Destroying horseshoes *via* heterodimensional cycles: generating bifurcations inside homoclinic classes

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To Paulo Rogério Sabini, in memoriam

Abstract

In this paper, we propose a model for the destruction of three-dimensional horseshoes *via* heterodimensional cycles. This model yields some new dynamical features. Among other things, it provides examples of homoclinic classes properly contained in other classes and it is a model of a new sort of heteroclinic bifurcations we call *generating*.

1 Introduction

A relevant problem in dynamics is to describe the transition from hyperbolic to non-hyperbolic regimes. An archetypal example of this transition can be found in the process of creation/destruction of horseshoes. In the destruction of horseshoes in two dimensions, the transition from the hyperbolic to the persistently non-hyperbolic regimes corresponds to the passage from non-critical to critical dynamics (creation of tangencies), see [8, Preface]. But in higher dimensions, there are transitions between these two persistent regimes which do not involve critical behavior. The dynamics is partially hyperbolic and the lack of hyperbolicity follows from coexistence in the same transitive piece of dynamics (a homoclinic class, see the precise definition below) of saddles having different *indices* (i.e., dimension of the unstable bundle).

The goal of this paper is the study of the destruction of three dimensional horseshoes via heterodimensional cycles which yields some new dynamical features in bifurcation theory. We construct a model diffeomorphism F having a horseshoe Λ with a uniformly hyperbolic splitting into three one-dimensional spaces $E^s \oplus E^c \oplus E^u$, where E^s and E^c are contracting directions (the contraction in E^s is stronger than the one in E^c) and E^u is expanding. This splitting is defined in a neighborhood U of the horseshoe Λ and there is some saddle $Q \in U$ (which does not belong to Λ) such that E^c is an expanding direction of Q (thus the saddle Q has index two). Hence the dynamics of F in U is partially

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hyperbolic and the non-wandering set of F in U is the (disjoint) union of the horseshoe Λ and the saddle Q. The partially hyperbolic splitting prevents both Hopf and homoclinic bifurcations (tangencies).

We consider a one-parameter family of diffeomorphisms $(F_t)_{t\in[-\epsilon,\epsilon]}$ with $F_1 = F$. We see that, for every $t > 0$, the diffeomorphism F_t has a horseshoe Λ_t, which is the continuation of $\Lambda = \Lambda_1$. Indeed the horseshoe Λ_t is the homoclinic class $H(P, F_t)$ of a saddle P, i.e. the closure of the transverse intersections of the invariant manifolds of P. Moreover, for $t > 0$, the non-wandering set of F_t in U is hyperbolic and equal to the disjoint union of Q (the saddle of index two above) and Λ_t (a hyperbolic transitive set whose saddles have index one). For $t = 0$, there is a bifurcation, the diffeomorphism F_0 has a heterodimensional cycle (see definition in Section 1.2) associated to the saddle P of the *old* horseshoe and the saddle Q: a transverse homoclinic point X_t of P (defined for all positive t and depending continuously on t) of the horseshoe becomes a heteroclinic point X_0 of P and Q (an intersection of the unstable manifold of P and the stable one of Q). Some relevant features in bifurcation theory of our model that we want to point out are the following:

1. The bifurcation occurs inside a homoclinic class: the heteroclinic point X_0 responsible for the bifurcation is a non-transverse heteroclinic point accumulated by transverse homoclinic points of P. We shall call *internal* these type of bifurcations (see Section 1.2 for a more explicit discussion on this topic).

2. At the bifurcation parameter $t = 0$, the homoclinic class of Q is properly contained in the homoclinic class of P.

3. Also, at the bifurcation parameter $t = 0$, there is an explosion of the dynamics, meaning that the old horseshoe $\Lambda_t = H(P, F_t)$ is a factor of $H(P_0, F_0)$. In other words, there is a continuous, surjective but non-injective map $\pi : H(P_0, F_0) \to \Lambda$, which is a semi-conjugacy between the corresponding dynamics. We call these type of bifurcation as *generating dynamics*, see Section 1.3. In fact, the dynamics of the non-hyperbolic homoclinic class $H(P_0, F_0)$ is extremely rich and contains infinitely many central intervals coming from explosions of *old* homoclinic points.

4. After the bifurcation, i.e. for $t < 0$, the homoclinic classes of P and Q coincide (phenomenon of intermingled homoclinic classes), and contain infinitely many *central segments* (parallel to the central direction).

We observe that an important tool of this paper is the study of 1-dimensional one-parameter families of *systems of iterated functions* which describe the dynamics in the central direction. These one dimensional maps are obtained by considering the quotient dynamics along the strong stable and strong unstable directions. In fact, most of the properties of the diffeomorphisms (as existence of periodic points, hyperbolicity, and creation of cycles) are obtained using these families.

We will give in Section 1.1 some historical account of bifurcation of horseshoes and discuss in Sections 1.2 and 1.3 some aspects of bifurcation theory related to our results. Our main results are stated in Section 1.4. The precise definition of the model is in Section 2.

1.1 Destruction of horseshoes

In dimension two, the creation of (dissipative) horseshoes, as the ones in Figure 1, accomplishes infinitely many bifurcations including homoclinic tangencies associated to saddles and loss of hyperbolicity of saddles. This last sort of bifurcation includes saddle-node and Hopf ones as well as a cascade of period doubling bifurcations, see [43]. Another relevant feature is that the creation of horseshoes yields, together with infinitely many orbit-creation, infinitely many orbit-annihilation of periodic points, see [28]. This situation is different than for the quadratic family of the interval, where periodic orbits are created monotonically, see [34]. For a survey of this subject see [38, Chapter 3].

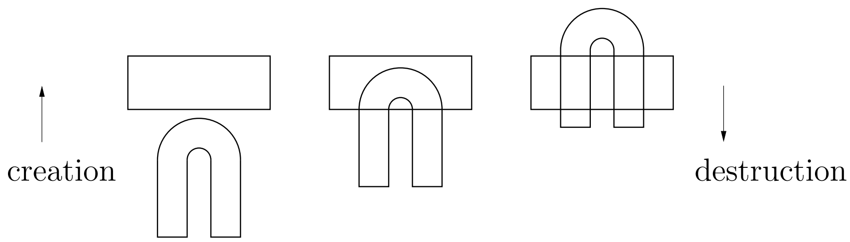

Figure 1: Creation/destruction of a horseshoe

Related to these constructions, there is the problem of the first bifurcation in the destruction of a horseshoe. Such a first bifurcation depends on global properties of the horseshoe as, in very rough terms, the curvature of the invariant manifolds, the symbolic dynamics (number of symbols) of the horseshoe, and the rate of expansion/contraction of the map. For instance, [2] gives examples where such a first bifurcation is a period doubling one. On the other hand, [29, 39] illustrate the case where the first bifurcation is a homoclinic bifurcation such that the non-transverse homoclinic point is a limit point (one calls these bifurcations *internal tangencies inside homoclinic classes*). In fact, a key aspect of this sort of bifurcations is whether or not the non-transverse intersection responsible for the failure of hyperbolicity is a limit point. Note that non-transverse homoclinic intersections are non-wandering points, the relevant fact here is that this non-transverse intersection is a limit point inside some homoclinic class.

A second prototypical example of creation/destruction of hyperbolic sets is given by the Hénon-like maps, that is, perturbations of the Hénon family

$$H_{a,b}(x,y) = (1 - a\,x^2 + y, b\,x).$$

Hénon and Hénon-like families have been extensively studied since the publication of [15] claiming that the non-wandering sets of these maps are hyperbolic for large a and small $b > 0$. This leads to the description of the boundary of hyperbolicity for these families. For Hénon families, [3, 4] showed that there is a *first bifurcation* parameter $a^\star$ (fix small b, $H_{a,b}$ is hyperbolic for all $a > a^\star$) corresponding to a homoclinic tangency. In [11], the same question was addressed for Hénon-like families, considering a geometrical approach. On the one hand, as in [29, 39], the orbit corresponding to the tangency belongs to the limit set (thus this set is not hyperbolic). On the other hand, for such a parameter $a^\star$ all periodic orbits are uniformly hyperbolic (in fact, we have more: all Lyapunov exponents of all measures supported on the non-wandering set are uniformly bounded away from zero), see [11].

Finally, a third mechanism for the destruction of horseshoes was proposed in [44, 41]: the so-called *saddle-node horseshoes* depicted in Figure 2 and whose dynamics is described in [14]. In this setting, a saddle of the horseshoe loses its hyperbolicity (it becomes a saddle-node) and it disappears thereafter. In some cases, this bifurcation may be isolated (see for instance [9], which is a reformulation of the derived from Anosov construction in [42] to the horseshoe context). But in some cases this sort of bifurcation leads to a string of bifurcations (including homoclinic tangencies), see [24, 26, 13] after [37].

Saddle-node horseshoes exemplify in a rather precise way the phenomena of annihilation and creation of dynamics. On the one hand, the saddle-node bifurcation carries the subsequent annihilation of infinitely many hyperbolic periodic orbits of arbitrarily large period of the *old* horseshoe. On the other hand, due to global aspects of the dynamics (creation of homoclinic tangencies) new periodic orbits are generated. An analysis of the balance between creation and annihilation of dynamics in terms of entropy can be found in [18, 31]. For a survey on this subject see [19].

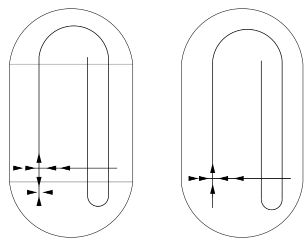

Figure 2: A saddle-node horseshoe

1.2 Heterodimensional cycles and homoclinic classes

Robustly transitive non-hyperbolic but partially hyperbolic systems were first obtained in [40], in dimension four, and later in [33], in dimension three. Next, motivated by the constructions in [17], [6] introduced the notion of *blender* which is the basis for the systematic construction of (non-hyperbolic) transitive sets persistently containing saddles of different indices. This corresponds to the concept of *unstable dimension variability* in [30].

The blender constructions are related to the existence of *heterodimensional cycles.* We say that a diffeomorphism F has a *heterodimensional cycle* if there are saddles P and Q of F having different indices whose invariant manifolds are related in a cyclic way ($W^s(P,F) \cap W^u(Q,F) \neq \emptyset$ and $W^u(P,F) \cap W^s(Q,F) \neq \emptyset$). This sort of cycles can only occur in dimension three or higher. These cycles were introduced in [36] and systematically studied in the series of papers [17, 16, 20, 21, 22, 23].

In this paper, we describe the destruction of three dimensional horseshoes via heterodimensional cycles. Our construction is mostly motivated by the results in [39, 11] (the destruction of two-dimensional horseshoes keeping the hyperbolicity of the periodic points, and exhibiting internal non-transverse intersection). Our model consists of a one-parameter family of partially hyperbolic diffeomorphisms $(F_t)_{t\in[-\epsilon,\epsilon]}$ within a fixed open set U. We see that, for every $t > 0$, the diffeomorphism F_t has a horseshoe Λ_t, which is the continuation of a hyperbolic horseshoe $\Lambda = \Lambda_1$ and the homoclinic class of a given saddle P. Moreover, for $t > 0$, the non-wandering set of F_t in U is hyperbolic and equal to the disjoint union of the point Q (a saddle of index two) and Λ_t (a hyperbolic transitive set whose saddles have index one). For $t = 0$, there is a bifurcation, the diffeomorphism F_0 has a heterodimensional cycle associated to a saddle P of the *old* horseshoe and the saddle Q: a transverse homoclinic point X_t of P of the horseshoe becomes a heteroclinic point X_0 of P and Q. This process is depicted in Figure 5. The arc $(F_t)_t$ is constructed in Section 2. A key point here is that, for the bifurcating parameter $t = 0$, there is a transitive set Λ_0 (the homoclinic class of P) containing the heteroclinic point X_0 and the saddles Q and P with different indices. The set Λ_0 can be viewed as a *partially hyperbolic horseshoe.*

Another relevant point here, is that this bifurcation leads a string of secondary bifurcations (in fact, for all $t < 0$ the diffeomorphism F_t is not hyperbolic) which do not include neither homoclinic tangencies nor Hopf bifurcations. These secondary bifurcations include saddle-node and flip ones as well as heterodimensional cycles.

As we said before, a relevant point in bifurcations via tangencies or heterodimensional cycles is whether or not the non-transverse orbits responsible for them are limit points. In the bifurcations considered in this paper, the non-transverse intersections belong to the homoclinic class of a saddle. Thus this sort of bifurcation can be better viewed as *internal non-transverse bifurcations of homoclinic classes.* We now discuss these bifurcations.

Recall that the *homoclinic class* of a saddle P of a diffeomorphism F, denoted by $H(P, F)$, is the closure of the transverse intersections of the invariant manifolds of the orbit of P. Homoclinic classes are transitive sets containing a dense subset of hyperbolic periodic points of the same index as P. The homoclinic class of a saddle P can be also defined as the closure of the saddles homoclinically related to P. Recall that two saddles P and Q are *homoclinically related* if the invariant manifolds of their orbits meet transversely: the stable manifold of the orbit of P transversely meets the unstable manifold of the orbit of Q and vice versa. These properties of homoclinic classes can be found, for instance, in [35]. We note that the homoclinic class of a saddle P may contain saddles (with the same or different index as P) which are not homoclinically related to it. In fact, the bifurcations considered in this paper provide examples of such homoclinic classes.

We say that the arc of diffeomorphisms $(F_t)_{t\in[-\epsilon,\epsilon]}$ exhibits an *internal non-transverse bifurcation of a homoclinic class at* $t = 0$ if, for every parameter t, there is a saddle P_t depending continuously on t (for simplicity, we omit the dependence of P_t on t) such that, for every $t \in (0, 1]$, the homoclinic class $H(P, F_t)$ is a hyperbolic set, and for $t = 0$, there is a non-transverse intersection between saddles of $H(P, F_0)$ which is inside of the class. This implies that the homoclinic class $H(P, F_0)$ is non-hyperbolic.

1.3 Annihilating and generating bifurcations

We consider two criteria for classifying internal non-transverse bifurcations of homoclinic classes. The first one follows [36] and takes into account the type of non-transverse intersection: *equidimensional,* if the saddles involved in the non-transverse intersection have the same index, and *heterodimensional* otherwise. For instance, the bifurcations in [39] depicted in Figure 3 are equidimensional while the ones studied in our paper are heterodimensional.

We introduce a second classification related to the symbolic dynamics of the bifurcating homoclinic class which is motivated by the paper [27] about symbolic extensions. Suppose that for the parameters $t \in (0, 1]$ corresponding to hyperbolic dynamics, there is a shift space (Σ, ς) with finite alphabet and a homeomorphism $\pi_t \colon \Sigma \to H(P, F_t)$ conjugating F_t in $H(P, F_t)$ to ς in Σ, i.e., $F_t \circ \pi_t = \pi_t \circ \varsigma$. The bifurcation at $t = 0$ *annihilates dynamics* if there is a continuous surjection $\pi_0 \colon \Sigma \to H(P, F_0)$ which is not one-to-one such that $F_0 \circ \pi_0 = \pi_0 \circ \varsigma$. The bifurcation at $t = 0$ *generates dynamics* if there is a continuous one-to-one map $\pi_0 \colon \Sigma \to H(P, F_0)$ which is not onto such that $F_0 \circ \pi_0 = \pi_0 \circ \varsigma$. In the case of generating bifurcations, the dynamics of the bifurcating homoclinic class is richer than the one of the shift (i.e., the dynamics of the homoclinic classes before the bifurcation). In the case of annihilating bifurcations we have the opposite situation. *Neutral* bifurcation are defined in the obvious way.

The bifurcations in [39, 32] described in Figure 3 are annihilating ones. Dynamically, these bifurcations correspond to the collision of two transverse homoclinic points of a horseshoe to a non-transverse one. This collision carries identifications in the symbolic level, therefore some annihilation of dynamics. More precisely, for every $t \in (0, 1]$, the diffeomorphisms F_t have a horseshoe Λ_t (the homoclinic class of a saddle P) conjugate to the complete shift of three symbols, $\varsigma \colon \Sigma \to \Sigma$ (we let $\Sigma = \{0, 1, 2\}^{\mathbb{Z}}$). For the bifurcation $t = 0$, the *horseshoe* $\Lambda_0 = H(P, F_0)$ has an internal homoclinic tangency which is inside it. Using the notation above, for $t = 0$, the map $\pi_0 \colon \Sigma \to \Lambda_0$ is onto but it fails to be one-to-one (i.e., π_0 is a semi-conjugacy): two different sequences of Σ correspond now to the same point (a non-transverse homoclinic point) of Λ_0.

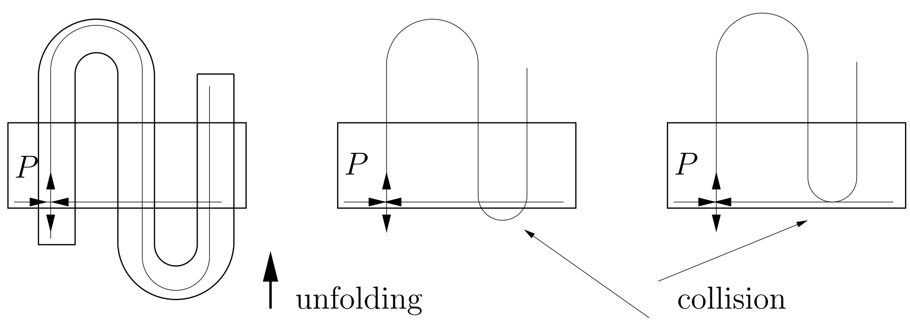

Figure 3: Annihilating bifurcation

Finally, the saddle-node horseshoes discussed in Section 1.1 and depicted in Figure 2 provide examples of neutral bifurcations.

The bifurcations in our paper are generating ones and are discussed in the next section.

1.4 Description of the dynamics

We next describe the dynamical features of our three dimensional model family F_t (see Theorem 1 for the precise statement). The homoclinic classes $H(P, F_t)$ are hyperbolic and conjugate, for $t > 0$, to the shift on Σ_{11}, the space of sequences of symbols $\{0, 1\}$ with forbidden block $\{1, 1\}$. For the bifurcation parameter $t = 0$, the homoclinic class $H(P, F_0)$ has an internal non-transverse bifurcation corresponding to a heterodimensional cycle: there is a saddle $Q \in H(P, F_0)$ of different index of P such that P and Q are related by a heterodimensional cycle and there are non-transverse intersections between the unstable manifold of P and the stable one of Q which are inside the homoclinic class $H(P, F_0)$. Moreover, using the notation above, for $t = 0$, the map $\pi_0 \colon \Sigma_{11} \to H(P, F_0)$ is one-to-one but it is not onto. In fact, in the next paragraph we see that the dynamics of $H(P, F_0)$ is extremely richer than the one of Σ_{11}.

In fact, π_0^{-1} naturally extends to an onto map $\varrho\colon H(P, F_0) \to \Sigma_{11}$, with $\varsigma \circ \varrho = \varrho \circ F_0$, which assigns infinitely many points of $H(P, F_0)$ to the same sequence, for an infinite number of sequences of Σ_{11}. In very rough terms, the dynamics of $H(P, F_0)$ explodes in the central direction: there are infinitely many central segments of $H(P, F_0)$ which are mapped by ϱ to the same sequence of Σ_{11}. These points correspond to an explosion of a transverse homoclinic point of P. We now clarify this issue.

We will prove that, for the bifurcating diffeomorphism F_0, there is an invariant central curve $\gamma \subset W^s(P, F_0) \cap W^u(Q, F_0)$ (see Figure 4) joining P and Q and contained in the homoclinic class of P. This implies, in particular, that $Q \in H(P, F_0)$. The segment γ *projects* to the same sequence of Σ_{11} (the sequence corresponding to the saddle P). Using this fact, we will get infinitely many central heteroclinic segments with a similar property as follows. Consider the strong unstable manifold $W^{uu}(Q, F_0)$ of Q (the unique invariant manifold tangent at Q to the strong unstable bundle E^u). We call a transverse intersection of the stable manifold $W^s(P, F_0)$ of P and the strong unstable manifold $W^{uu}(Q, F_0)$ of Q a *fake homoclinic point of* P. We prove that there are (infinitely many) heteroclinic curves η contained in the intersection between $W^s(P, F_0)$ and $W^u(Q, F_0)$ whose extremes are a fake homoclinic point of P and either a true transverse homoclinic point of P or a heteroclinic point (associated to P and Q). Each curve η is contained in the homoclinic class of P and projects to the same sequence of Σ_{11}. In Figure 4 there is depicted a heteroclinic curve η bounded by a fake homoclinic point Y of P and a heteroclinic point X.

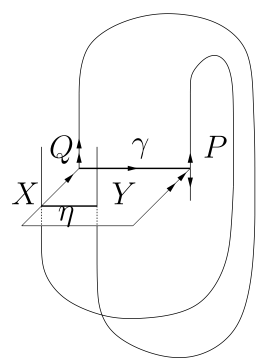

Figure 4: Fake homoclinic points and heteroclinic segments

Note that the coexistence of saddles of different indices in the same homoclinic class prevents its hyperbolicity. Thus, since $Q \in H(P, F_0)$, that homoclinic class is non-hyperbolic. Nevertheless, we prove that every periodic point of the non-hyperbolic homoclinic class $H(P, F_0)$ is hyperbolic, although the Lyapunov exponents of the periodic points accumulate to zero. It is interesting to compare this result with the destruction of hyperbolic sets in the Hénon family in [11] and of horseshoes with internal tangencies in [10] (see Figure 3), where the Lyapunov exponents of the periodic points of the non-hyperbolic horseshoe are uniformly bounded away from zero.

Finally, for the bifurcating diffeomorphism F_0, the homoclinic class $H(Q, F_0)$ is trivial and thus properly contained in the homoclinic class of P. This gives, as far as we know, the first example of two saddles whose homoclinic classes where one is properly contained in the other one: $H(Q,F_0)=\{Q\}\subset H(P,F_0)$. Recall, that for C^1-generic diffeomorphisms (i.e., diffeomorphisms in a residual subset of $\mathrm{Diff}^1(M)$) non-disjoint homoclinic classes coincide, see [12, 5]. For examples of *overlapping* homoclinic classes (each class is not contained in the other one and the classes have non-empty intersection) see [25].

We also study the dynamics arising from the unfolding of the cycle. Recall that the heteroclinic orbits associated to P and Q are generated as follows. We fix local invariant manifolds $W^s_{loc}(Q,F_t)$ of Q and $W^u_{loc}(P,F_t)$ of P. For every $t>0$, there is a transverse homoclinic point $X_t\in W^u_{loc}(P,F_t)$ of P, depending continuously on t. The points X_t converge to some heteroclinic point $X_0\in W^s_{loc}(Q,F_0)\cap W^u_{loc}(P,F_0)$, see Figure 5. The cycle associated to P and Q generates a string of secondary bifurcations for $t<0$. For instance, transverse homoclinic points of P become heteroclinic intersections between $W^u(P,F_t)$ and $W^s(Q,F_t)$, thus generating new heterodimensional cycles. In Figure 5, Y_t is a transverse homoclinic point for all $t\in(t_1,t_2]$ which generates a *secondary* heterodimensional cycle for t_1. Moreover, infinitely many saddle-node and flip bifurcations also occur throughout the unfolding.

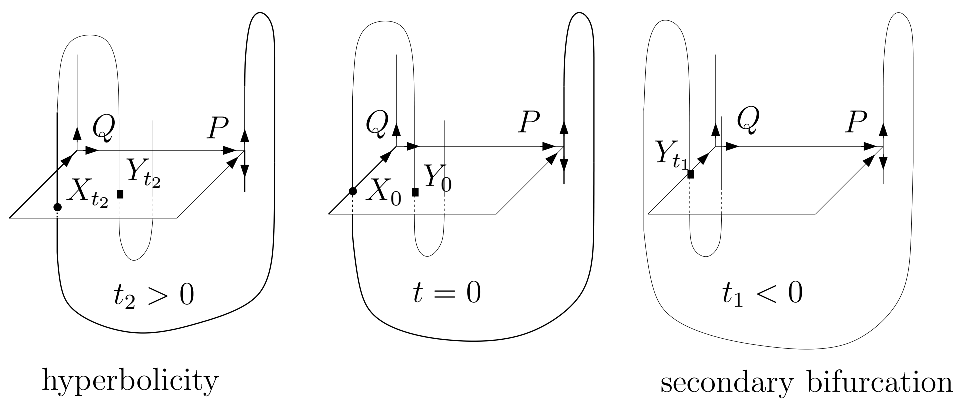

Figure 5: Generating bifurcation through a heterodimensional cycle

We also prove that, for every small $t<0$, the homoclinic classes of P and Q coincide. Therefore these classes are not hyperbolic. This follows from a much stronger fact: the two dimensional stable manifold of P is contained in the closure of the one-dimensional stable manifold of Q. This is a version of the so-called *distinctive property of blenders* in [8, Chapter 6], that follows using a *blender-like* construction motivated by [6, 17]. For instance, this construction gives that, for $t<0$, the whole central curve γ joining P and Q is contained in $H(Q,F_t)$. In fact, there are infinitely many curves joining a homoclinic point of Q and a homoclinic point of P contained in $H(Q,F_t)$. These curves are analogous of the central curves η above joining fake and true homoclinic points of P.

Finally, let us observe that there are two sort of heterodimensional cycles, *twisted* and *non-twisted*, according to the geometry of the invariant manifolds of the saddles in the cycle. This classification was proposed in [1]. The unfolding of twisted cycles generate both saddle-node and flip bifurcations of periodic points, while the non-twisted ones *a priori* only accomplish saddle-node bifurcations. The heterodimensional cycles we consider in this paper are twisted ones. Twisted heterodimensional cycles were conjectured to cause a *crisis* of chaotic attractors in [1], where numerical evidence of this is provided.

This paper is organized as follows. In Section 2, we construct the arc of diffeomorphisms $(F_t)_{t\in[-\epsilon,\epsilon]}$. Roughly speaking, the diffeomorphisms of the family $(F_t)_{t\in[-\epsilon,\epsilon]}$ are the skew-product of a hyperbolic linear dynamics of saddle type and two interval maps $f_0 = f_{0,t}$ and $f_{1,t}$. The central dynamics of the diffeomorphisms F_t is obtained considering (suitable) compositions of these two maps of the interval.

More precisely, we consider a pair of interval maps f_0 and f_1, the map f_0 has two fixed points 0 (repelling) and 1 (attracting) and f_1 is an affine contraction. The effect of the parameter t consists in considering translations of f_1 by t (i.e. $f_{1,t} = f_1 + t$). We consider compositions of the maps f_0 (which does not depend on t) and $f_{1,t}$ such that two consecutive compositions of $f_{1,t}$ are forbidden. That is the reason because we consider the shift space Σ_{11} to describe the hyperbolic dynamics (in fact, our dynamics is modeled over that shift space). In this way, we get a system $\mathfrak{F}_t$ of iterated functions whose relevant dynamical properties (as existence of periodic orbits, hyperbolicity-like properties, and explosion of the dynamics in the central direction) are translated to similar properties of F_t. In Section 3, which is the main technical part of the paper, we study these systems. In Section 3.1, we analyze the system $\mathfrak{F}_t$ for $t > 0$ and state its hyperbolic properties. In Section 3.2, we study the system for $t = 0$ and obtain the (non-uniform) hyperbolicity for the periodic points of $\mathfrak{F}_t$. In Section 3.3, we prove the existence of dense orbits for the system $\mathfrak{F}_t$ for t close to 0. Finally, in Section 3.4, we prove that the system $\mathfrak{F}_t$ satisfies an expanding property.

In Section 4, we translate properties of the system $\mathfrak{F}_t$ to the diffeomorphisms F_t in terms of existence of periodic and homoclinic points and cycles. Finally, in Sections 5, 6, and 7, we use the results about the system $\mathfrak{F}_t$ to prove the main result in this paper.

2 The model family of diffeomorphisms

In this section, we construct the model family F_t and state the main result.

Consider in $\mathbb{R}^3$ the cube $R = [0,1] \times [-\delta, 1+\delta] \times [0,1]$, for a small $\delta > 0$, and the sub-cubes $R_0 = [0,1] \times [-\delta, 1+\delta] \times [0,1/6]$, and $R_1 = [0,1] \times [-\delta, 1+\delta] \times [5/6,1]$ of R. We consider a family of *horseshoe maps* $F_t \colon R \to \mathbb{R}^3$, $t \in [-\epsilon, \epsilon]$, small $\epsilon > 0$, on the cube as follows. The restrictions $F_{i,t}$ of F_t to R_i, $i = 0, 1$, are defined by:

- $F_{0,t}(x,y,z) = F_0(x,y,z) = (\lambda_0 x, f(y), \beta_0 z)$, with $0 < \lambda_0 < 1/3$, $\beta_0 > 6$, and f is the time one map of a vector field to be defined in the sequel;
- $F_{1,t}(x,y,z) = (3/4 - \lambda_1 x, \sigma(1-y) + t, \beta_1(z - 5/6))$, with $0 < \lambda_1 < 1/3$, $3 < \beta_1 < 4$, and σ close to $1/4$; and
- points $(x,y,z) \notin R_0 \cup R_1$ are mapped by F_t outside of R, i.e., $F_t(x,y,z) \notin R$.

The map $f : \mathbb{R} \to \mathbb{R}$ is defined as the time one of the vector field

$$x' = x(1-x).$$

Note that f maps diffeomorphically the interval $[0,1]$ into itself. This map f is depicted in Figure 6.

Observe that $f(0) = 0$ and $f(1) = 1$, and, for every $y \neq 0$ and $n \in \mathbb{Z}$, we have

$$f^n(y) = \frac{1}{1 - \left(1 - \frac{1}{y}\right) e^{-n}}. \tag{1}$$

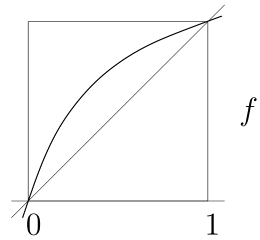

Figure 6: The central map f

We also have

$$(f^n)'(y) = \frac{\frac{e^{-n}}{y^2}}{\left(1-\left(1-\frac{1}{y}\right)e^{-n}\right)^2} = \frac{e^{-n}}{y^2}\,(f^n(y))^2\,. \tag{2}$$

Observe that $f'(0) = e$ and $f'(1) = 1/e$. Therefore, as $f(0) = 0$ and $f(1) = 1$, for every t, the point $Q = (0, 0, 0)$ is a fixed saddle of index 2 of F_t, and the point $P = (0, 1, 0)$ is a fixed saddle of index 1 of F_t.

By construction, $F_t(R_0)$ intersects both R_0 and R_1 (and such intersections are cubes). Moreover, $F_t(R_1)$ only intersects R_0 (and the intersection also is a cube). Figures 7 and 8 describes the dynamics of the diffeomorphisms F_t in the cube R. In this way, to each point X whose orbit is contained in the cube R, we associate a sequence $\iota(X) \in \Sigma = \{0, 1\}^{\mathbb{Z}}$ defined as follows: $\iota_k = j$ if $X_k = F_t^k(X) \in R_j$. By comments above (there is no transition from R_1 to R_1), the sequence $\iota(X)$ is in Σ_{11} (the subset of $\{0, 1\}^{\mathbb{Z}}$ of sequences with forbidden block 11). We say that $\iota(X)$ is the *itinerary* of X.

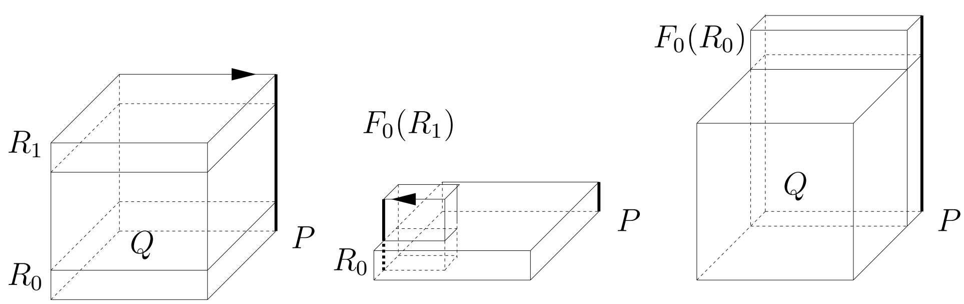

Figure 7: Bifurcating horseshoe

Theorem 1. *Consider the arc of diffeomorphisms $(F_t)_{t\in[-\epsilon,\epsilon]}$ above. The dynamics of F_t in the rectangle R satisfies the following properties:*

A) Hyperbolic dynamics ($t > 0$) : *For every $t > 0$ small enough, the limit set of F_t in R is the disjoint union of two hyperbolic sets: the saddle Q and the homoclinic class of P. Moreover, the dynamics of F_t in the homoclinic class $H(P, F_t)$ is conjugate to the shift $\varsigma\colon \Sigma_{11} \to \Sigma_{11}$.*

B) Bifurcating dynamics ($t = 0$) :

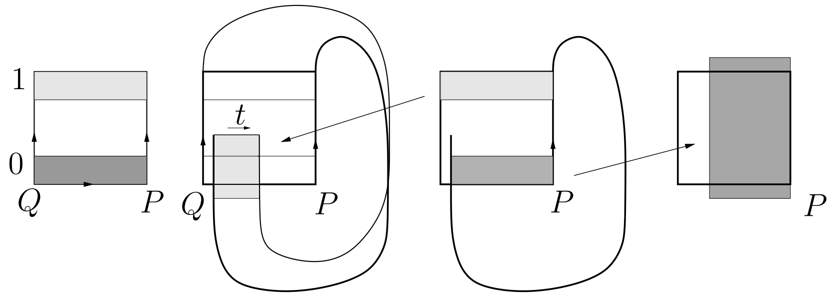

Figure 8: Projection of the twisted horseshoe

1. *The diffeomorphism F_0 has a heterodimensional cycle associated to the saddles P and Q. Moreover, there is a non-transverse intersection between $W^s(Q, F_0)$ and $W^u(P, F_0)$ whose orbit is contained in $H(P, F_0)$.*
2. *The homoclinic class of Q is trivial and is contained in the non-trivial homoclinic class of P. In particular, $H(P, F_0)$ is not hyperbolic.*
3. *There is a surjection*

$$\varrho\colon H(P, F_0) \to \Sigma_{11}, \qquad \text{with} \quad \varrho \circ F_0 = \varsigma \circ \varrho,$$

and infinitely many central segments $I = \{x\} \times [a, b] \times \{z\}$ of $W^s(P, F_0) \cap W^u(Q, F_0)$ such that every I is contained in the homoclinic class of $H(P, F_0)$ and $\varrho(y_1) = \varrho(y_2)$ for every pair of points $y_1, y_2 \in I$.

C) Robustly non-hyperbolic dynamics after the bifurcation $(t < 0)$: *For every $t < 0$ close to 0, the homoclinic classes $H(P, F_t)$ and $H(Q, F_t)$ coincide and so they are not hyperbolic. Moreover, these homoclinic classes contain infinitely many central segments of $W^s(P, F_0) \cap W^u(Q, F_0)$.*

Remark 2.1. Using the notation introduced in Section 1.2, by item 1 in B, the bifurcation at $t = 0$ is an internal non-transverse bifurcation of a homoclinic class. Using the terminology in Section 1.3, items A and 3 of B mean that the bifurcation is of the generating type.

Remark 2.2. In our construction, see Remark 4.3, we prove that the Hausdorff dimension of the homoclinic class $H(P, F_t)$ is at least one for every $|t|$ close to 0 (since for $t \leq 0$ the homoclinic class $H(P, F_t)$ contains intervals, it is obvious for $t \leq 0$). We observe that in our constructions, by shrinking R_0 and R_1 in the x and z-directions, we can take the expansion constants β_0 and β_1 in the z-direction arbitrarily large and the contraction constants λ_0 and λ_1 in the x-direction arbitrarily close to 0. This suggests the possibility of considering a model where the initial hyperbolic homoclinic class $H(P, F_{t_0})$, where t_0 is possibly bigger than ϵ, has Hausdorff dimension strictly less than one. Our construction indicates that the Hausdorff dimension of $H(P, F_t)$ increases as t decreases and becomes one before the bifurcation. In the context of heterodimensional cycles, the relation between fractal dimensions and bifurcations is not well understood. We note that for homoclinic bifurcations of surface diffeomorphisms these fractal dimensions play a key role for determining the dynamics, see [38]. Our construction may suggest some link between bifurcations and fractal dimensions in the heterodimensional setting.

3 Systems of iterated functions

Consider $\delta > 0$ and σ as in Section 2 and for small $|t|$, consider the maps

$$f_{0,t}, f_{1,t} \colon [-\delta, 1] \to \mathbb{R}$$

defined by

$$f_{0,t}(y) = f(y) \qquad \text{and} \qquad f_{1,t}(y) = \sigma\,(1-y) + t.$$

The dynamics of F_t in the central direction is modeled by a system of iterated functions generated by $f_{0,t}$ and $f_{1,t}$. More precisely, given any $X = (x_0^s, x_0^c, x_0^u) \in R$ whose forward orbit is contained in R, we consider its forward itinerary $\iota^+(X)$, with $\iota^+(X) \in \Sigma_{11}^+ \subset \{0,1\}^{\mathbb{N}}$, defined as follows

$$\iota_k^+(X) = j \qquad \text{if} \qquad X_k = F_t^k(X) \in R_j\,.$$

Consider $k \geq 0$ and let $X_k = F_t^k(X) = (x_k^s, x_k^c, x_k^u)$. By the definition of F_t, the central coordinate x_k^c, of X_k is

$$x_k^c = f_{i_{k-1},t} \circ \cdots \circ f_{i_0,t}(x_0^c), \quad \text{where } \iota^+(X) = (i_k)\,.$$

Given a sequence $(i_n) \in \Sigma_{1,1}^+$, for each given $k \geq 0$ we consider the k-block $\varrho_k = \varrho_k(i_n) = [i_0, i_1, \dots, i_k]$ associated to (i_n). To this block we associate the map

$$\Phi_{\varrho_k,t} = f_{i_k,t} \circ f_{i_{k-1},t} \circ \cdots \circ f_{i_0,t}\,,$$

which is defined in some interval (see the discussion below). We now consider the system of iterated functions (s.i.f.) $\mathfrak{F}_t$ generated by $f_{0,t}$ and $f_{1,t}$ defined as follows

$$\mathfrak{F}_t = \{\Phi_{\varrho_k,t} \colon \varrho_k \text{ is a block of size } k \text{ of } \Sigma_{11}^+ \text{ and } k \in \mathbb{N}\}.$$

The goal of this section is to obtain properties for the s.i.f. $\mathfrak{F}_t$ which will be translated into properties of the map F_t. When $t \geq 0$ the interval $[0,1]$ is invariant by $\mathfrak{F}_t$ (i.e., $\Phi_{\varrho_k,t}([0,1]) \subset [0,1]$). On the other hand, for $t < 0$ the interval $[0,1]$ is not anymore invariant. We will see that for $t < 0$, for each block ϱ_k, there is a maximal interval I_{ϱ_k} where $\Phi_{\varrho_k,t}$ is defined. This is one of the reason because the analysis of the dynamics of $\mathfrak{F}_t$ is different for $t \geq 0$ and $t < 0$.

In Section 3.1, we obtain in Proposition 3.1 a contraction property for the orbits of $\mathfrak{F}_t$ for $t > 0$ (except for the fixed point 0 corresponding to Q). In Section 5, we see that this property implies the hyperbolicity of the limit set of F_t: this set consists of the saddle Q and the homoclinic class of P. The contracting property of $\mathfrak{F}_t$ implies that the direction of the y-axis is a contracting direction of the homoclinic class of P. Since the direction of the x-axis is uniformly contracting and the direction of the z-axis is uniformly expanding we obtain the hyperbolicity of the homoclinic class of P for $t > 0$.

In Section 3.2, we prove that, for $t = 0$, all periodic points of $\mathfrak{F}_0$ are hyperbolic (in fact, any periodic point different from 0 is contracting). However, this hyperbolicity is not uniform.

In Section 3.3, for t close to 0, we get a subset S of the orbit of 1 by $\mathfrak{F}_t$ which is dense in $[t,1]$, see Proposition 3.7. We will see that the subset S corresponds to central coordinates of transverse homoclinic points of P (thus points in $H(P,F_t)$). This is the key step to show that $H(P, F_0)$ contains infinitely many central segments (see Section 6). This is also a key step to prove that for small $t \leq 0$ the homoclinic class of Q is contained in the one of P (see Section 7.1).

Finally, we prove, in Section 3.4, that, for $t < 0$, the s.i.f. $\mathfrak{F}_t$ satisfies an expanding property (see Proposition 3.15). We will see, in Section 7.2, how this property implies that the homoclinic class of P is contained in the homoclinic class of Q. Therefore $H(P, F_t) = H(Q, F_t)$, for every small $t < 0$.

3.1 Hyperbolic systems of iterated functions ($t > 0$)

In this section, for $t > 0$, we study the dynamics of the s.i.f. $\mathfrak{F}_t$ in the interval. We begin by observing that, for small $t > 0$, the interval $[t, 1]$ is invariant by $\mathfrak{F}_t$: $f_{0,t}([t, 1]) \subset [t, 1]$ and $f_{1,t}([t, 1]) \subset [t, 1]$, therefore $\Phi_{\varrho_k,t}([t, 1]) \subset [t, 1]$ for every block ϱ_k of Σ_{11}^+.

The main technical result of this section is the following:

Proposition 3.1 (Contraction property)**.** *Consider $t > 0$. For every $y \in [t, 1]$ and every sequence (i_n) of Σ_{11}^+ it holds that*

$$\lim_{k\to\infty} |\Phi'_{\varrho_k,t}(y)| = 0, \quad \textit{where } \varrho_k = \varrho_k(i_n).$$

We say that a point p is a *periodic point* of the system $\mathfrak{F}_t$ if there is a block ϱ_k of Σ_{11}^+ such that $\Phi_{\varrho_k,t}(p) = p$ and the concatenation of ϱ_k with itself is a block of Σ_{11}^+ (in other words, the block ϱ_k either starts or ends by 0). In this case, we say that ϱ_k is a *periodic block* of the periodic point p. Note that a point p may be periodic for different blocks. We say that p is a *contracting periodic point* for the s.i.f. $\mathfrak{F}_t$ if there are constants $C = C(p) > 0$ and $\varkappa = \varkappa(p) < 1$ such that $\Phi'_{\varrho_k,t}(p) \leq C\varkappa^k$. We say that the periodic points of $\mathfrak{F}_t$ are *uniformly contracting* if the constants C and $\varkappa$ can be taken the same for all periodic points. As an immediate consequence of the proof of Proposition 3.1 we will get the following:

Corollary 3.2. *Let $t > 0$, then the periodic points $p \in [t, 1]$ of the s.i.f. $\mathfrak{F}_t$ are uniformly contracting.*

Proof of the proposition. Note that this proposition is trivial if the sequence (i_n) contains finitely many 1's (i.e., there is j such that $i_k = 0$ for all $k \geq j$): just note that in this case $\Phi_{\varrho_k,t}(x) \to 1$ as $k \to \infty$, and that $f_{0,t} = f_0 = f$ contracts in a neighborhood of 1. Therefore, from now on, we consider sequences having infinitely many 1's. Moreover, since the sequence (i_n) has no two consecutive 1's, if $i_0 = 1$, we replace y by $f_{1,t}(y)$ and we can assume that the sequence starts with 0.

To each sequence $\iota = (i_n) \in \Sigma_{11}^+$ (with infinitely many 1's), we associate the sequence of positive integers $(\alpha_j(\iota))_{j\in\mathbb{N}}$ defined as follows: first, we consider the indices k_j such that $i_{k_j} = 1$ ($k_j < k_{j+1}$). By convention, we write $k_0 = 0$. We let

$$\alpha_j = \alpha_j(\iota) = k_{j+1} - k_j - 1, \quad j \geq 1.$$

This number is the number of consecutive 0's between two consecutive 1's (corresponding to $i_{k_{j+1}}$ and i_{k_j}). Note that the sequence (α_j) determines the sequence (i_n) and vice versa.

Note that, if the block ϱ_k contains m_k symbols 1's, we have

$$k \geq \sum_{i=0}^{m_k-1} (\alpha_i + 1).$$

We define

$$\alpha(\varrho_k) = k - \sum_{i=0}^{m_k-1} (\alpha_i + 1).$$

Using this notation, if ϱ_k contains m_k entries equal to 1 then

$$\Phi_{\varrho_k,t}(x) = f_0^{\alpha(\varrho_k)} \circ f_{1,t} \circ f_0^{\alpha_{m_k}} \circ \cdots \circ f_{1,t} \circ f_0^{\alpha_2} \circ f_{1,t} \circ f_0^{\alpha_1}(x).$$

Note that if the last entry of ϱ_k is $i_k = 1$ then $\alpha(\varrho_k) = 0$ and

$$\Phi_{\varrho_k,t}(x) = f_{1,t} \circ f_0^{\alpha_{m_k}} \circ \cdots \circ f_{1,t} \circ f_0^{\alpha_2} \circ f_{1,t} \circ f_0^{\alpha_1}(x).$$

We say that $f_{1,t} \circ f_0^{\alpha_i}$, $i = 1, \ldots, m_k$, are the *links* of $\Phi_{\varrho_k,t}$ and that $f_0^{\alpha(\varrho_k)}$ is the *remainder* of $\Phi_{\varrho_k,t}$.

Note that $(f_0^k)'(x) \to 0$ as $k \to \infty$ for $x \in [t, 1]$, $t > 0$, uniformly on x. Thus

$$0 < (f_0^{\alpha(\varrho_k)})'(x) \leq M(t), \quad \text{for all } x \in [t, 1]. \tag{3}$$

Recall that we are considering sequences (i_n) having infinitely many 1's. This implies that if m_k is the number of links of ϱ_k (i.e., the number of 1's) one has $m_k \to \infty$ as $k \to \infty$. We also have $f_{1,t} \circ f_0^{\alpha}(x) < 1$ for every $x \in [0, 1]$, in particular $x = 1$ has to be considered just in the first link, and for the purpose to estimate the limit below, it is irrelevant. Therefore, in view of (3), to prove the proposition, it is enough to see that, for every sequence (α_k) with $\alpha_k \geq 1$, one has that

$$\lim_{k\to\infty} (f_{1,t} \circ f_0^{\alpha_k} \circ \cdots \circ f_{1,t} \circ f_0^{\alpha_2} \circ f_{1,t} \circ f_0^{\alpha_1})'(x) = 0, \quad \text{for all } x \in [t, 1). \tag{4}$$

In order to estimate the derivative of $\Phi_{\varrho_k,t}$, we begin by estimating the derivatives of its links. Thus, we consider segments of orbits consisting of α consecutive iterations by f_0 followed by exactly one iteration by $f_{1,t}$. We now compute the derivative of such a composition. Since $f_{1,t}$ is affine we just need to estimate the derivative of f_0^{α}.

For future uses, see Section 3.2, we also compute some derivatives for $t = 0$.

Lemma 3.3. *Consider $t \geq 0$ and a point $y \in [t, 1)$, $y \neq 0$. Then*

$$|(f_{1,t} \circ f_0^{\alpha})'(y)| = \left(\frac{w}{y\,(1-y)}\right)\left(1 - \frac{w}{\sigma}\right), \quad \textit{where } f_0^{\alpha}(y) = 1 - w/\sigma.$$

Proof. Consider a point $y \in [t, 1)$, $y \neq 0$, and write

$$f_{1,t} \circ f_0^{\alpha}(y) = t + w, \quad \text{where} \quad f_0^{\alpha}(y) = 1 - w/\sigma.$$

Since $f_0^{\alpha}(y) \in [0, 1]$ and $w = 0$ (respectively $w = \sigma$) if, and only if, $y = 1$ (respectively $y = 0$) we have $w \in (0, \sigma)$. It follows from the definition of $f_{1,t}$ that

$$|(f_{1,t} \circ f_0^{\alpha})'(y)| = \sigma\,|(f_0^{\alpha})'(y)|. \tag{5}$$

Equation (2) and the definition of $f_0^{\alpha}(y)$ give the following:

$$|(f_0^{\alpha})'(y)| = \frac{e^{-\alpha}}{y^2}\,(f_0^{\alpha}(y))^2 = \frac{e^{-\alpha}}{y^2}\,\left(1 - \frac{w}{\sigma}\right)^2. \tag{6}$$

Claim 3.4.

$$e^{-\alpha} = \left(\frac{y}{y-1}\right)\left(1 - \frac{1}{1-w/\sigma}\right).$$

Proof. By definition of $f_0^\alpha(y)$ and Equation (1),

$$f_0^\alpha(y) = 1 - \frac{w}{\sigma} = \frac{1}{1-(1-1/y)\,e^{-\alpha}}.$$

Thus,

$$1-(1-1/y)\,e^{-\alpha} = \frac{1}{1-w/\sigma}$$

Then,

$$\frac{y-1}{y}\,e^{-\alpha} = 1 - \frac{1}{1-w/\sigma}.$$

This implies immediately the claim. □

By the claim, replacing $e^{-\alpha}$ in Equation (6), one gets

$$\begin{aligned}|(f_0^\alpha)'(y)| &= \left|\frac{1}{y^2}\left(\frac{y}{y-1}\right)\left(1-\frac{1}{1-w/\sigma}\right)\left(1-\frac{w}{\sigma}\right)^2\right| = \\ &= \left|\left(\frac{1}{y\,(y-1)}\right)\frac{w}{\sigma}\left(\frac{-1}{1-w/\sigma}\right)\left(1-\frac{w}{\sigma}\right)^2\right| = \frac{1}{\sigma}\left(\frac{w}{y\,(1-y)}\right)\left(1-\frac{w}{\sigma}\right).\end{aligned}$$

Hence, using Equation (5), we have

$$|(f_{1,t}\circ f_0^\alpha)'(y)| = \sigma\,|(f_0^\alpha)'(y)| = \left(\frac{w}{y\,(1-y)}\right)\left(1-\frac{w}{\sigma}\right).$$

This concludes the proof of the lemma. □

Recall that in Equation (4) we consider iterations of the form

$$\Phi_{\varrho_\ell,t}(y) = f_{1,t}\circ f_0^{\alpha_k}\circ\cdots\circ f_1\circ f_0^{\alpha_2}\circ f_1\circ f_0^{\alpha_1}(y),$$

for some block ϱ_ℓ ending with a 1 ($\ell = k + \sum_{j=1}^k \alpha_j$). We let $y = t + w_0$ and, for $j = 0,\ldots,k$, define inductively

$$(f_{1,t}\circ f_0^{\alpha_j})\circ\cdots\circ(f_{1,t}\circ f_0^{\alpha_2})(f_{1,t}\circ f_{0,t}^{\alpha_1})(t+w_0) = f_{1,t}\circ f_0^{\alpha_j}(t+w_{j-1}) = t+w_j.$$

Note that $w_i\in(0,\sigma)$ for every i. Lemma 3.3 gives the following estimate:

$$\begin{aligned}|(\Phi_{\varrho_\ell,t})'(t+w_0)| &= \frac{w_1\,(1-w_1/\sigma)}{(t+w_0)\,(1-(t+w_0))}\cdots\frac{w_k\,(1-w_k/\sigma)}{(t+w_{k-1})\,(1-(t+w_{k-1}))} = \\ &= \prod_{i=1}^k \frac{w_i\,(1-w_i/\sigma)}{(t+w_{i-1})\,(1-(t+w_{i-1}))}.\end{aligned}$$

Note that, rearranging the quotients, we have

$$|(\Phi_{\varrho_\ell,t})'(t+w_0)| = \left(\prod_{i=1}^{k-1}\frac{w_i\,(1-w_i/\sigma)}{(t+w_i)\,(1-(t+w_i))}\right)\frac{w_k\,(1-w_k/\sigma)}{(t+w_0)\,(1-(t+w_0))}. \tag{7}$$

We need the following claim, whose proof we postpone to the end of the proof of the proposition.

Claim 3.5. *For every $t \geq 0$ small enough and $w \in (0, \sigma)$ it holds that*

$$\frac{w\,(1 - w/\sigma)}{(t + w)\,(1 - (t + w))} \leq (1 - t).$$

Observe that, by Claim 3.5,

$$\prod_{i=1}^{k-1} \frac{w_i\,(1 - w_i/\sigma)}{(t + w_i)\,(1 - (t + w_i))} \leq (1 - t)^{k-1}. \tag{8}$$

Write now

$$C(t, w_0) = \frac{1}{(t + w_0)\,(1 - (t + w_0))}.$$

Note that, if $t > 0$ then $C(t, w_0)$ is upper bounded by some constant $C(t)$ independent of w_0; and for $t = 0$, $C(0, w_0)$ is upper bounded by a constant $C(w_0)$ that only depends on the initial point $w_0 = y \neq 0$. In both cases, these constants do not depend on k. The inequality in (8) and Equation (7) (recall that $w_k\,(1 - w_k/\sigma) \in (0, 1)$) imply that for the block ϱ_ℓ above with k links one has

$$|(\Phi_{\varrho_\ell, t})'(t + w_0)| \leq (1 - t)^{k-1}\, C(t, w_0). \tag{9}$$

This immediately implies Equation (4) for $t > 0$ and every $x \in [t, 1)$.

We now prove the claim.

Proof of Claim 3.5. The inequality in the claim can be written in the form

$$\begin{aligned} w - \frac{w^2}{\sigma} &\leq \left((t + w) - (t + w)^2\right)\,(1 - t) = \\ &= (t + w) - (t + w)^2 - t\,(t + w) + t\,(t + w)^2. \end{aligned}$$

Since $t\,(t + w)^2 \geq 0$, to get this inequality it is enough to check that

$$-\frac{w^2}{\sigma} \leq t - t^2 - 2\,t\,w - w^2 - t^2 - t\,w,$$

that is,

$$w\,\left(3\,t + w - \frac{w}{\sigma}\right) \leq t\,(1 - 2\,t) \quad \text{or equivalently} \quad w\,\left(3\,t - \left(\frac{1}{\sigma} - 1\right)\,w\right) \leq t\,(1 - 2\,t).$$

First, if $w \in [1/4, 1]$ and $t \geq 0$ is small enough, the left-hand side of the last inequality is negative while the right-hand side is non-negative. Thus, in this case, the inequality is obvious. For $w \in [0, 1/4]$, if t is small, then $w < (1 - 2\,t)/3$, therefore, as $\sigma^{-1} > 1$,

$$w\,\left(3\,t - \left(\frac{1}{\sigma} - 1\right)\,w\right) \leq 3\,t\,w < t\,(1 - 2\,t).$$

The proof of the claim is complete. □

The proof of the proposition is now complete. □

3.2 Hyperbolicity of periodic points (t=0)

As a consequence of the proof of Proposition 3.1, we now obtain in Proposition 3.6 that all periodic points $y \in (0,1]$ of the s.i.f. $\mathfrak{F}_0$ are contracting. Moreover, note that 0 is a periodic (fixed) point of $\mathfrak{F}_0$ associated to the periodic block $\varrho_1 = [0]$. The fact that the origin is an expanding fixed point for f_0 implies that the 0 is an expanding fixed point (i.e. $\Phi'_{\varrho_1,0}(0) > 1$) for $\mathfrak{F}_0$. Thus, all periodic points of $\mathfrak{F}_0$ are hyperbolic.

Proposition 3.6. *Let $y \in (0,1]$ be a periodic point of $\mathfrak{F}_0$ and ϱ_k a periodic block of it (i.e., $\Phi_{\varrho_k,0}(y) = y$). Then*

$$|(\Phi_{\varrho_k,0})'(y)| < 1.$$

Proof. If the block ϱ_k has only 0's it is immediate to check that $y = 1$. In this case, the hyperbolicity of y is obvious. Therefore, we can assume that the block ϱ_k contains some 1 and some 0. This means that changing the beginning of the orbit, we can assume that $y \in (0,\sigma]$ and that the block is of the form $[0,\dots,1]$ (recall that, by definition of periodic point of $\mathfrak{F}_0$, a periodic block ϱ_k associated to y ending by 1 it necessarily starts by a 0).

Write $y = w_0$ and assume that the block ϱ_k has ℓ links. Using the notation above, we have $w_\ell = w_0 = y$. For $t = 0$, Equation (7) becomes

$$|(\Phi_{\varrho_\ell,0})'(0 + w_0)| = \left(\prod_{i=1}^{\ell-1} \frac{w_i\,(1 - w_i/\sigma)}{w_i\,(1 - w_i)}\right) \frac{w_\ell\,(1 - w_\ell/\sigma)}{w_0\,(1 - w_0)} = \prod_{i=0}^{\ell-1} \frac{1 - w_i/\sigma}{1 - w_i} < 1.$$

This completes the proof of the proposition. □

3.3 Dense orbits for $\mathfrak{F}_t$ ($|t|$ close to 0)

In Sections 3.1 and 3.2, we consider $t \geq 0$, in such cases the interval $[t,1]$ is invariant by $\mathfrak{F}_t$. Now we also consider negative t and then the interval $[t,1]$ is not invariant for $\mathfrak{F}_t$. In what follows, we just consider points and blocks whose corresponding orbits remain in $[t,1]$, that is, points $y \in [t,1]$ and blocks $\varrho_k = [i_1, i_2, \dots, i_k]$ of Σ_{11}^+ such that $\Phi_{\varrho_j,t}(y) \in [t,1]$, for all $\varrho_j = [i_1,\dots,i_j]$ and $j \leq k$. In this case, we say that ϱ_k is an *admissible block* of y. The $\mathfrak{F}_t$-*orbit* of y is the set of points $\Phi_{\varrho_k,t}(y)$ where ϱ_k is an admissible block of y.

Consider small t and define

$$\sigma_t = f_{1,t} \circ f_0 \circ f_{1,t}(1).$$

Note that $\sigma_t \to \sigma$ as $t \to 0$ and $\sigma_0 = \sigma$. Thus $\sigma_t \in (0,1)$ if $|t|$ is small enough. The goal of this section is to prove the following proposition.

Proposition 3.7. *The orbit of σ_t by the system $\mathfrak{F}_t$ is dense in $[t,1]$, for every $|t|$ small enough.*

The proof of this proposition consists of several steps. We first get an abstract result about families of sequences which guarantees that the closure of their points contain an interval (see Proposition 3.8 in Section 3.3.1). Next, in Section 3.3.2, we obtain in Proposition 3.12 properties about the maps of the system $\mathfrak{F}_t$ which allows us to construct sequences contained in $\mathfrak{F}_t$-orbit of σ_t verifying the hypotheses of Proposition 3.8. Using this fact, we conclude the proof of Proposition 3.7 in Section 3.3.3.

3.3.1 Multisequences

We need some notations. We denote by $[\mathfrak{b}]_k$ a k-tuple of natural numbers $[\mathfrak{b}]_k = i_1, \dots, i_k$ and by $[\mathfrak{b}]_k, j$ the $(k+1)$-tuple $i_1, \dots, i_k, j$. We also denote by $[\mathbf{0}]_k$ the k-tuple of k consecutive 0's. Finally, $[\mathfrak{b}]_0$ and $[\mathbf{0}]_0$ denote empty tuples[1].

Proposition 3.8. *Consider a strictly increasing sequence of real numbers $(x_k)_{k\geq 0}$ converging to some x^+. For every $m \in \{1, 2, \dots\}$ and every m-tuple $[\mathfrak{b}]_m = i_1, i_2, \dots, i_m$ with $i_j \in \mathbb{N} = \{0, 1, \dots\}$ for all $j = 1, \dots, m$, consider a strictly increasing sequence of real numbers*

$$(x_{[\mathfrak{b}]_m,k})_{k\geq 0} = (x_{i_1,i_2,\dots,i_m,k})_{k\geq 0}, \quad x_{[\mathfrak{b}]_m,k} \in \mathbb{R},$$

such that:

- **P1) (Convergence)** *For every $m \geq 1$ and for every m-tuple $[\mathfrak{b}]_m$, one has that $x_{[\mathfrak{b}]_m,k} \to x_{[\mathfrak{b}]_m}$ as $k \to \infty$.*
- **P2) (Contraction)** *There are constants $C > 0$ and $\lambda \in (0, 1)$ such that*

$$\mathrm{diam}((x_{[\mathfrak{b}]_m,k})_k) = x_{[\mathfrak{b}]_m} - x_{[\mathfrak{b}]_m,0} \leq \lambda^m C.$$

- **P3) (Overlapping)** *There is $r \in \{1, 2, \dots\}$ such that, for every $m \geq 0$, every $h \geq 1$, and every m-tuple $[\mathfrak{b}]_m$, it holds*

$$x_{[\mathfrak{b}]_m,h,[\mathbf{0}]_r} < x_{[\mathfrak{b}]_m,(h-1)} \,.$$

Let

$$x^- = \lim_{k\to\infty} x_{[\mathbf{0}]_k}$$

and

$$S = \bigcup_{m\geq 1} S_m, \quad \textit{where} \quad S_m = \{x_{[\mathfrak{b}]_m} = x_{i_1,i_2,\dots,i_m} : (i_1, i_2, \dots, i_m) \in \mathbb{N}^m\}.$$

Then the set S is dense in the interval $[x^-, x^+]$.

A simpler form of Proposition 3.8 with $r = 1$ was proved in [16, Lemma 4.1].

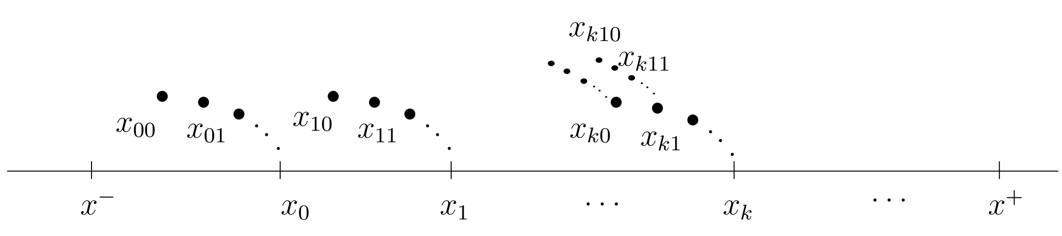

Figure 9: The multisequences in Proposition 3.8

Using the monotonicity of the sequences $(x_{[\mathfrak{b}]_m,k})_k$, it is immediate to check that the interval $[x^-, x^+]$ is the convex hull of the set S.

We say that the sequences $(x_{[\mathfrak{b}]_m,k})_k$ are sequences of *m-th generation*. The sequence $(x_k)_k$ is the sequence of generation zero.

[1]We use different notations for k-tuples (subsets of $\mathbb{N}^k$) and for k-blocks (which are blocks of Σ_{11}^+).

Proof of Proposition 3.8. Consider any point $y \in [x_0^-, x^+]$. We must construct a sequence of points z_k of S converging to y. The result is obvious if $y = x^{\pm}$. Thus we can focus on points in the interval (x^-, x^+). We need the following preparatory lemma:

Lemma 3.9. *Under the assumptions of Proposition 3.8, for every $m \geq 0$, m-tuple $[\mathfrak{b}]_m$, $h \in \mathbb{N}$, $x_{[\mathfrak{b}]_m}$, and w with*

$$x_{[\mathfrak{b}]_m,h} \leq w < x_{[\mathfrak{b}]_m,(h+1)}$$

there are $k \geq 0$ and $j \geq 0$ such that

$$x_{[\mathfrak{b}]_m,(h+1),[\mathbf{0}]_k,j} \leq w < x_{[\mathfrak{b}]_m,(h+1),[\mathbf{0}]_k,(j+1)}.$$

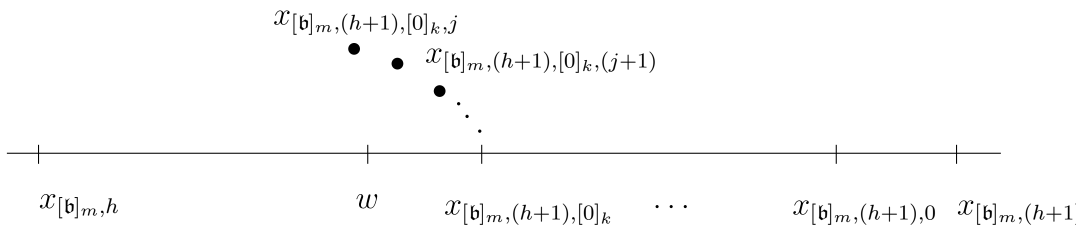

Figure 10: Lemma 3.9

Proof. Let r be as in (P3) and consider the points

$$x_{[\mathfrak{b}]_m,(h+1),[\mathbf{0}]_r} < x_{[\mathfrak{b}]_m,(h+1),[\mathbf{0}]_{r-1}} < \cdots < x_{[\mathfrak{b}]_m,(h+1),0} < x_{[\mathfrak{b}]_m,(h+1)}.$$

By hypothesis,

$$x_{[\mathfrak{b}]_m,(h+1),[\mathbf{0}]_r} < x_{[\mathfrak{b}]_m,h} < w < x_{[\mathfrak{b}]_m,(h+1)}.$$

Thus there is k, $0 < k \leq r$, with

$$x_{[\mathfrak{b}]_m,(h+1),[\mathbf{0}]_{k+1}} \leq w < x_{[\mathfrak{b}]_m,(h+1),[\mathbf{0}]_k}.$$

By (P1), one has

$$x_{[\mathfrak{b}]_m,(h+1),[\mathbf{0}]_k,n} \longrightarrow x_{[\mathfrak{b}]_m,(h+1),[\mathbf{0}]_k},$$

thus there is some $j \geq 0$ such that

$$x_{[\mathfrak{b}]_m,(h+1),[\mathbf{0}]_k,j} \leq w < x_{[\mathfrak{b}]_m,(h+1),[\mathbf{0}]_k,(j+1)}.$$

This completes the proof of the lemma. □

Lemma 3.10. *Consider any $y \in (x^-, x^+)$ such that $y \notin S$. Then there are sequences $(z_n^-)_n$ and $(z_n^+)_n$ contained in S such that*

1. *$z_n^- < y < z_n^+$;*
2. *$z_n^- = x_{[\mathfrak{b}_n]_{m_n},h_n}$ and $z_n^+ = x_{[\mathfrak{b}_n]_{m_n},(h_n+1)}$, for some (m_n)-tuple $\mathfrak{b}_n$ and $h_n \in \mathbb{N}$; and*
3. *$m_1 \geq 0$ and $m_{(n+1)} > m_n$, thus $m_n \geq n-1$.*

This lemma implies Proposition 3.8. To prove that S is dense in (x^-, x^+) it is enough to see that any point in $(x^-, x^+) \setminus S$ is accumulated by points in S. Thus we can assume that $y \neq x_{[\mathbf{0}]_k}$, for all k. Recall that $m_n \geq n-1$ and that, by Property (P2), every sequence $(x_{[\mathfrak{b}]_m,k})_k$ of generation m has diameter less than $\lambda^m C$. Thus

$$z_n^+ - z_n^- \leq \lambda^{m_n} C.$$

This immediately implies that $z_n^\pm \to y$, proving the proposition.

Proof. The proof of the lemma is by induction. Consider the decreasing sequence of points $(x_{[\mathbf{0}]_k})_k$, $x_{[\mathbf{0}]_k} \to x^-$, and the sequence of intervals $I_k = (x_{[\mathbf{0}]_k}, x_{[\mathbf{0}]_{k-1}})$, $k \geq 1$, where we let $x_{[\mathbf{0}]_0} = x^+$. Clearly, since $y \neq x_{[\mathbf{0}]_k}$ for all k, we have that $y \in I_k$, for some $k \geq 0$. By condition (P1), $x_{[\mathbf{0}]_{k-1},j} \to x_{[\mathbf{0}]_{k-1}}$ and since the sequence $(x_{[\mathbf{0}]_{k-1},j})_j$ is increasing and $y \notin S$, there is $j \geq 0$ such that

$$z_1^- = x_{[\mathbf{0}]_{k-1},j} < y < x_{[\mathbf{0}]_{k-1},(j+1)} = z_1^+.$$

We let $m_1 = k \geq 1$, $[\mathfrak{b}_1]_{m_1} = [\mathbf{0}]_{k-1}$, and $h_1 = j$.

Suppose now inductively defined the terms $z_1^\pm, \ldots, z_n^\pm$ (thus we have defined the numbers $m_1, \ldots, m_n$, the blocks $[\mathfrak{b}_1]_{m_1}, \ldots, [\mathfrak{b}_n]_{m_n}$ and numbers $h_1, \ldots, h_n$) satisfying the conclusions in the lemma. To define $z_{n+1}^\pm$ we apply Lemma 3.9 to the points y, $z_n^- = x_{[\mathfrak{b}_n]_{m_n},h_n}$ and $z_n^+ = x_{[\mathfrak{b}_n]_{m_n},(h_n+1)}$,

$$x_{[\mathfrak{b}_n]_{m_n},(h_n+1),[\mathbf{0}]_{\ell-1},j} < y < x_{[\mathfrak{b}_n]_{m_n},(h_n+1),[\mathbf{0}]_{\ell-1},(j+1)}.$$

It is enough to take

$$m_{n+1} = m_n + \ell\,, \quad [\mathfrak{b}_{n+1}]_{m_{n+1}} = [\mathfrak{b}_n]_{m_n}, (h_n+1), [\mathbf{0}]_{\ell-1}\,, \quad \text{and} \quad h_{n+1} = j\,.$$

This completes the proof of the lemma. □

The proof of Proposition 3.8 is now complete. □

3.3.2 Generation of multisequences

We now construct the multisequences to which we apply Proposition 3.8. Recall that, for small t, we define

$$\sigma_t = f_{1,t} \circ f_0 \circ f_{1,t}(1) \in (0,1).$$

A natural choice for the first sequence $(x_k)_k$ is to consider the increasing sequence $(f_0^k(\sigma_t))_k$, converging to $x^+ = 1$. In this way, the next generation of sequences is defined by taking iterates of the first one by the maps $f_0^n \circ f_{1,t} \circ f_0 \circ f_{1,t}$ with $n \geq 1$. Note that, since $f_0^k(\sigma_t) \to 1$, fixed n, we have

$$x_{n,k} = f_0^n \circ f_{1,t} \circ f_0 \circ f_{1,t}(x_k) \to f_0^n(\sigma_t) = x_n, \quad \text{as } k \to \infty.$$

This gives the sequences of second generation. Note that the map $f_{1,t} \circ f_0 \circ f_{1,t}$ preserves the orientation, and, since the sequence $(x_k)_k$ is increasing, its images are also increasing.

In order to have the contraction in (P2), we need to consider *large* k, considering iterates of f_0 close to 1, thus localized in a contracting region. In this way, we must truncate the sequence above and consider only its tail $f_0^k(\sigma_t)$, $k \geq k_0$, for some k_0. In this way, for large k_0, we get the desired contraction properties for the sequences. Unfortunately, this truncation has some side-effects, doing this we may be losing the overlapping condition in (P3) for the next generation of sequence. That is the main reason because in our construction we must consider r bigger than 1 in Proposition 3.8: the overlapping occurs after some generations. Let us now go into the details of our construction. We first introduce some notations.

Consider the fundamental domains of f_0

$$D_0(\sigma_t) = [f_0^{-1}(\sigma_t), \sigma_t] \quad \text{and} \quad D_1(\sigma_t) = [f_0^{-2}(\sigma_t), f_0^{-1}(\sigma_t)],$$

and the interval

$$D(\sigma_t) = D_1(\sigma_t) \cup D_0(\sigma_t) = [f_0^{-2}(\sigma_t), \sigma_t].$$

Given a positive integer k, we define the block

$$[\mathfrak{v}]_k := [\mathbf{0}]_k \,, 1, 0, 1 \,,$$

that is, $[\mathfrak{v}]_k$ is the concatenation of the blocks $[1, 0, 1]$ and $[\mathbf{0}]_k$.

Note that the blocks $[\mathfrak{v}]_k$ and $[\mathbf{0}]_k$ are blocks of Σ_{11}^+ and that the concatenation of two consecutive blocks $[\mathfrak{v}]_k$ and $[\mathfrak{v}]_r$ with k and $r > 0$ is also a block of Σ_{11}^+.

We denote by $[\mathfrak{v}]_{i_1,i_2,\dots,i_m}$ the concatenation of the blocks $[\mathfrak{v}]_{i_1}, [\mathfrak{v}]_{i_2}, \dots [\mathfrak{v}]_{i_m}$,

$$[\mathfrak{v}]_{i_1,i_2,\dots,i_m} = [\mathbf{0}]_{i_1} \,, 1, 0, 1, [\mathbf{0}]_{i_2} \,, 1, 0, 1, \dots, [\mathbf{0}]_{i_m} \,, 1, 0, 1 \,.$$

Clearly, the block $[\mathfrak{v}]_{i_1,i_2,\dots,i_m}$ is a block of Σ_{11}.

For each $k \geq 1$ and t close to 0, consider the map

$$\Phi_{[\mathfrak{v}]_k,t} \colon D(\sigma_t) \to \mathbb{R}, \quad \Phi_{[\mathfrak{v}]_k,t}(x) = f_{1,t} \circ f_0 \circ f_{1,t} \circ f_0^k(x).$$

Remark 3.11. Every map $\Phi_{[\mathfrak{v}]_k,t}$, $k \geq 0$, preserves the natural orientation in $\mathbb{R}$: just note that $f_{1,t}$ reverses the orientation (and it is applied twice) and f_0 preserves the orientation. Thus $\Phi_{[\mathfrak{v}]_k,t}$ maps strictly increasing sequences to strictly increasing sequences.

Next proposition is the key ingredient to obtain sequences of points contained in the orbit of σ_t satisfying the conditions in Proposition 3.8.

Proposition 3.12. *There exists a neighborhood (a, b) of $1/4$ such that, for all $\sigma \in (a, b)$, there are $t_0 > 0$, and $\lambda \in (0, 1)$, such that for every $t \in [-t_0, t_0]$ the following holds:*

- **G1)** $\Phi_{[\mathfrak{v}]_{2+i},t}(D(\sigma_t)) \subset D(\sigma_t)$, *for every $(2 + i)$-block $[\mathfrak{v}]_{2+i}$ of Σ_{11}^+ with $i \geq 0$;*
- **G2)** $\Phi^8_{[\mathfrak{v}]_2,t}(\sigma_t) \notin D_0(\sigma_t)$, *thus $\Phi^8_{[\mathfrak{v}]_2,t}(\sigma_t)$ is in the interior of $D_1(\sigma_t)$; and*
- **G3)** $0 < \Phi'_{[\mathfrak{v}]_{2+i},t}(x) < \lambda$ *for all $x \in D(\sigma_t)$ and $i \geq 0$.*

For the sake of clearness, the proposition means that, for each σ, we consider the affine maps $f_{1,t}(x) = \sigma\,(1 - x) + t$ and the system $\mathfrak{F}_t$ associated to f_0 and such a $f_{1,t}$, in particular, the maps $\Phi_{[\mathfrak{v}]_{2+i},t}, \Phi_{[\mathfrak{v}]_2,t} \in \mathfrak{F}_t$.

The statement of the proposition is also valid for other choices of blocks of length $m + i$, with $m \neq 2$, in (G1) and (G3), and it also is valid for $\Phi^r_{[\mathfrak{v}]_m,t}(\sigma_t)$ with $r \neq 8$ and $m \neq 2$ in (G2).

Proof. The computations of the iterates of functions were made with the assistance of the software Maple[2]. We just reproduce the final results of these computations. First, let us prove the proposition for $t = 0$ and $\sigma_0 = \sigma = 1/4$. Thus, for notational simplicity, in this proof we omit the parameter t.

It follows easily from the monotonicity of f_0^i and $f_{1,t} \circ f_0 \circ f_{1,t}$ that

$$\Phi_{[\mathfrak{v}]_2}(x) < \Phi_{[\mathfrak{v}]_{2+i}}(x) < 1/4, \quad \text{for all } i = 1, 2, \dots \text{ and } x \in [0, 1). \tag{10}$$

Thus, it is sufficient to prove (G1) considering $i = 0$. In fact, since $\Phi_{[\mathfrak{v}]_2}$ is injective and preserves the orientation it is enough to show that the extremal points of $D(1/4)$ satisfy

$$f_0^{-2}(1/4) \le \Phi_{[\mathfrak{v}]_2}(f_0^{-2}(1/4)) < \Phi_{[\mathfrak{v}]_2}(1/4) < 1/4.$$

As observed above, the last inequality is satisfied for all points in $(0, 1)$, in particular for $1/4$. The second inequality follows from the monotonicity of $\Phi_{[\mathfrak{v}]_2}$. For the first inequality, we have

$$\Phi_{[\mathfrak{v}]_2}(f_0^{-2}(1/4)) - f_0^{-2}(1/4) = \frac{13}{4}\,\frac{e^{-1}}{3 + 13\,e^{-1}} - \frac{1}{1 + 3\,e^{2}} > 0. \tag{11}$$

This proves (G1).

To prove (G2), one has to show that

$$f_0^{-2}(1/4) < \Phi^8_{[\mathfrak{v}]_2}(1/4) < f_0^{-1}(1/4). \tag{12}$$

The first inequality follows from the invariance of $D(1/4)$ by $\Phi_{[\mathfrak{v}]_2}$ in (G1), that is,

$$\Phi^8_{[\mathfrak{v}]_2}(1/4) - f_0^{-2}(1/4) > 0.$$

For the second inequality in (12), we have

$$f_0^{-1}(1/4) - \Phi^8_{[\mathfrak{v}]_2}(1/4) \ge 0.4752 - 0.1978 > 0. \tag{13}$$

It remains to show (G3). Applying Equations (1) and (2), and simplifying the result, we find

$$\Phi'_{[\mathfrak{v}]_2}(f_0^{-2}(1/4)) = \frac{(1 + 3\,e^2)^2\,e^{-3}}{(3 + 13\,e^{-1})^2} < 1. \tag{14}$$

Since $\Phi'_{[\mathfrak{v}]_2}(x)$ is a decreasing function, we have the Property (G3) for $i = 0$. A similar argument and the same computations (in fact, for $i > 0$ we add some iterations in the contracting region for f_0) show that (G3) also holds for all $i \in \mathbb{N}$. More precisely, call $f_0^{-2}(1/4) = y$ and note that

$$\Phi'_{[\mathfrak{v}]_{2+i}}(y) = \Phi'_{[\mathbf{0}]_i}(\Phi_{[\mathfrak{v}]_2}(y))\,\Phi'_{[\mathfrak{v}]_2}(y)$$

and that $\Phi_{[\mathfrak{v}]_2}(y)$ belongs to a region where $\Phi'_{[\mathfrak{v}]_{2+i}}$ is contracting for all $i \ge 0$. This completes the proof of the proposition for the case $t = 0$.

For $t \neq 0$, first note, that, by monotonicity, the corresponding of equation (10) holds for t close to 0,

$$\Phi_{[\mathfrak{v}]_2,t}(x) < \Phi_{[\mathfrak{v}]_{2+i},t}(x) < \sigma_t, \quad \text{for all } i = 1, 2, \dots \text{ and } x \in [0, 1).$$

[2]This refers to the computations in Equations (11), (13), and (14).

Now, to get condition (G1), we can proceed as in the case $t = 0$ just considering $i = 0$. But this property follows by continuity. Clearly, (G2) follows by continuity. Finally, to get (G3), we apply the arguments in the proof for $t = 0$ and it is enough to prove (G3) for $i = 0$: for $i > 0$ we add some iterations in the contracting region. Finally, condition (G3) for $i = 0$, follows by continuity. The proof of the proposition is now complete. $\square$

Proposition 3.13. *Consider σ and t_0 as in Proposition 3.12. For every $t \in [-t_0, t_0]$, given a block $[\mathfrak{b}]_n = i_1, \dots, i_n$, with $i_j \geq 0$, let*

$$x_{[\mathfrak{b}]_n}(t) = \Phi_{[\mathfrak{v}]_{2+i_1},t} \circ \cdots \circ \Phi_{[\mathfrak{v}]_{2+i_n},t}(\sigma_t).$$

Then the set

$$S(t) = \bigcup_{n\geq 1} S_n(t), \quad \textit{where} \quad S_n(t) = \{x_{[\mathfrak{b}]_n}(t) : [\mathfrak{b}]_n \in \mathbb{N}^n\}$$

is dense in $D_0(\sigma_t)$.

Remark 3.14. By construction, the points $x_{[\mathfrak{b}]_n}(t)$ are in the orbit of 1 (or of σ_t) for the system $\mathfrak{F}_t$.

Proof. To prove the proposition it is enough to see that the family of sequences $(x_{[\mathfrak{b}]_n,k}(t))_k$ satisfies the hypotheses in Proposition 3.8 with the constant r in (P3) taken as $r = 8$.

The proof of this proposition follows by using the properties of the system $\mathfrak{F}_t$ obtained in Proposition 3.12 which hold for all small $|t|$. Therefore, for notational simplicity, we assume that $t = 0$ and omit the parameter t. The proof for other values of t is identical.

Note that the points $x_{i,[\mathfrak{b}]_n}$ satisfy the following inductive definition rule,

$$x_{i,[\mathfrak{b}]_n} = \Phi_{[\mathfrak{v}]_{2+i}}(x_{[\mathfrak{b}]_n}). \tag{15}$$

That is, if we consider blocks $[\mathfrak{b}]_n = i_1, \dots, i_n$ and $[\mathfrak{b}]_{n+1} = i, i_1, \dots, i_n$, one has

$$x_{[\mathfrak{b}]_{n+1}} = \Phi_{[\mathfrak{v}]_{2+i}}(x_{[\mathfrak{b}]_n}).$$

This definition rule and condition (G1) in Proposition 3.12 imply that the points $x_{[\mathfrak{b}]_n}$ and $x_{i,[\mathfrak{b}]_n}$ are well defined and belong to $D(\sigma)$ for all $i \geq 0$ and every block $[\mathfrak{b}]_n = i_1, \dots, i_n$.

The proof of the proposition goes inductively. Note that

$$x_i = \Phi_{[\mathfrak{v}]_{2+i}}(\sigma), \quad i \geq 0.$$

Since $f_0^{2+i}(\sigma) \to 1$ as $i \to \infty$, we have

$$x_i = \Phi_{[\mathfrak{v}]_{2+i}}(\sigma) = f_1 \circ f_0 \circ f_1 \circ f_0^{2+i}(\sigma) \to f_1 \circ f_0 \circ f_1(1) = f_1 \circ f_0(0) = f_1(0) = \sigma = x^+.$$

Noting that $(f_0^{2+i}(\sigma))_i$ is a strictly increasing sequence, Remark 3.11 implies that $(x_i)_i$ is a strictly increasing sequence. Thus the zero generation sequence $(x_i)_i$ satisfies Proposition 3.8 (note that $x^+ = \sigma$).

We now prove inductively that the sequences satisfy conditions (P1) and (P2) in Proposition 3.8. Since $x_n \to \sigma = x^+$, the definition rule implies that, fixed i,

$$x_{i,n} = \Phi_{[\mathfrak{v}]_{2+i}}(x_n) \to \Phi_{[\mathfrak{v}]_{2+i}}(\sigma) = x_i, \quad \text{as } n \to \infty.$$

Again by Remark 3.11, every sequence $(x_{i,n})_n$ is strictly increasing. Thus the sequences of generation one satisfy (P1).

Condition (G3) and $x_n \in D(\sigma)$ for all n imply that

$$\operatorname{diam}(x_{i,n})_n \leq \lambda \operatorname{diam}((x_n)_n) < \lambda < 1,$$

where λ is as in Proposition 3.12. Thus the sequences of generation one satisfy (P2).

Assume now defined, for every $\ell \leq n$ and every ℓ-tuple $[\mathfrak{b}]_\ell$, the sequences $(x_{[\mathfrak{b}]_\ell,k})_k$ of ℓ-th generation satisfying (P1) and (P2):

(P1)$_\ell$ $(x_{[\mathfrak{b}]_\ell,k})_k$ is a strictly increasing sequence which converges to $x_{[\mathfrak{b}]_\ell}$; and

(P2)$_\ell$ $\operatorname{diam}(x_{[\mathfrak{b}]_\ell,k}) \leq \lambda^\ell$.

We claim that the sequences $(x_{i,[\mathfrak{b}]_n,k})_k$ of generation $n+1$,

$$x_{i,[\mathfrak{b}]_n,k} = \Phi_{[\mathfrak{v}]_{2+i}}(x_{[\mathfrak{b}]_n,k}),$$

satisfy:

(P1)$_{n+1}$ $(x_{i,[\mathfrak{b}]_n,k})_k$ is a strictly increasing sequence converging to $x_{i,[\mathfrak{b}]_n}$.

Note that, by (P1)$_n$, $x_{[\mathfrak{b}]_n,k} \to x_{[\mathfrak{b}]_n}$. Thus,

$$x_{i,[\mathfrak{b}]_n,k} = \Phi_{[\mathfrak{v}]_{2+i}}(x_{[\mathfrak{b}]_n,k}) \to \Phi_{[\mathfrak{v}]_{2+i}}(x_{[\mathfrak{b}]_n}) = x_{i,[\mathfrak{b}]_n}.$$

Finally, since $\Phi_{[\mathfrak{v}]_{2+i}}$ preserves the orientation and $(x_{[\mathfrak{b}]_n,k})_k$ is strictly increasing, the sequence $(x_{i,[\mathfrak{b}]_n,k})_k$ is also increasing.

(P2)$_{n+1}$ $\operatorname{diam}((x_{i,[\mathfrak{b}]_n,k})_k) \leq \lambda^{n+1}$.

Note that $x_{[\mathfrak{b}]_n,k} \in D(\sigma)$, recall condition (G3) in Proposition 3.12, and condition (P2)$_n$. By the definition of $x_{i,[\mathfrak{b}]_n,k}$ one has

$$\begin{aligned}\operatorname{diam}((x_{i,[\mathfrak{b}]_n,k})_k) &= \operatorname{diam}((\Phi_{[\mathfrak{v}]_{2+i}}(x_{[\mathfrak{b}]_n,k}))_k) \leq \\ &\leq \lambda \operatorname{diam}((x_{[\mathfrak{b}]_n,k})_k) \leq \lambda\,\lambda^n = \lambda^{n+1}.\end{aligned}$$

This implies that the sequences $(x_{i,[\mathfrak{b}]_n,k})_k$ satisfy (P1) and (P2) in Proposition 3.8. It remains to verify that these sequences also satisfy (P3) (overlapping condition).

Hypothesis (P3) also follows by induction. Bearing in mind the definition of the sequences in (15) and recalling that $x_{[\mathbf{0}]_0} = x^+ = \sigma$, condition (G2) in Proposition 3.12 implies that

$$x_{[\mathbf{0}]_8} = \Phi_{[\mathfrak{v}]_2}(x_{[\mathbf{0}]_{8-1}}) = \Phi^8_{[\mathfrak{v}]_2}(\sigma) < f_0^{-1}(\sigma).$$

Therefore, since $\Phi_{[\mathfrak{v}]_{2+j}}$ preserves the orientation, for all $j \geq 1$, one has

$$x_{j,[\mathbf{0}]_8} = \Phi_{[\mathfrak{v}]_{2+j}}(x_{[\mathbf{0}]_8}) < \Phi_{[\mathfrak{v}]_{2+j}}(f_0^{-1}(\sigma)) = \Phi_{[\mathfrak{v}]_{2+j-1}}(\sigma) = x_{j-1}.$$

This completes the claim for sequences of generation zero.

We now argue inductively, assume that (P3) holds for every sequence of generation n: for each block $[\mathfrak{b}]_n$ of length n and $j \geq 1$ one has

$$x_{[\mathfrak{b}]_n,j,[\mathbf{0}]_8} < x_{[\mathfrak{b}]_n,(j-1)}.$$

To see that (P3) holds for sequences of generation $(n+1)$ recall that $\Phi_{[\mathfrak{v}]_{2+i}}$ preserves the orientation, thus

$$x_{i,[\mathfrak{b}]_n,j,[\mathbf{0}]_8} = \Phi_{[\mathfrak{v}]_{2+i}}(x_{[\mathfrak{b}]_n,j,[\mathbf{0}]_8}) < \Phi_{[\mathfrak{v}]_{2+i}}(x_{[\mathfrak{b}]_n,(j-1)}) = x_{i,[\mathfrak{b}]_n,(j-1)}.$$

This completes the construction of the sequences satisfying Proposition 3.8.

It remains to check that the closure of these sequences contains the fundamental domain $D_0(\sigma) = [f_0^{-1}(\sigma), \sigma]$. Proposition 3.8 implies that the closure of these sequences contains the interval $[x^-, x^+]$, where $x^+ = \sigma$ and $x^- = \lim_{k\to\infty} x_{[\mathbf{0}]_k}$. Since the sequence $(x_{[\mathbf{0}]_k})_k$ is decreasing and by (G2) in Proposition 3.12 one has $x_{[\mathbf{0}]_8} < f_0^{-1}(\sigma)$, this ends the proof of the proposition. □

3.3.3 Density of orbits: Proof of Proposition 3.7

Consider small $|t|$ and the set $S(t)$ defined in Proposition 3.13. Note that, since the blocks considered there do not admit consecutive 1's, and finish with a 0, the set $S(t)$ is contained in the orbit $\mathcal{O}_{\mathfrak{F}_t}(\sigma_t)$ of σ_t by the system $\mathfrak{F}_t$. Proposition 3.13 implies that $S(t)$ is dense in $[f_0^{-1}(\sigma_t), \sigma_t]$. Let

$$\tilde{S}(t) = \bigcup_{n\geq 1} f_0^n(S(t)) \subset \mathcal{O}_{\mathfrak{F}_t}(\sigma_t).$$

Thus the set $\tilde{S}(t)$ is dense in $[\sigma_t, 1]$ (note that $f_0^n(\sigma_t) \to 1$ as $n \to \infty$ and that $[f_0^{-1}(\sigma_t), \sigma_t]$ is a fundamental domain of f_0).

Note that, $f_{1,t}(1) = t$ and that, for t close to 0, $f_{1,t}(\sigma_t) = b_t > 1/16$ if t is close to 0 and σ to $1/4$. This implies that $\hat{S}(t) = f_{1,t}(\tilde{S}(t))$ is dense in the interval $[t, b_t]$. Observe that if t is small then $f_0(t) < b_t$ (this fact is obvious for $t \leq 0$). Thus the interval $[t, b_t]$ contains a fundamental domain of f_0 in $[0,1]$ (in fact, for $t \leq 0$ it contains infinitely many fundamental domains). Considering now the images of $\hat{S}(t)$ by f_0^k, for $k \geq 0$, we get a set which is dense in $[t, 1]$. Finally, by construction, this set is contained in the orbit of σ_t by $\mathfrak{F}_t$. This completes the proof of the proposition. □

3.4 Expanding systems of iterated functions ($t < 0$)

Given small $\vartheta < 0$ and $t_0 < 0$, for $t \in [t_0, 0)$, we consider the restriction of the maps f_0 and $f_{1,t}$ to the interval $[\vartheta, 1]$. Recall that this interval is not invariant for $\mathfrak{F}_t$. Using a notation introduced in the beginning of this section, given a sequence $\iota = (i_n) \in \Sigma_{11}^+$ and $k \geq 0$, we consider the block $\varrho_k = \varrho_k(\iota) = [i_0, i_1, \dots, i_k]$ and the map

$$\Phi_{\varrho_k,t}(x) = f_{i_k,t} \circ f_{i_{k-1},t} \circ \cdots \circ f_{i_1,t} \circ f_{i_0,t}(x).$$

This means that for every sub-block ϱ_ℓ of ϱ_k of the form $\varrho_\ell = [i_0, i_1, \dots, i_\ell]$ the point $\Phi_{\varrho_\ell,t}(x)$ belongs to $[\vartheta, 1]$.

The goal of this section is to prove the following result:

Proposition 3.15. *There is small $t_0 < 0$ such that, for every interval $I \subset (0,1)$ and every $t \in (t_0, 0)$, there are a block $\varrho_k = [i_0, i_1, \dots, i_k]$ of Σ_{11}^+ and a point $x \in I$ such that $\Phi_{\varrho_k,t}(x) = 0$ and $\Phi_{\varrho_\ell,t}(x) \in [t, 1]$ for every sub-block ϱ_ℓ of ϱ_k of the form $\varrho_\ell = [i_0, i_1, \dots, i_\ell]$.*

The main step of the proof of this proposition is to define an expanding return map $\mathcal{R}_t$ in a fundamental domain of f_0 in $[0,1]$ (see Corollary 3.19). This map has discontinuities and it is of the form $\Phi_{\varrho,t}$ (here $\varrho = \varrho_x$ and depends on the point) for some block of Σ_{11}^+. An essential ingredient of the proof is to analyze the $\mathfrak{F}_t$-orbits of the discontinuities of this return map.

By shrinking ϑ and t_0 we can assume that, for all $t \in [t_0, 0]$, one has that

- $f_0([t,0]) \subset [\vartheta, 0]$,
- $f_{1,t}([\vartheta, 0]) = [\sigma + t, \sigma\,(1-\vartheta) + t] \subset [0, 1 + (2\,t\,e)/\sigma]$.

Since $f_{1,t}(1) = t$ and $f_{1,t}$ reverses the orientation, one has that $f_{1,t}(x) \in [t, 0]$ for all $x \in [1 + t/\sigma, 1]$. Hence, from the choice of ϑ and t_0 above,

$$f_0(f_{1,t}(x)) \in [\vartheta, 0], \quad \text{for all } x \in [1 + t/\sigma, 1].$$

Therefore,

$$f_{1,t}(f_0(f_{1,t}(x))) \in [0, 1 + (2\,t\,e)/\sigma], \quad \text{for all } x \in [1 + t/\sigma, 1] \text{ and } t \in [t_0, 0]. \tag{16}$$

Throughout this section we will shrink, if necessary, the size of $t_0 < 0$.

3.4.1 The return map

For $t < 0$, we define $\nu_t > 0$ by the relation

$$f_0\left(1 + \frac{\nu_t\, t}{\sigma}\right) = 1 + \frac{t}{\sigma}. \tag{17}$$

Remark 3.16. Note that $\nu_t \to e$ (the inverse of the derivative of f_0 at 1) as $t \to 0$.

We consider the fundamental domain $D_t = [1 + \nu_t\, t/\sigma, 1 + t/\sigma)$ of f_0. Note that, by definition,

$$f_{1,t}\left(1 + \frac{t}{\sigma}\right) = 0. \tag{18}$$

Therefore, since $f_{1,t}(x) \geq f_{1,t}(1 + t/\sigma)$ for every $x \in D_t$, one has $f_{1,t}(x) \in (0,1)$ for all $x \in D_t$. Moreover, if t is small enough and $x \in D_t$ then $f_{1,t}(x) < 1 + (\nu_t\, t)/\sigma$. Hence there is (exactly one) $n(x) > 0$ (the *return time of* x) such that

$$f_0^{n(x)}(f_{1,t}(x)) \in D_t.$$

We now define a *return map* $\mathcal{R}_t$ on D_t as follows,

$$\mathcal{R}_t \colon D_t \to D_t, \quad \mathcal{R}_t(x) = f_0^{n(x)} \circ f_{1,t}(x).$$

Note that $f_{1,t}(D_t) = (0, t - \nu_t\, t] = (0, (\nu_t - 1)\,|t|\,]$. Thus $f_{1,t}(D_t)$ contains infinitely many fundamental domains of f_0.

We consider the subset $\mathcal{D}$ of D_t of points d such that $\mathcal{R}_t(d) = 1 + (\nu_t\, t)/\sigma$. We have that $\mathcal{D} = \{d_1, d_2, \dots\}$, where $(d_i)_{i=1}^{\infty}$ is an increasing sequence with $d_i \to 1 + t/\sigma$. We say that $\mathcal{D}$ is the *discontinuity set of* $\mathcal{R}_t$.

We let $d_0 = 1 + \nu_t\, t/\sigma$ and consider the partition of D_t given by the intervals $D_{i,t} = (d_{i-1}, d_i]$, $i \geq 1$. It is immediate to check the following:

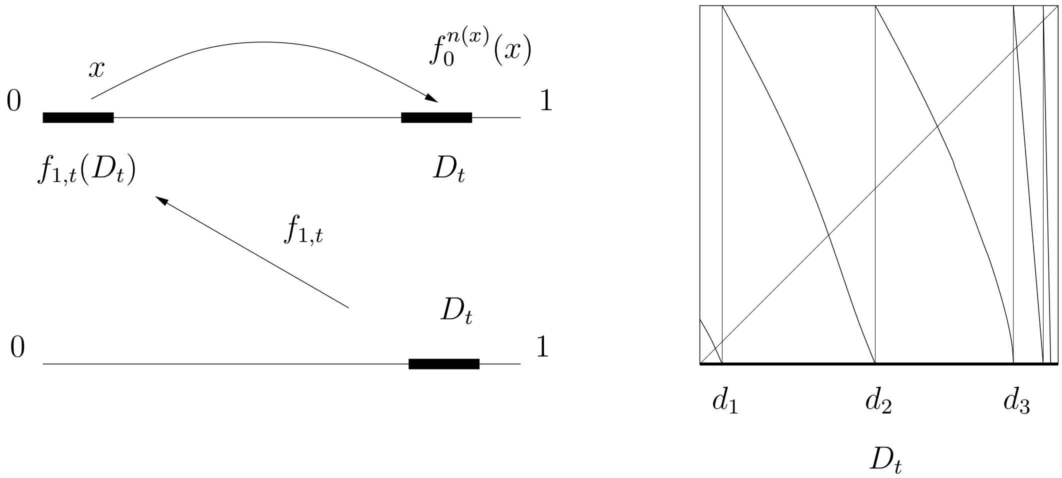

Figure 11: The return map $\mathcal{R}_t$

- $n(x) = n(d_i)$, for every $x \in D_{i,t}$;
- $n(d_{i+1}) = n(d_i) + 1$; and
- the restriction of $\mathcal{R}_t$ to each $D_{i,t}$ is strictly decreasing and, for $i \geq 2$, $\mathcal{R}_t(D_{i,t}) = D_t$.

Lemma 3.17. *Let d_i be a discontinuity of $\mathcal{R}_t$. Then there is a block ρ of Σ_{11}^+ of the form $[1, 0, \dots, 0, 1]$ with exactly two 1's such that $\Phi_{\rho,t}(d_i) = 0$.*

Proof. The definition of d_i and Equation (17) give

$$f_0^{n(d_i)}(f_{1,t}(d_i)) = 1 + \frac{\nu_t\, t}{\sigma} \qquad \text{and} \qquad f_0^{n(d_i)+1}(f_{1,t}(d_i)) = 1 + \frac{t}{\sigma}.$$

Finally, Equation (18) gives $f_{1,t}(f_0^{n(d_i)+1}(f_{1,t}(d_i))) = 0$. This composition corresponds to $\Phi_{\rho,t}$ for some a block $\rho = [1, 0, \dots, 0, 1]$ with $n(d_i)+1$ consecutive zeros as in the statement of the lemma, ending the proof of the lemma. $\square$

We consider the following parametrization φ_t of the interval D_t. Given $x \in D_t$, we write

$$\varphi_t \colon (1, \nu_t] \to D_t, \quad \varphi_t(\nu) = 1 + \nu\, t/\sigma.$$

Using this parametrization, we define the map

$$\Theta_t \colon (1, \nu_t] \to (1, \nu_t], \quad \Theta_t(\nu) = \varphi_t^{-1} \circ \mathcal{R}_t \circ \varphi_t(\nu).$$

Note that the maps Θ_t and $\mathcal{R}_t$ are conjugate. This formula gives

$$\mathcal{R}_t(1 + \nu\, t/\sigma) = 1 + \frac{\Theta_t(\nu)\, t}{\sigma}. \tag{19}$$

We consider the set of discontinuities of Θ_t

$$\Delta = \varphi_t^{-1}(\mathcal{D}) = \{\delta_i = \varphi_t^{-1}(d_i)), \quad i \in \mathbb{N}\}.$$

Note that $(\delta_i)_i$ is a decreasing sequence. We let $\Delta_i = [\delta_i, \delta_{i-1}) = \varphi_t^{-1}(D_i)$.

Proposition 3.18. *There is small $t_0 < 0$ such that for every $t \in [t_0, 0)$, every $k \in \mathbb{N}$, and every $\nu \in (1, \nu_t]$ such that $\Theta_t^i(\nu) \notin \Delta$ for all $i \in \{0, 1, \ldots, k-1\}$, it holds that*

$$|(\Theta_t^k)'(\nu)| > \frac{1}{3}\left(\frac{15}{12}\right)^k.$$

Using the conjugation φ_t above one immediately gets:

Corollary 3.19. *There is small $t_0 < 0$ such that for every $t \in [t_0, 0)$, every $k \in \mathbb{N}$, and every $x \in D_t$ such that $\mathcal{R}_t^i(x) \notin \mathcal{D}$ for all $i \in \{0, 1, \ldots, k-1\}$, it holds that*

$$|(\mathcal{R}_t^k)'(x)| \geq \frac{1}{3}\left(\frac{15}{12}\right)^k.$$

A straightforward consequence of this corollary is the following. Given an interval I denote by $|I|$ its length.

Corollary 3.20. *There is small $t_0 < 0$ such that for every $t \in [t_0, 0)$ and every interval I such that $\mathcal{R}_t^i(I) \cap \mathcal{D} = \emptyset$, for all $i \in \{0, 1, \ldots, k-1\}$, and $\mathcal{R}_t^k(I)$ is an interval with*

$$|\mathcal{R}_t^k(I)| \geq \frac{1}{3}\left(\frac{15}{12}\right)^k |I|.$$

Therefore, for every interval $I \subset D_t$, there is a first $k \geq 0$ such that $\mathcal{R}_t^k(I)$ contains a discontinuity.

Proof of Proposition 3.18. The first step in the proof of the proposition is to estimate the derivative of Θ_t:

Lemma 3.21. *For every $t < 0$ small enough*

$$|\Theta_t'(\nu)| = C_t(\nu)\,\frac{\Theta_t(\nu)}{\nu - 1}, \quad \textit{where} \quad C_t(\nu) = \frac{1 + \frac{\Theta_t(\nu)\,t}{\sigma}}{1 - (1-\nu)\,t}.$$

Therefore

$$\lim_{t \to 0^-} C_t(\nu) \to 1 \quad \textit{and} \quad \lim_{t \to 0^-} |\Theta_t'(\nu)| = \frac{\Theta_t(\nu)}{\nu - 1}.$$

Proof. Note first that

$$f_{1,t}(1 + (\nu/\sigma)\,t) = (1-\nu)\,t.$$

For the sake of simplicity of notation, we write $n(\nu) := n(\varphi_t(\nu))$. Then $f_0^{n(\nu)}(f_{1,t}(\varphi_t(\nu))) = \mathcal{R}_t(\varphi_t(\nu))$. From Equations (2) and (19), it follows

$$|(f_0^{n(\nu)})'(f_{1,t}(1 + (\nu/\sigma)\,t)| = |(f_0^{n(\nu)})'((1-\nu)\,t)| = \frac{e^{-n(\nu)}}{(1-\nu)^2\,t^2}\left(1 + \frac{\Theta_t(\nu)\,t}{\sigma}\right)^2. \qquad (20)$$

On the other hand, by Equations (1) and (19),

$$f_0^{n(\nu)}((1-\nu)\,t) = \frac{1}{1 - \left(1 - \frac{1}{(1-\nu)\,t}\right)e^{-n(\nu)}} = 1 + \frac{\Theta_t(\nu)\,t}{\sigma}.$$

This equation gives
$$e^{-n(\nu)} = \frac{\Theta_t(\nu)\, t^2\,(1-\nu)}{\sigma\left(1+\frac{\Theta_t(\nu)\,t}{\sigma}\right)\,((1-\nu)\,t-1)}.$$
Replacing this value of $e^{-n(\nu)}$ in Equation (20), one gets
$$|(f_0^{n(\nu)})'(f_{1,t}(1+(\nu/\sigma)\,t))| = \frac{\Theta_t(\nu)}{(\nu-1)}\,\frac{1+\frac{\Theta_t(\nu)\,t}{\sigma}}{\sigma\,(1-(1-\nu)\,t)}.$$
The lemma follows noting that the derivative of $f_{1,t}$ is constant and equal to $-\sigma$. $\square$

Given $\nu \in (1,\nu_t]$ write $\nu_0=\nu$ and define inductively $\nu_{i+1}=\Theta_t(\nu_i)$. Using Lemma 3.21 we write
$$|\Theta_t'(\nu_i)| = C_t(\nu_i)\,\frac{\nu_{i+1}}{\nu_i-1}.$$
Arguing inductively, we have
$$\begin{aligned} |(\Theta_t^k)'(\nu_0)| &= C_t(\nu_0)\,\frac{\nu_1}{\nu_0-1}\;C_t(\nu_1)\,\frac{\nu_2}{\nu_1-1}\;\cdots\;C_t(\nu_{k-1})\,\frac{\nu_k}{\nu_{k-1}-1} = \\ &= \frac{\nu_k}{\nu_0-1}\left(\prod_{i=0}^{k-1} C_t(\nu_i)\right)\left(\prod_{i=1}^{k-1}\frac{\nu_i}{\nu_i-1}\right). \end{aligned} \tag{21}$$
Recalling that $C_t(\nu)\to 1$ as $t\to 0^-$, we have that if t is close to 0^- then
$$C_t(\nu) > 5/6.$$
By Remark 3.16, if $t<0$ is close to 0 then $\nu_t<3$, thus $\nu_i\in(1,3]$. Since
$$\min_{x\in(1,3]}\frac{x}{x-1} = \frac{3}{2} \quad \text{thus} \quad \frac{\nu_i}{\nu_i-1} \geq \frac{3}{2}.$$
Finally, note that given any pair of numbers x and y in the interval $(1,3]$ one has
$$\frac{x}{y-1} \geq \frac{1}{2} \quad \text{thus} \quad \frac{\nu_k}{\nu_0-1} \geq \frac{1}{2}.$$
Putting together these facts, it follows from Equation (21) that
$$|(\Theta_t^k)'(\nu_0)| \geq \frac{1}{2}\left(\frac{5}{6}\right)^k\left(\frac{3}{2}\right)^{k-1} = \frac{1}{3}\left(\frac{5}{6}\right)^k\left(\frac{3}{2}\right)^k = \frac{1}{3}\left(\frac{15}{12}\right)^k.$$
This completes the proof of the proposition. $\square$

3.4.2 Proof of Proposition 3.15

To prove the proposition, note that $\mathcal{R}_t(x) = \Phi_{\varrho_x,t}(x)$ where ϱ_x is the block $[1,0,\dots,0]$ of Σ_{11}^+ with $n(x)\geq 1$ consecutive 0's. Also note that the concatenation of the blocks $[\varrho_x][\varrho_{\mathcal{R}_t(x)}]\dots[\varrho_{\mathcal{R}_t^k(x)}]$ is a block of Σ_{11}^+.

Lemma 3.22. *Consider an interval I of D_t such that $I, \mathcal{R}_t(I), \ldots, \mathcal{R}_t^k(I)$ do not contain discontinuities. Then there is a block $\varrho_k(I) := [1, \ldots, 0]$ of Σ_{11}^+ having exactly $k+1$ entries equal to 1 such that*

$$\mathcal{R}_t^{k+1}(x) = \Phi_{\varrho_k(I),t}(x), \quad \textit{for all } x \in I.$$

Proof. Suppose that I_0 is an interval contained in D_t such that $I_0 \cap \mathcal{D} = \emptyset$. Then $I_0 \subset D_{i,t}$ for some i. This implies that, for all $x \in I_0$, one has, for $n_i = n(x) = n(d_{i-1})$,

$$\mathcal{R}_t(x) = f_0^{n_i} \circ f_{1,t}(x) = \Phi_{\varrho[0],t}(x),$$

where $\varrho[0]$ is of form $\varrho[0] = [1, 0, \ldots, 0]$.

Consider $I = I_0$ an interval contained in D_t and write $I_j = \mathcal{R}_t^j(I_0)$. Suppose that $I_0, I_1, \ldots, I_k$ do not intersect the discontinuity set $\mathcal{D}$ of $\mathcal{R}_t$. Then, for each $j = 0, 1, \ldots, k$, we have $I_j \subset D_{t,i_j}$, for some i_j. Then, by the first step of the construction, for each j there is a block $\varrho[j] = [1, 0, \ldots 0]$ of Σ_{11}^+ such that, for all $x \in I_j = \mathcal{R}_t^j(I_0)$, one has that

$$\mathcal{R}_t(x) = \Phi_{\varrho[j],t}(x).$$

Therefore,

$$\mathcal{R}_t^{k+1}(x) = \Phi_{\varrho[k],t} \circ \Phi_{\varrho[k-1],t} \circ \cdots \circ \Phi_{\varrho[1],t} \circ \Phi_{\varrho[0],t}(x) = \Phi_{\varrho_k(I),t}(x),$$

for every $x \in I$. The block $\varrho_k(I)$ is the concatenation of $\varrho[0], \varrho[1], \ldots, \varrho[k]$ and this concatenation is a block of Σ_{11}^+. Finally, since we are concatenating $k+1$ blocks and each block has exactly one entry equal to 1, the block $\rho_k(I)$ satisfies the lemma. $\square$

Using Lemma 3.22, Corollary 3.20 can be read as follows:

Lemma 3.23. *For every interval $I \subset D_t$ there are $x \in I$ and a block ϱ of Σ_{11}^+ starting and ending by 1 such that $\Phi_{\varrho,t}(x) = 0$.*

Proof. By Corollary 3.20, there is a first k such that $\mathcal{R}_t^k(I)$ contains a discontinuity. Thus the intervals $I, \mathcal{R}_t(I), \ldots, \mathcal{R}_t^{k-1}(I)$ do not contain discontinuities. Lemma 3.22 gives a block $\varrho = [1, 0, \ldots, 0]$ of Σ_{11}^+ having k entries equal to 1 with

$$\mathcal{R}_t^k(x) = \Phi_{\varrho,t}(x), \quad \text{for all } x \in I.$$

By hypothesis, there is $\nu \in I$ with $\mathcal{R}_t^k(\nu) = \Phi_{\varrho,t}(\nu) = d_i$, for some discontinuity. By Lemma 3.17, $\Phi_{\rho,t}(d_i) = 0$ for some block $\rho = [1, 0, \ldots, 0, 1]$ with exactly two 1's. Consider the concatenation $[\tau] = [\varrho]\,[\rho]$, which is a block of Σ_{11}^+ starting and ending by 1, by construction $\Phi_{\tau,t}(\nu) = \Phi_{\rho,t} \circ \Phi_{\varrho,t}(\nu) = \Phi_{\rho,t}(d_i) = 0$. This completes the proof of the lemma. $\square$

End of the proof of Proposition 3.15. Note that given any interval $I \subset (0,1)$ then I contains some subinterval J satisfying one of the three possibilities: (i) $J \subset D_t$, (ii) $J \subset (0, 1 + (\nu_t\, t)/\sigma)$ (recall that $1 + (\nu_t\, t)/\sigma$ is the left extreme of D_t), and (iii) $J \subset (1 + t/\sigma, 1)$ (recall that $1 + t/\sigma$ is the right extreme of D_t).

First, if the interval J is contained in D_t the proposition follows from Lemma 3.23.

Second, if the interval J is in $(0, 1+(\nu_t\, t)/\sigma)$ there is some m such that $f_0^m(J) \cap D_t \neq \emptyset$. Then we apply the first case and we are done.

It remains to consider the case $J \subset (1+t/\sigma, 1)$. In this case, we have by Equation (16) and since $\nu_t \to e$ as $t \to 0^-$, that for every t close to 0^-,

$$f_{1,t} \circ f_0 \circ f_{1,t}(J) \subset (0, 1 + (2\,t\,e)/\sigma) \subset (0, 1 + (\nu_t\,t)/\sigma).$$

We are now in the second case and the result follows. This completes the proof of the proposition. $\square$

4 Dynamics of the model family (Dictionary $\mathfrak{F}_t$–F_t)

In this section, we state some basic dynamical properties of F_t. Recall the definition of F_t in Section 2 and note that if $X_0 = (x_0^s, x_0^c, x_0^u) \in R_i$, $i = 0, 1$, then

$$X_1 = (x_1^s, x_1^c, x_1^u) = F_t(X_0), \quad x_1^c = f_{i,t}(x_0^c). \tag{22}$$

Recall that $f_{0,t} = f$ and $f_{1,t}(x) = \sigma\,(1 - x) + t$.

Remark 4.1. For every t, $P = (0, 1, 0)$ and $Q = (0, 0, 0)$ are fixed saddles of F_t of indices 1 and 2, respectively. Moreover, we have that

$$\begin{aligned} W^s(P, F_t) &\supset [0,1] \times (0,1] \times \{0\}, \\ W^s(Q, F_t) &\supset [0,1] \times \{0\} \times \{0\}, \\ W^u(P, F_t) &\supset \{0\} \times \{1\} \times [0,1], \quad \text{and} \\ W^u(Q, F_t) &\supset \{0\} \times [0,1) \times [0,1]. \end{aligned}$$

Since $\{0\} \times \{1\} \times [5/6, 1] \subset R_1$ and $t = f_{1,t}(1)$, one has

$$\{3/4\} \times \{t\} \times [0, 1/2] \subset F_t(\{0\} \times \{1\} \times [5/6, 1]) \subset W^u(P, F_t),$$

we also have that, for $t > 0$, $X_t = (3/4, t, 0)$ is a transverse homoclinic point of P.

Moreover, for $t = 0$, $X_0 = (3/4, 0, 0)$ is a heteroclinic point ($X_0 \in W^u(P, F_0) \cap W^s(Q, F_0)$). Since $\{0\} \times (0,1) \times \{0\} \subset W^u(Q, F_t) \cap W^s(P, F_t)$, it follows that F_0 has a heterodimensional cycle associated to P and Q.

Similarly, for $t < 0$, the point $(3/4, 0, 0)$ is a transverse homoclinic point of Q.

We now translate some dynamical properties of the system $\mathfrak{F}_t$ to the diffeomorphism F_t. These properties follow in a standard way using the invariance by F_t of the x, y, and z-directions and from the uniform contraction in the x-direction and the uniform expansion in the z-direction. The proofs of properties (D1) and (D2) are omitted, for details we refer to [22, 23, 7].

Consider the maximal invariant set Γ_t of F_t in R, $\Gamma_t = \cap_{k \in \mathbb{Z}} F_t^k(R)$. To each point $X \in \Gamma_t$, we associate the sequence $\iota(X) = (\iota_j(X))_{j \in \mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$ defined by

$$\iota_j(X) = i \quad \text{if, and only if,} \quad F_t^j(X) \in R_i.$$

The sequence $\iota(X)$ is the *itinerary* of X. Since $F_t(R_1) \cap (R_1) = \emptyset$, the itinerary $\iota(X)$ does not have two consecutive 1's, so we have that $\iota(X) \in \Sigma_{11}$ for all $X \in \Gamma_t$.

D1) Let ϱ be an admissible periodic block of Σ_{11}^+ and define by ϱ_∞ the concatenation of ϱ with itself infinitely many times (which is a sequence of Σ_{11}). Suppose that $\Phi_{\varrho,t}(a) = a$. Then there is a periodic point (a^s, a, a^u) of F_t whose itinerary is ϱ_∞.

Given $\iota = (i_k)_k \in \Sigma_{11}$ a *central block* of ι is a block $\varrho = [i_{k^-}, \dots, i_0, \dots, i_{k^+}]$ for some $k^+ \geq 0$ and $k^- \leq 0$. We say that the point $A = (a^s, a, a^u)$ *has a ϱ-itinerary* if for all $\ell \in [k^-, k^+]$ one has $F_t^\ell(A) \in R_{i_\ell}$.

D2) Consider $\iota \in \Sigma_{11}$ and a central block ϱ of ι. Suppose that there is a point A having a ϱ-itinerary. Then there is a cube $C_\varrho(A) = I^s \times I^c \times I^u$ containing A such that every point of $C_\varrho(A)$ has a ϱ-itinerary and $|I^s| < (1/3)^{k^+}$ and $|I^u| < (1/3)^{-k^-}$.

In fact, given a point $A = (a^s, a^c, a^u)$ whose infinite forward itinerary is ι^+ then the points in $[0,1] \times \{a^c\} \times \{a^u\}$ have the same forward infinite itinerary ι^+. Similarly, if $A = (a^s, a^c, a^u)$ has infinite backward itinerary ι^- then the points in either $\{a^s\} \times \{a^c\} \times [0, 1/6]$, if $a^u \in [0, 1/6]$, or in $\{a^s\} \times \{a^c\} \times [5/6, 1]$, if $a^u \in [5/6, 1]$, have the same backward itinerary ι^-.

Given a block $\varrho = [i_0, i_1, \dots, i_k]$ of Σ_{11}^+, we say that ρ is an *initial sub-block of ϱ* if $\rho = [i_0, \dots, i_\ell]$ for some $\ell \leq k$.

D3) Let $\varrho = [i_0, \dots, i_k]$ be a block of Σ_{11}^+ such that, for every initial sub-block ρ of ϱ, one has $\Phi_{\rho,t}(1) \in [\vartheta, 1]$ ($\vartheta < 0$ as in Section 3.4).

1. Suppose that $\Phi_{\varrho,t}(1) > 0$. Then there is a transverse homoclinic point of P of the form $(a^s, \Phi_{\varrho,t}(1), 0)$, with $a^s \in [0,1]$, such that $\{a^s\} \times \{\Phi_{\varrho,t}(1)\} \times [0, 1/6]$ is contained in $W^u(P, F_t)$.
2. Suppose that $\Phi_{\varrho,t}(1) = 0$. Then F_t has a heterodimensional cycle associated to P and Q
3. If a point $X = (x^s, x^c, x^u) \in R$ is a (transverse) homoclinic point of P for F_t then there is a block ϱ of Σ_{11}^+ such that $x^c = \Phi_{\varrho,t}(1)$.

Proof of (D3). By definition of F_t, if $i_0 = 0$ there is a subinterval W of $\{0\} \times \{1\} \times [0, 1/6]$ consisting of points having a ϱ-itinerary. Similarly, if $i_0 = 1$ there is a sub-interval of $\{0\} \times \{1\} \times [5/6, 1]$ consisting of points having a ϱ-itinerary. In both cases, these segments are contained in $W^u(P, F_t)$ (recall Remark 4.1) and have length less than $(1/3)^{k+1}$. By definition, there is $a^s \in (0,1)$ with

$$\{a^s\} \times \{\Phi_{\varrho,t}(1)\} \times [0, 1/6] \subset F_t^{k+1}(W) \subset W^u(P, F_t), \quad k+1 \text{ is the length of } \varrho.$$

By Remark 4.1, $[0,1] \times (0,1] \times \{0\} \subset W^s(P, F_t)$, thus if $\Phi_{\varrho,t}(1) > 0$ we have that $(a^s, \Phi_{\varrho,t}(1), 0)$ is a transverse homoclinic point of P. This proves the first item of (D3). Item (3) of (D3) follows analogously and its proof is omitted.

The second item of (D3) follows noting that $[0,1] \times \{0\} \times \{0\}$ is contained in $W^s(Q, F_t)$. Thus if $\Phi_{\varrho,t}(1) = 0$ then $W^u(P, F_t) \cap W^s(Q, F_t) \neq \emptyset$. By Remark 4.1, $\{0\} \times (0,1) \times \{0\} \subset W^s(P, F_t) \cap W^u(Q, F_t)$, thus F_t has a heterodimensional cycle. $\square$

We now introduce some notations. Given points $x^s, x^u \in [0,1]$ and a (non-trivial) subinterval I of $[\vartheta, 1]$, define the following subsets of the cube R:

- *s-strip*, $S(I, x^u) = [0,1] \times I \times \{x^u\}$;
- *lower u-strip*, $S^-(x^s, I) = \{x^s\} \times I \times [0, 1/6]$;

- *upper u-strip,* $S^+(x^s, I) = \{x^s\} \times I \times [5/6, 1]$; and
- *complete u-strip,* $S^\pm(x^s, I) = S^+(x^s, I) \cup S^-(x^s, I)$.

For $|t|$ small, consider the subset Υ_t of $H(P, F_t)$ consisting of points of the form $(a^s, \Phi_{\varrho,t}(1), 0)$ such that $a^s \in [0, 1]$, $\Phi_{\varrho,t}(1) \in [0, 1]$, $\Phi_{\rho,t}(1)$ is in $[\vartheta, 1]$ for every initial sub-block ρ of ϱ, and $\{a^s\} \times \{\Phi_{\varrho,t}(1)\} \times [0, 1/6] \subset W^u(P, F_t)$.

As an immediate consequence of (D3) and of Proposition 3.7, we have the following:

Remark 4.2. Consider $|t|$ close to 0. Every s-strip $S(I, x^u)$, with $I \subset [t, 1]$ and $x^u \in [0, 1/6]$, intersects $W^u(P, F_t)$. In particular, if $S(I, x^u) \subset W^s(P, F_t)$ then $S(I, x^u)$ contains a transverse homoclinic point of P for F_t.

Remark 4.3. Note that the Hausdorff dimension $HD(H(P, F_t))$ of the homoclinic class $H(P, F_t)$ is at least the Hausdorff dimension of its projection in the central direction (see, for instance, [38, Chapter 4]). By Remark 4.2, the central projection of this class contains the interval $[t, 1]$ if $t \geq 0$ or $[0, 1]$ if $t < 0$. Thus $HD(H(P, F_t)) \geq 1$ for every small $|t|$.

By the definition of F_t, the images of u-strips $S^+(x^s, I)$ and $S^-(x^s, I)$ satisfy

$$\{x^+\} \times f_{1,t}(I) \times [0, 1/6] \subset F_t(S^+(x^s, I)), \quad \text{where } x^+ = 3/4 - \lambda_1 x^s,$$
$$\{x^-\} \times f_0(I) \times [0, 1] \subset F_t(S^-(x^s, I)), \quad \text{where } x^- = \lambda_0 x^s.$$

Therefore we have the following,

Remark 4.4. Consider small $t < 0$. Then for every point $x_0 \in [0, 1]$ and every interval $I \in [t, 1]$ one has that

- The image by F_t of an upper u-strip $S^+(x_0, I)$ contains a lower u-strip of the form $S^-(x_0^+, f_{1,t}(I))$;
- The image by F_t of the lower u-strip $S^-(x_0, I)$ contains a complete u-strip of the form $S^\pm(x_0^-, f_0(I))$.

Arguing inductively and applying Remark 4.4, we get the following:

D4) Consider a complete u-strip $S^\pm(x^s, I)$, where I is an interval of $\subset [\vartheta, 1]$. For every point $y \in I$ and every block $\varrho = [i_0, \dots, i_k]$ of Σ_{11}^+ such that, for every initial sub-block ρ of ϱ, the point $\Phi_{\rho,t}(y) \in [\vartheta, 1]$, there are a sub-interval J of I and a point $\bar{x}^s \in (0, 1)$ such that

$$F_t^{k+1}(S^\pm(x^s, I)) \supset \begin{cases} S^-(\bar{x}^s, \Phi_{\varrho,t}(J)), & \text{if } i_k = 1, \\ S^\pm(\bar{x}^s, \Phi_{\varrho,t}(J)), & \text{if } i_k = 0. \end{cases}$$

5 Hyperbolic dynamics for positive t

The goal of this section is to prove item A of Theorem 1.

5.1 Basic geometrical properties of the dynamics ($t \geq 0$)

Given small $t \geq 0$ consider the subset Ω_t of Γ_t consisting of non-wandering points of F_t, $\Omega_t = \Gamma_t \cap \Omega(F_t)$. We will prove that for small $t > 0$ the set Ω_t is hyperbolic (the disjoint union of the saddle Q and $H(P, F_t)$) and properly contained in Γ_t. Observe that, for every t, the set Γ_t contains the interval $\{0\} \times [0, 1] \times \{0\}$.

Theorem 5.1. *For every $t > 0$, the set $\Omega_t \setminus \{Q\}$ is hyperbolic and the restriction of F_t to $\Omega_t \setminus \{Q\}$ is conjugate to the shift $\varsigma\colon \Sigma_{11} \to \Sigma_{11}$. Moreover, $\Omega_t \setminus \{Q\} = H(P, F_t)$.*

Note that this result implies item A in Theorem 1. The proof of this theorem has two steps. The first one (which holds for $t \geq 0$) is topological and has two parts. In Section 5.1.1, we see that the itinerary map $\iota\colon \Gamma_t \to \Sigma_{11}$ in Section 4 is onto and defines a semi-conjugacy between the dynamics of F_t restricted to Γ_t and the shift map on Σ_{11}, i.e., $\iota \circ F_t = \varsigma \circ \iota$. In Section 5.1.2, we localize the non-wandering points of F_t.

In the second part (which holds just for $t > 0$), we see that hyperbolic estimates in Section 3.1 for the system $\mathfrak{F}_t$ imply the hyperbolicity of $H(P, F_t)$. Using the hyperbolic estimates, we prove that the restriction of ι to $H(P, F_t)$ is in fact a conjugacy with the shift on Σ_{11}, see Section 5.2. This also will imply that, for $t > 0$, the non-wandering set is the disjoint union of the homoclinic class of P and the saddle Q.

5.1.1 The semi-conjugacy

Lemma 5.2. *For every $t \geq 0$, the map $\iota\colon \Gamma_t \to \Sigma_{11}$ is onto and satisfies $\iota \circ F_t|_{\Gamma_t} = \varsigma \circ \iota$.*

We observe that this map is not injective. For instance, the segments $\{0\} \times [0, 1] \times \{0\}$ and $\{3/4\} \times [t, \sigma + t] \times \{0\}$ are contained in Γ_t, and each interval consists of points with the same itinerary. In fact, in the proof of the lemma, we show that points in the same central segment of Γ_t have the same itinerary, see Scholium 5.3.

Proof. The semi-conjugacy property $\iota \circ F_t|_{\Gamma_t} = \varsigma \circ \iota$ follows from the definition of ι. To see that ι is onto define *lower and upper u-segments* in $\tilde{R} = [0, 1]^3$ as segments of the form

$$\{a\} \times \{b\} \times [0, 1/6] \subset R_0, \qquad \{a\} \times \{b\} \times [5/6, 1] \subset R_1, \quad a \in [0, 1] \quad \text{and} \quad b \in [0, 1],$$

respectively. By construction, the image by F_t of a lower u-segment contains both a lower and an upper u-segment. Similarly, the image by F_t of an upper u-segment just contains a lower u-segment. Using that F_t is uniformly expanding in the z-direction (recall (D2)), given any u-segment V in R_k, $k = 0, 1$, and a sequence $(i_n) \in \Sigma_{11}^+$, with $i_0 = k$, there is a unique point in V whose forward itinerary is the one given by (i_n).

Using the facts above, it is easy to see (recall (D2)) that given a sequence $\mathcal{J}^+ = (i_n)$ the points whose forward itinerary is $\mathcal{J}^+$ form a s-strip $S^s(\mathcal{J}^+) = S([0, 1], c]) = [0, 1] \times [0, 1] \times \{c\}$, for some $c \in [0, 1]$.

Consider now backward itineraries $\mathcal{J}^- = (i_n)_{n \leq 0}$ and negative orbits, using the expansion of F_t^{-1} in the x-direction, one gets that given a sequence $\mathcal{J}^- = (i_n)_{n \leq 0}$ the points whose backward itinerary is $\mathcal{J}^-$ forms either an upper or a lower u-strip $S^u(\mathcal{J}^-)$ of the form $S^u(\mathcal{J}^-) = S(a, [0, 1])$, recall (D2) again.

Given now a complete itinerary $\mathcal{J} = (i_n) \in \Sigma_{11}$, we consider the s-strip $S^s(\mathcal{J}^+)$ associated to $\mathcal{J}^+$ and the u-strip $S^u(\mathcal{J}^-)$ associated to $\mathcal{J}^-$ and consider their intersection, which is a central segment consisting of points whose itinerary is $\mathcal{J}$. ☐

Scholium 5.3. *Two points* $X = (x^s, x^c, x^u)$ *and* $Y = (y^s, y^c, y^u)$ *have the same itinerary if, and only if,* $x^s = y^s$ *and* $x^u = y^u$.

5.1.2 Localization of the non-wandering set

Lemma 5.4. *Consider small* $t \geq 0$. *Then every non-wandering point* $Y = (y^s, y^c, y^u) \in \Gamma_t$ *with* $Y \neq Q$ *satisfies* $t \leq y^c \leq 1$.

Proof. Consider the auxiliary sub-cube $\tilde{R} = [0,1] \times [0,1] \times [0,1]$ of R. We claim that if a point $Y \in \Gamma_t$ has some forward iterate $F_t^k(Y)$ in $\tilde{R}$ then the future iterates of $F_t^k(Y)$ remains in $\tilde{R}$. Just note that, by Equation (22), if $Y_0 = (y_0^s, y_0^c, y_0^u) \in \tilde{R} \cap \Gamma_t$ then $Y_1 = (y_1^s, y_1^c, y_1^u)$ satisfies $y_1^c = f_{i,t}(y_0^c)$, $i = 0, 1$, and $f_{i,t}([0,1]) \subset [0,1]$ for $t \geq 0$.

It also follows from the definition of F_t and Equation (22), that every point $Y = (y^s, y^c, y^u)$ with $y^c > 1$ or with $-\delta \leq y^c < 0$ (δ as in the definition of the cube in the model family) has either some iterate outside R or has some iterate Y' in $\tilde{R}$. In this last case, by the observation above if $Y \in \Gamma_t$, then the whole forward iterate of Y' remains in $\tilde{R}$. This immediately implies that every non-wandering point $Y \in \Gamma_t$ belongs to $\tilde{R}$.

Note that the proof is complete for $t = 0$. Thus consider $t > 0$ and note that the previous arguments imply that we can focus on the points in the cube $\tilde{R}$.

For all $x \in (0, t]$, if $t > 0$ is small then $f_{0,t}^n(x) = f_0^n(x) > t$, for some n, and $f_{1,t}(x) = \sigma\,(1 - x) + t > t$. By Equation (22), every $Y_0 = (y_0^s, y_0^c, y_0^u) \in \tilde{R}$ with $y_0^c < t$ satisfies $y_i^c \geq t$, for all $i > i_0$ for some i_0. Therefore such a point cannot be non-wandering. Thus every $Y \in \Gamma_t \cap \Omega_t$ satisfies $y^c \in [t, 1]$.

To finish the proof of the lemma it remains to see that the only non-wandering point with central coordinate equal to 0 is Q. Observe that any point Y with central coordinate equal to 0 different from Q either has some iterate outside R (in this case we are done) or it has a first iterate Y_i in R_1. In this last case, $y_{i+1}^c > t$ and so $y_{i+k} > t$ for all $k \geq 1$. Thus the assertion follows from the arguments above. □

For each $t \geq 0$, consider the cubes $R_{0,t} = [0,1] \times [t,1] \times [0,1/6]$ and $R_{1,t} = R_1$ and let $R_t = R_{0,t} \cup R_1$. As a direct consequence of Lemma 5.4 we get

Lemma 5.5. *For* $t > 0$, $\Omega_t \subset \{Q\} \cup \Lambda_t$, *where* $\Lambda_t = \cap_{k \in \mathbb{Z}} F_t(R_t)$.

Note that to prove Theorem 5.1 it remains to check that Λ_t is hyperbolic and coincide with $H(P, F_t)$, and that the restriction of F_t to Λ_t is conjugate to the shift ς.

To prove the hyperbolicity of the set Λ_t it is enough to focus on the invariant y-direction: the x-direction is a uniformly contracting direction of F_t and the z-direction is a uniformly expanding direction of F_t. Since the dynamics of F_t in the y-direction is given by the system $\mathfrak{F}_t$, a consequence of Proposition 3.1 is that, for $t > 0$, the y-direction is a uniformly contraction for F_t (restricted to the set Λ_t.)

Finally, the conjugacy to the shift will follows using the hyperbolicity of Λ_t. Recall that Lemma 5.2 gives a semi-conjugacy between the dynamics of F_t on Γ_t and the shift ς defined on Σ_{11}, for $t \geq 0$.

5.2 Hyperbolicity of the homoclinic class $H(P, F_t)$ for $t > 0$

In this section, we conclude the proof of item A of Theorem 1 (i.e., the proof of Theorem 5.1).

Proposition 5.6. *For every $t > 0$ small enough, the set Λ_t is hyperbolic for F_t and equal to $H(P, F_t)$. Moreover, the map $F_t|_{H(P,F_t)}$ is conjugate to the shift (ς, Σ_{11}).*

Proof. By construction, the set Λ_t has a DF_t invariant splitting $T_{\Lambda_t}M = E^s \oplus E^c \oplus E^u$ where the bundles are one-dimensional and parallel to the axis x, y, and z, respectively. The first bundle is uniformly contracting and the last one is uniformly expanding. We now prove that the central bundle is also uniformly contracting, if $t > 0$. Equation (22) yields that the central dynamics of F_t is given by system $\mathfrak{F}_t$. Proposition 3.1 implies that this system is contracting. Therefore we have

Lemma 5.7. *For every point $X \in \Lambda_t$, $t > 0$, and every vector $v \in E^c_X$, one has*

$$\lim_{n\to\infty} |D_X F_t^n(v)| \to 0, \quad \textit{for every } t > 0 \textit{ small enough.}$$

To prove the hyperbolicity of the central bundle E^c is enough to see that there is $C > 0$ such that for all $n \geq 0$

$$|D_X F_t^n(v)| \leq C\,\frac{1}{2^n}, \quad \text{for every unitary vector } v \in E^c_X \text{ and } X \in \Lambda_t. \tag{23}$$

This claim follows from Lemma 5.7 using standard arguments that we recall for completeness. We first claim that there is m such that for all $X \in \Lambda_t$ and all unitary vector $v \in E^c_X$ there is $m(X) \leq m$ such that $|D_X F_t^{m(X)}(v)| \leq 1/2$. Indeed, by Lemma 5.7, for all $X \in \Lambda_t$ there is $n(X)$ such that $|D_X F_t^{n(X)}(v)| < 1/3$, for every unitary vector $v \in E^c_X$. Thus, for each $X \in \Lambda_t$ there is a neighborhood $B(X)$ of X such that

$$|D_Y F_t^{n(X)}(v)| < 1/2, \quad \text{for every } Y \in B(X) \text{ and every unitary vector } v \in E^c_Y.$$

It follows from the compactness of Λ_t the existence of $B(X_i)$, $i = 1, \dots, N_0$, such that $\Lambda_t \subset \cup_{i=1}^{N_0} B(X_i)$. Take

$$M_0 = \sup\{\|D_Y F_t^n | E^c\| \colon Y \in B(X_i), 1 \leq n \leq n(X_i), i = 1, \dots N_0\},$$

We fix $m_0 > 2$ and m such that

$$M_0\,\frac{1}{2^{m_0}} < \frac{1}{2} \qquad \text{and} \qquad m > m_0 \cdot \sup\{n(X_i) \colon i = 1, \dots, N_0\}. \tag{24}$$

Given any $X \in \Lambda_t$, there is $1 \leq i_1 \leq N_0$ such that $X \in B(X_{i_1})$. Consider $n_{i_1}, \dots, n_{i_k}, n_{i_{k+1}}$, where $n_{i_j} = n(X_{i_j})$, satisfying

a) $F_t^{n_{i_1}+\dots+n_{i_j}}(X) \in B(X_{i_{j+1}})$, $\quad 1 \leq j \leq k$; and

b) $n_{i_1} + \dots + n_{i_k} \leq m < n_{i_1} + \dots + n_{i_k} + n_{i_{k+1}}$.

Note that item (b) and the definition of m in Equation (24) imply that $k \geq m_0$. Then, for $L_j = n_{i_1} + \dots + n_{i_j}$, $1 \leq j \leq k+1$, we have

$$|D_X F_t^m(v)| = \|D_{F_t^{L_k}(X)} F_t^{m-L_k} | E^c\| \cdot \prod_{j=1}^{k} \|D_{F_t^{L_{j-1}}(X)} F_t^{n_{i_j}} | E^c\| < \frac{M_0}{2^k} \leq \frac{M_0}{2^{m_0}} < \frac{1}{2}, \tag{25}$$

for every $X \in \Lambda_t$ and every unitary vector $v \in E^c_X$.

Given $n \in \mathbb{N}$ and a point $X \in \Lambda_t$, we write $n = k\,m + r$, $r \in [0, m-1]$. Note that $k \geq n/m - 1$. Let $C' = \sup\{\|D_Y F_t^r|E^c\| : r = 1, \dots, m\}$. By Equation (25), we have that

$$|D_X F_t^n(v)| \leq \frac{1}{2^k}\,\|D_{F_t^{n-r}(X)} F_t^r|E^c\| \leq C'\,\frac{1}{2^{(n/m-1)}} \leq C\,\frac{1}{2^n}\,,$$

where $C = 2\,C'/2^{1/m}$. This concludes the proof of Equation (23). Thus, the proof of the hyperbolicity of Λ_t, $t > 0$, is complete.

Lemma 5.8. *The restriction $\iota|_{\Lambda_t} \colon \Lambda_t \to \Sigma_{11}$ is a bijection for $t > 0$ small enough.*

Proof. Lemma 5.2 claims that $\iota\colon \Lambda_t \to \Sigma_{11}$ is onto and Scholium 5.3 claims that two points $X = (x^s, x^c, x^u)$ and $Y = (y^s, y^c, y^u)$ in Λ_t having the same itinerary satisfy $x^s = y^s$ and $x^u = y^u$.

It is easy to prove that if two points X and Y as above, $X, Y \in \Lambda_t$, have the same itinerary then the whole central segment $I = \{x^s\} \times [x^c, y^c] \times \{x^u\}$ is contained in Λ_t and consists of points with the same itinerary. The contraction in the central direction implies that the length of $F_t^{-i}(I)$ increases exponentially. Since the central diameter of R is bounded this is not possible. The proof of the lemma is now complete. ☐

The fact that $\iota|_{\Lambda_t}$ is a topological conjugacy follows using standard arguments analogous to the proof of the conjugacy between the usual Smale's linear horseshoe and the complete shift of two symbols.

To finish the proof of the proposition, it remains to see that $\Lambda_t \subset H(P, F_t)$. The conjugacy with the shift implies that the transverse homoclinic points of P are dense in Λ_t. For this it is enough to observe that every sequence (different from the zero sequence) starting with a sequence of zeros and ending with a sequence of zeros corresponds to a transverse homoclinic point of P and that these sequences are dense in Σ_{11}. ☐

6 Generating bifurcations: explosion of dynamics

In this section, we prove item B of Theorem 1, which follows from the results for the system $\mathfrak{F}_0$ in Section 3.3. We first observe that arguing as in Section 5.2 and using the results in Section 3.2, one gets that every periodic point of $H(P, F_0)$ different from Q is hyperbolic and has index one. Note that (1) in item B, existence of a heterodimensional cycle, was obtained in Remark 4.1. Moreover, it is obvious that $H(Q, F_0)$ is trivial (see statement (2) in item B). Now let us recall that $\Gamma_0 = \cap_{k \in \mathbb{Z}} F_0^k(R)$.

Theorem 6.1. *Assume that $\sigma \in (a, b)$ is as in Proposition 3.12. Then $H(P, F_0) = \Gamma_0$ and thus $H(P, F_0)$ contains infinitely many central intervals.*

Using the semi-conjugacy in Section 5.1.1, this theorem implies (3) in item B of Theorem 1.

The inclusion $H(P, F_0) \subset \Gamma_0$ follows by construction recalling (3) in (D3) and that $\Phi_{\varrho,0}(1) \geq 0$ for every block ϱ of Σ_{11}^+. The main step of the proof of Theorem 6.1 is the following:

Lemma 6.2. *Let $A = (a^s, a^c, a^u) \in \Gamma_0$ with $a_c > 0$. Then there are sequences of points $A_n = (a^s, a^c, a^u_n)$, $A_n \to A$, and of positive numbers ϵ_n, $\epsilon_n \to 0$, such that the disks*

$$D_n = [a^s - \epsilon_n, a^s + \epsilon_n] \times [a^c - \epsilon_n, a^c + \epsilon_n] \times \{a^u_n\}$$

satisfy

- *$A_n \in D_n$ and $D_n \subset W^s(P, F_t)$,*
- *D_n intersects transversely $W^u(P, F_t)$.*

This lemma implies that every disk D_n contains a point X_n of $H(P, F_t)$. By construction, $X_n \to A$. Thus $A \in H(P, F_t)$. It is not difficult to check that the points of Γ_0 with positive central coordinate form a dense subset of Γ_0, thus it follows that $\Gamma_0 \subset H(P, F_0)$.

Proof. By replacing A by some iterate we can assume that $A \in R_0$ (and thus $a^u \in [0, 1/6]$). Consider the segment $I^u(A) = \{a^s\} \times \{a^c\} \times [0, 1/6]$. By (D2), the infinite backward itinerary of every point of $I^u(A)$ is the ones $\iota^-(A)$ of A.

Using the expansion in the z-direction and considering positive iterations of $I^u(A)$, we obtain a sequence $A_n = (a^s, a^c, a^u_n)$ with $F_0^n(A_n) = (\bar{a}^s_n, \bar{a}^c_n, 0)$, where $\bar{a}^s_n \in (0, 1)$ and $\bar{a}^c_n = \Phi_{\varrho,0}(a^c) > 0$ for some block of Σ^+_{11} (recall that $\Phi_{\varrho,0}(0, 1) \subset (0, 1)$). We consider small $\bar{\epsilon}_n$ such that

$$\hat{D}_n = [\bar{a}^s_n - \bar{\epsilon}_n, \bar{a}^s_n + \bar{\epsilon}_n] \times [\bar{a}^c_n - \bar{\epsilon}_n, \bar{a}^c_n + \bar{\epsilon}_n] \times \{0\} \subset W^s(P, F_0).$$

By construction, $F_0^{-n}(\hat{D}_n)$ contains a disk D_n as the one in the lemma.

We need to prove that D_n contains some transverse homoclinic point of P. Consider the stable segment $I^s(A_n) = [a^s - \epsilon_n, a^s + \epsilon_n] \times \{a^c\} \times \{a^u_n\} \subset D_n$ and its backwards iterates. Using the expansion of F_0^{-1} in the x-direction, we get a negative iterate m such that

$$F_0^m(I^s(A_n)) = [0, 1] \times \{\tilde{a}^c_m\} \times \{\tilde{a}^u_m\}, \quad \text{with } \tilde{a}^c_m > 0 \text{ and } \tilde{a}^u_m \in [0, 1/6].$$

This implies that the disk D_n contains a sub-disk $\tilde{D}^n$ such that $F_0^m(\tilde{D}_n)$ is a s-strip $S = S(I, x^u)$ in $W^s(P, F_t)$ with $I \subset (0, 1)$ and $x^u \in [0, 1/6]$. By Remark 4.2, the s-strip S contains a transverse homoclinic point of P, ending the proof of the lemma. □

This finishes the proof of the theorem. □

7 Dynamics for $t < 0$. Non-hyperbolic classes

In this section, we prove that for small $t < 0$ the homoclinic classes of P and Q coincide. In fact, arguing as in Section 5.1, it is not hard to prove that these classes are the maximal invariant set of F_t in R (we are not going to prove this fact). As a consequence, these homoclinic classes contains infinitely many central segments. We give a different proof of this fact in Section 7.1 (see Scholium 7.2).

7.1 $H(Q, F_t) \subset H(P, F_t)$

Theorem 7.1. *Assume that $\sigma \in (a, b)$ is as in Proposition 3.12 and take small $t < 0$. Then $H(Q, F_t) \subset H(P, F_t)$.*

The proof of this theorem just consists in a small modification of the arguments in the proof of Theorem 6.1, so we omit some details.

Proof. It is enough to see that every transverse homoclinic point X of Q is accumulated by points in $H(P, F_t)$. Replacing X by some iterate of it, we can assume that $X = (x^s, 0, 0)$ and that $F_t^{-1}(X) \notin [0, 1] \times \{0\} \times \{0\} = W^s_{loc}(Q, F_t)$. This implies that $\tilde{X} = F_t^{-1}(X) = (\tilde{x}^s, \tilde{x}^c, \tilde{x}^u)$ belongs to R_1 and $\tilde{x}^c = 1 - |t|/\sigma \in (0, 1)$. For each $\epsilon > 0$, consider the disk

$$V_\epsilon = [\tilde{x}^s - \epsilon, \tilde{x}^s + \epsilon] \times [\tilde{x}^c - \epsilon, \tilde{x}^c) \times \{\tilde{x}^u\} \subset W^s(P, F_t).$$

There is now $n < 0$ such that $F_t^n(V_\epsilon)$ contains a s-strip $S(f_0^n([\tilde{x}^c - \epsilon, \tilde{x}^c)), \bar{x}^u)$, where $f_0^n([\tilde{x}^c - \epsilon, \tilde{x}^c)) \subset (0, 1)$ and $\bar{x}^u$ is close to 0. By Remark 4.2, $S(f_0^n([\tilde{x}^c - \epsilon, \tilde{x}^c)), \bar{x}^u)$ intersects transversely $W^u(P, F_t)$. This implies that V_ϵ meets $H(P, F_t)$ for all $\epsilon > 0$. Hence $\tilde{X}$ (thus X) belongs to $H(P, F_t)$, ending the proof of the theorem. □

Scholium 7.2. *The arguments above imply that every point in $\{0\} \times (0, 1) \times \{0\}$ is contained in $H(P, F_t)$ for small $t < 0$. In fact, the same argument also works for central segments contained in $W^s(P, F_t) \cap W^u(Q, F_t)$ whose extremes are homoclinic points of Q and P. Note that there are infinitely many segments of that type.*

7.2 $H(P, F_t) \subset H(Q, F_t)$

Theorem 7.3. *For every $t < 0$ small enough, the homoclinic class $H(P, F_t)$ of P is contained in the homoclinic class $H(Q, F_t)$ of Q.*

Proof. To prove the theorem, consider a transverse homoclinic point X of P. Replacing X by some forward iterate of it, we can assume that $X = (x^s, x^c, 0)$, $x^s \in [0, 1]$ and $x^c > 0$. The definition of F_t and the geometry of the cycle imply that there is small $\epsilon > 0$ such that either

$$\{x^s\} \times [x^c - \epsilon, x^c) \times [0, 1/6] \subset W^u(Q, F_t), \text{ or}$$
$$\{x^s\} \times (x^c, x^c + \epsilon] \times [0, 1/6] \subset W^u(Q, F_t).$$

Assume that the first case (the other case is similar) holds. We will prove that, for every big n, the forward orbit of $W_n = \{x^s\} \times [x^c - 1/n, x^c) \times [0, 1/n]$ transversely meets $[0, 1] \times \{0\} \times \{0\} \subset W^s(Q, F_t)$. Therefore $W_n \cap H(Q, F_t) \neq \emptyset$. Since W_n can be taken arbitrarily small and close to X, this will conclude the proof of the theorem.

We need the following two lemmas:

Lemma 7.4. *Consider small $t < 0$. For every small $\epsilon > 0$ and every disk $W = \{a^s\} \times [a^c - \epsilon, a^c + \epsilon] \times [0, \epsilon]$, with $a^c > 0$, there is $m > 0$ such that $F_t^m(W)$ contains a complete u-strip $S^\pm(\bar{a}^s, I)$, where $I \subset (0, 1)$.*

Proof. To prove the lemma, we consider forward iterations of W by F_t. We need that some of these iterations contain a well located disk (i.e., a disk whose central coordinates are in $(0, 1)$). For this we need the following claim.

Claim 7.5. *Consider $t < 0$. Then there is $x_0^t \in [0,1]$ such that $(x_0^t, f_{1,t}(f_0(t)), 0)$ is a transverse homoclinic point of P and $\{x_0^t\} \times \{f_{1,t}(f_0(t))\} \times [0,1] \subset W^u(P, F_t)$, where $f_{1,t}(f_0(t)) \in (0,1)$.*

The lemma follows immediately from the claim considering forward iterates of W, noting that F_t expands in the z-direction and preserves the product structure and using the λ-lemma. More precisely, the λ-lemma implies that $F_t^m(W)$ accumulates to $W^u(P, F_t)$, therefore, for large m, $F_t^m(W)$ contains disks close to $\{x_0^t\} \times \{f_{1,t}(f_0(t))\} \times [0,1]$. Since $f_{1,t}(f_0(t)) = f_{1,t}(f_0(f_{1,t}(0))) \in (0,1)$ this provides the announced complete u-strip with $I \subset (0,1)$.

Proof of the claim. By Remark 4.1, $\{0\} \times \{1\} \times [0,1] \subset W^u(P, F_t)$. We have

$$\{3/4\} \times \{t\} \times [0, 1/6] \subset F_t(\{0\} \times \{1\} \times [5/6, 1]) \subset W^u(P, F_t).$$

Note that

$$\{\lambda_0\, 3/4\} \times \{f_0(t)\} \times [0,1] \subset F_t(\{3/4\} \times \{t\} \times [0, 1/6]) \subset W^u(P, F_t).$$

Finally,

$$\{x_0^t\} \times \{f_{1,t}(f_0(t))\} \times [0, 1/6] \subset F_t(\{\lambda_0\, 3/4\} \times \{f_0(t))\} \times [5/6, 1]) \subset W^u(P, F_t),$$

for some $x_0^t \in [0,1]$. Note that, if t is small enough,

$$f_{1,t}(f_0(t)) = \sigma\,(1 - f_0(t)) + t \in (0,1).$$

The claim follows noting that $[0,1] \times (0,1] \times \{0\}$ is contained in $W^s(P, F_t)$. $\square$

The proof of the lemma is now complete. $\square$

Lemma 7.6. *Consider small $t < 0$ and any complete u-strip $S = S^{\pm}(a, I)$, where $I \subset (0,1)$. Then there is ℓ such that $F_t^{\ell}(S)$ transversely meets $W^s_{loc}(Q, F_t) = [0,1] \times \{0\} \times \{0\}$.*

Proof. By Proposition 3.15 there are a point $x \in I$ and a block $\varrho = [i_0, i_1, \ldots, i_k]$ of Σ_{11}^+ such that $\Phi_{\varrho,t}(x) = 0$. Moreover, we can assume that $i_k = 0$ (recall that $f_0(0) = 0$). By shrinking I if necessary, condition (D4) implies that there is $\bar{a} \in (0,1)$ such that

$$F^{k+1}(S^{\pm}(a, I)) \supset \{\bar{a}\} \times \{\Phi_{\varrho,t}(I)\} \times [0,1] \supset \{\bar{a}\} \times \{0\} \times [0,1].$$

This concludes the proof of the lemma taking $\ell = k + 1$. $\square$

We are now ready to finish the proof the theorem. Consider W_n as above. By Lemma 7.4, there is a positive iterate F_t^m of W_n which contains a complete u-strip $S = S^{\pm}(\bar{x}^s, I)$ with $I \subset (0,1)$. By Lemma 7.6 there is ℓ such that $F_t^{\ell}(S)$ transversely meets $W^s_{loc}(Q, F_t)$. Therefore $F_t^{m+\ell}(W_n)$ transversely meets $W^s_{loc}(Q, F_t)$. This conclude the proof of the theorem. $\square$

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