HETERODIMENSIONAL TANGENCIES

L. J. DÍAZ
PUC-RIO

lodiaz@mat.puc-rio.br
www.mat.puc-rio.br/~lodiaz

JOINT WORK: A. NOGUEIRA (PUC-RIO)
E.R. PUJALS (IMPA)

$\boxed{\begin{array}{l} f : M \circlearrowleft \text{ DIFFEO} \\ p \text{ SADDLE OF } f \end{array}}$

HOMOCLINIC CLASS OF P :

$H(P, f)$

CLOSURE $(W^s(P) \pitchfork W^u(P))$

PROPERTIES :
- TRANSITIVE
  — dense ORBIT —
- PERIODIC POINTS OF THE SAME INDEX AS P ARE DENSE IN $H(P, f)$

index = $\dim E^u$

... IN GENERAL : $H(P, f)$ MAY FAIL TO BE HYPERBOLIC

EQUIVALENT DEFINITION:

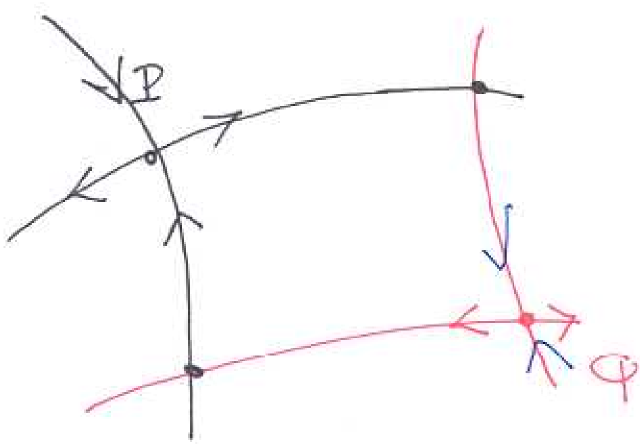
Homoclinic Relation

P, Q SADDLES

$P \underset{h}{\sim} Q$ HOMOCLINICALLY RELATED

$$W^s(P) \pitchfork W^u(Q) \neq \emptyset \neq W^u(P) \pitchfork W^s(P)$$

(P, q SAME INDEX)



H(p) = closure of q hom. related to p.

... Rmk dim $M \geq 3$

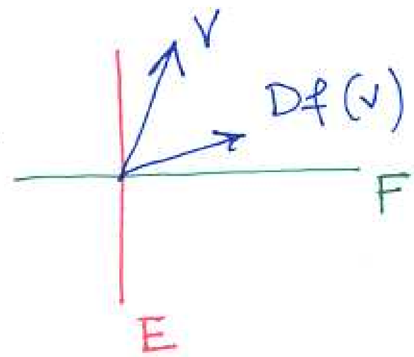
there are homoclinic classes which contain
(NON-HYPERBOLIC)

PERSISTENTLY saddles having different indices

$\hookrightarrow$ P_f SADDLE $\exists$ NBHOOD $\mathcal{U}$ of f (in $\mathrm{Diff}^r(M), r \geq 1$)

$\forall h \in \mathcal{U}$ $\exists$ Q_h (SADDLE)

- $\mathrm{index}(P_h) \neq \mathrm{index}(Q_h)$
- .. $Q_h \in H(P_h, h)$



DOMINATED SPLITTING

$T_\Lambda M = E \oplus F$

$u \in E$
$v \in F$
$|u| = |v| = 1$

$\exists \ell \geq 1$ $\|D_x f^\ell(u)\| \leq 2 \|D_x f^\ell(v)\|$

PARTIALLY HYPERBOLIC E OR F is HYPERBOLIC

Thm (Bonatti, D., Pujals)

GENERIC DICHOTOMY FOR HOMOCLINIC CLASSES

$\mathrm{Diff}^1(M)$., GENERICALLY (i.e. $\exists$ R residual of $\mathrm{Diff}^1(M)$ s.t)

$\forall f \in R$, $\forall$ saddle p_f of f

CR

(I) $H(p_f, f)$ HAS A DOMINATED SPLITTING

(dim M = 3 PARTIALLY HYPERBOLIC
M = 2 UNIF HYPERBOLIC)

(II) $H(p_f, f) \subset$ closure $\underbrace{\left\{ \begin{matrix} \text{SINKS} \\ \text{SOURCES} \end{matrix} \text{ OF } f \right\}}_{\text{THIS SET IS } \infty}$

(NEWHOUSE COEXISTENCE PHENOMENON)

Rmk: dim M ≥ 3 BOTH SITUATIONS OCCUR

(more examples in this talk)

dim M = 2 $\boxed{???}$

Conj: $H(p_f, f)$ HYPERB. OPEN AND DENSE

Thm (Bonatti, D) $\dim M = 3$.

Let $\mathcal{U} \subset \mathrm{Diff}^1(M^3)$ OPEN SUCH THAT

$\forall f \in \mathcal{U}$ $\exists p_f$ (depending cont on f)

SADDLE

- $H(p_f, f)$ does not admit any dominated splitting

  – PERSISTENTLY NONDOMINATED CLASS.

- • $H(p_f, f)$ contains saddles Q_f, R_f

  $\mathrm{Jac}(Q_f) > 1$ & $\mathrm{Jac}(R_f) < 1$

$\Rightarrow \exists \mathcal{R} \subset \mathcal{U}$ RESIDUAL $\forall f \in \mathcal{R}$

f has ∞-many MINIMAL SETS

CANTOR

(every point has a dense orbit)

$\boxed{\text{Rmk : There exist such an } \mathcal{U}.}$

$\boxed{\text{Goal}}$: give a natural method for constructing sets like $\mathcal{U}$.

(WAY) $\Longrightarrow$ $\boxed{\text{HETERODIM. TANGENCIES}}$

TYPICAL WAY FOR OBTAINING NON HYPERBOLIC CLASSES

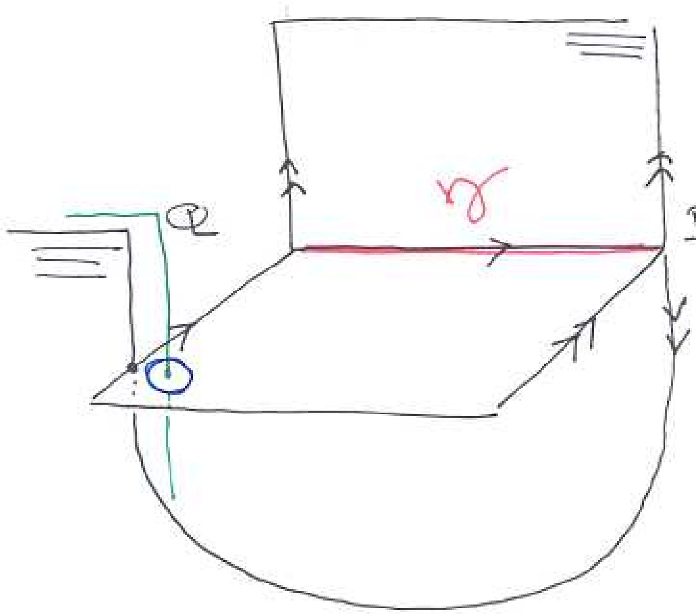
(PERSISTENTLY)

$\Rightarrow$ $\boxed{\text{Heterodim. Cycles}}$

SIMPLEST CYCLE P index 1
Q index 2

$W^s(P) \cap W^u(Q)$ transv

$W^u(P) \cap W^s(Q)$ quasi-transv



$\Rightarrow$ UNFOLDING...

CREATION OF HOMOCLINIC POINTS

OF P O

Q

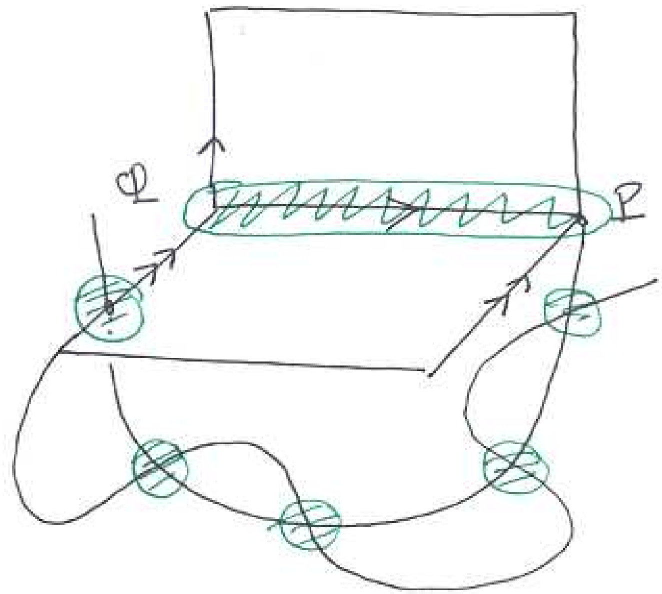
$\boxed{\text{MILD CONDITIONS on } \gamma}$ (small distortion)

$H(P,f) = H(Q,f)$ ROBUSTLY (D.)

$(\gamma \subset H(P,f), H(Q,f)$

⟼ examples verifying [BDI]
[BD]

Rmk: the proof only involves the dynamics of f on a $\boxed{\text{NEIGHBOURHOOD OF THE CYCLE}}$



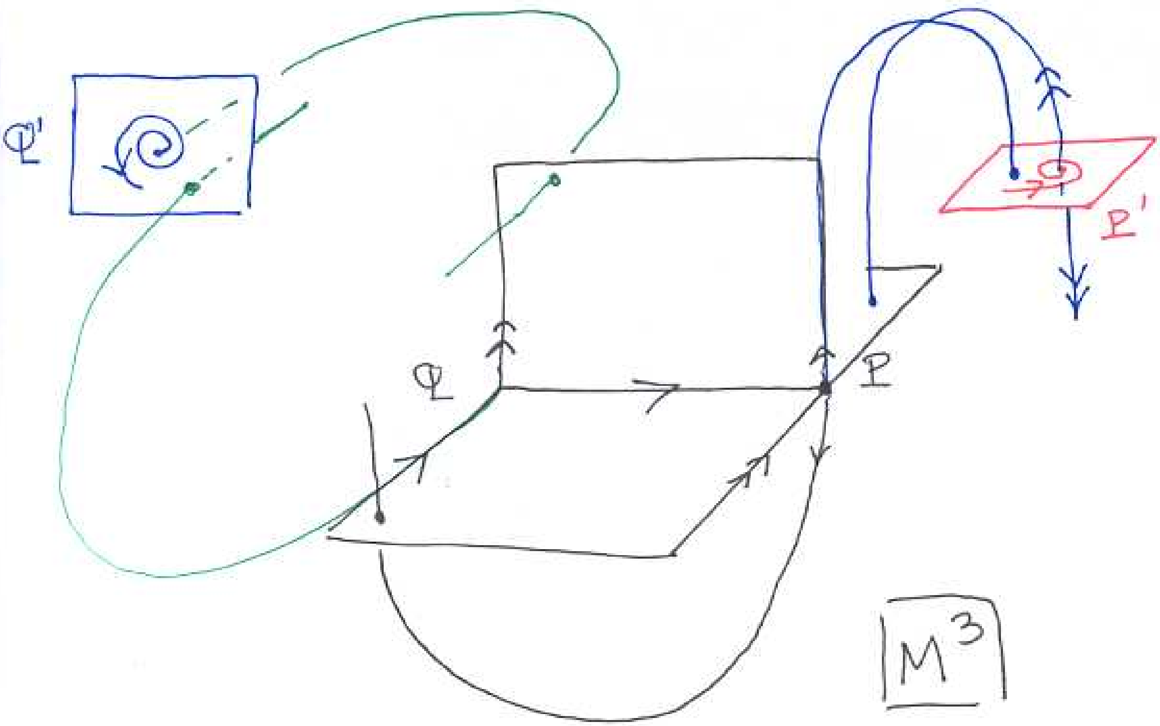
THERE ARE NO CONDITIONS ON THE DYNAMICS OUTSIDE THIS NEIGHBOURHOOD

⟼ EXTRA HYPOTHESES

P' WITH COMPLEX EIGENVALUES RELATED TO P

Q' WITH COMPLEX EIGENVALUES RELATED TO Q

THIS PREVENTS DOMINATION



if $E \oplus F$ dominated

P' complex eigenv $\Rightarrow$ dim $E = 2$.

Q' complex eigenv $\Rightarrow$ dim $F = 2$

?

$\Rightarrow$ non-dominated class.

NOW: A NATURAL WAY FOR OBTAINING NON DOMINATED CLASSES

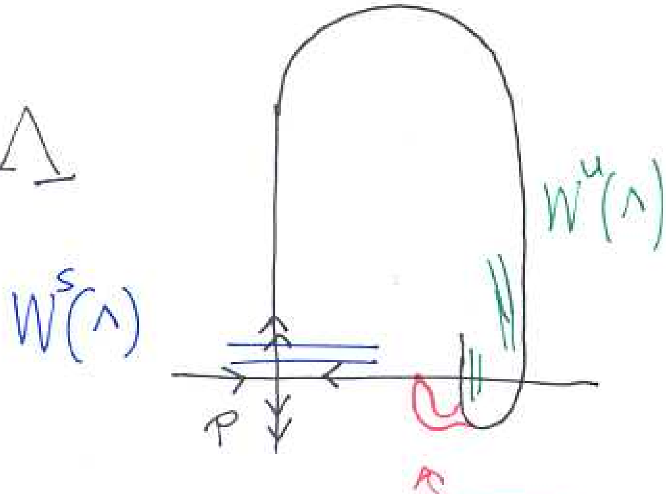
$\Downarrow$

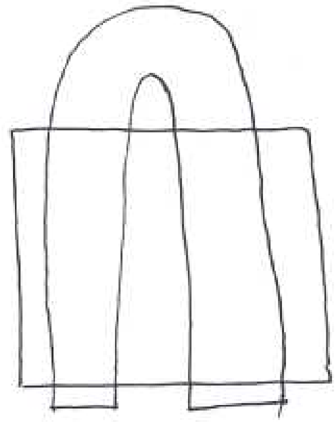
HETERODIM TANGENCES

PREVIOUS RESULT

(NEWHOUSE) $\boxed{\text{PERSISTENCE OF TANGENCIES}}$ $\mathrm{Diff}^2(M)$

Λ





$\boxed{\Lambda \text{ thick horseshoe}}$

↳ CREATION OF A HOMOCLINIC TANGENCY

(PERTURBATION)

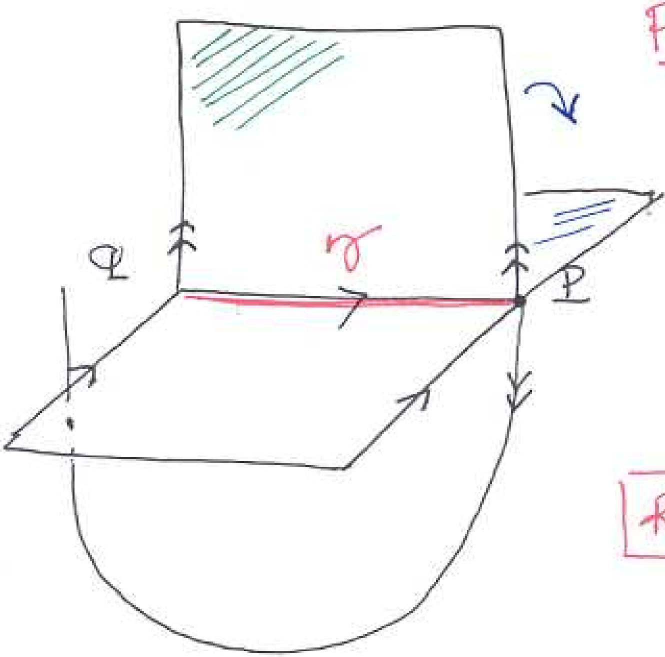
$\Longrightarrow$ PERSISTENCE OF TANGENCIES ASSOCIATED TO THE HOM. CLASS OF p (OR TO THE CONTINUATION OF Λ)

$\Downarrow$ (next)

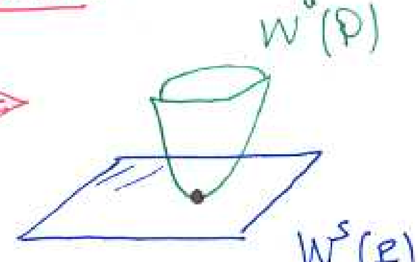
$\boxed{\infty\text{-many sinks}}$ (LOCALLY GENERICALLY)

MODEL

$\mathrm{Diff}^1(M)$



Perturbation $\Rightarrow$



(PARABOLIC CAT.)

$\boxed{\text{Restriction on } \gamma \text{ mild}}$

ROLES:

(NEWHOUSE)

THICK HORSESHOES

MODEL

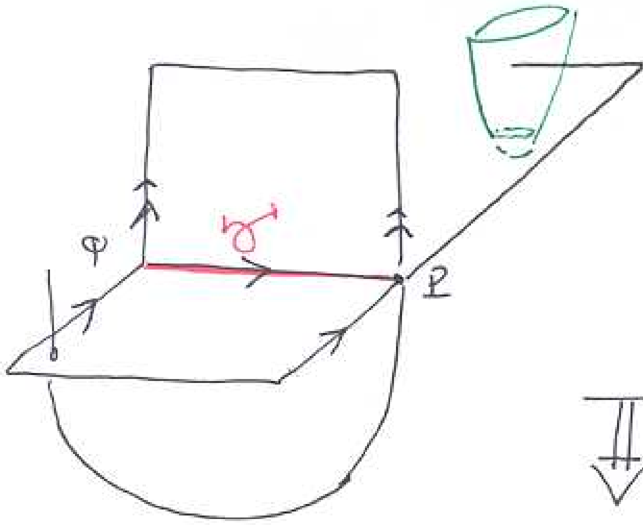
$H(P,f) = H(Q,f)$

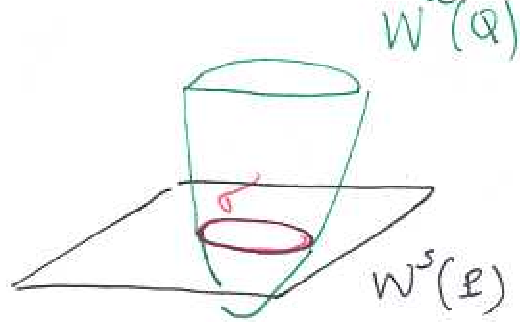
— BLENDERS —

HOMOCLINIC TANG.

HETERODIM. PARABOLIC TANGENCY.

UNFOLDING...





$\Downarrow$ Same arguments proving that

$\gamma \subset H(P,f)$

$\Downarrow$

$\sigma \subset H(P,f)$

$\longmapsto$ This prevents the existence of a dominated splitting

[PERSISTENT PROPERTY]

$\Rightarrow$

$$(\mathrm{BDP})\quad \left.\begin{array}{l} \exists\ \mathcal{U} \text{ open} \\ \mathcal{R} \text{ residual in } \mathcal{U} \\ f \in \overline{\mathcal{U}} \end{array}\right\} \forall g \in \mathcal{R} \quad H(P_g, g) \subseteq \mathrm{Clos}\left\{\begin{array}{l} \text{sinks} \\ \text{sources} \end{array}\right\}$$

$$\oplus\ \mathrm{Jac}(P) > 1 \ \&\ \mathrm{Jac}(Q) < 1 \underset{(\mathrm{BD})}{\Rightarrow} \subseteq \mathrm{Clos}\left\{\begin{array}{l} \text{minimal} \\ \text{Cantor} \end{array}\right\}$$

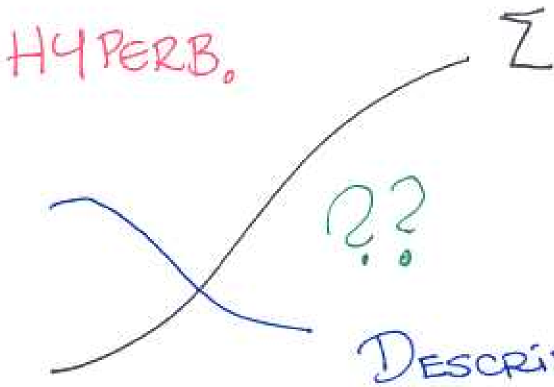
KEY:

The ideas in the model hold in much more general context

MORE MOTIVATIONS:

NEWHOUSE - PALIS 70's

TRANSITION FROM
HYPERBOLIC DYNAMICS
TO...
(boundary of hyperbol)



cod 1 | DIFFEOS. WITH A HOM. TANGENCY

DESCRIBE PARAM. FAMILIES CROSSIN Σ ($\pitchfork$).

Here

TRANSITION FROM

$\boxed{\text{PARTIALLY HYPERBOLIC DYNAMICS}}$

TWO BUNDLES $E^c \oplus E^u$

THREE BUNDLES $E^s \oplus E^c \oplus E^u$ (more interesting)

$\Downarrow \leftarrow$ $\boxed{\text{HETERODIM TANGENCIES}}$

NON DOMINATED DYNAMICS

STATEMENT OF THEOREM

(1) ROBUSTLY TRANSITIVE SETS (RTS)

U open

$$\Lambda_f(\bar{U}) = \bigcap_{\mathbb{Z}} f^i(\bar{U}) \quad \text{(MAXIMAL INVARIANT SET)}$$

$\Lambda_f(\bar{U})$ is RTS if $\exists\ \mathcal{U}_f$ OPEN IN $\mathrm{Diff}^1(M)$

S.T. $\forall\, g \in \mathcal{U}_f$

$\Lambda_g(\bar{U})$ is TRANSITIVE AND CONTAINED IN U.

INTERESTING CASES

- $\Lambda_f(\bar{U})$ NON-HYPERBOLIC
- $\bar{U} = M$ (ROB. TRANS. DIFF)

EXAMPLES $\bar{U} \neq M$: MAXIMAL INVARIANT SET IN A NEIGH. OF A HETERODIM CYCLE

$\boxed{\text{PROPERTIES OF RTS}}$ $\mathrm{Diff}^1(M)$

(1) PARTIAL HYPERBOLICITY (D. PUJALS URES)
(dim 3)

DOMINATED SPLITTING (BONATTI, D, PUJALS)
(dim ≥ 4)

HYPERBOLICITY (MAÑE)
(dim 2)

(2) (RELATIVE) $\boxed{\text{HOMOCLINIC CLASSES}}$ (CONNECTING LEMMA)

(3) OPEN AND DENSELY (BONATTI, D, PUJALS, ROCHA)
THE INDICES OF THE SADDLES OF $\Lambda_f(\bar{U})$ FORM AN INTERVAL IN $\mathbb{N}$ $\boxed{I(U)}$

(4) DENSELY THERE ARE HETERODIMENSIONAL CYCLES (i.e. p index k

(CONNECTING LEMMA)

q index $\neq k$

$W^s(p) \cap W^u(q) \neq \phi$

$W^u(p) \cap W^s(q) \neq$)

THM (D, NOGUEIRA, RUJALS)

$f \in \mathrm{Diff}^1(M^3)$

$f \in \mathcal{I}(U)$ [THERE ARE SADDLES OF INDICES 1 AND 2]

$p \in \Lambda_f(\bar{U})$ index 2

q index 1

WITH A HETERODIM TANGENCY [Rmk: The point of tangency does not belong to U]

$\Rightarrow \exists\, V$ open in $\mathrm{Diff}^1(M)$

$\exists\, R$ residual in V

$\forall\, h \in R \quad H(p,h) \subset \text{closure} \begin{Bmatrix} \text{SINKS} \\ \text{SOURCES} \end{Bmatrix}$

$\boxed{\infty\text{-MANY}}$

(+) $\begin{matrix} \mathrm{Jac}(P) > 1 \\ \mathrm{Jac}(Q) < 1 \end{matrix}$ OR $\begin{matrix} < 1 \\ > 1 \end{matrix}$

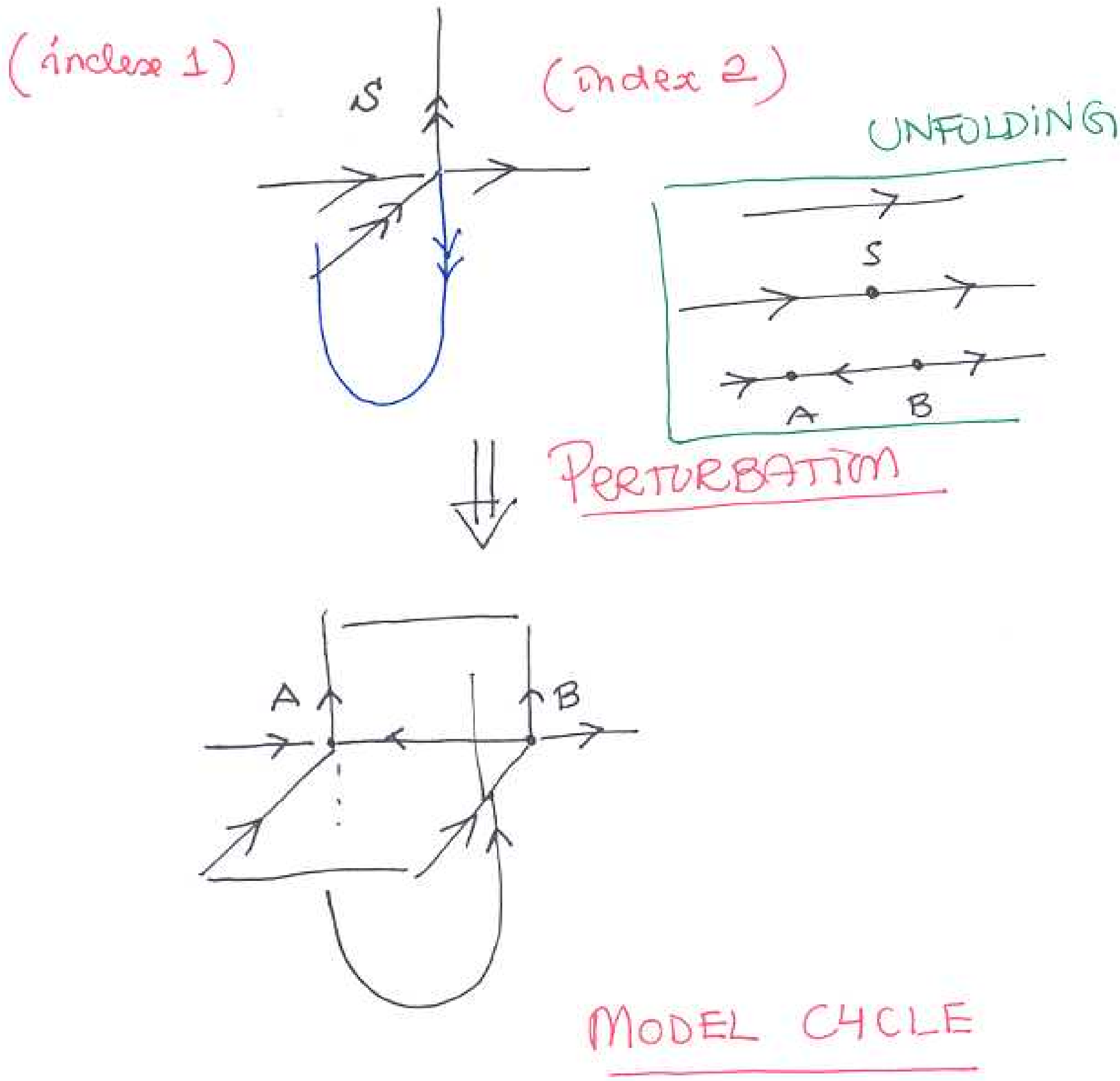
$\Rightarrow \quad H(p,h) \subset \text{closure} \begin{Bmatrix} \text{MINIMAL} \\ \text{CANTOR} \\ \text{SETS} \end{Bmatrix}$

$\boxed{\text{IDEA OF THE PROOF:}}$ THIS IMPLIES THAT (AFTER PERT) THERE IS A HETERODIMENSIONAL TANGECY ASSOCIATED TO A MODEL CYCLE.

STEPS

(1) TO OBTAIN A SADDLE-NODE HETERODIM CYCLE

S SADDLE-NODE $S \in \Lambda_f(U)$



...

SAME RESULT FOR HOMOCLINIC CLASSES