E^{ss} and any eigenvalue β corresponding to E^{uu} . Note that if the the periodic point P is hyperbolic then its index is either $\dim(E^{uu})$ or $\dim(E^{uu}) + 1$. In the first case, $W^u(P, f) = W^{uu}(P, f)$ and $W^{ss}(P, f) \subset W^s(P, f)$. In the second one, $W^s(P, f) = W^{ss}(P, f)$ and $W^{uu}(P, f) \subset W^u(P, f)$.

As E^c has dimension one, if the periodic point P is not hyperbolic, either $\lambda_c = 1$ or $\lambda_c = -1$. In the first case, we say that P is a *saddle-node*, in the second one P is a *flip*.

The goal of this section is to prove:

Theorem 3.3. *Let f be a diffeomorphism with a co-index one cycle with real central eigenvalues. Then there are diffeomorphisms arbitrarily C^1 -close to f having strong homoclinic intersections associated to saddle-nodes or to flips.*

The proof of this theorem has two steps. The first step (which is the main one) is the proposition below:

Proposition 3.4. *Let f be a diffeomorphism having a co-index one cycle with real central eigenvalues. Then there are a constant $C > 1$ and a sequence f_n of diffeomorphisms, $f_n \rightarrow f$ (in the C^1 -topology), such that every f_n has a periodic point A_n such that:*

- *The orbit of A_n has a partially hyperbolic splitting $E^{ss} \oplus E^c \oplus E^{uu}$, where E^c is one-dimensional, E^{ss} is uniformly contracting, and E^{uu} is uniformly expanding.*
- *The sequence of periods $\pi(A_n)$ of A_n satisfies $\pi(A_n) \rightarrow \infty$ as $n \rightarrow \infty$.*
- *The central eigenvalue $\lambda^c(A_n)$ of $Df_n^{\pi(A_n)}$ corresponding to E^c satisfies $|\lambda^c(A_n)| \in [1/C, C]$.*
- *The periodic point A_n has a quasi-transverse strong homoclinic intersection.*

After proving Proposition 3.4, for large n , one performs a C^1 -perturbation of the diffeomorphism f_n along the orbit of A_n in order to transform A_n into a saddle-node or a flip. This perturbation preserves the strong homoclinic intersection. This perturbation roughly is a composition with a homothety of radius $(|\lambda_c(A_n)|)^{1/\pi(A_n)}$ along the orbit of A_n .

In this way, one gets a central eigenvalue of modulus one. Note that the sequence $(|\lambda_c(A_n)|)$ is bounded and $\pi(A_n) \rightarrow \infty$, thus $(|\lambda_c(A_n)|)^{1/\pi(A_n)} \rightarrow 1$ as $n \rightarrow \infty$. Hence the size of the perturbation can be taken arbitrarily small. This gives a sequence of diffeomorphisms $g_n \rightarrow f$ (in the C^1 -topology), such that every g_n has a periodic saddle-node or flip with a strong homoclinic intersection. This concludes the proof of Theorem 3.3.

To prove Proposition 3.4 we need the following preparatory ingredients and results:

⁵The existence and uniqueness of $W^{ss}(P, f)$ and $W^{uu}(P, f)$ follows from [HPS]. These manifolds are the *strong stable* and *strong unstable manifolds* of P .

- *Simple cycles* (see Section 3.1). We prove in Proposition 3.6 that, after a perturbation, every co-index one cycle with real central eigenvalues has local coordinates where the dynamics of the cycle is affine and partially hyperbolic (with one-dimensional central direction). Hence simple cycles have an affine structure associated to its partially hyperbolic splitting. We consider the unfolding of simple cycles preserving such an affine structure.
- *Model unfolding families* (see Section 3.2). We construct three-parameter families of affine maps defined on cubes of \mathbb{R}^n satisfying the following key property: for every unfolding of a simple cycle preserving its affine structure there is a model unfolding family which describes such an unfolding of the cycle (and vice-versa). See Remark 3.7.
- *One-dimensional reductions* (see Section 3.3). The model families preserve a co-dimension one bundle (corresponding to the sum of the strong stable and the strong unstable bundles). Then one can consider the one-dimensional quotient dynamics describing the central dynamics. This leads to systems of iterated functions defined on the central direction. In Section 3.4, for these one-dimensional reductions, we obtain periodic points with two different periodic itineraries.
- *From one-dimensional reductions to model unfolding families*. We finish the proof of Proposition 3.4 in Section 3.5. We see how the existence (for the one-dimensional reductions) of periodic points with two different itineraries is translated to the existence of periodic points with quasi-transverse strong homoclinic intersections for the model unfolding family.

3.1 Simple cycles

In this section, we consider a diffeomorphism f with a co-index one cycle with real central eigenvalues. We obtain, after a C^1 -perturbation, a new co-index one cycle (associated to the same initial saddles, thus with real central eigenvalues) and local coordinates at these saddles such that the dynamics in a neighborhood of the cycle is affine. Let us explain this point more precisely. We first describe simple cycles in non-technical and non-formal terms. For that we begin by introducing some notations.

Consider a cycle associated to saddles P and Q , say with index $(P) + 1 = \text{index}(Q)$. We fix small neighborhoods U_P and U_Q of the orbits of P and Q and heteroclinic points $X \in W^s(P, f) \cap W^u(Q, f)$ and $Y \in W^u(P, f) \cap W^s(Q, f)$. After a perturbation, we can assume that the intersection between $W^s(P, f)$ and $W^u(Q, f)$ at X is transverse and that the intersection between $W^u(P, f)$ and $W^s(Q, f)$ at Y is quasi-transverse. Then there are neighborhoods U_X of X and U_Y of Y and natural numbers n and m such that

$$f^n(U_X) \subset U_P, \quad f^{-n}(U_X) \subset U_Q, \quad f^m(U_Y) \subset U_Q, \quad \text{and} \quad f^{-m}(U_Y) \subset U_P.$$

We say that

- the set $V = U_P \cup U_Q \cup (\cup_{i=-n}^n f^i(U_X)) \cup (\cup_{i=-m}^m f^i(U_Y))$ is a *neighborhood of the cycle*,
- $2n$ and $2m$ are *transition times* from U_Q to U_P and from U_P to U_Q , respectively, and
- the maps $\mathfrak{T}_1 = f^{2m}$ and $\mathfrak{T}_2 = f^{2n}$ are *transition maps* from U_P to U_Q and from U_Q to U_P (these maps are defined on small neighborhoods \tilde{U}_Y of $f^{-n}(Y)$ and \tilde{U}_X of $f^{-m}(X)$), see Figure 3.

Our goal is to prove that, after a C^1 -perturbation, we can choose the neighborhoods U_P , U_Q , U_X , and U_Y and the numbers n and m such that there are local coordinates at P and Q such that (in these coordinates),

- $f^{\pi(P)}$ and $f^{\pi(Q)}$ are linear maps, and
- the transitions $\mathfrak{T}_1 = f^{2m}: \tilde{U}_Y \rightarrow U_Q$ and $\mathfrak{T}_2 = f^{2n}: \tilde{U}_X \rightarrow U_P$ are affine maps.

If the conditions above are satisfied, we say that the cycle is a *simple cycle*. The precise (somewhat technical) definition is given below. The elements in the definition are depicted in Figure 3.

Definition 3.5 (Simple cycle). A co-index one cycle of a diffeomorphism f associated to periodic saddles P and Q is *simple* if it satisfies conditions (S1)–(S4) below.

S1) The cycle has real central eigenvalues.

S2) There are local charts U_P and U_Q centered at P and Q where the expressions of $f^{\pi(P)}$ and $f^{\pi(Q)}$ are linear. Moreover, there is a partially hyperbolic splitting $E^{ss} \oplus E^c \oplus E^{uu}$, defined over the orbits of P and Q , which in these local charts is of the form

$$E^{ss} = \mathbb{R}^s \times \{(0, 0^u)\}, \quad E^c = \{0^s\} \times \mathbb{R} \times \{0^u\}, \quad E^{uu} = \{(0^s, 0)\} \times \mathbb{R}^u,$$

where s and u are the dimensions of $W^s(Q, f)$ and $W^u(P, f)$, respectively.

We extend the splitting $E^{ss} \oplus E^c \oplus E^{uu}$ to the neighborhood $U_P \cup U_Q$ as constant bundles.

S3) There is a quasi-transverse heteroclinic point $Y_P \in W^s(Q, f) \cap W^u(P, f)$ in the chart U_P ,

$$\dim(T_{Y_P}W^s(Q, f) + T_{Y_P}W^u(P, f)) = n - 1,$$

such that (in these local coordinates):

1. The point Y_P is of the form $Y_P = (0^s, 0, a^u)$, where $a^u \in \mathbb{R}^u$. Moreover, there is a neighborhood $C^s(Y_P)$ of Y_P in $W^s(Q, f) \cap U_P$ contained in $\mathbb{R}^s \times \{(0, a^u)\}$.
2. There is $\ell > 0$ such that $Y_Q = f^\ell(Y_P)$ belongs to the chart U_Q around Q and $Y_Q = (a^s, 0, 0^u)$, where $a^s \in \mathbb{R}^s$. Moreover, there is a neighborhood $C^u(Y_Q)$ of Y_Q in $W^u(P, f) \cap U_Q$ contained in $\{(a^s, 0)\} \times \mathbb{R}^u$.
3. There is a neighborhood U_{Y_P} of Y_P , $U_{Y_P} \subset U_P$, such that $f^\ell(U_{Y_P}) \subset U_Q$ and

$$\mathfrak{T}_1 = f^\ell: U_{Y_P} \rightarrow f^\ell(U_{Y_P})$$

is an affine map preserving the splitting $E^{ss} \oplus E^c \oplus E^{uu}$. Moreover, the map \mathfrak{T}_1 is uniformly contracting in the E^{ss} direction, uniformly expanding in the E^{uu} direction, and an isometry in the central direction E^c .

S4) There is a point $X_Q \in U_Q$ in the transverse intersection $W^u(Q, f) \pitchfork W^s(P, f)$ such that (in these local coordinates):

1. The point X_Q is of the form $X_Q = (0^s, b_Q, 0^u)$, with $b_Q > 0$. Moreover, there is $\varepsilon > 0$ such that the segment $I = \{0^s\} \times [b_Q - \varepsilon, b_Q + \varepsilon] \times \{0^u\}$ containing X_Q is contained in $W^u(Q, f) \pitchfork W^s(P, f)$.

2. There is $r > 0$ such that $X_P = f^r(X_Q)$ is in the chart U_P and $X_P = (0^s, b_P, 0^u)$, where $b_P < 0$. Moreover, the curve $J = f^r(I)$ satisfies

$$J = f^r(I) = \{0^s\} \times [b_P - \varepsilon, b_P + \varepsilon] \times \{0^u\} \subset U_P.$$

3. There is a neighborhood U_{X_Q} of X_Q , $U_{X_Q} \subset U_Q$, such that $f^r(U_{X_Q}) \subset U_P$ and

$$\mathfrak{T}_2 = f^r : U_{X_Q} \rightarrow f^r(U_{X_Q})$$

is an affine map preserving the splitting $E^{ss} \oplus E^c \oplus E^{uu}$. Moreover, the map \mathfrak{T}_2 is uniformly contracting in the E^{ss} direction, uniformly expanding in the E^{uu} direction, and an isometry in the central direction E^c .

We call the affine maps \mathfrak{T}_1 and \mathfrak{T}_2 the *transitions* of the simple heterodimensional cycle.

This definition means the dynamics in a neighborhood of a simple cycle is given by linear maps (the dynamics nearby the saddles), by affine maps (the dynamics corresponding to the transitions), and by suitable compositions of these maps.

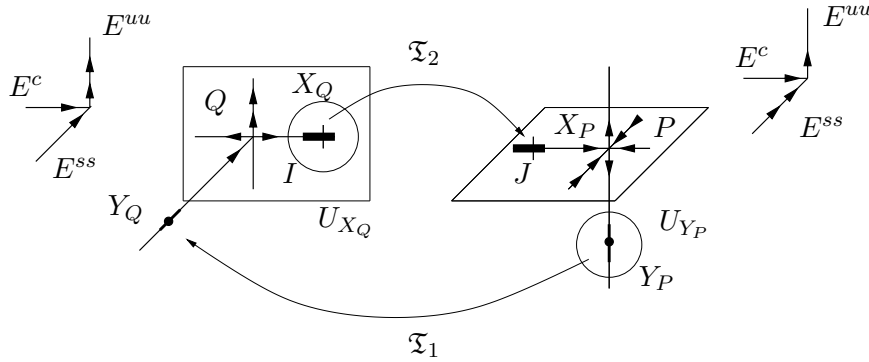


Figure 3: Elements of a simple cycle

Proposition 3.6. *Let f be a diffeomorphism having a co-index one cycle with real central eigenvalues associated to the saddles P and Q . Then any C^1 -neighborhood \mathcal{U} of f contains diffeomorphisms g with simple cycles associated to P and Q .*

Proof: This proposition is almost the same as [BDPR, Lemma 3.2], the only novelty here is that we require the transition maps \mathfrak{T}_1 and \mathfrak{T}_2 to be isometries in the central direction. So we just prove this step and refer to [BDPR, Lemma 3.2] for details.

Using [BDPR, Lemma 3.2], we can assume (after a perturbation) that there are defined maps \mathfrak{T}_1 and \mathfrak{T}_2 preserving the partially hyperbolic splitting $E^{ss} \oplus E^c \oplus E^{uu}$. Let A and B be the derivatives of $f^{\pi(P)}$ at P and of $f^{\pi(Q)}$ at Q in the local charts. Note that one can replace \mathfrak{T}_1 by $A^{m_1} \circ \mathfrak{T}_1 \circ B^{m_2}$, for any positive m_1 and m_2 . This corresponds to replace the quasi-transverse heteroclinic points Y_P and Y_Q by $f^{-m_1 \pi(P)}(Y_P)$ and $f^{m_2 \pi(Q)}(Y_Q)$, respectively, thus replacing the transition time ℓ by a larger number.

We can choose m_1 and m_2 arbitrarily large in such a way the modulus of the derivative of \mathfrak{T}_1 in the central direction remains (upper and lower) bounded. Now, after a C^1 -perturbation of f in

a small neighborhood of the segment of orbit $Y_P, f(Y_P), \dots, f^\ell(Y_P) = Y_Q$ (a small multiplication in the central direction), we can assume that the modulus of the derivative of \mathfrak{T}_1 in the central direction is exactly one. Thus, after a new perturbation, we have that the action of \mathfrak{T}_1 in the central direction is an isometry.

The proof for the transition \mathfrak{T}_2 is completely similar.

Finally, the expansion (resp., contraction) of \mathfrak{T}_1 and \mathfrak{T}_2 in the E^{uu} (resp., E^{ss}) follows observing that A and B are expanding (resp., contracting) in these directions, so it is enough to take large ℓ and r (i.e., to increase m_1 and m_2 above). This completes the sketch of the proof of the proposition. \square

3.2 Model unfolding families

By Proposition 3.6, every co-index one cycle with real central eigenvalues is approximated by simple cycles. The key property is that the dynamics in a neighborhood of these cycles is affine. We now focus on simple cycles and describe the dynamics of nice perturbations of them: the unfolding of simple cycles preserving their affine structures. This leads us to consider 3-parameter families of *model unfolding maps* $F_{\lambda, \beta, t}^{\pm, \pm}$, where the parameters $\lambda \in (0, 1)$ and $\beta \in (1, \infty)$ correspond to the central eigenvalues of the cycle and the parameter $t \in \mathbb{R}$ corresponds to the unfolding of the cycle. The parameters \pm describe the orientation of the transitions \mathfrak{T}_1 and \mathfrak{T}_2 of the simple cycle in the central direction.

We now define *model unfolding families*. Consider two copies Δ_P and Δ_Q of the unitary cube $[-1, 1]^s \times [-1, 1] \times [-1, 1]^u$ and sub-cubes $\Sigma_P \subset \Delta_P$ and $\Sigma_Q \subset \Delta_Q$, defined as follows. Consider small $\delta > 0$ and points $b_Q \in (0, 1)$ and $a_P^u \in (\mathbb{R}^u \setminus \{0^u\})$ such that

$$[b_Q - \delta, b_Q + \delta] \subset (0, 1) \quad \text{and} \quad [a_P^u - \delta, a_P^u + \delta]^u \subset (-1, 1)^u \setminus \{0^u\},$$

where $[a_P^u - \delta, a_P^u + \delta]^u$ is the u -cube of edges of length 2δ centered at a_P^u (we use this notation from now on). Then

- Σ_P is the cube $[-1, 1]^s \times [-\delta, \delta] \times [a_P^u - \delta, a_P^u + \delta]^u$ contained in Δ_P ;
- Σ_Q is the cube $[-1, 1]^s \times [b_Q - \delta, b_Q + \delta] \times [-1, 1]^u$ contained in Δ_Q .

We also fix linear maps

- $A^s, B^s, T_1^s, T_2^s: \mathbb{R}^s \rightarrow \mathbb{R}^s$, which are contractions (i.e., their norms are strictly less than one);
- $A^u, B^u, T_1^u, T_2^u: \mathbb{R}^u \rightarrow \mathbb{R}^u$, which are expansions (i.e., their inverse maps are contractions).

We now define a family of maps on the disjoint union $\Delta_P \amalg \Delta_Q$ (the resulting maps will be the model ones) as follows:

1. Given $\lambda \in (0, 1)$ and $\beta > 1$, consider the linear maps

$$\mathcal{A}_\lambda, \mathcal{B}_\beta, \tilde{\mathcal{T}}_1^+, \tilde{\mathcal{T}}_1^-, \tilde{\mathcal{T}}_2^+, \tilde{\mathcal{T}}_2^-: \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad n = s + u + 1,$$

defined by

- $\mathcal{A}_\lambda(x^s, x^c, x^u) = (A^s(x^s), \lambda x^c, A^u(x^u))$;

- $\mathcal{B}_\beta(x^s, x^c, x^u) = (B^s(x^s), \beta x^c, B^u(x^u))$;
- $\tilde{\mathcal{T}}_i^\pm(x^s, x^c, x^u) = (T_i^s(x^s), \pm x^c, T_i^u(x^u))$; $i = 1, 2$.

2. Fix $b_P \in (-1, 0)$ with $[b_P - \delta, b_P + \delta] \subset (-1, 0)$ and let \mathcal{T}_2^\pm be the affine map defined by

$$\mathcal{T}_2^\pm(x^s, x^c, x^u) = \tilde{\mathcal{T}}_2^\pm(x^s, x^c, x^u) + (0^s, -b_Q + b_P, 0^u).$$

Note that $\mathcal{T}_2^\pm(0^s, b_Q, 0^u) = (0^s, b_P, 0^u)$.

3. Take $a_Q^s \in \mathbb{R}^s \setminus \{0^s\}$ such that $[a_Q^s - \delta, a_Q^s + \delta]^s \subset [-1, 1]^s \setminus \{0^s\}$ (as above, $[a_Q^s - \delta, a_Q^s + \delta]^s$ is the s -cube of edges of length 2δ centered at a_Q^s). For small $t \in \mathbb{R}$, let $\mathcal{T}_{1,t}^\pm$ be the affine map defined by

$$\mathcal{T}_{1,t}^\pm(x^s, x^c, x^u) = \tilde{\mathcal{T}}_1^\pm(x^s, x^c, x^u) + (a_Q^s, t, -T_1^u(a_P^u)).$$

Note that, for $t = 0$, $\mathcal{T}_{1,t}^\pm = \mathcal{T}_{1,0}^\pm$ and $\mathcal{T}_{1,t}^\pm(0^s, 0, a_P^u) = (a_Q^s, t, 0^u)$.

The map $\mathcal{T}_{1,t}^\pm$ is a perturbation of \mathcal{T}_1^\pm obtained considering a translation of size t in the central direction.

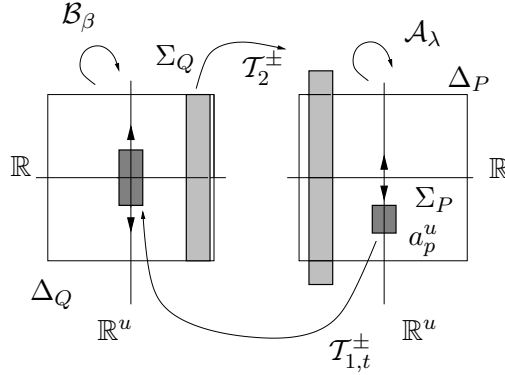


Figure 4: Model maps

Given $\lambda \in (0, 1)$, $\beta > 1$, and small $t \in \mathbb{R}$, we consider the map $F_{\lambda, \beta, t}^{\pm, \pm}$ defined on a subset $S_{Q, P, t}$ of $\Delta_Q \amalg \Delta_P$,

$$F_{\lambda, \beta, t}^{\pm, \pm}: S_{Q, P, t} \rightarrow \Delta_Q \amalg \Delta_P,$$

obtained as follows:

- if $x \in \Delta_P \setminus \Sigma_P$ and $\mathcal{A}_\lambda(x) \in \Delta_P$, then $F_{\lambda, \beta, t}^{\pm, \pm}(x) = \mathcal{A}_\lambda(x) \in \Delta_P$;
- if $x \in \Delta_Q \setminus \Sigma_Q$ and $\mathcal{B}_\beta(x) \in \Delta_Q$, then $F_{\lambda, \beta, t}^{\pm, \pm}(x) = \mathcal{B}_\beta(x) \in \Delta_Q$;
- if $x \in \Sigma_P$ and $\mathcal{T}_{1,t}^\pm(x) \in \Delta_Q$, then $F_{\lambda, \beta, t}^{\pm, \pm}(x) = \mathcal{T}_{1,t}^\pm(x) \in \Delta_Q$;
- if $x \in \Sigma_Q$ and $\mathcal{T}_2^\pm(x) \in \Delta_P$, then $F_{\lambda, \beta, t}^{\pm, \pm}(x) = \mathcal{T}_2^\pm(x) \in \Delta_P$.

The set $S_{Q,P,t}$ is the maximal subset of $\Delta_Q \amalg \Delta_P$ where $F_{\lambda,\beta,t}^{\pm,\pm}$ can be defined according to the rules above. Note that the map $F_{\lambda,\beta,t}^{\pm,\pm}$ is bi-valuated in some points.

For a fixed $\lambda \in (0, 1)$ and $\beta > 1$, we say that $F_{\lambda,\beta,0}^{\pm,\pm}$ is a *model map* and the 3-parameter family $F_{\lambda,\beta,t}^{\pm,\pm}$ is a *model unfolding family*. The maps \mathcal{T}_1^\pm and \mathcal{T}_2^\pm are the *transitions* of the model family. Finally, $\mathcal{T}_{1,t}^\pm$ is the *unfolding map*.

Standing notation: Here $F_{\lambda,\beta,t}^{+,+}$ means that we consider the maps $\mathcal{T}_{1,t}^+$ and $\mathcal{T}_{2,t}^+$, for $F_{\lambda,\beta,t}^{-,+}$ we choose $\mathcal{T}_{1,t}^-$ and $\mathcal{T}_{2,t}^+$, for $F_{\lambda,\beta,t}^{+,-}$ we take $\mathcal{T}_{1,t}^+$ and $\mathcal{T}_{2,t}^-$, and $F_{\lambda,\beta,t}^{-,-}$ means that we consider $\mathcal{T}_{1,t}^-$ and $\mathcal{T}_{2,t}^-$. As the role of \pm is rather unessential in this section, for notational simplicity, we will omit these scripts. The first script \pm , relative to $\mathcal{T}_{1,t}^\pm$, will play an important role in Section 3.4. The effect of these scripts only affect the orientation of central dynamics: for the choices $(++)$ and $(--)$ provide isometries preserving the orientation (the identity), and the choices $(+-)$ and $(-+)$ give isometries reversing the orientation (minus the identity).

Remark 3.7 (Model unfolding families and simple cycles).

- We denote by P the point $(0^s, 0, 0^u) \in \Delta_P$ and by Q the point $(0^s, 0, 0^u) \in \Delta_Q$. These points are saddles of $F_{\lambda,\beta,t}$ of indices u and $u + 1$, respectively. The map $F_{\lambda,\beta,0}$ has a co-index one cycle with real central eigenvalues associated to P and Q : it is enough to note that:

$$(0^s, 0, a_P^u), (a_Q^s, 0, 0^u) \in W^u(P, F_{\lambda,\beta,0}) \cap W^s(Q, F_{\lambda,\beta,0})$$

and

$$(0^s, b_Q, 0^u), (0^s, b_P, 0^u) \in W^s(P, F_{\lambda,\beta,0}) \cap W^u(Q, F_{\lambda,\beta,0}).$$

Also note that the intersections at $(0^s, 0, a_P^u)$ and $(a_Q^s, 0, 0^u)$ are quasi-transverse and the intersections at $(0^s, b_Q, 0^u)$ and $(0^s, b_P, 0^u)$ are transverse.

- Observe that if f is a diffeomorphism with a simple cycle then there is a model map $F_{\lambda,\beta,0}$ such that the dynamics of f in a neighborhood of the cycle is, after a finite time re-scaling, the one of $F_{\lambda,\beta,0}$. More precisely, there are a model map $F_{\lambda,\beta,0}$, local coordinates around the saddles P and Q in the cycle, and heteroclinic points X_P, X_Q, Y_P and Y_Q such that:
 - i) The point X_P corresponds to the point $(0^s, 0, a_P^u)$ of the model, X_Q corresponds to $(a_Q^s, 0, 0^u)$, Y_P corresponds to $(0^s, b_P, 0^u)$, and Y_Q corresponds to $(0^s, b_Q, 0^u)$.
 - ii) The transitions \mathfrak{T}_1 and \mathfrak{T}_2 of the simple cycle are the transitions \mathcal{T}_1 and \mathcal{T}_2 of the model map.
 - iii) Suppose that the central eigenvalues of the simple cycle are λ and β . Assume first that these eigenvalues are both positive. Then the dynamics of $f^{\pi(P)}$ in a neighborhood of the saddle P is the one of the model map in the cube Δ_P , for some linear map \mathcal{A}_λ (for appropriate A^s and A^u). Similarly, the dynamics of $f^{\pi(Q)}$ in a neighborhood of Q is the one of the model map in the cube Δ_Q , for some linear map \mathcal{B}_β (for appropriate B^s and B^u). If the central eigenvalue λ is negative, we consider $f^{2\pi(P)}$ and the dynamics is given by some \mathcal{A}_{λ^2} . Analogously, if $\beta < -1$, we consider $f^{2\pi(Q)}$ and the dynamics is given by some \mathcal{B}_{β^2} .

In this case, we say that $F_{\lambda,\beta,0}$ is a *model map* for the simple cycle of f . If $\lambda < 0$ (resp. $\beta < 0$) we replace λ by λ^2 (resp. β^2).

- Suppose that the diffeomorphism f has a simple cycle and that $F_{\lambda,\beta,0}$ is a model map of it. Then, for every λ' and β' close to λ and β and for every t close to 0, there is a diffeomorphism g , which is C^1 -close to f , such that the dynamics of g in a neighborhood of the cycle of f (up to the same re-scaling as the one of f) is given by $F_{\lambda',\beta',t}$.

The main result about model families is the following:

Proposition 3.8. *Let $F_{\lambda,\beta,0}^{\pm,\pm}$ be a model map. Then there are a constant $C > 0$ and sequences $\lambda_n \rightarrow \lambda$, $\beta_n \rightarrow \beta$, and $t_n \rightarrow 0$, such that, for every n , the map*

$$F_n = F_{\lambda_n,\beta_n,t_n}^{\pm,\pm}$$

has a periodic point A_n of period m_n , $m_n \rightarrow \infty$, such that

- the central eigenvalue $\lambda_c(A_n)$ of $DF_n^{m_n}(A_n)$ satisfies $1/C < |\lambda_c(A_n)| < C$,
- the periodic point A_n has a quasi-transverse strong homoclinic intersection.

In view of the previous comments, this proposition implies Proposition 3.4.

We will prove Proposition 3.8 in Section 3.4. The proof involves a one-dimensional reduction associated to model families. Note that every model unfolding family $F_{\lambda,\beta,t}$ preserves the foliation \mathcal{F}^{su} generated by the hyperplane $E^{ss} \oplus E^{uu}$ (indeed the model family preserves any foliation tangent to E^{ss} , or to E^c , or to E^{uu} , or tangent to the sum of any pair of these bundles). This fact allows us to consider the quotient dynamics of $F_{\lambda,\beta,t}$ by the leaves of \mathcal{F}^{su} , which defines a one-dimensional dynamics. The study of this one-dimensional reduction and its dynamics is the goal of the next section.

3.3 One-dimensional dynamics associated to model families

Let I_P and I_Q be two copies of $[-1,1]$. Denote by P and Q , respectively, the point 0 in the segments I_P and I_Q . For any $\lambda \in (0,1)$, $\beta > 1$, consider the linear maps

$$f_\lambda: I_P \rightarrow \mathbb{R}, \quad f_\lambda(x) = \lambda x \quad \text{and} \quad g_\beta: I_Q \rightarrow \mathbb{R}, \quad g_\beta(x) = \beta x.$$

Consider $b_Q \in (0,1)$, $b_P \in (-1,0)$, and $\delta > 0$ as in Section 3.2. Denote by J_Q the segment $[b_Q - \delta, b_Q + \delta]$ in I_Q and by J_P the segment $[-\delta, +\delta]$ in I_P . Finally, consider the isometries θ_2^\pm and $\theta_{1,t}^\pm$ defined by

$$\theta_2^\pm: J_Q \rightarrow I_P, \quad \theta_2^\pm(b_Q + x) = \pm x + b_P$$

and

$$\theta_{1,t}^\pm: J_P \rightarrow I_Q, \quad \theta_{1,t}^\pm(x) = \pm x + t.$$

Given a model unfolding family $F_{\lambda,\beta,t}^{\pm,\pm}$, the family of maps $(g_\beta^n \circ \theta_{1,t}^\pm \circ f_\lambda^m \circ \theta_2^\pm)_{n,m \geq 0}$ is the one-dimensional family associated to $F_{\lambda,\beta,t}^{\pm,\pm}$. Note that each map $(g_\beta^n \circ \theta_{1,t}^\pm \circ f_\lambda^m \circ \theta_2^\pm)$ is defined on a sub-interval (this interval may be empty) of J_Q .

The goal of this section is to prove the following proposition:

Proposition 3.9. *Let $F_{\lambda,\beta,t}^{\pm,\pm}$ be a model unfolding family. Then there is $N > 0$ with the following property: For every pair of natural numbers n and m , $n, m \geq N$, any parameter t , and any point $a \in J_Q$ satisfying*

$$g_\beta^n \circ \theta_{1,t}^\pm \circ f_\lambda^m \circ \theta_2^\pm(a) = a$$

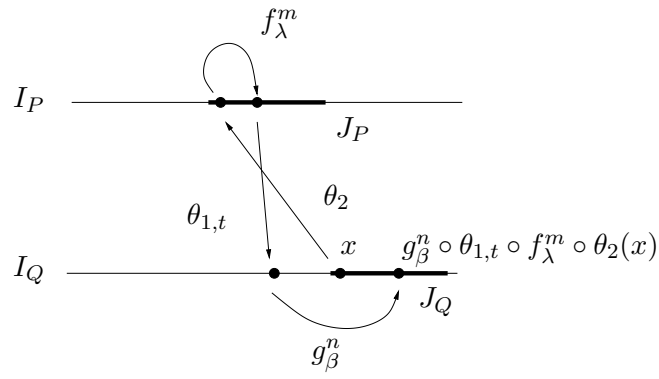


Figure 5: One-dimensional families

there is a periodic point $A_{m,n} \in \Sigma_Q$ of $F_{\lambda,\beta,t}^{\pm,\pm}$ of period $m+n+2$ of the form

$$A_{m,n} = (a^s, a, a^u)$$

such that:

- The central eigenvalue $\lambda^c(A_{m,n})$ of $D(F_{\lambda,\beta,t}^{\pm,\pm})^{m+n+2}(A_{m,n})$ is $\pm\lambda^m\beta^n$ (where \pm is the product of the signals associated to the maps $\theta_{1,t}^{\pm}$ and θ_2^{\pm}).
- Suppose, in addition, that there are $m', n' \in \mathbb{N}$ larger than N such that $(m', n') \neq (m, n)$ and

$$g_{\beta}^{n'} \circ \theta_{1,t}^{\pm} \circ f_{\lambda}^{m'} \circ \theta_2^{\pm}(a) = a.$$

Then there is a point B , $B \neq A_{m,n}$, $B \in W^{uu}(A_{m,n}, F_{\lambda,\beta,t}^{\pm,\pm}) \cap W^{ss}(A_{m,n}, F_{\lambda,\beta,t}^{\pm,\pm})$ (i.e., the periodic point $A_{m,n}$ has a strong homoclinic intersection). Moreover, this intersection is quasi-transverse.

By Proposition 3.9, to prove Proposition 3.8 (thus Proposition 3.4) it is enough to see that given any simple cycle and a model unfolding family $F_{\lambda,\beta,t}^{\pm,\pm}$ of it, there are a point $a \in J_Q$ and sequences of parameters $t_k \rightarrow 0$, of eigenvalues $\lambda_k \rightarrow \lambda$ and $\beta_k \rightarrow \beta$, and of natural numbers $n_k, m_k, n'_k, m'_k \rightarrow \infty$, $(m_k, n_k) \neq (m'_k, n'_k)$, such that

$$g_{\beta_k}^{n_k} \circ \theta_{1,t_k}^{\pm} \circ f_{\lambda_k}^{m_k} \circ \theta_2^{\pm}(a) = a \quad \text{and} \quad g_{\beta_k}^{n'_k} \circ \theta_{1,t_k}^{\pm} \circ f_{\lambda_k}^{m'_k} \circ \theta_2^{\pm}(a) = a.$$

We now prove Proposition 3.9

Proof: Recall that the map $F_{\lambda,\beta,t}$ (we omit the superscripts \pm) coincides with the linear maps $\mathcal{A}_{\lambda} = (A^s, f_{\lambda}, A^u)$ in $\Delta_P \cap \mathcal{A}_{\lambda}^{-1}(\Delta_P)$ and $\mathcal{B}_{\beta} = (B^s, g_{\beta}, B^u)$ in $\Delta_Q \cap \mathcal{B}_{\beta}^{-1}(\Delta_Q)$. Since A^s and $(B^u)^{-1}$ are contractions, there is large N such that

$$\|(B^u)^{-N}\| < \delta \quad \text{and} \quad \|(A^s)^N\| < \delta,$$

where δ is as in the definition of the model family.

We say that a subset C^v of $\Delta_P \cup \Delta_Q$ is a *vertical cylinder* at the point $X = (x^s, x, x^u) \in C^v$ if there is some compact set $K^s \subset [-1, 1]^s$ such that $C^v = K^s \times \{x\} \times [-1, 1]^u$. Similarly, a set $C^h \subset \Delta_P \cup \Delta_Q$ is a *horizontal cylinder* at $X \in C^h$ if there is some compact set $K^u \subset [-1, 1]^u$ such that $C^h = [-1, 1]^s \times \{x\} \times K^u$.

Lemma 3.10. *Let $a \in J_Q$ such that there are $m, n > N$ such that the point*

$$b = g_\beta^n \circ \theta_{1,t}^\pm \circ f_\lambda^m \circ \theta_2^\pm(a) \in I_Q$$

is well defined. Then there are points $A \in \Sigma_Q$ and $B \in \Sigma_Q$ of the form $A = (a^s, a, a^u)$ and $B = (b^s, b, b^u)$ and a horizontal cylinder C^h at A such that:

- *The map $(F_{\lambda,\beta,t})^{n+m+2}$ is defined (and continuous) on C^h , and*
- *$(F_{\lambda,\beta,t})^{n+m+2}(C^h)$ is a vertical cylinder C^v at B .*

In fact, the orbit of the point A in the lemma has the following itinerary:

- $A \in \Sigma_Q$ and $F_{\lambda,\beta,t}(A) = \mathcal{T}_2(A) \in \Delta_P$,
- $F_{\lambda,\beta,t}^{i+1}(A) = (\mathcal{A}_\lambda)^i \circ \mathcal{T}_2(A) \in \Delta_P$, for all $i = 1, \dots, m$, and $F_{\lambda,\beta,t}^{m+1}(A) \in \Sigma_P$,
- $F_{\lambda,\beta,t}^{m+2}(A) = \mathcal{T}_{1,t} \circ (\mathcal{A}_\lambda)^m \circ \mathcal{T}_2(A) \in \Delta_Q$,
- $F_{\lambda,\beta,t}^{j+m+2}(A) = (\mathcal{B}_\beta)^j \circ \mathcal{T}_{1,t} \circ (\mathcal{A}_\lambda)^m \circ \mathcal{T}_2(A) \in \Delta_Q$, for all $j = 1, \dots, n$.

Proof: Consider the $(s+u)$ -disk $D = [-1, 1]^s \times \{a\} \times [-1, 1]^u$. By the choice of N , as $m > N$, and since $(T_2^u)^{-1}$, T_2^s , $(A^u)^{-1}$, and A^s are linear contractions, the map

$$(F_{\lambda,\beta,t})^{1+m} = \mathcal{A}_\lambda^m \circ \mathcal{T}_2$$

is defined on a horizontal cylinder H at some point A_0 of the form $A_0 = (a_0^s, a, a_0^u)$ whose image is a vertical cylinder V_{m+1} of the form

$$V_{m+1} = (F_{\lambda,\beta,t})^{1+m}(H) = K_{m+1}^s \times \{a_{m+1}\} \times [-1, 1]^u,$$

where

$$a_{m+1} = f_\lambda^m \circ \theta_2(a) \quad \text{and} \quad K_{m+1}^s = (A^s)^m \circ T_2^s([-1, 1]^s).$$

Since $m > N$, one has that the set K_{m+1}^s is contained in $[-\delta, \delta]^s$. Therefore the intersection $V'_{m+1} = V_{m+1} \cap \Sigma_P$ is of the form

$$V'_{m+1} = K_{m+1}^s \times \{a_{m+1}\} \times [a_P^u - \delta, a_P^u + \delta]^u.$$

Note that, by construction, the set $(F_{\lambda,\beta,t})^{-(1+m)}(V'_{m+1})$ is a horizontal cylinder $H' \subset H$ at some point A_1 of the form (a_1^s, a, a_1^u) (in fact, we can take $A_0 = A_1$).

As T_1^u is a linear expansion, the set $V'_{m+2} = \mathcal{T}_{1,t}(V'_{m+1})$ contains a set V_{m+2} of the form

$$V_{m+2} = T_1^s(K_{m+1}^s) \times \{a_{m+2}\} \times [-\delta, \delta]^u, \quad \text{where } a_{m+2} = \theta_{1,t}(a_{m+1}).$$

By the choice of N and since $n > N$, the map $(F_{\lambda,\beta,t})^n = \mathcal{B}_\beta^n$ is defined from a horizontal cylinder H_n of the form

$$H_n = [-1, 1]^s \times \{a_{m+2}\} \times K_n^u \subset [-1, 1]^s \times \{a_{m+2}\} \times [-\delta, \delta]^u$$

onto a vertical cylinder at some point $B = (b^s, b, b^u)$, where

$$b = g_\beta^n(a_{m+2}) = g_\beta^n(\theta_{1,t}(a_{m+1})) = g_\beta^n \circ \theta_{1,t} \circ f_\lambda^m \circ \theta_2(a).$$

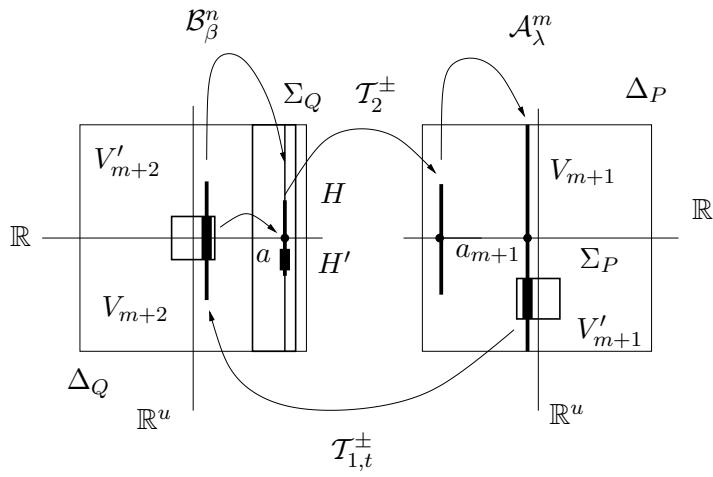


Figure 6: Disks in the proof of Lemma 3.10 (projection along the \mathbb{R}^s -direction)

Consider now the intersection

$$W = H_n \cap V_{m+2} = T_1^s(K_{m+1}^s) \times \{a_{m+2}\} \times K_n^u.$$

Finally, by construction, $(F_{\lambda,\beta,t})^n(W)$ is a vertical cylinder $C^v(B)$ at the point $B = (b^s, b, b^u)$ and $(F_{\lambda,\beta,t})^{-(m+2)}(W)$ is an horizontal cylinder $C^h(A)$ at some point A of the form $A = (a^s, a, a^u)$. This concludes the proof of the lemma. \square

Scholium 3.11. *Under the hypothesis of Lemma 3.10 and assuming that $a = b$, there is a horizontal cylinder $C^h(A)$ at A such that $C^v(A) = (F_{\lambda,\beta,t})^{(n+m+2)}(C^h(A))$ is a vertical cylinder at A crossing $C^h(A)$ in a Markovian way.*

We are now ready to finish the proof of Proposition 3.9. By Scholium 3.11, if $a = b$, then there is a horizontal cylinder $C^h(A)$ at A such that $C^v(A) = (F_{\lambda,\beta,t})^{(n+m+2)}(C^h(A))$ is a vertical cylinder at A crossing $C^h(A)$ in a Markovian way. Hence $(F_{\lambda,\beta,t})^{m+n+2}$ has a unique periodic point $A_{m,n} = (a^s, a, a^u) \in C^v(A) \cap C^h(A)$.

Since the transitions maps \mathcal{T}_1, t and \mathcal{T}_2 are isometries in the (invariant) central bundle E^c , one has that the central eigenvalue of $D(F_{\lambda,\beta,t})^{n+m+2}(A_{m,n})$ is $\pm\lambda^m\beta^n$. This concludes the proof of the first part of the corollary.

To prove the second part of the corollary, note first that the vertical disk $\{(a^s, a)\} \times [-1, 1]^u$ is contained in the unstable manifold of $A_{m,n}$: by construction, Lemma 3.10 implies that $(F_{\lambda,\beta,t})^{-(n+m+2)}$ maps $\{(a^s, a)\} \times [-1, 1]^u$ inside itself in a (linear) contracting way. Similarly, $(F_{\lambda,\beta,t})^{n+m+2}$ maps the horizontal disk $[-1, 1]^s \times \{(a, a^u)\}$ inside itself as a linear contraction. Thus the disk $[-1, 1]^s \times \{(a, a^u)\}$ is contained in the stable manifold of $A_{m,n}$.

Take now the integers $m', n' \in \mathbb{N}$ in the second part of the corollary. The first part of the corollary gives a periodic point $A_{m',n'}$ of $F_{\lambda,\beta,t}$ of period $m'+n'+2$ of the form $A_{m',n'} = (\bar{a}^s, a, \bar{a}^u)$. By construction, $A_{m',n'} \neq A_{m,n}$. As a consequence, by the comments before, the horizontal and the vertical disks through $A_{m',n'}$ intersect transversely (in a su -hyperplane $\mathbb{R}^s \times \{a\} \times \mathbb{R}^u$) the vertical and the horizontal disks through $A_{m,n}$, respectively,

$$\begin{aligned} (\{(a^s, a)\} \times [-1, 1]^u) \cap ([-1, 1]^s \times \{(a, \bar{a}^u)\}) &\neq \emptyset, \\ (\{(\bar{a}^s, a)\} \times [-1, 1]^u) \cap ([-1, 1]^s \times \{(a, a^u)\}) &\neq \emptyset. \end{aligned}$$

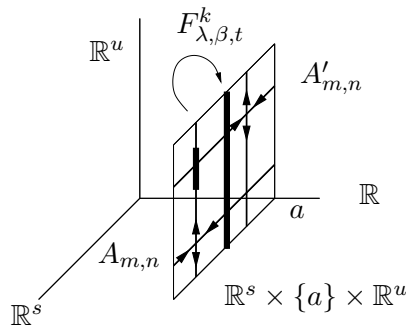


Figure 7: Strong homoclinic intersections

This implies that

$$W^{uu}(A_{m,n}, F_{\lambda,\beta,t}) \cap W^{ss}(A'_{m,n'}, F_{\lambda,\beta,t}) \neq \emptyset, \quad W^{uu}(A'_{m,n'}, F_{\lambda,\beta,t}) \cap W^{ss}(A_{m,n}, F_{\lambda,\beta,t}) \neq \emptyset.$$

Thus there is a cycle between the strong stable and unstable manifolds of $A'_{m,n'}$ and $A_{m,n}$. Moreover, these strong manifolds meet quasi-transversely. The λ -lemma now gives a point $X \neq A_{m,n}$ in $W^{uu}(A_{m,n}, F_{\lambda,\beta,t}) \cap W^{ss}(A'_{m,n'}, F_{\lambda,\beta,t})$. Moreover, this intersection is quasi-transverse. See Figure 7. The proof of Proposition 3.9 is now complete. \square

3.4 Periodic points for one-dimensional maps associated to model families

In this section, we consider one-parameter families of maps of the form

$$g_\beta^n \circ \theta_{1,t}^\pm \circ f_\lambda^m \circ \theta_2^\pm, \quad n, m \geq 0.$$

We prove that given an initial system $(g_\beta^n \circ \theta_{1,0}^\pm \circ f_\lambda^m \circ \theta_2^\pm)_{n,m \geq 0}$ there are λ' and β' close to λ and β , t close to 0, and large n, m, n', m' , $(m, n) \neq (m', n')$, such that the maps $g_{\beta'}^n \circ \theta_{1,t}^\pm \circ f_{\lambda'}^m \circ \theta_2^\pm$ and $g_{\beta'}^{n'} \circ \theta_{1,t}^\pm \circ f_{\lambda'}^{m'} \circ \theta_2^\pm$ have a common fixed point a . Considering these maps as associated one-dimensional maps of model unfolding families, one gets the assumptions of Proposition 3.9 (after some perturbation).

3.4.1 The orientation preserving case

In the section, we prove Proposition 3.8 when $\lambda > 0$, $\beta > 0$ and the transition T_1 preserves the orientation of the central bundle.

Lemma 3.12. *For any $\varepsilon > 0$ and $K > 0$, there are $\beta_0 \in (\beta - \varepsilon, \beta + \varepsilon)$, $\lambda_0 \in (\lambda - \varepsilon, \lambda + \varepsilon)$, $t \in (0, \varepsilon)$, and natural numbers n, m, n' larger than K such that:*

1. $n < n'$,
2. $\lambda_0^{m+1} b_P + t = \beta_0^{-n} b_Q$, and
3. $\lambda_0^m b_P + t = \beta_0^{-n'} b_Q$.

Proof: We first claim that there are β_1 arbitrarily close to β and n, m arbitrarily large such that

$$\lambda^m (1 - \lambda) |b_P| = \beta_1^{-n} b_Q.$$

To prove this claim, one first takes β_2 close to β such that $\frac{\log \lambda}{\log \beta_2}$ is irrational. This allows us to choose n, m arbitrarily large such that

$$\frac{\lambda^m (1 - \lambda) |b_P|}{\beta_2^{-n} b_Q}$$

is arbitrarily close to 1. A small modification of β_2 gives the announced β_1 .

Next, we fix $n' > n$ such that $\beta_3^{-(n'-n)}$ is very small for all β_3 close to β_1 . This allows to choose β_0 in a small neighborhood of β_1 such that

$$\lambda^m (1 - \lambda) |b_P| = \beta_0^{-n} (1 - \beta_0^{-(n'-n)}) b_Q.$$

Take

$$t = \beta_0^{-n} b_Q + \lambda^{m+1} |b_P| \quad \text{and} \quad \lambda_0 = \lambda,$$

This choice of t immediately gives equality (2) in the lemma:

$$\beta_0^{-n} b_Q = t - \lambda^{m+1} |b_P| = t + \lambda^{m+1} b_P.$$

Similarly, the choices of n' and t give

$$\beta_0^{-n'} b_Q = \beta_0^{-n} b_Q + \lambda^{m+1} |b_P| - \lambda^m |b_P| = t + \lambda^m b_P.$$

This completes the proof of the lemma. \square

Lemma 3.12 can be written in terms of the one-dimensional maps associated to the model unfolding maps $F_{\lambda, \beta, t}^{+, \pm}$ (where $\mathcal{T}_{1, t}$ preserves the orientation).

Corollary 3.13. *For any $\varepsilon > 0$ and $K > 0$, there are $\beta_0 \in (\beta - \varepsilon, \beta + \varepsilon)$, $\lambda_0 \in (\lambda - \varepsilon, \lambda + \varepsilon)$, $t \in (0, \varepsilon)$, and natural numbers n, m, n' larger than K such that*

- $g_\beta^n \circ \theta_{1, t}^+ \circ f_\lambda^{m+1} \circ \theta_2^\pm(b_Q) = b_Q$,
- $g_\beta^{n'} \circ \theta_{1, t}^\pm \circ f_\lambda^m \circ \theta_2^\pm(b_Q) = b_Q$, where $n' > n$.

Proof: Note that $\theta_2^\pm(b_Q) = b_P$ and $\theta_{1, t}^+(x) = x + t$. Thus the first equality in the corollary is equivalent to

$$b_Q = g_\beta^n \circ \theta_{1, t}^+ \circ f_\lambda^{m+1} \circ \theta_2^\pm(b_Q) = \beta^{-n} (\lambda^{m+1} b_p + t), \quad \beta^n b_Q = \lambda^{m+1} b_p + t.$$

This identity is exactly (2) in Lemma 3.12. The second identity in the corollary follows from (3) and (1) in Lemma 3.12 arguing in the same way. \square

3.4.2 The orientation reversing case

In the section, we prove Proposition 3.8 when $\lambda > 0$, $\beta > 0$, and the transition T_1 reverses the orientation of the central bundle. The proof follows essentially as the one in the orientation preserving case. The main difficulty in this case is to get a result similar to Lemma 3.12.

Lemma 3.14. *For any $\varepsilon > 0$ and $K > 0$, there are $\beta_0 \in (\beta - \varepsilon, \beta + \varepsilon)$, $\lambda_0 \in (\lambda - \varepsilon, \lambda + \varepsilon)$, and natural numbers n, m, n', m' larger than K such that:*

1. $n < n'$ and $m < m'$,
2. $\lambda_0^m |b_P| = \beta_0^{-n} b_Q$, and
3. $\lambda_0^{m'} |b_P| = \beta_0^{-n'} b_Q$.

Proof: We first take $a > 1$ close to 1 (in particular, $\log a$ is much smaller than $|\log \lambda|$ and $\log \beta$) such that

$$r(\log a) = \log \left(\frac{b_Q}{|b_P|} \right), \quad \text{for some } r \in \mathbb{N}.$$

We replace λ and β by some λ_0 and β_0 , arbitrarily close to λ and β , of the form

$$\log \lambda_0 = -h \log a \quad \text{and} \quad \log \beta_0 = k \log a, \quad h, k \in \mathbb{N}.$$

We claim that one can take $h, k \in \mathbb{N}$ being relatively prime integers. Otherwise (i.e., if for this choice of a , h and k are not relatively prime numbers), we replace a by $a^{\frac{1}{k}}$, h by $hk + 1$ and k by k^2 . In this case, $hk + 1$ and k^2 are relatively prime. Clearly, $\log a^{\frac{1}{k}}$ divides $\log \left(\frac{b_Q}{|b_P|} \right)$, β_0 is not modified, and the new λ_0 is close to λ (if k is big enough).

As h and k are relatively prime numbers, there are m_0 and n_0 with $-m_0 h + n_0 k = 1$. By the definition of h and k , this choice of m_0 and n_0 gives $\lambda_0^{m_0} \beta_0^{n_0} = a$. Hence, since $\log \left(\frac{b_Q}{|b_P|} \right) = r \log a$, taking $n = r n_0$ and $m = r m_0$, one gets

$$\lambda_0^m \beta_0^n = \frac{b_Q}{|b_P|}.$$

This gives (2) in the lemma.

To get (3) in the lemma, note that the ratio $\frac{\log \lambda_0}{\log \beta_0}$ is rational by construction, thus there are $n_1, m_1 > 0$ such $\lambda_0^{m_1} \beta_0^{n_1} = 1$. Taking $m' = m + m_1$ and $n' = n + n_1$ one gets (3). Note that, by construction, $n' > n$ and $m' > m$. This completes the proof of the lemma. \square

As in the orientation preserving case, Lemma 3.14 can be formulated in terms of the one-dimensional maps associated to the family $F_{\lambda, \beta, t}^{-, \pm}$.

Corollary 3.15. *For any $\varepsilon > 0$ and $K > 0$, there are $\beta_0 \in (\beta - \varepsilon, \beta + \varepsilon)$ and $\lambda_0 \in (\lambda - \varepsilon, \lambda + \varepsilon)$, and natural numbers n, m, n', m' larger than K such that:*

- $n < n'$ and $m < m'$,
- $g_\beta^n \circ \theta_{1,0}^- \circ f_\lambda^m \circ \theta_2^\pm(b_Q) = b_Q$,
- $g_\beta^{n'} \circ \theta_{1,0}^- \circ f_\lambda^{m'} \circ \theta_2^\pm(b_Q) = b_Q$.

3.5 Strong homoclinic intersections for model unfolding maps

In this section, we prove Proposition 3.8. As in the previous section, there are two cases (the preserving and reversing orientation cases).

3.5.1 The transition $\mathcal{T}_{1,t}$ preserves the orientation of the central bundle

When the transition $\mathcal{T}_{1,t}$ preserves the orientation, using Corollary 3.13 and Proposition 3.9, we get strong homoclinic intersections associated to periodic points of the model unfolding family:

Proposition 3.16. *For every model unfolding family $(F_{\lambda,\beta,t}^{+,\pm})$ there is a constant $C > 1$ satisfying the following property:*

For any $\varepsilon > 0$ and $K > 0$, there are $\beta_0 \in (\beta - \varepsilon, \beta + \varepsilon)$, $\lambda_0 \in (\lambda - \varepsilon, \lambda + \varepsilon)$, $t \in (0, \varepsilon)$, and natural numbers n, m larger than K , such that $F_{\lambda_0,\beta_0,t}^{+,\pm}$ has a periodic point $A_{m+1,n}$, of period $m + n + 3$, such that

- *the central eigenvalue $\lambda^c(A_{m+1,n})$ of $A_{m+1,n}$ satisfies $|\lambda^c(A_{m+1,n})| \in [1/C, C]$,*
- *the periodic point $A_{m+1,n}$ has a quasi-transverse strong homoclinic intersection.*

Proof: Proposition 3.9 and Corollary 3.13 give λ_0, β_0 , and t such that $F_{\lambda_0,\beta_0,t}^{+,\pm}$ has a periodic point $A_{m+1,n}$ (of period $n + m + 3$) with a quasi-transverse strong homoclinic intersection and whose central eigenvalue has modulus $|\lambda_0^{m+1} \beta_0^n|$.

Thus it remains to choose the constant C (bounding the modulus of the central eigenvalue). By Lemma 3.12, there is $n' > n$ such that

$$\lambda_0^{m+1} b_P + t = \beta_0^{-n} b_Q \quad \text{and} \quad \lambda_0^m b_P + t = \beta_0^{-n'} b_Q.$$

As a consequence, one has

$$\lambda_0^m (\lambda_0 - 1) b_P = \beta_0^{-n} (1 - \beta_0^{n-n'}) b_Q, \quad \lambda_0^m \beta_0^n = \frac{(1 - \beta_0^{n-n'}) b_Q}{(1 - \lambda_0) |b_P|}.$$

Hence, as $n' > n$,

$$\left(\frac{1 - \beta_0^{-1}}{1 - \lambda_0} \right) \frac{b_Q}{|b_P|} \leq \lambda_0^m \beta_0^n < \left(\frac{1}{1 - \lambda_0} \right) \frac{b_Q}{|b_P|}.$$

Since λ_0, β_0 are close to λ, β , it is enough to choose a constant C satisfying

$$C \geq \frac{2}{\lambda} \max \left\{ \left(\frac{1}{1 - \lambda} \right) \frac{b_Q}{|b_P|}, \left(\frac{1 - \lambda}{1 - \beta^{-1}} \right) \frac{|b_P|}{b_Q} \right\}.$$

This concludes the proof of the proposition. □

3.5.2 The transition $\mathcal{T}_{1,t}$ reverses the orientation of the central bundle

If $\mathcal{T}_{1,t}$ reverses the orientation, using Corollary 3.15 and Proposition 3.9, we get a quasi-transverse strong homoclinic intersections associated to saddles of the model unfolding family for $t = 0$ (note that in this case, the parameter t is not modified):

Proposition 3.17. *For every model unfolding family $(F_{\lambda,\beta,t}^{\pm})$ there is a constant $C > 1$ satisfying the following property:*

For any $\varepsilon > 0$ and $K > 0$, there are $\beta_0 \in (\beta - \varepsilon, \beta + \varepsilon)$, $\lambda_0 \in (\lambda - \varepsilon, \lambda + \varepsilon)$, and natural numbers n, m larger than K , such that $F_{\lambda_0, \beta_0, 0}^{\pm}$ has a periodic point $A_{m+1, n}$, of period $m + n + 3$, such that

- *the central eigenvalue $\lambda^c(A_{m+1, n})$ of $A_{m+1, n}$ satisfies $|\lambda^c(A_{m+1, n})| \in [1/C, C]$,*
- *the periodic point $A_{m+1, n}$ has a strong homoclinic intersection.*

The proof of this proposition follows exactly as the one of Proposition 3.16, thus it is omitted.

3.5.3 End of the proof of Proposition 3.4. Proof of Theorem 3.3

Proposition 3.16 implies immediately Proposition 3.4 when the transition $\mathcal{T}_{1,t}$ preserves the orientation of the central bundle. Finally, Proposition 3.17 implies Proposition 3.4 when the transition $\mathcal{T}_{1,t}$ reverses the central orientation. Thus the proof of Proposition 3.4 is now complete.

Recall that in the introduction of this section, we derived Theorem 3.3 from Proposition 3.4. This completes the proof of Theorem 3.3.

4 Robust cycles at strong homoclinic intersections

The goal of this section is to prove the following result:

Theorem 4.1. *Let f be a diffeomorphism with a quasi-transverse strong homoclinic intersection associated to a saddle-node or to a flip. Then every C^1 -neighborhood \mathcal{U} of f contains an open set of diffeomorphisms with C^1 -robust heterodimensional cycles.*

The main step of the proof of this theorem is to see that any diffeomorphisms with a quasi-transverse strong homoclinic intersection associated to a saddle-node or to a flip can be approximated by diffeomorphisms exhibiting *blenders*, see Lemma 4.4. We first analyze the case when the strong homoclinic intersection is associated to a saddle-node. The flip case is derived from this case (see Section 4.2).

4.1 Proof of Theorem 4.1: the saddle-node case

In this section, we will sketch the proof of Theorem 4.1 for strong homoclinic intersections associated to saddle-nodes. This is a rather folkloric result in partially hyperbolic dynamics.

4.1.1 Affine saddle-node cycles

Consider a diffeomorphism f having a quasi-transverse strong homoclinic intersection Y associated to a saddle-node P . First, as in the case of co-index one cycles, we perform a series of C^1 -perturbations to the diffeomorphism f in order to linearize the dynamics in neighborhoods of the saddle-node P and of the homoclinic orbit of Y . This first part of the construction (A1–A3) is similar to the one for co-index one cycles in Section 3.1. Let us describe a bit more precisely this construction.

A1) By a C^1 -perturbation of f , we can assume that there is a local chart $[-1, 1]^{s+1+u}$ at the saddle-node P such that the expression of $f^{\pi(P)}$ is of the form

$$f^{\pi(P)} = (A^s, id, A^u): \mathbb{R}^s \times \mathbb{R} \times \mathbb{R}^u \rightarrow \mathbb{R}^s \times \mathbb{R} \times \mathbb{R}^u,$$

where A^s and A^u are linear maps with $\|A^s\| < 1$ and $\|(A^u)^{-1}\| < 1$. (This step corresponds to the first part of (S2)).

A2) Consider the closure Λ_Y of the orbit of the homoclinic point Y (i.e., the union of the orbits of P and Y). By the previous item and since the intersection of $W^{uu}(P, f)$ and $W^{ss}(P, f)$ at Y is quasi-transverse, one has that f is partially hyperbolic in the set Λ_Y , having a partially hyperbolic splitting of the form

$$T_{\Lambda_Y} M = E^{ss} \oplus E^c \oplus E^{uu},$$

where the central bundle E^c is one-dimensional, $\dim E^{ss} = s$, and $\dim E^{uu} = u$. This splitting can be extended to a small neighborhood of Λ_Y . (This step corresponds to the second part of (S2)).

A3) Using the local coordinates above, we define local strong unstable and strong stable manifolds of P by

$$W_{loc}^{uu}(P, f) = \{(0^s, 0)\} \times [-1, 1]^u \quad \text{and} \quad W_{loc}^{ss}(P, f) = [-1, 1]^s \times \{(0, 0^u)\}.$$

Then there are points $W = (0^s, 0, w^u)$ and $Z = (z^s, 0, 0^u)$ in the orbit of Y and $m_0 > 0$, such that $W \in W_{loc}^{uu}(P, f)$, $Z \in W_{loc}^{ss}(P, f)$, and $f^{m_0}(W) = Z$. Moreover, for every $i \in \{1, \dots, m_0 - 1\}$, $f^i(W)$ does not belong to the local chart.

After shrinking the local chart and replacing W by some negative iterate of it and Z by some positive iterate of it (so that m_0 is replaced by a larger number), we can perform a perturbation of f along the segment of orbit $f(W), f^2(W), \dots, f^{m_0-1}(W)$ in such a way the expression of f^{m_0} in a neighborhood of W (in the local coordinates) is of the form

$$f^{m_0}(x^s, x, x^u) = (K^s(x^s) + z^s, \pm x, K^u(x^u - w^u)),$$

where K^s and K^u are linear maps with $\|K^s\| < 1$ and $\|(K^u)^{-1}\| < 1$. (This step corresponds to (S3)).

A4) By construction, the stable-unstable hyperplane $\Pi^{su} = \mathbb{R}^s \times \{0\} \times \mathbb{R}^u$ is locally invariant by $f^{\pi(P)}$ in a neighbourhood of P and by f^{m_0} in a neighborhood of W . Hence there is an iterate of f whose restriction to the hyperplane Π^{su} has a Smale linear horseshoe Σ_0 containing P, W , and Z . Moreover, after replacing the homoclinic point W by a homoclinic point corresponding to an even iterate of it by f^{m_0} (if necessary), we can assume that (in a neighborhood of W) f^{m_0} is the identity in the central direction:

$$f^{m_0}(x^s, x, x^u) = (K^s(x^s) + z^s, x, K^u(x^u - w^u)).$$

A5) Since the horseshoe Σ_0 contains infinitely many strong homoclinic intersections of P , there is a strong homoclinic point G of P , $G = (g^s, 0, 0^u) \in W_{loc}^{ss}(P, f)$, whose f -orbit is disjoint

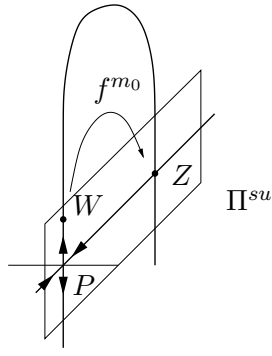


Figure 8: A partially hyperbolic horseshoe

from the one of W . We also consider a point $H = (0^s, 0, h^u) = f^{-m}(G) \in W_{loc}^{uu}(P, f)$, for some big $m \in \mathbb{N}$.

More precisely, after changing linearly the coordinates around P , shrinking the local chart, and replacing (if necessary) Z and G by some forward iterates of them and W and H by some backwards iterates of them, there are natural numbers n_2 and n_3 , $f^{n_2}(W) = Z$ and $f^{n_3}(H) = G$, a cube $C = [-1, 1]^{s+1+u}$, and three *horizontal sub-cubes* C_1 , C_2 , and C_3 of the form

$$C_i = [-1, 1]^{s+1} \times C_i^u, \quad i \in \{1, 2, 3\},$$

where C_1^u , C_2^u , and C_3^u are disjoint u -disks contained in $[-1, 1]^u$, such that:

- the cube C_1 contains the points P , Z , and G , the cube C_2 contains W , and the cube C_3 contains H ;
- $f^{\pi(P)}(C_1)$ is a vertical sub-cube of C which crosses C_1 , C_2 , and C_3 in a Markovian way;
- $f^i(C_2)$ is disjoint from C for every $i = 1, \dots, n_2 - 1$, and $f^{n_2}(C_2)$ is a vertical sub-cube $C_2^s \times [-1, 1]^{1+u}$ of C containing Z ;
- the expression of the restriction of f^{n_2} to C_2 in these local coordinates is

$$f^{n_2}(x^s, x^c, x^u) = (T^s(x^s) + z^s, x^c, T^u(x^u - w^u)),$$

where T^s and T^u are linear maps with $\|T^s\| < 1$ and $\|(T^u)^{-1}\| < 1$;

- there is n_3 such that $f^i(C_3)$ is disjoint from C for every $i = 1, \dots, n_3 - 1$, and $f^{n_3}(C_3)$ is a vertical sub-cube of $C_3^s \times [-1, 1]^{1+u}$ of C containing G ;
- the expression of the restriction of f^{n_3} to C_3 in these local coordinates is affine,

$$f^{n_3}(x^s, x^c, x^u) = (L^s(x^s) + g^s, x^c, L^u(x^u - h^u)),$$

where L^s and L^u are linear maps with $\|L^s\| < 1$ and $\|(L^u)^{-1}\| < 1$;

- the orbit of G is disjoint from C_2 and the orbit of Z is disjoint from C_3 .

A6) Consider the map $F: C_1 \cup C_2 \cup C_3 \rightarrow C$ whose restrictions to C_1 , C_2 and C_3 are $f^{\pi(P)}$, f^{n_2} , and f^{n_3} , respectively. We write $F = (F^{su}, id)$ (i.e., we write the central coordinate in the last position). Observe that F^{su} is a linear horseshoe map conjugate to a complete shift of three symbols.

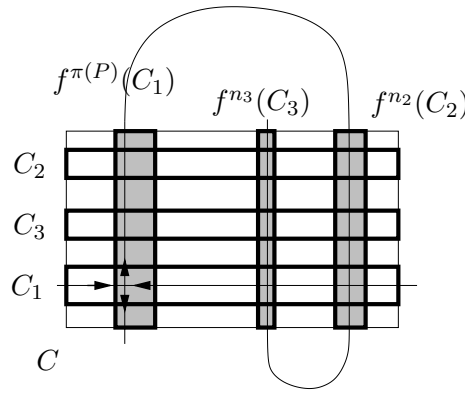


Figure 9: The horseshoe map F^{su} in the hyperplane $\mathbb{R}^s \times \{0\} \times \mathbb{R}^u$

4.1.2 Local perturbations

We now consider a local C^1 -perturbation f_φ of f such that its associated map F_φ defined on the cubes C_1, C_2 and C_3 (defined exactly as F) is of the form

$$F_\varphi = (F^{su}, \varphi),$$

where $\varphi: [-1, 1] \rightarrow [-1, 1]$ is a Morse-Smale map C^1 -close to the identity having exactly two fixed points close to 0, the repelling point 0 and the attracting point $-\delta$ (small $\delta > 0$), these points have eigenvalues close to one. In this way, we get a diffeomorphism f_φ having a pair of twin horseshoe (of indices u and $u + 1$) as depicted in Figure 10. More precisely, the following properties hold:

- The point $P_\delta = (0^s, -\delta, 0^u)$ is a periodic saddle of period $\pi(P)$ and index u of F_φ . The points $Z_\delta = (z^s, -\delta, 0^u)$ and $G_\delta = (g^s, -\delta, 0^u)$ are strong homoclinic intersections of P_δ . Similarly, P is a periodic saddle of period $\pi(P)$ and index $u + 1$ of F_φ and the points Z and G are strong homoclinic intersections of P .
- The restriction of F_φ to the hyperplane $[-1, 1]^s \times \{0\} \times [-1, 1]^u$ is the linear horseshoe map F^{su} . Consider the cube

$$C(\delta) = [-1, 1]^s \times [-\delta/2, \delta/2] \times [-1, 1]^u.$$

Then the maximal invariant set Γ of F_φ in $C(\delta)$ is a hyperbolic basic set of index $u + 1$.

We now modify f in $f^{n_2-1}(C_2)$ and $f^{n_3-1}(C_3)$ to get a two-parameter family of diffeomorphisms $f_{t,r}$ (see Figure 10) such that:

- the restriction of $f_{t,r}^{n_2}$ to C_2 is $f_{t,r}^{n_2}(x^s, x, x^u) = f^{n_2}(x^s, x, x^u) + (0, t, 0)$;
- the restriction of $f_{t,r}^{n_3}$ to C_3 is $f_{t,r}^{n_3}(x^s, x, x^u) = f^{n_3}(x^s, x, x^u) + (0, r, 0)$.

For small t and r , denote by $F_{t,r}$ the map defined on $C_1 \cup C_2 \cup C_3$ as follows:

- $F_{t,r}(x^s, x, x^u) = F_\varphi(x^s, x, x^u)$, if $(x^s, x, x^u) \in C_1$,

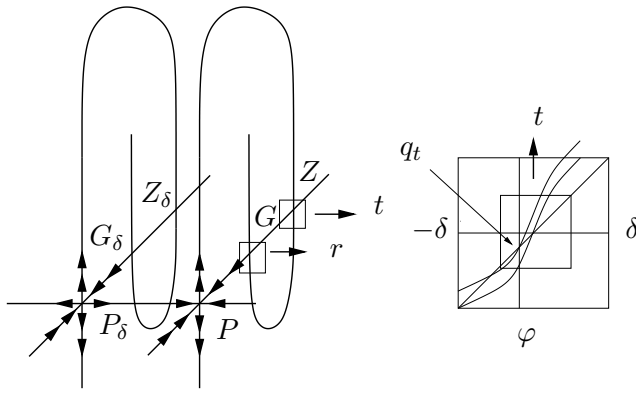


Figure 10: Twin horseshoes for f_φ

- $F_{t,r}(x^s, x, x^u) = f_{t,r}^{n_2}(x^s, x, x^u) = F_\varphi(x^s, x, x^u) + (0, t, 0)$, if $(x^s, x, x^u) \in C_2$, and
- $F_{t,r}(x^s, x, x^u) = f_{t,r}^{n_3}(x^s, x, x^u) = F_\varphi(x^s, x, x^u) + (0, r, 0)$, if $(x^s, x, x^u) \in C_3$.

Remark 4.2. For every small t and r , the maps $f_{t,r}$ and $F_{t,r}$ satisfy the following properties:

1. The perturbation $f_{t,r}$ do not modify the orbits of the periodic points P_δ and P of f_φ .
2. The vertical disk $\{(g^s, -\delta + r)\} \times [-1, 1]^u$ is contained in the unstable manifold of P_δ (of dimension u) of $F_{t,r}$.
3. For $i = 1, 2, 3$, consider the cubes $C_i(\delta) = C_i \cap C(\delta)$. Denote by $\Lambda_{t,r}$ the maximal invariant set of $F_{t,r}$ in $C_1(\delta) \cup C_2(\delta)$. Since this set does not depend on r we just write $\Lambda_t = \Lambda_{t,r}$.
 - For $t = 0$, Λ_0 is a basic set of F_φ of index $(u + 1)$ contained in Γ (the maximal invariant set of F_φ in $C(\delta)$). Hence, Λ_t is a hyperbolic basic set of $F_{t,r}$ of index $(u + 1)$, which is the continuation of Λ_0 .
 - The map $F_{t,r}$ has a unique fixed point $Q_{t,r}$ in $C_2(\delta)$. Since this point does not depend on r we write $Q_t = Q_{t,r}$. Note that $Q_t = (q^s, q_t, q^u)$, where $q_t < 0$ if and only if $t > 0$, see Figure 10.

4.1.3 End of the proof of Theorem 4.1 (saddle-node case)

Next proposition implies Theorem 4.1 when the strong homoclinic intersection is associated to a saddle-node:

Proposition 4.3. For every small $t > 0$ and r such that $-\delta + r \in (q_t, 0)$, the diffeomorphism $f_{t,r}$ has a robust heterodimensional cycle associated to the hyperbolic set Λ_t (of index $u + 1$) and the hyperbolic saddle P_δ (of index u).

Proof: First, note that $F_{t,r}$ coincides with φ on $\{0^s\} \times [-1, 1] \times \{0^u\}$. By definition of φ , this implies that $W^s(P_\delta, f_{t,r})$ meets transversely $W^u(P, f_{t,r}) \subset W^u(\Lambda_t, f_{t,r})$ along the center curve $\{0^s\} \times (-\delta, 0) \times \{0^u\}$ bounded by the periodic points P and P_δ (see Figure 10). Note that this intersection is robust.

To get a (robust) heterodimensional cycle associated to P_δ and Λ_t , it remains to prove that $W^u(P_\delta, f_{t,r})$ meets $W^s(\Lambda_t, f_{t,r})$ in a robust way. Observe that this property cannot be obtained from a transversality argument: the sum of the dimensions of these manifolds is $s + u < s + u + 1$ (the dimension of the ambient manifold). We will use here that, for any small $t > 0$, the set Λ_t is a *blender* and the unstable manifold of P_δ transversely meets the *characteristic region of the blender* (see [BDV₂, Chapter 6.2] for a discussion of the notion of blender). Let us explain this point more precisely. We begin by recalling the meaning of blender. A simple argument in [BDV₁] proves the following:

Lemma 4.4 (The lemma in page 717 in [BDV₁]). *For every small $t > 0$ and every ρ_1 and ρ_2 with $q_t < \rho_1 < \rho_2 < 0$ there is a backward iterate by $F_{t,r}$ of the local stable manifold of P which meets transversely the vertical strip $\{x^s\} \times (\rho_1, \rho_2) \times [-1, 1]^u$, for any $x^s \in [-1, 1]$.*

Let Λ_t be the maximal invariant set Λ_t of $F_{t,r}$ in the cube $C(\delta)$. This set is hyperbolic and has index $(u + 1)$. Observe that the local stable manifold of P is contained in $W_{loc}^s(\Lambda_t, F_{t,r})$. An immediate consequence of Lemma 4.4 now is the following:

Corollary 4.5. *For every small $t > 0$, the stable manifold of Λ_t intersects any vertical disk $\{(x^s, x)\} \times [-1, 1]^u$ with $x^s \in [-1, 1]^s$ and $x \in [q_t, 0]$.*

Keeping in mind these results we prove Proposition 4.3. We first see that $F_{t,r}$ has a heterodimensional cycle associated to the hyperbolic set Λ_t and P_δ . By item 2 in Remark 4.2, the vertical disk $\{(g^s, -\delta + r)\} \times [-1, 1]^u$ is contained in the unstable manifold of P_δ for $F_{t,r}$. Since, by hypothesis, $-\delta + r \in (q_t, 0)$, Corollary 4.5 gives a heterodimensional cycle associated to Λ_t and P_δ for every $f_{t,r}$.

Let us now explain why this heterodimensional cycle is robust. This follows from the proof of [BD₁, Lemma 1.11] (in fact, the proof of Lemma 4.4 is a simplified version of the arguments in [BD₁]). We will outline this proof in the next paragraphs.

For every diffeomorphisms h which is C^1 -close to $f_{t,r}$, one considers *almost vertical strips* \mathcal{S} , that is, $(u + 1)$ -disks such that:

- they are tangent to a small cone field around the center-unstable direction $E^c \oplus E^u$ and are foliated by u -disks tangent to a small unstable cone field around the unstable direction;
- they cross the cube $C(\delta)$ from the bottom to the top.

The *central width* $w(\mathcal{S})$ of an almost vertical strip \mathcal{S} is (roughly) defined as follows: $w(\mathcal{S})$ is the minimum size of a curve tangent to the central direction going from one vertical boundary of the strip \mathcal{S} to the other vertical boundary of the strip (the *vertical boundary* of \mathcal{S} is the part of the boundary of \mathcal{S} tangent to the unstable cone field).

Finally, if the vertical strip \mathcal{S} is in between the local stable manifolds of the continuations of the saddles P and Q_t , we say that the strip is in the *characteristic region* of the blender.

Given a map ψ close to f , considering the restrictions of $\psi^{\pi(p)}$ and ψ^{n_2} to the cubes $C_1(\delta)$ and $C_2(\delta)$, one gets a map Ψ which is C^1 -close to $F_{t,r}$. One proves the following,

There is $\kappa > 1$ such that, for every vertical strip \mathcal{S} in the characteristic region, either the image by Ψ of \mathcal{S} intersects the local stable manifold of the continuation of P or it contains a vertical strip \mathcal{S}' in the characteristic region whose central-width satisfies $w(\mathcal{S}') > \kappa w(\mathcal{S})$.

As the central widths of the vertical strips in the characteristic region are uniformly bounded from above, there is some iterate by Ψ of the initial strip \mathcal{S} which meets the local stable manifold of the continuation of P . This completes the outline of the proof. For further details and precise definitions, see the proof of [BD₁, Lemma 1.11]⁶. \square

4.2 Proof of Theorem 4.1: the flip case

The first step of this case is analogous to the saddle-node one: the goal is to get an affine flip cycle after a perturbation of a flip point with a strong homoclinic intersection. These flip cycles generate saddle-node cycles:

Remark 4.6. *In the case that the diffeomorphism f has a flip with a strong homoclinic intersection, one can perform perturbations similar to the ones in Section 4.1.1. The only difference is that, in the flip case, in item (A1) one has $(-id)$ in the central direction instead of the identity. As a consequence, the resulting associated map F in item (A6) is now of the form $(F^{su}, -id)$.*

Using the horseshoe Σ_0 in item (A5) and considering periodic points of even period in this horseshoe, one immediately gets (after an arbitrarily small perturbation) a saddle-node with a strong homoclinic intersection.

The flip case follows now from the saddle-node case in the previous section. The proof of Theorem 4.1 is now complete.

5 Cycles with non-real central eigenvalues

The aim of this section is to prove the following theorem:

Theorem 5.1. *Let f be a diffeomorphism having a co-index one cycle associated to saddles P and Q . Then every C^1 -neighborhood \mathcal{U} of f contains a diffeomorphism g with a co-index one cycle with real central eigenvalues. Moreover, the new cycle can be taken associated to saddles P'_g and Q'_g homoclinically related to the continuations P_g and Q_g of P and Q .*

This result concludes the proof of Theorem 1: every co-index one cycle generates (by a C^1 -perturbations) cycles with real central eigenvalues, Theorem 2.1 now gives robust heterodimensional cycles.

The organization of this section is the following. In Section 5.1, we see that every co-index one cycle associated to a pair of saddles with non-trivial homoclinic classes generates cycles with real central eigenvalues, (see Theorem 5.3). In Section 5.2, we see that if f has a co-index one cycle associated to a pair of saddles P and Q such that the central eigenvalue of the cycle associated to P is non-real, one can assume (after a perturbation) that the homoclinic class of Q is non-trivial (see Proposition 5.7). Using these preparatory results, in Section 5.3 we prove Theorem 5.1 for cycles whose central eigenvalues are all non-real. Finally, in Section 5.4, we consider cycles having only one real eigenvalue.

⁶Indeed, this robust intersection between vertical strips and a local stable manifold of a hyperbolic set is called *distinctive property of blenders* in [BDV₂].

5.1 Cycles associated to saddles with non-trivial homoclinic classes

We begin this section with a definition.

Definition 5.2 (Periodic points with real multipliers). *Let f be a diffeomorphism and P a periodic point of f . We say that P has real multipliers if every eigenvalue λ of $Df^{\pi(P)}(P)$ is real and has multiplicity one and every eigenvalue σ of $Df^{\pi(P)}(P)$ with $\sigma \neq \lambda$ satisfies $|\lambda| \neq |\sigma|$.*

Theorem 5.3. *Let f be a diffeomorphism with a co-index one cycle associated to periodic points P and Q . Suppose that the homoclinic class of P is non-trivial. Then every C^1 -neighborhood \mathcal{U} of f contains a diffeomorphism $g \in \mathcal{U}$ having a hyperbolic periodic point P'_g such that:*

- *there is a co-index one cycle associated to P'_g and the continuation Q_g of Q ;*
- *the saddle P'_g has real multipliers and is homoclinically related to P_g .*

The main step for proving Theorem 5.3 is the following:

Proposition 5.4. *Let K be a non-trivial hyperbolic basic set of a diffeomorphism f . Assume that f has a heterodimensional cycle associated to a pair of periodic points P and Q , where $P \in K$. Then every C^1 -neighborhood \mathcal{U} of f contains a diffeomorphism g such that:*

- *The continuation K_g of K contains a periodic point P'_g with real multipliers and homoclinically related to P_g .*
- *The diffeomorphism g has a heterodimensional cycle associated to the continuations P_g and Q_g .*

Proof: Consider the set Σ of periodic points of K . As K is a basic set, the periodic orbits in Σ are homoclinically related. With the notation in [BDP, Section 1.4], the derivative of f induces on Σ a *periodic linear system with transitions*. [BDP, Lemmas 4.16 and 1.9] now imply that, for any $\varepsilon > 0$, there are a periodic orbit $\gamma = \{Y, \dots, f^{\pi(Y)-1}(Y)\}$ of K and an ε -perturbation A of the derivative Df (considered as a linear cocycle) along γ (i.e., $\|Df(f^j(Y)) - A(f^j(Y))\| < \varepsilon$, for every $0 \leq j \leq \pi(Y) - 1$) such that the eigenvalues of the linear map

$$A(\gamma) = A(f^{\pi(Y)-1}(Y)) \circ \dots \circ A(Y),$$

are all real and different in modulus and have multiplicity one. We need the following lemma:

Lemma 5.5 (Franks' Lemma, [Fr], [Ma₂]). *Consider a C^1 -diffeomorphism f and an f -invariant finite set Σ . Let A be an ε -perturbation of the derivative Df of f along Σ . Then, for every neighborhood V of Σ , there is a diffeomorphism g C^1 - ε -close to f such that*

- $g(x) = f(x)$, if $x \in \Sigma$ or if $x \notin V$,
- $Dg(x) = A(x)$, for all $x \in \Sigma$.

This lemma allows us to consider a C^1 -perturbation g of f , supported on an arbitrarily small neighborhood V of the orbit γ of Y , such that (i) g is equal to f outside V and over the orbit γ and (ii) $Dg(X) = A(X)$, for every $X \in \gamma$. As the perturbation g of f is arbitrarily small, the periodic point Y of g is homoclinically related to P (the orbit of P is not modified). Taking

$P'_g = Y$, one has $Dg^{\pi(P'_g)}(P'_g) = A(\gamma)$. Thus the periodic point P'_g has real multipliers. For the details of the previous construction see, for instance, [AD, Lemma 3.4] which is a just a dynamical reformulation of [BDP, Lemmas 4.16 and 1.9].

To conclude the proof of the proposition it remains to check that the diffeomorphism g can be taken having a cycle associated to Q_g and P_g . As the initial f has a heterodimensional cycle, there are heteroclinic points $X_1 \in W^u(Q, f) \cap W^s(P, f)$ and $X_2 \in W^s(Q, f) \cap W^u(P, f)$. Since the neighborhood V of the orbit γ above can be chosen arbitrarily small, we can assume that it is disjoint from the orbits of the heteroclinic points X_1 and X_2 (note that the distance between the closure of the orbits of X_1 and X_2 and γ is strictly positive). As a consequence, $X_1 \in W^u(Q_g, g) \cap W^s(P_g, g)$ and $X_2 \in W^s(Q_g, g) \cap W^u(P_g, g)$ (note that in the previous construction we have $P_g = P$ and $Q_g = Q$). Thus the diffeomorphism g has a heterodimensional cycle associated to P_g and Q_g . \square

Proof of Theorem 5.3: Suppose that the indices of P and Q are u and $u+1$. Fix heteroclinic points $X_1 \in W^u(Q, f) \cap W^s(P, f)$ and $X_2 \in W^s(Q, f) \cap W^u(P, f)$. Up to a C^1 -perturbation, one may assume that the intersection at X_1 is transverse.

Since the homoclinic class of P is non-trivial, it contains a non-trivial basic set K containing P . By Proposition 5.4, we can assume (after a C^1 -perturbation of f) that K contains a periodic point P' homoclinically related to P and whose multipliers are real. Therefore, for every g close to f , P_g and P'_g are homoclinically related and the multipliers of P'_g are real. Thus to prove the theorem it is enough to find g close to f with a cycle associated to Q_g and P'_g .

The λ -lemma and the fact that P and P' are homoclinically related imply that $W^s(P', f)$ C^1 -approaches any compact disk in $W^s(P, f)$, thus $W^s(P', f)$ meets transversally $W^u(Q, f)$ in a point X'_1 close to X_1 . Therefore, for every g close to f , $W^s(P'_g, g)$ meets transversally $W^u(Q_g, g)$.

As P' and P are homoclinically related there are a sequence of points $(Y_i)_i$ and a sequence of natural numbers $(m_i)_i$ such that $(Y_i)_i$ converges to some point $Y \in W^u(P', f)$ and $(f^{m_i}(Y_i))_i$ converges to $X_2 \in W^s(Q, f) \cap W^u(P, f)$. This implies that the saddles P' and Q satisfy the hypotheses of the lemma below (taking $P' = A_f$ and $Q = B_f$):

Lemma 5.6 (Hayashi's Connecting Lemma, [Ha]). *Let f be a C^1 -diffeomorphism and A_f and B_f a pair of hyperbolic saddles of f . Suppose that there are sequences of points T_i and of natural numbers n_i such that T_i accumulates to $W_{loc}^u(A_f, f)$ and $f^{n_i}(T_i)$ accumulates to $W^s(B_f, f)$. Then there is g arbitrarily C^1 -close to f such that $W^u(A_g, g) \cap W^s(B_g, g) \neq \emptyset$.*

This lemma implies that there is g arbitrarily close to f such that $W^u(P'_g, g) \cap W^s(Q_g, g) \neq \emptyset$. Since $W^s(P'_g, g) \cap W^u(Q_g, g) \neq \emptyset$, the diffeomorphism g has a co-index one cycle associated to Q_g and P'_g . By construction, this cycle satisfies the conclusions of the theorem. \square

5.2 Non-real central eigenvalues and homoclinic intersections

In this section, we see that every diffeomorphisms having a co-index one cycle with some non-real central eigenvalue can be approximated by a diffeomorphism with a cycle involving a saddle whose homoclinic class is non-trivial.

Proposition 5.7. *Let f be a diffeomorphism with a a co-index one cycle associated to saddles P and Q . Assume that the central eigenvalue of P is non-real. Then every C^1 -neighborhood of f contains a diffeomorphism g having a co-index one cycle associated to the saddles P_g and Q_g and such that the homoclinic class of Q_g is non trivial.*

Proof: For fixing the ideas, assume that the indices of P and Q are u and $u + 1$, respectively. We write, $n = s + 1 + u$, n is the dimension of the ambient manifold. First, after a perturbation, we can assume that $W^s(P, f) \cap W^u(Q, f)$ contains some transverse intersection point X and that $W^s(P, f) \cap W^u(Q, f)$ contains some quasi-transverse intersection point Y .

By a C^1 -perturbation in a small neighborhood of P (preserving the quasi-transverse and transverse heteroclinic points Y and X), one can assume that there are local coordinates $[-1, 1]^{s-1} \times [-1, 1]^2 \times [-1, 1]^u$ at P such that the expression of $f^{\pi(P)}$ in those coordinates is of the form

$$f^{\pi(P)}(x^{s-1}, x^c, x^u) = (A^{s-1}(x^{s-1}), A^c(x^c), A^u(x^u)),$$

where $A^{s-1}: \mathbb{R}^{s-1} \rightarrow \mathbb{R}^{s-1}$, $A^c: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, and $A^u: \mathbb{R}^u \rightarrow \mathbb{R}^u$ are linear maps with $\|A^{s-1}\| < 1$, $\|A^c\| < 1$, and $\|(A^u)^{-1}\| < 1$. Moreover, the linear map A^c is the composition of a homothety and a rotation, that is,

$$A^c = a \begin{pmatrix} \cos 2\pi\theta & -\sin 2\pi\theta \\ \sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix}, \quad 0 < |a| < 1, \quad a \in \mathbb{R}.$$

One also can assume that the angle $\theta \in [0, 1]$ is irrational. This step is analogous to (S2) in Definition 3.5 of simple cycle. The next steps are analogous to (S3) and (S4).

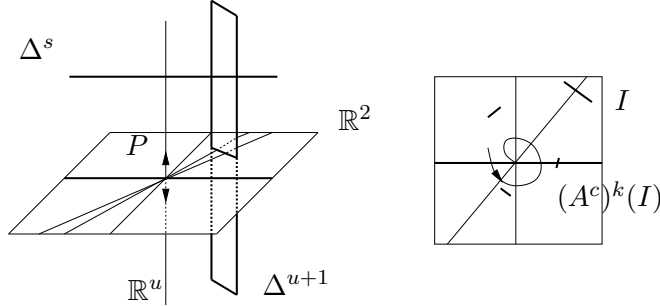


Figure 11: Creation of homoclinic points

By a C^1 -perturbation of f in a neighborhood of the quasi-transverse heteroclinic point Y , we can assume that $W^s(Q, f)$ contains an s -disk Δ^s containing a backward iterate $f^{-i}(Y)$ of Y , for some large $i > 0$, of the form (in the local coordinates at P)

$$\Delta^s = [-1, 1]^{s-1} \times [-1, 1] \times \{0\} \times \{y^u\}.$$

Similarly, by a C^1 -perturbation of f in a neighborhood of the transverse heteroclinic point X , one can assume that $W^u(Q, f)$ contains a $(u + 1)$ -disk containing a forward iterate $f^j(X)$ of X , for some large $j > 0$, of the form (in local coordinates)

$$\Delta^{u+1} = \{0^{s-1}\} \times I \times [-1, 1]^u,$$

where I is a segment in $[-1, 1]^2$. Furthermore, we can assume that I is transverse to the radial vector field $x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}$.

Since the rotation angle θ of A^c is irrational, there is some $k > 0$ such that $(A^c)^k(I)$ intersects transversely the segment $[-1, 1] \times \{0\}$. This implies that $f^{k\pi(P)}(\Delta^{u+1})$ intersects transversely

Δ^s . Since $\Delta^s \subset W^s(Q, f)$ and $\Delta^{u+1} \subset W^u(Q, f)$, this gives some transverse homoclinic point of Q . Thus the homoclinic class of Q is non-trivial. This ends the proof of the proposition. \square

Scholium 5.8. *Under the notation of the proof of Proposition 5.7, it follows that the curve $\gamma = \{0^{s-1}\} \times I \times \{0^u\}$ contained in the transverse intersection of $W^s(P, f)$ and $W^u(Q, f)$ is contained in the homoclinic class of Q .*

Consider any $Z \in \gamma$, $Z = (0^{s-1}, z, 0^u)$, $z \in I$, and any open sub-interval J of I containing z . Since rotation θ of A^c is irrational, there are infinitely many $k_i > 0$ such that $(A^c)^{k_i}(J) \cap [-1, 1] \times \{0\} \neq \emptyset$. As A^u is an expansion, for any neighborhood $V^u(J)$ of J in $W^u(Q, f)$ there is some large k_i such that $f^{k_i}(V^u(J))$ intersects Δ^s transversely. This gives transverse homoclinic points of Q arbitrarily close to $\{0^{s-1}\} \times J \times \{0^u\}$. As J can be taken arbitrarily small, one gets transverse homoclinic points of Q arbitrarily close to Z . This completes the argument.

5.3 Cycles with non-real central eigenvalues

Next lemma implies Theorem 5.1 for cycles whose central eigenvalues are all non-real.

Lemma 5.9. *Let f be a diffeomorphism having a co-index one cycle associated to saddles P and Q . Suppose that the central eigenvalues of the cycle associated to P and Q are both non-real. Then every C^1 -neighborhood \mathcal{U} of f contains a diffeomorphism g having a co-index one cycle associated to saddles P'_g and Q'_g such that*

- *the saddles P'_g and Q'_g have real multipliers, and*
- *P'_g is homoclinically related to P_g and Q'_g is homoclinically related to Q_g .*

Note that since P'_g and Q'_g have real multipliers, the co-index one cycle given by the lemma has real central eigenvalues.

Proof: Applying twice Proposition 5.7 (interchanging the roles of P and Q), we get a diffeomorphism ϕ close to f with a cycle associated to the saddles P_ϕ and Q_ϕ and such that the homoclinic classes of P_ϕ and Q_ϕ are both non-trivial.

Since the homoclinic class of P_ϕ is non-trivial, by Theorem 5.3, there is a diffeomorphism φ close to ϕ (thus close to f) with a cycle associated to Q_φ and a periodic point P'_φ which is homoclinically related to P_φ and has real multipliers. Note that since φ is close to ϕ , the homoclinic class of Q_φ is non-trivial

Finally, applying again Theorem 5.3, now to P'_φ and Q_φ (which has a non-trivial homoclinic class), we get a diffeomorphism g close to φ (thus close to f) with a co-index one cycle associated to P'_g and a saddle Q'_g with real multipliers which is homoclinically related to Q_g . The lemma follows noting that, for g close to φ , the saddle P'_g has real multipliers and is homoclinically related to P_g . \square

5.4 Cycles having only one real central eigenvalue

In this section, we consider cycles having only one real central eigenvalue. We prove that these cycles generate (by perturbations) new heterodimensional cycles associated to saddles with non-trivial homoclinic classes.

Lemma 5.10. *Let f be a diffeomorphism having a co-index one cycle associated to saddles P and Q . Suppose that cycle has only one real central eigenvalue. Then every C^1 -neighborhood \mathcal{U} of f contains diffeomorphisms g with a co-index one cycle associated to the saddles P_g and Q_g and such that the homoclinic classes of P_g and Q_g are both non-trivial.*

Using Theorem 5.3 and arguing as in the proof of Lemma 5.9, Lemma 5.10 implies Theorem 5.1 for cycles having only one real central eigenvalue.

Proof: Let us assume, for instance, that the indices of P and Q are u and $(u + 1)$ and that the central eigenvalue of the cycle corresponding to P is non-real (thus the central eigenvalue corresponding to Q is real). By Proposition 5.7, we can assume that the homoclinic class of Q is non-trivial. Thus to prove the lemma we need to generate simultaneously homoclinic points of P and a heterodimensional cycle (associated to the continuations of P and Q). This is done by considering local perturbations preserving prescribed compact parts of the invariant manifolds of P and Q . Thus we need to control some compact parts of these invariant manifolds after the perturbations.

5.4.1 Local coordinates

The first step is to select local coordinates and consider perturbations such that the resulting dynamics is linear or affine. This step is analogous to the definition of simple cycles (Definition 3.5): after a series of perturbations, one can assume that the cycle is in linear form in neighborhoods of P and Q (see conditions (1)–(9) below). We next explain this construction. The elements in our construction are depicted in Figure 12. First, for notational simplicity, let us assume in what follows that P and Q are fixed saddles ($\pi(P) = \pi(Q) = 1$).

Dynamics in a neighborhood of P . Arguing exactly as in Proposition 5.7, we fix local coordinates $[-1, 1]^{s-1} \times [-1, 1]^2 \times [-1, 1]^u$ at P such that after a perturbation the following holds (item (2) follows from $W^s(P, f) \cap W^u(Q, f) \neq \emptyset$ and item (3) from $W^u(P, f) \cap W^s(Q, f) \neq \emptyset$):

1. The local expression of f is

$$f(x^{s-1}, x^c, x^u) = (A^{s-1}(x^{s-1}), A^c(x^c), A^u(x^u)),$$

where $A^{s-1}: \mathbb{R}^{s-1} \rightarrow \mathbb{R}^{s-1}$, $A^c: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, and $A^u: \mathbb{R}^u \rightarrow \mathbb{R}^u$ are linear maps with $\|A^{s-1}\|, \|A^c\|, \|(A^u)^{-1}\| < 1$ and A^c is the composition of a homothety and a rotation of irrational angle θ .

Using these coordinates, we define the local stable and unstable manifolds of P ,

$$W_{loc}^s(P, f) = [-1, 1]^{s-1} \times [-1, 1]^2 \times \{0^u\} \quad \text{and} \quad W_{loc}^u(P, f) = \{(0^{s-1}, 0, 0)\} \times [-1, 1]^u.$$

2. The stable manifold $W^s(Q, f)$ contains a s -disk of the form

$$\Delta^s = [-1, 1]^{s-1} \times [-1, 1] \times \{0\} \times \{y^u\} = [-1, 1]^s \times \{0\} \times \{y^u\},$$

where $Y = (0^s, 0, 0, y^u) \in W_{loc}^u(P, f) \cap W^s(Q, f)$ is a heteroclinic point. Moreover, we can assume that the disk Δ^s is contained in a fundamental domain of $W^s(Q, f)$ and thus it is disjoint from all its iterates.

3. The unstable manifold $W^u(Q, f)$ contains the $(u + 1)$ -disk

$$\Delta^{u+1} = \{0^{s-1}\} \times I \times [-1, 1]^u,$$

where I is a segment in $[-1, 1]^2$ transverse to the radial vector field. By the proof of Scholium 5.8, the heteroclinic segment

$$\gamma = \{0^{s-1}\} \times I \times \{0^u\} \subset \Delta^{u+1} \cap W_{loc}^s(P, f)$$

is contained in the homoclinic class of Q . Thus, we can assume that disk Δ^s in (2) contains some transverse homoclinic point of Q

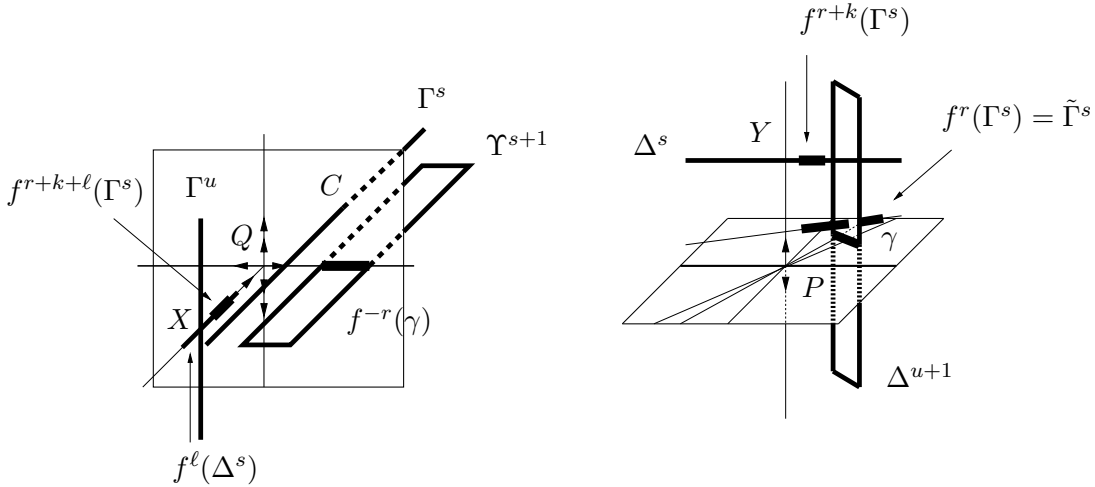


Figure 12: Heteroclinic intersections

Dynamics in a neighborhood of Q . As in the case of cycles with real central eigenvalues in Section 3.1, after a perturbation, we can choose coordinates $[-1, 1]^s \times [-1, 1] \times [-1, 1]^u$ at Q such that:

4. The expression of f is of the form

$$f(x^s, x, x^u) = (B^s(x^s), \beta x, B^u(x^u)),$$

where $\beta > 1$ and $B^s: \mathbb{R}^s \rightarrow \mathbb{R}^s$ and $B^u: \mathbb{R}^u \rightarrow \mathbb{R}^u$ are linear maps with $\|B^s\| < 1$ and $\|(B^u)^{-1}\|^{-1} < \beta$.

Using these coordinates, we define the local stable and unstable manifolds of Q as above. We also define the local center unstable manifold of Q by

$$W_{loc}^{cu}(Q, f) = \{0^s\} \times [-1, 1] \times \{0^u\}.$$

5. Let γ be the heteroclinic segment in (3). There are (arbitrarily) large $r > 0$ and an interval $[a, b] \subset (0, 1]$ such that

$$f^{-r}(\gamma) = \{0^s\} \times [a, b] \times \{0^u\} \subset W_{loc}^{cu}(Q, f).$$

In order to put $f^{-r}(\gamma)$ within $W_{loc}^{cu}(Q, f)$ note that, after shrinking γ and perturbing locally the dynamics, we can assume that there is some backward iterate of γ intersecting transversely the strong unstable foliation of Q . Thus the negative iterates of γ C^1 -approach $W_{loc}^{cu}(Q, f)$. Finally, after a new perturbation, we can put some large backward iterate of γ inside $W_{loc}^{cu}(Q, f)$.

6. In the local coordinates at Q , we can assume that the $(s+1)$ -disk Υ^{s+1} satisfies

$$\Upsilon^{s+1} = [-1, 1]^s \times [a, b] \times \{0^u\} \subset W^s(P, f).$$

Moreover, using now the coordinates at P ,

$$f^r(\Upsilon^{s+1}) \subset [-1, 1]^{s+1} \times \{0^u\} = W_{loc}^s(P, f).$$

Note that $f^r(\Upsilon^{s+1})$ contains the heteroclinic curve γ . Furthermore, by shrinking the local chart at Q , we can assume that $f^i(\Upsilon^{s+1})$ is disjoint from this local chart for every $i \in \{1, \dots, r-1\}$. Thus, since $f^r(\Upsilon^{s+1}) \subset W_{loc}^s(P, f)$, the whole forward orbit of Υ^{s+1} is disjoint from the local chart of Q .

Homoclinic and heteroclinic intersections. Consider the local coordinates at P and the s -disk $\Delta^s \subset W^s(Q, f)$ in (2) in the local chart of P . Since the rotation angle of A^c is irrational there are infinitely many $k > 0$ such that $f^{-k}(\Delta^s)$ (we consider iterations in the local chart of P) meets transversely the $(u+1)$ -disk $\Delta^{u+1} \subset W^u(Q, f)$ in (3) (recall Scholium 5.8). We choose large k and a transverse intersection point $H \in f^{-k}(\Delta^s) \cap \Delta^{u+1}$. The transverse homoclinic point H of Q can be taken arbitrarily close to γ . Thus, now in the local coordinates at Q , the point $C = f^{-r}(H) = (0^s, c, c^u)$ is close to $f^{-r}(\gamma)$. Thus we can assume that $c \in (a, b)$ (recall the definition of r and $f^{-r}(\gamma)$ in (5)) and c^u close to 0^u (for that it suffices to take large k). This construction can be summarized as follows:

7. In the local coordinates at Q , the stable manifold of Q contains the the s -disk

$$\Gamma^s = [-1, 1]^s \times \{(c, c^u)\} \subset W^s(Q, f),$$

where c^u is close to 0^u and $c \in (a, b)$ (here $[a, b]$ is the interval in (5)). The point $C = (0^s, c, c^u)$ is a transverse homoclinic point of Q . Moreover, the disk Γ^s can be chose satisfying the following properties:

- (i) There is a small s -disk $\tilde{\Gamma}^s \subset f^{-k}(\Delta^s)$ containing H such that $\Gamma^s = f^{-r}(\tilde{\Gamma}^s)$. Thus $f^{k+r}(\Gamma^s) \subset \Delta^s$. Moreover, we can assume that $\tilde{\Gamma}^s, f(\tilde{\Gamma}^s), \dots, f^k(\tilde{\Gamma}^s) \subset \Delta^s$ are contained in the local chart of P and are disjoint from $W_{loc}^u(P, f)$. Thus the disk $f^{r+i}(\Gamma^s) = f^i(\tilde{\Gamma}^s)$ is contained in the the local chart of P and disjoint from $W_{loc}^u(P, f)$ for every $i = 0, 1, \dots, k$.
- (ii) Since $f^{r+k}(\Gamma^s)$ is disjoint from $W_{loc}^u(P, f)$ the heteroclinic point $Y \in W_{loc}^u(P, f) \cap W^s(Q, f)$ in item (2) does not belong to $f^{r+k}(\Gamma^s)$.

(iii) The disk $f^i(\Gamma^s)$ is disjoint from the local chart of Q , for every $i \in \{1, \dots, r-1\}$.

8. Consider the heteroclinic point $Y \in W_{loc}^u(P, f) \cap W^s(Q, f)$, $Y \in \Delta^s$, in item (2). Let $\ell > 0$ be the smallest integer such that

$$X = f^\ell(Y) = (x^s, 0, 0^u) \in [-1, 1]^s \times \{(0, 0^u)\} = W_{loc}^s(Q, f).$$

By shrinking the local chart at Q , we can assume that $f^i(\Delta^s)$ does not intersect this chart for every $0 \leq i \leq \ell - 1$. Since $\Delta^s \subset W_{loc}^s(Q, f)$ and contains Y , we can assume that $f^\ell(\Delta^s) \subset W_{loc}^s(Q, f)$. Note that $X \in f^\ell(\Delta^s)$ and thus $X \in W_{loc}^s(Q, f)$.

9. There is a u -disk $\Gamma^u \subset W^u(P, f)$ in the local chart of Q containing X of the form

$$\Gamma^u = \{(x^s, 0)\} \times [-1, 1]^u$$

and such that Γ^u is disjoint from all its iterates (i.e., the disk Γ^u is contained in a fundamental domain of $W^u(P, f)$) and $f^{-\ell}(\Gamma^u) \subset W_{loc}^u(P, f)$. Moreover, $\{X\} = \Gamma^u \cap W_{loc}^s(Q, f)$.

5.4.2 Local perturbations

We will perform a local perturbation of f in a neighborhood of the disk Γ^u in (9) to get a diffeomorphism g such that there are (simultaneously) transverse homoclinic points of P_g and intersections between $W^u(P_g, g)$ and $W^s(Q_g, g)$. Since for every g close to f the invariant manifolds $W^s(P_g, g)$ and $W^u(Q_g, g)$ have transverse intersections, it follows that g has a heterodimensional cycle associated to P_g and Q_g such that the homoclinic class of P_g is non-trivial. Since the homoclinic class of Q_g is non-trivial (see items (3) or (7)), this will imply the lemma.

A transverse homoclinic point of P will be obtained as an intersection of the $(s+1)$ -disk $\Upsilon^{s+1} \subset W^s(P, f)$ in (6) and some positive iterate of $\Gamma^u \subset W^u(P, f)$. The perturbation is such that, for the resulting diffeomorphism g , the saddles P and Q are not modified, $\Upsilon^{s+1} \subset W^s(P, g)$ and $\Gamma^u \subset W^u(P, g)$. So such an intersection will provide a point in $H(P, g)$. The heteroclinic intersection between $W^u(P, g)$ and $W^s(Q, g)$ is obtained as the intersection of some positive iterate of Γ^u (in fact, the same as before) and the s -disk Γ^s in (7). Once more, for the resulting g one also has $\Gamma^s \subset W^s(Q, g)$.

The main difficulty for performing this perturbation is to modify the positive orbit of Γ^u without altering the fact that Γ^s and Υ^{s+1} are contained in the stable manifolds of Q and P , respectively. For solving this difficulty, we claim that:

Claim 5.11.

- *The closure of the forward orbit of Υ^{s+1} is disjoint from Γ^u .*
- *The closure of the forward orbit of Γ^s is disjoint from Γ^u .*

Proof: The first assertion follows noting that, by construction, the positive iterates of Υ^{s+1} are disjoint from the local chart at Q (item (6)) while Γ^u is contained in this local chart (item (9)).

The second part of the claim follows from the observations below:

- The iterates $f^i(\Gamma^s)$, $i \in \{1, \dots, r+k\}$ are disjoint from the local chart at Q (item 7(iii) implies the assertion for $i = 0, \dots, (r-1)$ and item 7(i) for $i = r, \dots, r+k$). Hence, since Γ^u is contained in the local chart of Q (recall (9)), these iterates are disjoint from Γ^u . Moreover, $f^{r+k}(\Gamma^s)$ is contained in $(\Delta^s \setminus W_{loc}^u(P, f))$ (see (7(i) and 7(ii)).

- As $f^{r+k}(\Gamma^s) \subset \Delta^s$, item (8) implies that the iterates $f^{r+k+i}(\Gamma^s) \subset f^i(\Delta^s)$, $i \in \{1, \dots, \ell - 1\}$, are disjoint from the local chart at Q . Thus, by (9), these iterates are disjoint from Γ^u . Furthermore, by item 7(ii),

$$Y \notin f^{r+k}(\Gamma^s)$$

and, by item (8),

$$f^{r+k+\ell}(\Gamma^s) \subset f^\ell(\Delta^s) \subset W_{loc}^s(Q, f).$$

Recalling that, by (9),

$$X = f^\ell(Y) \notin f^{r+k+\ell}(\Gamma^s) \quad \text{and} \quad \{X\} = \Gamma^u \cap W_{loc}^s(Q, f),$$

one has that

$$f^{r+k+\ell}(\Gamma^s) \cap \Gamma^u = \emptyset.$$

- We claim that the positive iterates of $f^{r+k+\ell}(\Gamma^s) \subset f^\ell(\Delta^s)$ do not meet the point $X \in \Gamma^u$. Assume, by contradiction, that $X = f^\ell(Y) \in f^{r+k+\ell+i}(\Gamma^s) \subset f^{\ell+i}(\Delta^s)$ for some $i > 0$. Since $X = f^\ell(Y) \in f^\ell(\Delta^s)$ (item (8)) this implies that $f^i(\Delta^s) \cap \Delta^s \neq \emptyset$ for some $i > 0$, contradicting that Δ^s is disjoint from its forward iterates (item (2)).

The claim now follows from the comments above recalling that $f^{r+k+\ell}(\Gamma^s) \subset W_{loc}^s(Q, f)$ and that $\{X\} = \Gamma^u \cap W_{loc}^s(Q, f)$ (item (9)). \square

Bearing in mind the previous comments, we are now ready to perform the announced perturbation of f . By Claim 5.11, we can choose a small neighborhood V of Γ^u disjoint from the closures of the positive orbits of Γ^s and of Υ^{s+1} . Let f_t be a diffeomorphism which coincides with f outside V and such that, in the local coordinates at Q , satisfies

$$f_t(x^s, x, x^u) = f(x^s, x^c, x^u) + (0^s, t, 0^u),$$

for every (x^s, x, x^u) in a small neighborhood U of Γ^u (contained in V). The definition of the perturbation and the choice of V imply that $\Upsilon^{s+1} \subset W^s(P, f_t)$ and $\Gamma^s \subset W^s(Q, f_t)$.

Let $c \in (a, b)$ be as in item (7). Then for every $t = \beta^{-m} c$, large $m > 0$, the diffeomorphism f_t is C^1 -close to f . If m is big enough, using the expansion in the u -direction, one has

$$\Gamma_t^u(m) = \{((B^s)^m(x^s), c)\} \times [-1, 1]^u \subset f_t^m(\Gamma^u) \subset W^u(P, f_t).$$

As $c \in (a, b)$, the segment $\Gamma_t^u(m)$ meets transversely the disk $\Upsilon^{s+1} = [-1, 1]^s \times [a, b] \times \{0^u\}$. Since $\Upsilon^{s+1} \subset W^s(P, f_t)$ this implies that the homoclinic class of P (for f_t) is non-trivial.

Similarly, we have that $\Gamma_t^u(m)$ intersects the disk $\Gamma^s = [-1, 1]^s \times \{(c, c^u)\}$. Since $\Gamma^s \subset W^s(Q, f_t)$, this implies that $W^u(P, f_t)$ meets $W^s(Q, f_t)$. This ends the proof of the lemma. \square

6 Applications to generic dynamics

The aim of this section is to prove Corollaries 2 and 3 and Theorems 2, 3, and 4 about C^1 -generic dynamics. We begin by collecting some properties of chain recurrence and homoclinic classes of C^1 -generic diffeomorphisms we will use systematically.

6.1 Homoclinic and chain recurrence classes of C^1 -generic diffeomorphisms

There is a residual subset \mathcal{R} of $\text{Diff}^1(M)$ such that every diffeomorphism $f \in \mathcal{R}$ satisfies the following properties:

- G1)** The chain recurrence set and the non-wandering set of f coincide. Moreover, these sets are equal to the closure of the hyperbolic periodic points of f . See [BC, Corollaries 1.2].
- G2)** Every chain recurrence class $\mathcal{C}(f)$ of f containing a periodic point P_f is the homoclinic class of P_f . See [BC, Remarque 1.10]. In particular, since the recurrence classes define a partition of the chain recurrence set, two homoclinic classes of $f \in \mathcal{R}$ which are non-disjoint coincide (this result was previously stated in [CMP]).
- G3)** Every homoclinic class of f containing saddles of indices a and b , $a < b$, also contains a saddle of index c for every $c \in (a, b) \cap \mathbb{N}$. See [ABCDW, Theorem 1].
- G4)** Every isolated chain recurrence class $\mathcal{C}(f)$ of f is robustly isolated: there are neighborhoods \mathcal{U} of f in $\text{Diff}^1(M)$ and V of the class $\mathcal{C}(f)$ in M such that, for every $h \in \mathcal{U}$, the intersection $\mathcal{R}(h) \cap V$ is a unique chain recurrence class of h . See [BC, Corollaire 1.13]. Note that, by item (G1), every isolated chain recurrence class of f is a homoclinic class.
- G5)** Given any pair of saddles P_f and Q_f of f , there is a neighborhood \mathcal{U}_f of f in \mathcal{R} such that either $H(P_g, g) = H(Q_g, g)$ for all $g \in \mathcal{U}_f$, or $H(P_g, g) \cap H(Q_g, g) = \emptyset$ for all $g \in \mathcal{U}_f$. This follows from the arguments in [BC] and a genericity argument. For an explicit formulation of this result (and its complete proof) see [ABCDW, Lemma 2.1].

6.2 Proof of Theorem 2

In this section, we prove Theorem 2: *There is an open and dense subset \mathcal{O} of the set of tame diffeomorphisms such that every $f \in \mathcal{O}$ either it satisfies the Axiom A plus the no-cycles condition or it has a C^1 -robust heterodimensional cycle.* Theorem 2 is a direct consequence of the following local version of it:

Proposition 6.1. *Let \mathcal{U} be an open subset of $\text{Diff}^1(M)$ and V an open subset of M such that, for every diffeomorphism $g \in \mathcal{U}$, the intersection of the chain recurrence set $\mathcal{R}(g)$ of g and the closure of V is a unique chain recurrence class $\mathcal{C}(g)$. Then there is an open and dense subset \mathcal{U}_V of \mathcal{U} such that, for every $g \in \mathcal{U}_V$, the chain recurrence class $\mathcal{C}(g)$ is either a hyperbolic basic set or it has a robust heterodimensional cycle.*

We postpone the proof of this proposition and prove Theorem 2 assuming it.

Proof of Theorem 2: We first observe the following:

Lemma 6.2. *Let $f \in \mathcal{R}$ be a tame diffeomorphism. Then for every chain recurrence class $\mathcal{C}(f)$ of f there are neighborhoods V of $\mathcal{C}(f)$ in M and \mathcal{U}_f of f in $\text{Diff}^1(M)$ such that, for every $g \in \mathcal{U}_f$, the intersection of the chain recurrence set of g and V is a chain recurrence class $\mathcal{C}(g)$ of g . Moreover, the chain recurrence class $\mathcal{C}(g)$ is a homoclinic class.*

Remark 6.3. *Lemma 6.2 means that we can apply Proposition 6.1 to chain recurrence classes of tame diffeomorphisms.*

Proof: This lemma follows from the generic conditions (G1)–(G5) above. Consider a chain recurrence class $\mathcal{C}(f)$ of a tame diffeomorphism $f \in \mathcal{R}$. By (G1), the chain recurrence class $\mathcal{C}(f)$ contains a periodic point, say P_f . By (G2), $\mathcal{C}(f)$ is the homoclinic class of P_f . Finally, since f is tame, the chain recurrence class $\mathcal{C}(f)$ is isolated. By (G3), $\mathcal{C}(f)$ is in fact robustly isolated. This completes the proof of the first part of the lemma.

Putting these results together and noting that every homoclinic class is contained in some chain recurrence class, one has that for every tame diffeomorphism $f \in \mathcal{R}$ homoclinic classes and chain recurrence classes coincide and are robustly isolated. This implies the lemma. \square

The theorem now follows from Proposition 6.1. Suppose that f is a tame diffeomorphism that cannot be C^1 -approximated by diffeomorphisms with robust heterodimensional cycles. By Lemma 6.2 and Proposition 6.1, each chain recurrence class of f is a hyperbolic basic set. Thus the chain recurrence set of f (consisting of finitely many chain recurrence classes which are basic sets) is hyperbolic. This implies that the diffeomorphism f is Axiom A.

We claim that f also verifies the no-cycles condition. Suppose, by contradiction, that f has a cycle associated to two basic sets (chain recurrence classes), say $\mathcal{C}_1(f)$ and $\mathcal{C}_2(f)$, of the spectral decomposition of its non-wandering set. Then, using this cycle, one has that, for any arbitrarily small neighborhoods V_1 of $\mathcal{C}_1(f)$ and V_2 of $\mathcal{C}_2(f)$, there is a diffeomorphism g arbitrarily close to f having a periodic point Q_g whose orbit intersects V_1 and V_2 . Thus the intersection of the chain recurrence set of g containing Q_g and V_1 is not a chain recurrence class contained in V_1 . Since the neighborhood V_1 of $\mathcal{C}_1(f)$ can be taken arbitrarily small, the chain recurrence class $\mathcal{C}_1(f)$ of the tame diffeomorphism f does not satisfy the conclusion of Proposition 6.1, contradicting Remark 6.3. The proof of Theorem 2 is now complete. \square

We are left to prove Proposition 6.1.

Proof of Proposition 6.1: Given open sets \mathcal{U} and V as in the proposition, consider a diffeomorphism $g \in \mathcal{U}$ and its chain recurrence class $\mathcal{C}(g)$ contained in V . By (G1), after a perturbation, we can assume that the chain recurrence class $\mathcal{C}(g)$ contains a hyperbolic periodic point D_g . By (G2), we can assume that $\mathcal{C}(g)$ is the homoclinic class $H(D_g, g)$ of D_g . Since this proposition is local, we can assume that the continuation D_h of D_g is defined for every $h \in \mathcal{U}$. Now, according to [Ma2], either $\mathcal{C}(g)$ is hyperbolic or there is a perturbation h of g such that h has a non-hyperbolic periodic point in V . After a new perturbation, we can assume that h has two saddles with different indices in V (the non-hyperbolic point splits into two hyperbolic periodic points of different indices). The hypotheses of the proposition imply that the orbits of these two periodic points are both contained in V and belong to the same chain recurrence class $\mathcal{C}(h)$. By (G2), we can assume that $\mathcal{C}(h) = H(D_h, h)$.

Since hyperbolic periodic points persist by C^1 -perturbations, the arguments above give a dense open subset \mathcal{U}_1 of \mathcal{U} ,

$$\mathcal{U}_1 = \mathcal{U}_{hyp} \amalg \mathcal{U}_2,$$

such that

- the set \mathcal{U}_{hyp} is open and, for every $g \in \mathcal{U}_{hyp}$, the chain recurrence class $\mathcal{C}(g)$ is hyperbolic;
- the set \mathcal{U}_2 is open and every $g \in \mathcal{U}_2$ has two periodic saddles of different indices in $\mathcal{C}(g)$.

To prove the proposition it remains to see that, for every diffeomorphism $g \in \mathcal{U}_{hyp}$, the chain recurrence class $\mathcal{C}(g)$ is a basic set (Lemma 6.4) and that the diffeomorphisms of \mathcal{U}_2 having robust cycles are dense in \mathcal{U}_2 (Lemma 6.6).

Lemma 6.4. *Let $g \in \mathcal{U}_{hyp}$. Then the chain recurrence class $\mathcal{C}(g)$ is a basic set equal to $H(D_g, g)$.*

Proof: Just note that the hypotheses of the proposition imply that $\mathcal{C}(g) = H(D_g, g)$ robustly. Using now that the chain recurrence set has associated a filtration⁷, it follows that $\mathcal{C}(g)$ is the maximal invariant set in some neighborhood of it. Since $\mathcal{C}(g)$ is hyperbolic and is a homoclinic class (thus it is transitive and the periodic points are dense in the class), one immediately has that $\mathcal{C}(g)$ is a basic set. \square

We now state a well known consequence of the Hayashi's Connecting Lemma (Lemma 5.6) that we will use repeatedly in this section. For completeness, we will include its proof:

Lemma 6.5. *Let \mathcal{U} be an open set of $\text{Diff}^1(M)$ such that, for every $f \in \mathcal{U}$, there are saddles P_f and Q_f with different indices depending continuously on f . Suppose that there is a dense subset \mathcal{D} of \mathcal{U} such that $H(P_f, f) = H(Q_f, f)$, for all $f \in \mathcal{D}$. Then there is a dense subset \mathcal{H} of \mathcal{U} consisting of diffeomorphisms f having a heterodimensional cycle associated to P_f and Q_f .*

Proof: Suppose that the indices of P_f and Q_f are p and q , $p < q$. Take $f \in \mathcal{D}$ and note that the homoclinic class of P_f is a transitive set. Thus there is $x \in H(P_f, f)$ whose forward orbit accumulates to P_f and Q_f . Hence there are sequences of natural numbers k_i and m_i such that

- $f^{m_i}(x)$ converges to some point of $W_{loc}^u(Q_f, f)$,
- $f^{k_i}(x)$ converges to some point of $W_{loc}^s(P_f, f)$,
- $k_i - m_i > i$.

Taking in Lemma 5.6 $T_i = f^{m_i}(x)$, $n_i = k_i - m_i$, $P_f = B_f$, and $Q_f = A_f$, we get a diffeomorphism h arbitrarily close to f such that $W^u(Q_h, h)$ intersects $W^s(P_h, h)$. Let n be the dimension of the ambient manifold. As the sum of the dimensions of $W^u(Q_h, h)$ and $W^s(P_h, h)$ is

$$q + (n - p) = n + q - p > n,$$

we can assume (after a new perturbation, if necessary) that the intersection between $W^u(Q_h, h)$ and $W^s(P_h, h)$ is transverse. This implies that there is an open and dense subset \mathcal{I} of \mathcal{U} such that, for all $g \in \mathcal{I}$, $W^u(Q_g, g)$ and $W^s(P_g, g)$ have a non-empty transverse intersection.

Consider now a diffeomorphism f in the set $\mathcal{I} \cap \mathcal{D}$ (which is a dense subset of \mathcal{U}). Since $H(P_f, f) = H(Q_f, f)$ we can argue as above and apply again Lemma 5.6 to the saddles P_f and Q_f . Now P_f plays the role of A_f and Q_f the role of B_f . This gives a diffeomorphism h arbitrarily close to f (thus $h \in \mathcal{I}$) such that $W^u(P_h, h)$ intersects $W^s(Q_h, h)$. As $W^s(P_h, h) \cap W^u(Q_h, h) \neq \emptyset$ (recall that $h \in \mathcal{I}$), one has that h has heterodimensional cycle. As h can be chosen arbitrarily close to $f \in \mathcal{I} \cap \mathcal{D}$ and the previous arguments hold for all $f \in \mathcal{I} \cap \mathcal{D}$, this completes the proof of the lemma. \square

Lemma 6.6. *There is an open and dense subset \mathcal{U}_{rcyc} of \mathcal{U}_2 such that every diffeomorphism $f \in \mathcal{U}_{rcyc}$ has a robust heterodimensional cycle.*

⁷There are two submanifolds with boundary M_1 and M_2 of the same dimension as the ambient manifold M such that $M_2 \subset M_1$, $g(M_i)$ is contained in the interior of M_i , $i = 1, 2$, and $W = (M_1 \setminus M_2)$ is a neighborhood of $\mathcal{C}(g)$ such that $\mathcal{C}(g)$ is the maximal invariant set of g in W (see [Co] for details).

Proof: Recall that Theorem 1 claims that every co-index one cycle generates robust heterodimensional cycles. Thus to prove the lemma it is enough to see that the diffeomorphisms having a co-index one cycle are dense in \mathcal{U}_2 . We first see that the diffeomorphisms $g \in \mathcal{U}_2$ such that the chain recurrence class $\mathcal{C}(g)$ contains saddles having consecutive indices form a residual (thus dense) subset of \mathcal{U}_2 . Thereafter, using Lemma 6.5, we will get the density of diffeomorphisms having co-index one cycles in \mathcal{U}_2 .

By (G2) there is a residual subset \mathcal{G}_2 of \mathcal{U}_2 such that, for every $f \in \mathcal{G}_2$, $\mathcal{C}(f) = H(D_f, f)$ for some periodic point D_f . Using (G4), we can assume that for every $f \in \mathcal{G}_2$ the indices of the periodic points of $H(D_f, f)$ form an interval of \mathbb{N} . Thus, by (G2) and the definition of \mathcal{U}_2 , we can assume that there are saddles P_f and Q_f with $\text{index}(P_f) = \text{index}(Q_f) - 1$ such that

$$\mathcal{C}(f) = H(P_f, f) = H(Q_f, f), \quad \text{for all } f \in \mathcal{G}_2.$$

Thus we can apply Lemma 6.5 to the open set \mathcal{U}_2 , the dense (in fact, residual) subset \mathcal{G}_2 of \mathcal{U}_2 , and the saddles P_f and Q_f . This gives a dense subset \mathcal{D}_2 of \mathcal{U}_2 consisting of diffeomorphisms f with a heterodimensional cycle associated to P_f and Q_f . Noting that this heterodimensional cycle has co-index one, we conclude the proof of the lemma. \square

Let \mathcal{U}_{rcyc} the subset of \mathcal{U}_2 of diffeomorphisms with a robust heterodimensional cycle. By Lemma 6.6, this set is open and dense in \mathcal{U}_2 . By Lemma 6.4, to prove the Proposition 6.1 it is enough to take $\mathcal{U}_V = \mathcal{U}_{hyp} \amalg \mathcal{U}_{rcyc}$. \square

6.3 Proof of Corollary 3

Recall that Corollary 3 claims the following: *There is a residual subset \mathcal{R} of $\text{Diff}^1(M)$ such that, for every diffeomorphism $f \in \mathcal{R}$ and every isolated chain recurrence class $\mathcal{C}(f)$ of f , there are two possibilities: either $\mathcal{C}(f)$ is hyperbolic or it has a robust heterodimensional cycle.*

This corollary is a direct consequence of Proposition 6.1. Consider the residual subset \mathcal{R} of $\text{Diff}^1(M)$ satisfying properties (G1)–(G5) above. By (G4), C^1 -generically isolated chain recurrence classes are robustly isolated. Hence any isolated class $\mathcal{C}(f)$ of $f \in \mathcal{R}$ verifies Lemma 6.2. Thus we can apply Proposition 6.1 to $\mathcal{C}(f)$, proving the corollary.

6.4 Proof of Theorem 3

Consider the residual subset \mathcal{R} of $\text{Diff}^1(M)$ satisfying (G1)–(G5). We prove the following result which implies Theorem 3.

Consider a diffeomorphism $f \in \mathcal{R}$ and a chain recurrence class $\mathcal{C}(f)$ of f containing two saddles P_f and Q_f of different indices. Then there are diffeomorphisms arbitrarily close to f having robust heterodimensional cycles.

Note that, by (G2), $\mathcal{C}(f) = H(P_f, f) = H(Q_f, f)$. Thus, by (G5), for every $g \in \mathcal{R}$ close to f , one has $\mathcal{C}(g) = H(P_g, g) = H(Q_g, g)$. Since the saddles P_g and Q_g have different indices, condition (G3) implies that there are saddles A_g and B_g in $\mathcal{C}(g) = H(P_g, g) = H(Q_g, g)$ having consecutive indices. Moreover, again by (G5) and (G2), for every $h \in \mathcal{R}$ close to g , one has

$$\mathcal{C}(h) = H(P_h, h) = H(Q_h, h) = H(A_h, h) = H(B_h, h).$$

Lemma 6.5 now gives φ arbitrarily close to h (thus arbitrarily close to f) with a co-index one cycle associated to A_φ and B_φ . By Theorem 1, this co-index one cycle generates robust heterodimensional cycles. The proof of the result is now complete.

6.5 Proof of Theorem 4

Recall that Theorem 4 claims that *every diffeomorphism with a co-index 1 cycle is in the closure of an open set of $\text{Diff}^1(M)$ of diffeomorphisms which do not satisfy the shadowing property*. The proof of Theorem 4 follows using the arguments in [AD, Theorem 1] (on its turn, these arguments are an adaptation of the ones in [BDT, YY]). Let us sketch these arguments.

The main step of the proofs in [AD, BDT] can be summarized as follows. Suppose that a diffeomorphism f has co-index one cycle with real central eigenvalues as follows. The cycle is associated to saddles P_f and Q_f , of indices u and $u + 1$, and there are a neighborhood of the cycle V and a partially hyperbolic splitting over V of the form $E^{ss} \oplus E^c \oplus E^u$, where E^c is one-dimensional and $\dim(E^{uu}) = u$. Then there is a C^1 -open set \mathcal{N} whose closure contains f such that for every $g \in \mathcal{N}$ the unstable manifold of P_g accumulates to $W_{loc}^s(Q_g, g)$ nicely: there are a small unstable cone field around E^{uu} , a sequence of points $x_n \in W^u(P_g, g)$ converging to some point $x \in W_{loc}^s(Q_g, g)$, and a sequence of u -disks D_n such that

- the disks are contained in the unstable manifold of P_g ;
- every disk D_n is tangent to the unstable cone-field; and
- the disk D_n is centered at x_n and its size is uniformly bounded from below (it contains a u -ball centered at x_n of uniform size).

Then, given $g \in \mathcal{N}$, we first select large m in such a way x_m is close to $x \in W_{loc}^s(Q_g, g)$. Thereafter we take large n such that $g^n(x)$ is close to Q_g and $g^{-n}(x_m)$ is close to P_g (recall that $x_m \in W^u(P_g, g)$). We now consider the following finite pseudo-orbit of g with three-jumps:

$$z_{n+1} = Q_g, \quad z_k = g^k(x), \text{ if } 0 \leq k \leq n, \quad z_k = g^k(x_m), \text{ if } -n \leq k < 0, \quad z_{-n-1} = P_g.$$

The partially hyperbolic assumption prevents this pseudo-orbit from being shadowed by a true g -orbit (this is proved in [AD, Lemma 3.12], see also [BDT, YY] where the similar arguments are used). Therefore \mathcal{N} is an open set of diffeomorphisms which do not satisfy the shadowing property.

We are now ready to finish the proof in our case. Suppose that f has a co-index one cycle. Then, by a perturbation of f , one gets a diffeomorphism h with a simple cycle (first, using Theorem 5.1, one obtains a cycle with real central eigenvalues and thereafter one uses Proposition 3.6). We now can apply the arguments above.

We note that the accumulation property above ($W^u(P_g, g)$ accumulates nicely to $W_{loc}^s(Q_g, g)$) can be obtained directly using the arguments in Proposition 4.3.

6.6 Proof of Corollary 2

In this section, we prove Corollary 2: *existence of robust cycles implies approximation by co-index one cycles*.

Suppose that \mathcal{U} is a C^1 -open set of diffeomorphisms f having robust heterodimensional cycles, say associated to hyperbolic transitive sets Γ_f and Σ_f . Suppose that the indices of these sets are p and q , $p < q$, respectively. Thus the set Γ_f is contained in the homoclinic class of a saddle P_f of index p and the set Σ_f is contained in the homoclinic class of a saddle Q_f of index q . Note that the saddles P_f and Q_f depends continuously on f .

Lemma 6.7. *Under the hypotheses above, there a residual subset \mathcal{G} of \mathcal{U} such that, for every $f \in \mathcal{G}$, the homoclinic classes of P_f contains saddles of indices $p, p + 1, \dots, q$.*

Proof: We first see that there is a residual subset \mathcal{G} of \mathcal{U} such that the homoclinic classes of P_f and Q_f are equal for all $f \in \mathcal{G}$. Therefore, by (G3) (we can assume that \mathcal{G} is contained in \mathcal{R}), the homoclinic class $H(P_f, f) = H(Q_f, f)$ contains saddles of indices $p, p + 1, \dots, q$. Thus to prove the lemma it is enough to get the generic equality of these homoclinic classes.

Using the that \mathcal{U} is an open set of diffeomorphisms with robust cycles and that the sum of dimensions of $W^s(P_f, f)$ and $W^u(Q_f, f)$ is greater than the dimension of the ambient manifold, one immediately gets an open and dense subset \mathcal{V} of \mathcal{U} of diffeomorphisms f such that $W^s(P_f, f)$ and $W^u(Q_f, f)$ have a non-empty transverse intersection. The λ -lemma now implies that,

$$W^u(P_f, f) \subset \text{closure}(W^u(Q_f, f)), \quad \text{for every } f \in \mathcal{V}.$$

Consider $f \in \mathcal{V}$. We claim that every heteroclinic point $x \in W^u(\Gamma_f, f) \cap W^s(\Sigma_f, f)$ is non-wandering. Fix $x \in W^u(\Gamma_f, f) \cap W^s(\Sigma_f, f)$ and a neighborhood U of x . By the λ -lemma, one has that $W^s(Q_f, f)$ and $W^u(P_f, f)$ intersect U . The λ -lemma also implies that

$$W^u(Q_f, f) \subset \text{closure}(\cup_{n \geq 0} f^n(U)).$$

As $W^u(P_f, f)$ is contained in the closure of $W^u(Q_f, f)$, one has that $W^u(P_f, f)$ is contained in the closure of the forward orbit of U . As $W^u(P_f, f) \cap U \neq \emptyset$, there is $k > 0$ with $f^k(U) \cap U \neq \emptyset$. Since this holds for every neighborhood U of x , this point is non-wandering.

Consider the residual subset $\mathcal{G} = \mathcal{R} \cap \mathcal{V}$ of \mathcal{U} . By (G1), the previous construction implies that, for every $f \in \mathcal{G}$, every $x \in W^u(\Gamma_f, f) \cap W^s(\Sigma_f, f)$ is a chain recurrent point. It is now immediate to see that the points P_f, Q_f , and x are in the same chain recurrence class. By (G2), one has $H(P_f, f) = H(Q_f, f)$ for all $f \in \mathcal{G}$. The proof of the lemma is now complete. \square

We have that, for every $f \in \mathcal{G}$, the homoclinic class $H(P_f, f) = H(Q_f, f)$ contains a saddle of index $p + 1$. Fix now $f \in \mathcal{G}$ and a saddle R_f of index $p + 1$ in $H(P_f, f)$. By (G5), for every $g \in \mathcal{G}$ close to f the saddle R_g belongs to $H(P_g, g)$. By (G2), $H(R_h, h) = H(P_h, h)$, for every $h \in \mathcal{G}$ close to g . Lemma 6.5 gives φ arbitrarily close to g (thus to f) with a cycle associated to P_h and R_h . By construction, this heterodimensional cycle has co-index one. This ends the proof of the corollary.

References

- [Ab] F. Abdenur, *Generic robustness of spectral decompositions*, Ann. Scient. Éc. Norm. Sup., **36**, 212–224, (2003).
- [ABCD] F. Abdenur, Ch. Bonatti, S. Crovisier, and L.J. Díaz, *Generic diffeomorphisms on compact surfaces*, Fund. Math., **187**, 127–159, (2005).
- [ABCDW] F. Abdenur, Ch. Bonatti, S. Crovisier, L.J. Díaz, and L. Wen, *Periodic points and homoclinic classes*, Ergodic Th. and Dynam. Syst., **26**, 1–22, (2006).
- [AD] F. Abdenur and L.J. Díaz, *Pseudo-orbit shadowing in the C^1 -topology*, Discrete Contin. Dyn. Syst., **17**, 223–245, (2007).

- [AS] R. Abraham and S. Smale, *Nongenericity of Ω -stability*, Global Analysis I, Proc. Symp. Pure Math. A.M.S., **14**, 5–8, (1968).
- [As] M. Asaoka. *A simple construction of C^1 -Newhouse domains for higher dimensions*, preprint.
- [BC] Ch. Bonatti and S. Crovisier, *Réurrence et généricité*, Inventiones Math., **158**, 33–104, (2004).
- [BD₁] Ch. Bonatti and L.J. Díaz, *Persistent non-hyperbolic transitive diffeomorphisms*, Ann. Math., **143**, 357–396, (1995).
- [BD₂] Ch. Bonatti and L.J. Díaz, *Connexions hétéroclines et généricité d’une infinité de puits ou de sources*, Ann. Scient. Éc. Norm. Sup., **32**, 135–150, (1999).
- [BD₃] Ch. Bonatti and L.J. Díaz, *On maximal transitive sets of generic diffeomorphisms*, Publ. Math. Inst. Hautes Études Sci., **96**, 171–197, (2002).
- [BDP] Ch. Bonatti, L.J. Díaz, and E.R. Pujals, *A C^1 -generic dichotomy for diffeomorphisms: Weak forms of hyperbolicity or infinitely many sinks or sources*, Ann. of Math., **158**, 355–418, (2003).
- [BDPR] Ch. Bonatti, L.J. Díaz, E.R. Pujals, and J. Rocha, *Robustly transitive sets and heterodimensional cycles*, Astérisque, **286**, 187–222, (2003).
- [BDT] Ch. Bonatti, L.J. Díaz, and G. Turcat, *Pas de Shadowing Lemma pour les dynamiques partiellement hyperboliques*, C. R. Acad. Sci. Paris Sér. I Math., **330**, 587–592, (2000).
- [BDV₁] Ch. Bonatti, L.J. Díaz, and M. Viana, *Discontinuity of the Hausdorff dimension of hyperbolic sets*, C. R. Acad. Sci. Paris Sér. I Math., **320**, 713–718, (1995).
- [BDV₂] Ch. Bonatti, L.J. Díaz, and M. Viana, *Dynamics beyond uniform hyperbolicity*, Encyclopaedia of Mathematical Sciences (Mathematical Physics), **102**, Springer Verlag, (2005).
- [CMP] C. Carballo, C. Morales and M.J. Pacifico, *Homoclinic classes for generic C^1 vector fields*, Ergodic Th. and Dynam. Sys., **23**, (2003), 403–415, (2003).
- [Co] C. Conley, *Isolated invariant sets and the Morse index*, CBMS Regional Conference Series in Mathematics, **38**, AMS, Providence, R.I., (1978).
- [Dí₁] L.J. Díaz, *Robust nonhyperbolic dynamics and heterodimensional cycles*, Ergodic. Th. and Dynam. Sys., **15**, 291–315, (1995).
- [Dí₂] L.J. Díaz, *Persistence of cycles and non-hyperbolic dynamics at the unfolding of heterodimensional cycles*, Nonlinearity, **8**, 693–715, (1995).
- [DNP] L.J. Díaz, A. Nogueira, and E.R. Pujals, *Heterodimensional tangencies*, Nonlinearity, **19**, 2543–2566, (2006).

- [DR₁] L.J. Díaz and J. Rocha, *Nonconnected heterodimensional cycles: bifurcation and stability*, *Nonlinearity*, **5**, 1315–1341, (1992).
- [DR₂] L.J. Díaz and J. Rocha, *Partially hyperbolic and transitive dynamics generated by heteroclinic cycles*, *Ergodic Th. and Dynam. Syst.*, **25**, 25–76, (2001).
- [DR₃] L.J. Díaz and J. Rocha, *Heterodimensional cycles, partial hyperbolicity and limit dynamics*, *Fund. Math.*, **174**, 127–186, (2002).
- [DU] L.J. Díaz and R. Ures, *Persistent homoclinic tangencies and the unfolding of cycles*, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **11**, 643–659, (1994).
- [Fr] J. Franks, *Necessary conditions for stability of diffeomorphisms*, *Trans. A.M.S.*, **158**, 301–308, (1971).
- [Ha] S. Hayashi, *Connecting invariant manifolds and the solution of the C^1 -stability and Ω -stability conjectures for flows*, *Ann. of Math.*, **145**, 81–137, (1997).
- [HPS] M.W. Hirsch, C.C. Pugh, and M. Shub, *Invariant manifolds*, *Lecture Notes in Mathematics*, **583**, (1997).
- [Ma₁] R. Mañé, *Contributions to the stability conjecture*, *Topology*, **17**, 386–396, (1978).
- [Ma₂] R. Mañé, *An ergodic closing lemma*, *Ann. of Math.*, **116**, 503–540, (1982).
- [Mo] C. G. Moreira, *Personal communication and Dynamical Systems Seminary (I.M.P.A.): C^1 -stable intersections of regular Cantor sets do not exist*, April 2006.
- [Ne₁] S. Newhouse, *Nondensity of axiom $A(a)$ on S^2* , *Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, 1968)*, 191–202, Amer. Math. Soc., Providence, R.I., (1970).
- [Ne₂] S. Newhouse, *Hyperbolic Limit Sets*, *Trans. Amer. Math. Soc.*, **167**, 125–150, (1972).
- [Ne₃] S. Newhouse, *The abundance of wild hyperbolic sets and nonsmooth stable sets for diffeomorphisms*, *Publ. Math. I. H.E.S.*, **50**, 101–151, (1979).
- [NP] S. Newhouse and J. Palis, *Cycles and bifurcation theory*, *Asterisque*, **31**, 43–140, (1976).
- [Ro] N. Romero, *Persistence of homoclinic tangencies in higher dimensions*, *Ergodic Th. Dynam. Sys.*, **15**, 735–757, (1995).
- [Pa] J. Palis, *A global view of dynamics and a conjecture on the denseness and finitude of attractors*, *Astérisque*, **261**, 335–347, (2000).
- [PT] J. Palis and F. Takens, *Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations. Fractal dimensions and infinitely many attractors*, *Cambridge Studies in Advanced Mathematics*, **35**, Cambridge University Press, (1993).
- [PV] J. Palis and M. Viana, *High dimension diffeomorphisms displaying infinitely many sinks*, *Ann. of Math.*, **140**, 207–250, (1994).

- [PS] E. Pujals and M. Sambarino, *Homoclinic tangencies and hyperbolicity for surface diffeomorphisms*, Ann. of Math., **151**, 961–1023, (2000).
- [Sh] M. Shub, *Topological transitive diffeomorphism on T^4* , Lect. Notes in Math., **206**, 39 (1971).
- [Si] R.C. Simon, *A 3-dimensional Abraham-Smale example*, Proc. A.M.S., **34**(2), 629–630, (1972).
- [Sm₁] S. Smale, *Stable manifolds for differential equations and diffeomorphisms*, Ann. Scuola Norm. Sup. Pisa, **17**, 97–116, (1963).
- [Sm₂] S. Smale, *Differentiable dynamical systems*, Bull. A.M.S., **73**, 747–817, (1967).
- [Ur] R. Ures, *Abundance of hyperbolicity in the C^1 topology*, Ann. Sci. École Norm. Sup., **28**, 747–760, (1995).
- [YY] G-C. Yuan and J. A. Yorke, *An open set of maps for which every point is absolutely nonshadowable*, Proc. Amer. Math. Soc., **128**(3), 909–918, (2000).

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